A TOPOLOGICAL APPROACH TO INDUCTION THEOREMS IN SPRINGER THEORY

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ABSTRACT. We give a self-contained account of a construction due to Rossmann, which lifts Springer's action of a Weyl group on the cohomology of a Springer fiber to an action on its homotopy type. We use this construction to produce a generalization of an "induction theorem" of Alvis and Lusztig, which relates the Springer representations attached to a reductive group to those attached to a Levi subgroup. Our generalization applies to more general centralizers and to representations of Weyl groups on mod p cohomology.

1. INTRODUCTION

The Springer correspondence is a connection between the geometry of the space of unipotent elements $\mathcal{N}$ in an algebraic group G and the representation theory of the Weyl group W associated to G. Representations of W are constructed on the cohomology of the fibers of Springer's resolution of singularities $\tilde{\mathcal{N}} \to \mathcal{N}$. It has been clear for a long time that representations can be constructed in this way over any ground ring, but the focus of the theory has overwhelmingly been on representations in characteristic zero. Recently, Juteau [J] began a systematic study of the modular version of this theory, extending the Fourier transform techniques used by Hotta, Kashiwara, and Brylinski [HK, Br] and studied the extent to which the Brauer-Cartan (or "cde") triangle can be seen in the geometry of $\mathcal{N}$.

This paper is another contribution to the modular gap in Springer theory. Our perspective is topological: where Juteau approaches modular representations via the techniques of Hotta, Kashiwara, and Brylinski, we follow the approach of Grothendieck, Brieskorn, and Slodowy. We use this construction to produce "induction theorems" which relate the Springer correspondences for G and for a subgroup of G.

1.1. **Modular representations via Rossmann's homotopy action.** A major part of this paper is a review of and advocacy for an approach to Springer theory due to Rossman [R]. Write $F_\nu \subset \tilde{\mathcal{N}}$ for the fiber of Springer's resolution over $\nu \in \mathcal{N}$. The existence of the W-action on $H^*(F_\nu)$, constructed first by Springer in 1976, came as a surprise especially because in simple examples it is easy to see that it cannot possibly be induced by an action of W on the space F_ν. For a long time it was unknown whether the action on $H^*(F_\nu)$ could be lifted to an action on the homotopy type of F_ν (though early progress was made in [KL]). Rossmann gave an affirmative solution to this problem in 1993.

In the first five sections of this paper we give a self-contained account of Rossmann's construction. We produce an action of the braid group B_W associated to W on a $\dim(G)$-dimensional manifold-with-partial-boundary $\mathcal{U} \times C_\nu$ which is homotopy equivalent to F_ν, and show that the kernel of the projection $B_W \to W$ acts by transformations homotopic to the identity. This is a little weaker than producing a space homotopy equivalent to F_ν on which W acts—this is still an open problem as far as I know. Nevertheless, the construction does give an action of W on every functorial homotopy invariant of F_ν, in particular, on mod p cohomology.

The spaces $\mathcal{U}$ and C_ν are easy to describe: $\mathcal{U}$ is the contractible universal cover of the set of regular elements in a small ball around $1 \in W\backslash\backslash T$, and C_ν is a regular neighborhood with boundary of F_ν in $\tilde{\mathcal{N}}$. Rossmann's construction follows a standard construction (originally due to Grothendieck, Brieskorn, and Slodowy [B, S]) of the Springer representations via nearby cycles. In that approach, the essential point is that a Springer fiber has the same cohomology as the appropriate Milnor fibers of the projection $f : G \to W\backslash\backslash T$. (This is expressed sheaf-theoretically as an isomorphism between the nearby cycles sheaf of f and the direct image of the constant sheaf under the Springer map $\tilde{\mathcal{N}} \to \mathcal{N}$. See [G] for a nice explanation.) The essential point of Rossmann's refined construction is that these Milnor fibers are actually homeomorphic to C_ν.

1.2. **Induction theorems.** Let $H \subset G$ be a connected reductive subgroup of G. Suppose that ν lies in H. Then we may study the Springer representations for the pair $\nu \in H$ and for the pair $\nu \in G$. A basic question is how the representations of W_H on $H^*(F_{\nu,H})$ and of W_G on $H^*(F_{\nu,G})$ are related. In the case where H is a Levi subgroup, Alvis and Lusztig [AL, L] showed that the characters of the W_G-modules $\mathrm{Ind}_{W_H}^{W_G} H^*(F_{\nu,H})$ and $H^*(F_{\nu,G})$ are the same.

Theorem 1.1 (Alvis-Lusztig, [AL, L]). *Let G be a reductive group, let L be a Levi subgroup, and let ν be a unipotent element of L. Let $W_L \subset W_G$ denote the Weyl groups of L and G, and let $F_{\nu,L}$ and $F_{\nu,G}$ denote the Springer fibers associated to ν in L and G. There is a filtration by W_G-submodules of*

$$\mathrm{Ind}_{W_L}^{W_G}(\bigoplus_{i \in \mathbb{Z}} H^i(F_{\nu,L}; \mathbb{Q}))$$

so that the associated graded module satisfies

$$\mathrm{gr}^k(\mathrm{Ind}_{W_L}^{W_G}(\bigoplus_{i \in \mathbb{Z}} H^i(F_{\nu,L}; \mathbb{Q}))) \cong H^k(F_{\nu,G}; \mathbb{Q}).$$

In this paper we use Rossmann's construction to give a topological explanation for this. Our proof is reminiscent of topological work on the "fundamental lemma" in the Langlands program [GKM, N]; we use localization in circle-equivariant cohomology to make comparisons. This technique, using $\mathbb{Z}/p$-equivariant localization instead, applies in the modular setting as well. We will show the following:

Theorem 1.2. *Let G be a reductive group, let $\nu \in G$ be a unipotent element, and let $g \in G$ be an element of order p (p a prime) which is contained in the connected component of the centralizer of ν. Let $Z \subset G$ be the centralizer of g in G; note that ν is contained in Z. Suppose, furthermore, that Z is connected (this is true, for instance, if G is simply connected.) Let W_Z and W_G denote the Weyl groups*

of Z and G, and let $F_{\nu,Z}$ and $F_{\nu,G}$ be the Springer fibers associated to ν in Z and G. There is a filtration by W_G-modules of

$$\mathrm{Ind}_{W_Z}^{W_G}(\bigoplus_{i\in\mathbb{Z}} H^i(F_{\nu,Z};\mathbb{F}_p))$$

whose associated graded module satisfies

$$\mathrm{gr}^k(\mathrm{Ind}_{W_Z}^{W_G}(\bigoplus_{i\in\mathbb{Z}} H^i(F_{\nu,Z};\mathbb{F}_p))) \cong H^k(F_{\nu,G};\mathbb{F}_p).$$

2. Springer glossary

For our purposes, Springer theory is the geometry of the spaces and maps in this diagram:

$$(1)\qquad \begin{array}{ccccc} \tilde{\mathcal{N}} & \overset{\tilde{j}}{\hookrightarrow} & \tilde{G} & \overset{\tilde{f}}{\longrightarrow} & T \\ {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{N} & \overset{j}{\hookrightarrow} & G & \overset{f}{\longrightarrow} & W\backslash\backslash T \end{array}$$

Let us briefly indicate what these spaces are; for more details we refer to [G].

- G is a fixed complex reductive Lie group.
- T is the "universal maximal torus." That is, T is an algebraic torus which is naturally identified with $B/[B,B]$ for every Borel subgroup $B \subset G$.
- W is the Weyl group of G, and $W\backslash\backslash T$ is the space of W-orbits in T.
- $\tilde{G}$ is the Grothendieck alteration of G: the space of pairs (x,B) where $B \subset G$ is a Borel subgroup and x is an element of B.
- $\mathcal{N}$ is the set of unipotent elements of G.
- $\tilde{\mathcal{N}}$ is the Springer resolution of $\mathcal{N}$: the space of pairs (x,B) where B is a Borel subgroup and x is an element of $[B,B]$.

The maps j and $\tilde{j}$ are just the inclusions. The maps p and q are given by $(x,B) \mapsto x$ and π is the quotient map. The map $\tilde{f}$ carries (x,B) to the image of x in $B/[B,B] \cong T$, and f is the unique map making the right-hand square commute. Alternatively, letting G act on itself by conjugation, one may define f to be an appropriate quotient map $G \to G\backslash\backslash G$ composed with a canonical isomorphism $W\backslash\backslash T \cong G\backslash\backslash G$. (Here $G\backslash\backslash G$ must denote the so-called GIT or geometric invariant theory quotient, which identifies conjugacy classes which cannot be separated by a polynomial class function on G.)

Each of the spaces $\tilde{G}, G, T$ and $W\backslash\backslash T$ have dense open subsets of *regular semisimple* elements; we write them as $\tilde{G}^{\mathrm{rss}} \subset \tilde{G}$, $G^{\mathrm{rss}} \subset G$, etc. Explicitly, we have:

- G^{rss} is the set of regular semisimple elements in G.
- $\tilde{G}^{\mathrm{rss}}$ is the set of pairs $(x,B) \in \tilde{G}$ where x is a regular semisimple element.
- T^{rss} is the set of regular semisimple elements in T.
- $(W\backslash\backslash T)^{\mathrm{rss}}$ is the set of W-orbits of regular semisimple elements in T.

Each of the maps $\tilde{f}, f, q, \pi$ restricts to a submersion on the set of regular elements, and the square

$$
\begin{array}{ccc}
\tilde{G}^{\mathrm{rss}} & \longrightarrow & T^{\mathrm{rss}} \\
\downarrow & & \downarrow \\
G^{\mathrm{rss}} & \longrightarrow & (W\backslash\backslash T)^{\mathrm{rss}}
\end{array}
$$

is cartesian. In particular, the vertical arrows are covering spaces with deck group W.

Remark 2.1. If X is a subset of one of $\tilde{G}, G, T$ or $W\backslash\backslash T$ we will write X^{rss} for the intersection of X with $\tilde{G}^{\mathrm{rss}}, G^{\mathrm{rss}}, T^{\mathrm{rss}}$ or $(W\backslash\backslash T)^{\mathrm{rss}}$, respectively.

3. Metric structures and a trivialization of $\tilde{f}$

Fix a maximal compact subgroup K of G, and endow G with a Hermitian metric which is left and right invariant under K. This structure provides, for each Borel subgroup, a canonical diffeomorphism $B \cong T \times [B, B]$. Indeed, $B \cap K$ is a maximal compact torus of B, and its Zariski closure in B gives a section of the principal $[B, B]$-bundle $B \to T$. Alternatively, the diffeomorphism $B \cong T \times [B, B]$ can be obtained by exponentiating the orthogonal decomposition of the Lie algebra $\mathfrak{b} = [\mathfrak{b}, \mathfrak{b}] \oplus [\mathfrak{b}, \mathfrak{b}]^{\perp}$. Let us write $[B, B]^{\perp}$ for the connected subgroup of B whose Lie algebra is $[\mathfrak{b}, \mathfrak{b}]^{\perp}$, or equivalently for the Zariski closure of $B \cap K$.

Denote the projection $B \to [B, B]$ by $\mathbf{u}_B$ and the projection $B \to [B, B]^{\perp}$ by $\mathbf{s}_B$ (the $\mathbf{u}$ and $\mathbf{s}$ stand for "unipotent" and "semisimple"). The $\mathbf{u}_B$ assemble to a trivialization of $\tilde{f}$. That is, the map

$$
\tilde{G} \to \tilde{\mathcal{N}} \times T : (x, B) \mapsto ((\mathbf{u}_B(x), B), \tilde{f}(x, B))
$$

is a diffeomorphism and commutes with the projections to T.

Remark 3.1. We will actually only make use of the fact that $\tilde{f} : \tilde{G} \to T$ is a submersion—this is a weaker statement because $\tilde{f}$ is not proper.

The Hermitian structure on G induces one on the universal torus T, via the isomorphism $T \cong [B, B]^{\perp}$; the metric is independent of B. Let us write $d(x, y)$ for the induced distance function on these spaces. Let us also define a metric on the space $W\backslash\backslash T$, by setting

$$
d(W \cdot x, W \cdot y) = \inf_{w \in W} d(w \cdot x, y)
$$

for the distance between the W-orbit of x and the W-orbit of y.

For each $\delta > 0$, write $B(\delta, T)$ for the open δ-ball centered at the identity in T, and write $B(\delta, W\backslash\backslash T)$ for the open δ-ball centered at the identity orbit in $W\backslash\backslash T$:

$$
B(\delta, T) := \{x \in T \mid d(x, 1) < \delta\},
$$
$$
B(\delta, W\backslash\backslash T) := \{W \cdot x \in W\backslash\backslash T \mid d(W \cdot x, 1) < \delta\}.
$$

Note that we have $d_T(x, 1) = d_{W\backslash\backslash T}(W \cdot x, 1)$ for all $x \in T$. Thus the image of $B(\delta, T)$ under π is $B(\delta, W\backslash\backslash T)$.

4. Some spaces near a unipotent element

Fix a unipotent element $\nu \in \mathcal{N}$, and let ϵ and δ be positive real numbers with $\delta \ll \epsilon \ll 1$. Define the following spaces:

- Let $F_\nu \subset \tilde{\mathcal{N}}$ be the Springer fiber over ν:

$$F_\nu := p^{-1}(\nu).$$

- Let $E_\nu(\epsilon, \delta) \subset G$ be the intersection of a closed ϵ-ball around ν with the inverse image of an open δ-ball around $1 \in W \backslash\backslash T$. If B is any Borel subgroup containing x, then we may write this as

$$E_\nu := \{x \in G \mid d(x, \nu) \leq \epsilon \text{ and } d(\mathbf{s}_B(x), 1) < \delta\}.$$

- Let $D_\nu(\epsilon, \delta) \subset \tilde{G}$ be the inverse image of E_ν under q:

$$D_\nu := \{(x, B) \in \tilde{G} \mid d(x, \nu) \leq \epsilon \text{ and } d(\mathbf{s}_B(x), 1) < \delta\}.$$

- Let $C_\nu(\epsilon) \subset \tilde{\mathcal{N}}$ be the intersection of D_ν with $\tilde{\mathcal{N}}$:

$$C_\nu := \{(x, B) \in \tilde{\mathcal{N}} \mid d(x, \nu) \leq \epsilon\}.$$

(Since $\tilde{\mathcal{N}} \to \mathcal{N}$ is a resolution of singularities, a standard argument shows that C_ν deformation retracts onto F_ν. See Proposition 4.1.)

E_ν and D_ν fit into the following commutative square:

$$
\begin{array}{ccc}
D_\nu & \longrightarrow & B(\delta, T) \\
\downarrow & & \downarrow \\
E_\nu & \longrightarrow & B(\delta, W \backslash\backslash T)
\end{array}
$$

This square becomes cartesian when the maps are restricted to the regular parts of each space:

$$
\begin{array}{ccc}
D_\nu^{\mathrm{rss}} & \longrightarrow & B(\delta, T)^{\mathrm{rss}} \\
\downarrow & & \downarrow \\
E_\nu^{\mathrm{rss}} & \longrightarrow & B(\delta, W \backslash\backslash T)^{\mathrm{rss}}
\end{array}
$$

4.1. **The action of the compact centralizer.** Write $K(\nu) = K \cap Z_G(\nu)$ for the maximal compact subgroup of the centralizer of ν which is contained in K. The distance from ν with respect to the hermitian metric on G is invariant under $K(\nu)$, and therefore $K(\nu)$ acts on the spaces $F_\nu, E_\nu(\epsilon, \delta), D_\nu(\epsilon, \delta)$, and $C_\nu(\epsilon)$. Moreover, the fibers of $D_\nu \to B(\delta, T)$ and $E_\nu \to B(\delta, W \backslash\backslash T)$ are preserved by this action. We also have

Proposition 4.1. *For ϵ sufficiently small, the inclusion $F_\nu \to C_\nu(\epsilon)$ admits a $K(\nu)$-equivariant deformation retraction.*

Proof. The function

$$\rho : \tilde{\mathcal{N}} \to \mathbb{R} : (x, B) \mapsto d(x, \nu)^2$$

that measures distance from ν is $K(\nu)$-invariant and proper, and has 0 as an isolated critical value. Thus, its gradient vector field integrates to a $K(\nu)$-equivariant deformation retraction from $C_\nu(\epsilon) = \rho^{-1}[0, \epsilon^2]$ to $F_\nu = \rho^{-1}(0)$, for ϵ sufficiently small. $\qquad\square$

5. The key lemma and its consequences

This section is devoted to the proof of the following lemma:

Lemma 5.1. *For some $\epsilon_0 > 0$ and $\delta_0 = \delta_0(\epsilon) > 0$, the map $D_\nu(\epsilon, \delta) \to B(\delta, T)$ is a trivial fiber bundle whenever $\epsilon < \epsilon_0$ and $\delta < \delta_0(\epsilon)$. Moreover, the trivialization $D_\nu(\epsilon, \delta) \cong C_\nu(\epsilon) \times B(\delta, T)$ can be chosen $K(\nu)$-equivariant.*

We will use the following very special cases of Thom's isotopy lemmas:

Theorem 5.2 (Thom's first isotopy lemma)**.** *Let M be a manifold with boundary, let $B \subset \mathbb{R}^n$ be an open ball, and let $u : M \to B$ be a smooth proper surjective mapping. Suppose that u is submersive on both the interior and the boundary of M. Then u is a topologically trivial fiber bundle.*

Theorem 5.3 (Thom's second isotopy lemma)**.** *Let M and N be manifolds with boundary, and let $v : N \to M$ be a proper map which exhibits N as a locally trivial fiber bundle over M. Let $B \subset \mathbb{R}^n$ be an open ball, and let $u : M \to B$ be a smooth proper surjective mapping. Suppose that u is submersive on both the interior and the boundary of M, so that per the first isotopy lemma, u admits a trivialization over B. Then the map $v : N \to M$ admits a trivialization over B: for a given point $b \in B$, and letting w denote the natural map $(u \circ v)^{-1}(b) \to u^{-1}(b)$, there are homeomorphisms making the following diagram commute:*

$$
\begin{array}{ccc}
N & \xrightarrow{\ \cong\ } & B \times (u \circ v)^{-1}(b) \\
{\scriptstyle v}\downarrow & & \downarrow{\scriptstyle 1_B \times w} \\
M & \xrightarrow[\ \cong\ ]{} & B \times u^{-1}(b)
\end{array}
$$

Furthermore, all maps in this square commute with the projections to B.

Proofs of the full isotopy lemmas may be found in [M], however, we remark that the special cases we consider here may be proved by much more elementary means.

Proof of Lemma 5.1. Let $D_\nu^\circ(\epsilon, \delta) \subset D_\nu(\epsilon, \delta)$ be the open set of those $(x, B) \in D_\nu(\epsilon, \delta)$ with $d(x, \nu) < \epsilon$, and let $\partial D_\nu(\epsilon, \delta) \subset D_\nu(\epsilon, \delta)$ denote the closed complement of those (x, B) with $d(x, \nu) = \epsilon$. Define $C_\nu^\circ(\epsilon) \subset C_\nu(\epsilon)$ and $\partial C_\nu(\epsilon)$ similarly. Let ρ denote the function $\tilde{G} \to \mathbb{R}$ defined by

$$\rho(x, B) = d(x, \nu)^2.$$

Since ρ and $\rho|_{\tilde{\mathcal{N}}}$ are real algebraic and smooth, their critical values are isolated by Sard's theorem. It follows that there exists a positive real number ϵ_0 such that neither ρ nor $\rho|_{\tilde{\mathcal{N}}}$ have critical values in the open interval $(0, \epsilon_0)$. In particular, $\partial D_\nu(\epsilon, \delta)$ (and $\partial C_\nu(\epsilon)$) is a manifold for every ϵ in $(0, \epsilon_0)$.

By the first isotopy lemma, the fact that $D_\nu(\epsilon, \delta) \to B(\delta, T)$ is a trivial fiber bundle is a consequence the following claim: there exists δ_0 such that the projections $D_\nu^\circ(\epsilon, \delta) \to B_\delta(T)$ and $\partial D_\nu(\epsilon, \delta) \to B_\delta(T)$ are submersions for all $\epsilon < \epsilon_0$ and $\delta < \delta_0$. Furthermore, applying Thom's second isotopy lemma to the case where $M = D_\nu(\epsilon, \delta)$, $N = K(\nu) \times D_\nu(\epsilon, \delta)$, and $v : N \to M$ is the action map gives a $K(\nu)$-equivariant trivialization. Thus, let us prove the claim.

Since $q : \tilde{G} \to T$ is a submersion (by Section 3) and $D_\nu^\circ(\epsilon, \delta)$ is open in $\tilde{G}$, the map $D_\nu^\circ(\epsilon, \delta) \to B(\delta, T)$ is also a submersion for any positive ϵ and δ. We are left with

showing that $q|_{\partial D_\nu(\epsilon,\delta)}$ is a submersion for $\epsilon < \epsilon_0$ and δ sufficiently small. To show that $q|_{\partial D_\nu(\epsilon,\delta)}$ does not have critical values in a sufficiently small neighborhood of $1 \in T$, it suffices to show (since $q|_{\partial D_\nu(\epsilon,\delta)}$ is proper and the set of critical values is closed) that 0 itself is not critical—i.e., that at any point $(x, B) \in \tilde{\mathcal{N}} = q^{-1}(0)$, the vertical tangent space to q at (x, B) and the tangent space to $\partial D_\nu(\epsilon,\delta)$ at (x, B) are transverse.

Let us call these tangent spaces T_1 and T_2, respectively. Since $\partial D_\nu(\epsilon,\delta)$ is of real codimension one in $\tilde{G}$, it is equivalent to showing that T_2 does not contain T_1. But T_1 is the tangent space to $\tilde{\mathcal{N}}$ and T_2 is the kernel of $d\rho_{x,B}$, so if $T_1 \subset T_2$, then (x, B) is a critical point of $\rho|_{\tilde{\mathcal{N}}}$, which contradicts the assumption that $\epsilon < \epsilon_0$. This completes the proof. $\square$

We immediately have the following corollaries:

Corollary 5.4. *If ϵ_0 and δ_0 are as in Lemma 5.1, then for $\epsilon < \epsilon_0$ and $\delta < \delta_0(\epsilon)$, the map $E_\nu(\epsilon,\delta)^{\mathrm{rss}} \to B(\delta, W\backslash\backslash T)^{\mathrm{rss}}$ is a $K(\nu)$-equivariant locally trivial fiber bundle.*

In particular, the Borel construction of $E_\nu(\epsilon,\delta)$ with respect to any subgroup $K' \subset K(\nu)$ may be exhibited as a fiber bundle over $B(\delta, W\backslash\backslash T)^{\mathrm{rss}}$, whose fibers are Borel constructions of the K'-space $C_\nu(\epsilon)$. Indeed, letting $\mathrm{E}K'$ denote a contractible space on which K' acts freely, the map $K'\backslash(E_\nu(\epsilon,\delta) \times \mathrm{E}K') \to B(\delta, W\backslash\backslash T)^{\mathrm{rss}}$ that takes the K'-orbit of (x, y) to $f(x)$ is well defined and provides such a fibration. We will use this fact in the next section.

Corollary 5.5. *Let A be a commutative ring. With ϵ and δ as above, let ϕ denote the map $E_\nu^{\mathrm{rss}} \to B(\delta, W\backslash\backslash T)^{\mathrm{rss}}$. The higher direct image sheaves $R^i\phi_*A$ are locally constant on $B(\delta, W\backslash\backslash T)^{\mathrm{rss}}$. Moreover, the stalks are isomorphic to the cohomology sheaves $H^i(F_\nu; A)$ with coefficients in A, and the monodromy action $\pi_1(B(\delta, W\backslash\backslash T)^{\mathrm{rss}}) \to \mathrm{Aut}(H^i(F_\nu; A))$ factors through a quotient $\pi_1(B(\delta, W\backslash\backslash T)^{\mathrm{rss}}) \to W$.*

It is shown in [H] that when $A = \mathbb{Q}_\ell$, this action of W on $H^i(F_\nu)$ coincides with the original action of Springer after tensoring with the sign representation of W.

5.1. **Weyl group action on Springer fibers at the level of homotopy.** Let us indicate how this result lifts Springer's action of W on the cohomology of F_ν to an action of W on the homotopy type of F_ν. Let $\mathcal{U}$ be the universal cover of $B_\delta(W\backslash\backslash T)^{\mathrm{rss}}$ (equivalently: of $B_\delta(T)^{\mathrm{rss}}$); it is well known that $\mathcal{U}$ is contractible and that the deck group of $\mathcal{U}$ is B_W, the braid group associated to W (this was finally proved for all reflection groups in [D]). It follows that B_W acts on the pullback $E_\nu^{\mathrm{rss}} \times_{B_\delta(W\backslash\backslash T)^{\mathrm{rss}}} \mathcal{U}$ of E_ν^{rss} to $\mathcal{U}$. In fact, it acts freely. The lemma provides a homeomorphism between this pullback and $C_\nu \times \mathcal{U}$, and by Proposition 4.1, this space is homotopy equivalent to the Springer fiber F_ν. To complete the construction, we need to check that the subgroup B'_W of "pure braids" (i.e., the kernel of the projection $B_W \to W$) acts on $\mathcal{U} \times C_\nu$ by transformations homotopic to the identity. This also follows from the lemma: $C_\nu \times \mathcal{U}/B'_W$ is trivial over $\mathcal{U}/B'_W \cong B_\delta(T)^{\mathrm{rss}}$.

6. INDUCTION THEOREMS

In this section we will prove Theorems 1.1 and 1.2. To do this we will need to consider the spaces F, E, D, C defined in Section 4 for different reductive groups G. To keep them straight we need to complicate the notation introduced in Sections 2 and 4 with subscripts.

- Write $T_G, W_G, \mathcal{N}_G$, and $\tilde{\mathcal{N}}_G$ for the universal torus, unipotent cone, and Springer resolution associated to the complex connected reductive algebraic group G.
- Write $F_{\nu,G}, E_{\nu,G}(\epsilon,\delta), D_{\nu,G}(\epsilon,\delta)$, and $C_{\nu,G}(\epsilon)$ for the spaces associated to G defined in Section 4.
- If G is endowed with a hermitian metric, write K_G for the maximal compact which preserves it. Write $K_G(\nu)$ for the intersection of K_G with the centralizer of the unipotent element ν.

Proof of Theorem 1.1. Since any maximal torus of L is also a maximal torus of G, we may identify the universal tori T_L and T_G, so we will omit the subscript. For appropriate ϵ and δ, $\delta \ll \epsilon \ll 1$, we have by Corollary 5.4 a commutative square

$$
\begin{array}{ccc}
E_{\nu,L}(\epsilon,\delta)^{\mathrm{rss}} & \longrightarrow & E_{\nu,G}(\epsilon,\delta)^{\mathrm{rss}} \\
\downarrow & & \downarrow \\
B(\delta, W_L \backslash\backslash T)^{\mathrm{rss}} & \longrightarrow & B(\delta, W_G \backslash\backslash T)^{\mathrm{rss}}
\end{array}
$$

(Let us fix ϵ and δ, and for the rest of the proof omit them from the notation for $E_{\nu,L}, E_{\nu,G}$.) Let $\mathrm{U}(1) \subset G$ be a compact circle whose centralizer in G is L. Then $\mathrm{U}(1) \subset K_G(\nu)$ and the $\mathrm{U}(1)$-fixed points $E_{\nu,G}^{\mathrm{U}(1)}$ coincide with $E_{\nu,L}$. We will now deduce the theorem by standard "localization" techniques in $\mathrm{U}(1)$-equivariant cohomology, and the degeneration of the Borel spectral sequence.

   We may replace the top row of the diagram above by the Borel constructions with respect to $\mathrm{U}(1)$. That is, letting $X_{h\mathrm{U}(1)}$ denote the Borel construction of a $\mathrm{U}(1)$-space X, we have

$$
\begin{array}{ccc}
(E_{\nu,L}^{\mathrm{rss}})_{h\mathrm{U}(1)} & \longrightarrow & (E_{\nu,G}^{\mathrm{rss}})_{h\mathrm{U}(1)} \\
\downarrow & & \downarrow \\
B(\delta, W_L \backslash\backslash T)^{\mathrm{rss}} & \longrightarrow & B(\delta, W_G \backslash\backslash T)^{\mathrm{rss}}
\end{array}
$$

Let us denote the vertical maps by χ_L and χ_G, and the bottom horizontal map by ι. Define a locally constant sheaf $\mathcal{F}_L$ on $B(\delta, W_L \backslash\backslash T)^{\mathrm{rss}}$ and $\mathcal{F}_G$ on $B(\delta, W_G \backslash\backslash T)^{\mathrm{rss}}$ by

$$
\mathcal{F}_L = \bigoplus_{i \in \mathbb{Z}} R^i \chi_{L*} \mathbb{Q},
$$

$$
\mathcal{F}_G = \bigoplus_{i \in \mathbb{Z}} R^i \chi_{G*} \mathbb{Q}.
$$

The fibers of $\mathcal{F}_L$ (resp. $\mathcal{F}_G$) are isomorphic to the equivariant cohomology groups $H^*_{\mathrm{U}(1)}(F_{\nu,L}; \mathbb{Q})$ (resp. $H^*_{\mathrm{U}(1)}(F_{\nu,G}; \mathbb{Q})$). Fix an isomorphism of graded rings $H^*_{\mathrm{U}(1)}(pt; \mathbb{Q}) \cong \mathbb{Q}[t]$, where t has degree 2; then $\mathcal{F}_L$ and $\mathcal{F}_G$ are sheaves of $\mathbb{Q}[t]$-modules. Since $E_{\nu,L}^{\mathrm{rss}}$ is the $\mathrm{U}(1)$-fixed set in $E_{\nu,G}^{\mathrm{rss}}$, the localization theorem for $\mathrm{U}(1)$-equivariant cohomology (see e.g. [Q, Theorem 4.4]) provides an isomorphism of graded $\mathbb{Q}[t, t^{-1}]$-modules

$$
\mathcal{F}_G \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, t^{-1}] \cong \iota_* \mathcal{F}_L \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, t^{-1}].
$$

Translating this into the language of W_G-modules, we have

$$H^*_{\mathrm{U}(1)}(F_{\nu,G},\mathbb{Q}) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t,t^{-1}] \cong \mathrm{Ind}^{W_G}_{W_L} H^*_{\mathrm{U}(1)}(F_{\nu,L};\mathbb{Q}) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t,t^{-1}].$$

Since $\mathrm{U}(1)$ acts trivially on $F_{\nu,L}$, the group on the right side is isomorphic to $\mathrm{Ind}^{W_G}_{W_L} H^*(F_{\nu,L};\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[t,t^{-1}]$.

Consider the Leray spectral sequence for the composition of maps

$$(E^{\mathrm{rss}}_{\nu,G})_{h\mathrm{U}(1)} \to \mathrm{BU}(1) \times B(\delta, W_G\backslash\backslash T)^{\mathrm{rss}} \to B(\delta, W_G\backslash\backslash T)^{\mathrm{rss}}.$$

This is a spectral sequence in the category of locally constant sheaves on $B(\delta, W_G\backslash\backslash T)^{\mathrm{rss}}$, whose stalk at a point is the Borel spectral sequence

$$E^{ij}_2 = H^i_{\mathrm{U}(1)}(pt) \otimes H^j(F_{\nu,G};\mathbb{Q}) \implies H^{i+j}_{\mathrm{U}(1)}(F_{\nu,G};\mathbb{Q}).$$

In particular, we may regard the Borel spectral sequence as a spectral sequence of W_G-modules. Furthermore, since $H^*(F_{\nu,G};\mathbb{Q})$ is known to vanish in odd degrees (in fact, this is true for any coefficient ring, [DPL]), this spectral sequence degenerates. It follows that the localized spectral sequence

$$E^{ij}_2[t^{-1}] := \mathbb{Q}[t,t^{-1}] \otimes H^j(F_{\nu,G};\mathbb{Q}) \implies H^*_{\mathrm{U}(1)}(F_{\nu,G};\mathbb{Q}) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t,t^{-1}]$$

also degenerates. In other words,

$$H^*_{\mathrm{U}(1)}(F_{\nu,G};\mathbb{Q}) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t,t^{-1}] \cong \mathrm{Ind}^{W_G}_{W_L} H^*(F_{\nu,L};\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[t,t^{-1}]$$

admits a filtration whose associated graded is $E^{ij}_2[t^{-1}]$. The conclusion of the theorem follows from restricting to the $i+j=0$ line of $E^{ij}_2[t^{-1}]$.   □

Proof of Theorem 1.2. Let $\mathbb{Z}/p \subset G$ be the subgroup generated by g. As in the proof of Theorem 1.1, we have $\mathbb{Z}/p \subset K_G(\nu)$, and E^{rss}_ν is the set of $\mathbb{Z}/p$-fixed points of $E^{\mathrm{rss}}_{\nu,G}$. Thus as before, this time by applying the localization theorem for $\mathbb{Z}/p$-equivariant cohomology (Theorem 4.2 of [Q]), we have an isomorphism of W_G-modules:

$$H^*_{\mathbb{Z}/p}(F_{\nu,G},\mathbb{F}_p) \otimes_{H^*_{\mathbb{Z}/p}(\mathrm{pt})} H^*_{\mathbb{Z}/p}(\mathrm{pt})[t^{-1}]$$
$$\cong \mathrm{Ind}^{W_G}_{W_Z} H^*_{\mathbb{Z}/p}(F_{\nu,Z};\mathbb{F}_p) \otimes_{H^*_{\mathbb{Z}/p}(\mathrm{pt})} H^*_{\mathbb{Z}/p}(\mathrm{pt})[t^{-1}].$$

Here t denotes a polynomial generator in degree 2 if p is odd, and degree 1 if $p=2$. Once again since $\mathbb{Z}/p$ acts trivially on $F_{\nu,Z}$, we have $\mathrm{Ind}^{W_G}_{W_Z} H^*_{\mathbb{Z}/p}(F_{\nu,Z};\mathbb{F}_p) \otimes_{H^*_{\mathbb{Z}/p}(\mathrm{pt})}$ $H^*_{\mathbb{Z}/p}(\mathrm{pt})[t^{-1}] \cong \mathrm{Ind}^{W_G}_{W_Z} H^*(F_{\nu,Z};\mathbb{F}_p) \otimes_{\mathbb{F}_p} H^*_{\mathbb{Z}/p}(\mathrm{pt})[t^{-1}]$, and we may complete the proof by showing that the Borel spectral sequence

$$E^{ij}_2 = H^*_{\mathbb{Z}/p}(\mathrm{pt}) \otimes H^j(F_{\nu,G};\mathbb{F}_p) \implies H^{i+j}_{\mathbb{Z}/p}(F_{\nu,G};\mathbb{F}_p)$$

degenerates.

We may do this as follows. Since g is in the connected component of the identity of the centralizer of ν, we may find a circle subgroup $\mathrm{U}(1) \subset K_G(\nu)$ with $g \in \mathrm{U}(1)$. Then we have a cartesian square

$$
\begin{array}{ccc}
(F_{\nu,G})_{h\mathbb{Z}/p} & \xrightarrow{\;a\;} & (F_{\nu,G})_{h\mathrm{U}(1)} \\
{\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\
B(\mathbb{Z}/p) & \xrightarrow[\;d\;]{} & \mathrm{BU}(1)
\end{array}
$$

The morphisms in this square are fiber bundles with fiber $F_{\nu,G}$. Since both the cohomology groups $H^*(F_{\nu,G}, \mathbb{F}_p)$ and $H^*(\mathrm{BU}(1); \mathbb{F}_p)$ vanish in odd degrees, the derived pushforward sheaf $Rc_*\mathbb{F}_p$ is formal, i.e., isomorphic to a direct sum of its cohomology sheaves. By the proper base change theorem, the same is true for $Rb_*\mathbb{F}_p$. This implies that the Leray spectral sequence of the map b and the sheaf $\mathbb{F}_p$ degenerates. But this Leray spectral sequence is exactly the Borel spectral sequence we are considering. This completes the proof. $\qquad\square$

Remark 6.1. If g is not contained in the identity component of the centralizer of ν, we may still conclude from the existence of the Borel spectral sequence that

$$\mathrm{gr}^k(\mathrm{Ind}_{W_Z}^{W_G}(\bigoplus_{i\in\mathbb{Z}} H^i(F_{\nu,Z}; \mathbb{F}_p)))$$

is a subquotient of $H^k(F_{\nu,G}; \mathbb{F}_p)$, even if this spectral sequence does not degenerate.

Remark 6.2. Let us give an example (without proof) where the hypotheses of Theorem 1.2 apply, and one where they do not. Let $G = \mathrm{Sp}(4)$, $Z = \mathrm{SL}(2) \times \mathrm{SL}(2)$ (note that this is not a Levi subgroup) and $p = 2$. Then Z is the centralizer of

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We may take for ν the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

but not the subregular unipotent

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Acknowledgments

References

[AL]   D. Alvis and G. Lusztig. *On Springer's correspondence for simple groups of type E_n ($n = 6, 7, 8$).* Math. Proc. Camb. Phil. Soc. 92, pp. 65–72, 1982. MR662961 (83k:20040)

[B]   E. Brieskorn. *Singular elements of semi-simple algebraic groups.* Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, pp. 279–284, 1971. MR0437798 (55:10720)

[Br]   J.-L. Brylinski. *Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques.* Astérisque 140-141, pp. 3–134, 1986. MR864073 (88j:32013)

[DPL]   C. De Concini, G. Lusztig, and C. Processi. *Homology of the zero set of a nilpotent vector field on a flag manifold.* J. Amer. Math. Soc., vol. 1, pp. 15–34, 1988. MR924700 (89f:14052)

[D]   P. Deligne. *Les immeubles des groupes de tresses généralisés.* Invent. Math. 17, pp. 273–302, 1972. MR0422673 (54:10659)

[G]   M. Grinberg. *A generalization of Springer theory using nearby cycles.* Represent. Theory, vol. 2, pp. 410–431, 1998. MR1657203 (2000a:20099)

[GKM] M. Goresky, R. Kottwitz, and R. Macpherson. *Homology of affine Springer fibers in the unramified case.* Duke Math. J. 121, pp. 509–561, 2004. MR2040285 (2005a:14068)

[H]   R. Hotta. *On Springer's representations.* J. Fac. Sci., Tokyo 28, pp. 863–876, 1981. MR656061 (83h:20038)

[HK]   R. Hotta and M. Kashiwara. *The invariant holonomic system on a semisimple Lie algebra.* Invent. Math. 75, pp. 327–358, 1984. MR732550 (87i:22041)

[J]   D. Juteau. *Modular Springer correspondence and decomposition matrices.* Ph.D Thesis, Institut de Mathématiques de Jussieu, 2007.

[KL]   D. Kazhdan and G. Lusztig. *A topological approach to Springer's representations.* Adv. in Math. 38, pp. 222–228, 1980. MR597198 (82f:20076)

[L]   G. Lusztig. *An induction theorem for Springer's representations.* Adv. Studies in Pure Math., 40, in "Representation theory of algebraic groups and quantum groups", pp. 253–259, 2004.

[M]   J. Mather. *Notes on topological stability*, 1970. Available at www.math.princeton.edu/facultypapers/mather/.

[N]   B. Ngô. *Fibration de Hitchin et structure endoscopique de la formule des traces.* in "International Congress of Mathematicians. Vol. II", pp. 1213–1225, 2006. MR2275642 (2008f:14023)

[Q]   D. Quillen. *The spectrum of an equivariant cohomology ring, I.* Ann. of Math., vol. 94, pp. 549–572, 1971 MR0298694 (45:7743)

[R]   W. Rossmann. *Picard-Lefshetz theory for the coadjoint quotient of a semisimple Lie algebra.* Invent. Math. 121, pp. 531–578, 1995. MR1353308 (96j:32052)

[S]   P. Slodowy. *Simple singularities and simple algebraic groups.* Lecture Notes in Mathematics 815, Springer-Verlag, 1980. MR584445 (82g:14037)

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