

## HARISH-CHANDRA BIMODULES FOR QUANTIZED SLODOWY SLICES

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*Dedicated to the memory of Peter Slodowy*

ABSTRACT. The Slodowy slice is an especially nice slice to a given nilpotent conjugacy class in a semisimple Lie algebra. Premet introduced noncommutative quantizations of the Poisson algebra of polynomial functions on the Slodowy slice.

In this paper, we define and study Harish-Chandra bimodules over Premet's algebras. We apply the technique of Harish-Chandra bimodules to prove a conjecture of Premet concerning primitive ideals, to define projective functors, and to construct "noncommutative resolutions" of Slodowy slices via translation functors.

### 1. GEOMETRY OF SLODOWY SLICES

**1.1. Introduction.** Let  $\mathfrak{g}$  be a semisimple Lie algebra, and  $\mathcal{U}\mathfrak{g}$  the universal enveloping algebra of  $\mathfrak{g}$ . For any nilpotent element  $e \in \mathfrak{g}$ , Slodowy used the Jacobson-Morozov theorem to construct a slice to the conjugacy class of  $e$  inside the nilpotent variety of  $\mathfrak{g}$ . This slice,  $\mathcal{S}$ , has a natural structure of an affine algebraic Poisson variety.

More recently, Premet [P1] has defined, following earlier works by Kostant [Ko], Kawanaka [Ka], and Moeglin [Mœ], for each character  $c$  of the center of  $\mathcal{U}\mathfrak{g}$ , a filtered associative algebra  $A_c$ . The family of algebras  $A_c$  may be thought of as a family of quantizations of the Slodowy slice  $\mathcal{S}$  in the sense that, for any  $c$ , one has a natural Poisson algebra isomorphism  $\text{gr } A_c \cong \mathbb{C}[\mathcal{S}]$ . For further developments see also [BK1], [BGK], [Lo1], [P2]-[P3].

The algebras  $A_c$  have quite interesting representation theory which is similar, in a sense, to the representation theory of the Lie algebra  $\mathfrak{g}$  itself. It is well known that category  $\mathcal{O}$  of Bernstein-Gelfand-Gelfand plays a key role in the representation theory of  $\mathfrak{g}$ . Unfortunately, there seems to be no reasonable analogue of category  $\mathcal{O}$  for the algebra  $A_c$ , apart from some special cases; cf. [BK1], [BGK].

In this paper, we propose to remedy the above mentioned difficulty by introducing a category of (weak) Harish-Chandra *bimodules* over the algebra  $A_c$ . Our definition of weak Harish-Chandra bimodules actually makes sense for a wide class of algebras; cf. Definition 4.1.1. We show in particular that, in the case of enveloping algebras, a weak Harish-Chandra  $\mathcal{U}\mathfrak{g}$ -bimodule is a Harish-Chandra bimodule

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in the conventional sense (used in representation theory of semisimple Lie algebras for a long time) if and only if the corresponding  $\mathcal{D}$ -module on the flag variety has regular singularities; cf. Proposition 6.6.1. Motivated by this result, we use “micro-local” technique developed in §6 to introduce a notion of “regular singularities” for  $A_c$ -bimodules.<sup>1</sup> We then define Harish-Chandra  $A_c$ -bimodules as weak Harish-Chandra bimodules with regular singularities.

Let  $\mathcal{J}_c$  be the maximal ideal of the center of the algebra  $\mathcal{U}\mathfrak{g}$  corresponding to a *regular* central character  $c$ . There is an associated block  $\mathcal{O}_c$ , of the category  $\mathcal{O}$  of Bernstein-Gelfand-Gelfand, formed by the objects  $M \in \mathcal{O}$  such that the  $\mathcal{J}_c$ -action on  $M$  is nilpotent. One may also consider an analogous category of Harish-Chandra  $\mathcal{U}\mathfrak{g}$ -bimodules. Specifically, one considers Harish-Chandra  $\mathcal{U}\mathfrak{g}$ -bimodules  $K$  such that the left  $\mathcal{J}_c$ -action on  $K$  is nilpotent and such that one has  $K\mathcal{J}_c = 0$ . It is known that the resulting category is, in fact, *equivalent* to the category  $\mathcal{O}_c$ ; see [BG]. Thus, one may expect the category of Harish-Chandra  $A_c$ -bimodules to be the right substitute for a category  $\mathcal{O}(A_c)$  that may or may not exist.

For any algebra  $A$ , a basic example of a weak Harish-Chandra  $A$ -bimodule is the algebra  $A$  itself, viewed as the diagonal bimodule. Sub-bimodules of the diagonal bimodule are nothing but two-sided ideals of  $A$ . This shows that the theory of (weak) Harish-Chandra  $A$ -bimodules is well suited for studying ideals in  $A$ , primitive ideals, in particular.

We construct an analogue of the Whittaker functor from the category of Harish-Chandra  $\mathcal{U}\mathfrak{g}$ -bimodules to the category of Harish-Chandra  $A_c$ -bimodules. Among our most important results are Theorem 4.1.4 and Theorem 4.2.2, which describe key properties of that functor.

We use the above results to provide an alternative proof of a conjecture of Premet that relates finite dimensional  $A_c$ -modules to primitive ideals  $I \subset \mathcal{U}\mathfrak{g}$  such that the associated variety of  $I$  equals  $\overline{\text{Ad}G(e)}$ , the closure of the conjugacy class of the nilpotent  $e \in \mathfrak{g}$ . This conjecture was proved in a special case by Premet [P3], using reduction to positive characteristic, and in full generality by Losev [Lo1], using deformation quantization, and later also by Premet [P4]. Our approach is totally different from the approaches used by Losev or Premet and is, in a way, more straightforward. Some results closely related to ours were also obtained by Losev [Lo2].

Finally, we introduce translation functors on representations of the algebra  $A_c$ . In §6, we use those functors to construct “noncommutative resolutions” of the Slodowy slice by means of a noncommutative Proj-construction. Similar construction has been successfully exploited earlier, in other situations, by Gordon and Stafford [GS], and by Boyarchenko [Bo].

A different approach to “noncommutative resolutions” of Slodowy slices was also proposed by Losev in an unpublished manuscript.

*Remark 1.1.1.* We expect that, in the special case of *subregular* nilpotent elements, our construction of noncommutative resolution reduces to that of Boyarchenko. In more detail, Boyarchenko considered noncommutative resolutions of certain noncommutative algebras introduced by Crawley-Boevey and Holland [CBH]. These algebras are quantizations of the coordinate ring of a Kleinian singularity. By a

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<sup>1</sup>Our definition of regular singularities works more generally, for modules over an arbitrary filtered  $\mathbb{C}$ -algebra  $A$  having the property that  $\text{gr} A$  is a finitely generated commutative algebra such that the Poisson scheme  $\text{Spec}(\text{gr} A)$  admits a symplectic resolution.

well-known result of Brieskorn and Slodowy [Sl], the Slodowy slice to the subregular nilpotent in a simply laced Lie algebra  $\mathfrak{g}$  is isomorphic, as an algebraic variety, to the Kleinian singularity associated with the Dynkin diagram of  $\mathfrak{g}$ . Furthermore, it is expected (although no written proof of this seems to be available, except for type  $\mathbf{A}$ , see [Hod]) that the algebras  $A_c$  are, in that case, isomorphic to the algebras constructed by Crawley-Boevey and Holland. Thus, our noncommutative resolutions should correspond, via the isomorphism, to those considered by Boyarchenko.

1.2. Let  $\mathfrak{g}$  be a complex semisimple Lie algebra and  $G$  the adjoint group of  $\mathfrak{g}$ .

From now on, we fix an  $\mathfrak{sl}_2$ -triple  $\{e, h, f\} \subset \mathfrak{g}$ , equivalently, a Lie algebra imbedding  $\mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$  such that  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto e$  and  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto h$ . Slodowy (as well as Harish-Chandra) showed (see e.g. [Sl, §7.4]), that the affine linear space  $e + \text{Ker}(\text{ad } f)$  is a transverse slice to the  $\text{Ad } G$ -conjugacy class of  $e$ , a nilpotent element of  $\mathfrak{g}$ .

The Killing form on  $\mathfrak{g}$  provides an  $\text{Ad } G$ -equivariant isomorphism  $\kappa : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ , where  $\mathfrak{g}^*$  is the vector space dual to  $\mathfrak{g}$ . Put  $\chi := \kappa(e) \in \mathfrak{g}^*$ . Write  $\mathcal{N} \subset \mathfrak{g}^*$  for the image of the set of nilpotent elements of  $\mathfrak{g}$ , resp.  $\mathcal{S} \subset \mathfrak{g}^*$  for the image of the set  $e + \text{Ker } \text{ad } f$ , under the isomorphism  $\kappa$ . Thus,  $\mathcal{S}$  is a transverse slice at  $\chi$ , to be called the *Slodowy slice*, to the coadjoint orbit  $\mathcal{O} := \text{Ad } G(\chi)$ .

Below, we mostly restrict our attention to the scheme theoretic intersection  $\mathcal{S} := \mathcal{S} \cap \mathcal{N}$ . Extending some classic results of Kostant, Premet proved the following, [P1, Theorem 5.1].

**Proposition 1.2.1.** *The scheme  $\mathcal{S}$  is reduced, irreducible, and Gorenstein. Moreover, it is a normal complete intersection in  $\mathcal{S}$  of dimension  $\dim \mathcal{N} - \dim \mathcal{O}$ .  $\square$*

1.3. **The Lie algebra  $\mathfrak{m}$ .** The  $\text{ad } h$ -action on  $\mathfrak{g}$  yields a weight decomposition

$$(1.3.1) \quad \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i) \quad \text{where} \quad \mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}.$$

Note that  $e$  is a nonzero element of  $\mathfrak{g}(2)$ . Hence, the assignment  $\mathfrak{g}(-1) \times \mathfrak{g}(-1) \rightarrow \mathbb{C}$ ,  $x, y \mapsto \chi([x, y])$ , gives a skew-symmetric nondegenerate bilinear form on  $\mathfrak{g}(-1)$ . We choose and fix  $\ell \subset \mathfrak{g}(-1)$ , a Lagrangian subspace with respect to that form.

Following Kawanaka [Ka] and Moeglin [Mœ], one puts

$$\mathfrak{m} := \ell \oplus \left( \bigoplus_{i \leq -2} \mathfrak{g}(i) \right).$$

Thus,  $\mathfrak{m}$  is a nilpotent Lie subalgebra of  $\mathfrak{g}$  such that the linear function  $\chi$  vanishes on  $[\mathfrak{m}, \mathfrak{m}]$ . Let  $M$  be the unipotent subgroup of  $G$  with Lie algebra  $\mathfrak{m}$ . By a standard easy computation one finds (cf. (1.4.1)),

$$(1.3.2) \quad \dim M = \frac{1}{2} \dim \mathcal{O}.$$

Write  $V^\perp \subset \mathfrak{g}^*$  for the annihilator of a vector subspace  $V \subset \mathfrak{g}$ . We will prove:

**Proposition 1.3.3.** (i) *Any  $\text{Ad } G$ -orbit in  $\mathfrak{g}^*$  meets the affine space  $\chi + \mathfrak{m}^\perp$  transversely.*

(ii) *We have  $\mathcal{O} \cap (\chi + \mathfrak{m}^\perp) = \text{Ad } M(\chi)$  is a closed Lagrangian submanifold in the coadjoint orbit  $\mathcal{O} = \text{Ad } G(\chi)$ , a symplectic manifold with Kirillov's symplectic structure.*

(iii) *The scheme theoretic intersection  $\mathcal{N} \cap (\chi + \mathfrak{m}^\perp)$  is a reduced complete intersection in  $\chi + \mathfrak{m}^\perp$ ; the  $M$ -action induces an  $M$ -equivariant isomorphism  $M \times \mathcal{S} \xrightarrow{\sim} \mathcal{N} \cap (\chi + \mathfrak{m}^\perp)$ .*

*Remark 1.3.4.* Using the isomorphism in the last line above, we may write

$$(1.3.5) \quad \mathcal{S} \cong [\mathcal{N} \cap (\chi + \mathfrak{m}^\perp)]/\text{Ad } M.$$

This formula means that the variety  $\mathcal{S}$  may be obtained from  $\mathcal{N}$  by a Hamiltonian reduction with respect to the natural  $M$ -action on  $\mathcal{N}$ .

**Conjecture 1.3.6.** There exists a Borel subalgebra  $\mathfrak{b} \subset \mathfrak{g}$  such that  $\mathfrak{b}^\perp \cap \text{Ad } M(\chi) = \{\chi\}$ .

*Remark 1.3.7.* We recall that, for any Borel subalgebra  $\mathfrak{b}$ , (each irreducible component of) the set  $(\mathfrak{b}^\perp \cap \mathbb{O})_{\text{red}}$  is known to be a Lagrangian subvariety in  $\mathbb{O}$ ; see [CG, Theorem 3.3.6].

Thus, Conjecture 1.3.6 says that there exists a Borel subalgebra  $\mathfrak{b}$  such that the two Lagrangian subvarieties  $\mathfrak{b}^\perp \cap \mathbb{O}$  and  $\text{Ad } M(\chi)$  meet at a single point  $\chi$ .

Let  $\varpi : \mathfrak{g}^* \rightarrow \mathfrak{m}^*$  be the canonical projection induced by restriction of linear functions from  $\mathfrak{g}$  to  $\mathfrak{m}$ . The map  $\varpi$  may be thought of as a moment map associated with the  $\text{Ad } M$ -action on  $\mathfrak{g}^*$ . We put  $\chi_{\mathfrak{m}} := \chi|_{\mathfrak{m}} = \varpi(\chi)$ . Observe that  $\chi_{\mathfrak{m}} \in \mathfrak{m}^*$  is a fixed point of the coadjoint  $M$ -action on  $\mathfrak{m}^*$ , and we have  $\varpi^{-1}(\chi_{\mathfrak{m}}) = \chi + \mathfrak{m}^\perp$ .

We write  $\mathcal{O}_Y$  for the structure sheaf of a scheme  $Y$ . Let  $\mathcal{O}_\chi$  denote the localization of the polynomial algebra  $\mathbb{C}[\mathfrak{m}^*]$  at the point  $\chi_{\mathfrak{m}}$ .

**Corollary 1.3.8.** *Let  $Y$  be a  $G$ -scheme, and let  $f : Y \rightarrow \mathfrak{g}^*$  be a  $G$ -equivariant morphism such that  $\chi \in f(Y)$ . Then, we have:*

- (i) *The point  $\chi_{\mathfrak{m}}$  is a regular value of the composite map  $\varpi \circ f$ .*
- (ii) *Given a  $G$ -equivariant coherent  $\mathcal{O}_Y$ -module  $M$ , the localization  $(\varpi \circ f)^* \mathcal{O}_\chi \otimes_{\mathcal{O}_Y} M$  is a flat  $(\varpi \circ f)^* \mathcal{O}_\chi$ -module.*
- (iii) *If  $Y$  is reduced and irreducible, then  $f^{-1}(\mathbb{S})$  is a reduced complete intersection in  $Y$ , of dimension  $\dim f^{-1}(\mathbb{S}) = \dim Y - \dim \mathbb{O}$ . □*

*Proof.* The transversality statement of Proposition 1.3.3(i) may be equivalently reformulated as follows:

*For any  $x \in \mathfrak{g}^*$ , the point  $\chi_{\mathfrak{m}}$  is a regular value of the composite map  $\text{Ad } G(x) \hookrightarrow \mathfrak{g}^* \xrightarrow{\varpi} \mathfrak{m}^*$ .*

The above statement insures that, for any point  $y \in Y$  such that  $\varpi(f(y)) = \chi_{\mathfrak{m}}$ , the differential  $d\varpi \circ df : T_y Y \rightarrow T_{\chi_{\mathfrak{m}}} \mathfrak{m}^*$ , of the map  $\varpi \circ f$ , is a surjective linear map. This yields part (i) of the corollary. Parts (ii)–(iii) follow from (i) combined with Proposition 1.3.3(i). □

**1.4. Proof of Proposition 1.3.3.** First, we recall a few well-known results. Write  $\mathfrak{g}^x \subset \mathfrak{g}$  for the centralizer of an element  $x \in \mathfrak{g}$ . To prove formula (1.3.2), we compute

$$\dim \mathfrak{m} = \dim \left( \bigoplus_{i < 0} \mathfrak{g}_i \right) - \frac{1}{2} \dim \mathfrak{g}(-1) = \frac{1}{2} [\dim \mathfrak{g} - \dim \mathfrak{g}(0)] - \frac{1}{2} \dim \mathfrak{g}(-1).$$

By  $\mathfrak{sl}_2$ -theory, one has that  $\dim \mathfrak{g}(0) + \dim \mathfrak{g}(1) = \dim \mathfrak{g}^e$ . Hence, we find

$$(1.4.1) \quad \dim \mathfrak{m} = \frac{1}{2} [\dim \mathfrak{g} - \dim \mathfrak{g}(0) - \dim \mathfrak{g}(1)] = \frac{1}{2} [\dim \mathfrak{g} - \dim \mathfrak{g}^e] = \frac{1}{2} \dim \mathbb{O}. \quad \square$$

Observe next that, for the element  $f$  of our  $\mathfrak{sl}_2$ -triple, we have  $\mathfrak{g}^f \subset \bigoplus_{i \leq 0} \mathfrak{g}(i)$ , by  $\mathfrak{sl}_2$ -theory. It follows that  $\mathbb{S} \subset \chi + \mathfrak{m}^\perp$ . Also, it is easy to see that the set

$\chi + \mathfrak{m}^\perp$  is stable under the coadjoint  $M$ -action. Moreover, it was proved in [GG, Lemma 2.1] that the  $M$ -action in  $\chi + \mathfrak{m}^\perp$  is free, and the action map induces an  $M$ -equivariant isomorphism of algebraic varieties,

$$(1.4.2) \quad M \times \mathbb{S} \xrightarrow{\sim} \chi + \mathfrak{m}^\perp,$$

where  $M$  acts on  $M \times \mathbb{S}$  via its action on the first factor by left translations.

We may exponentiate the Lie algebra map  $\mathfrak{sl}_2 \rightarrow \mathfrak{g}$  to a rational group homomorphism  $SL_2 \rightarrow G$ . Restricting the latter map to the torus  $\mathbb{C}^\times \subset SL_2$ , of diagonal matrices, one gets a morphism  $\gamma : \mathbb{C}^\times \rightarrow G$ ,  $t \mapsto \gamma_t$ . Following Slodowy, one defines a  $\bullet$ -action of  $\mathbb{C}^\times$  on  $\mathfrak{g}$  by

$$(1.4.3) \quad \mathbb{C}^\times \ni t : x \mapsto t \bullet x := t^2 \cdot \text{Ad } \gamma_{t^{-1}}(x), \quad x \in \mathfrak{g}.$$

Since  $\text{Ad } \gamma_t(e) = t^2 \cdot e$ , the  $\bullet$ -action fixes  $e$ . Dualizing, one gets a  $\bullet$ -action of  $\mathbb{C}^\times$  on  $\mathfrak{g}^*$  that fixes the point  $\chi \in \mathfrak{g}^*$ . It is easy to see that each of the spaces,  $\chi + \mathbb{S}$  and  $\chi + \mathfrak{m}^\perp$ , is  $\bullet$ -stable and, moreover, the  $\bullet$ -action contracts the space  $\chi + \mathfrak{m}^\perp$  to  $\chi$ .

*Proof of Proposition 1.3.3.* Observe first that the statement of part (i) is clear for the orbit  $\mathbb{O} = \text{Ad } G(\chi)$ , since the space  $\chi + \mathfrak{m}^\perp$  contains a transverse slice to  $\mathbb{O}$ . Below, we will use the identification  $\kappa : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ , so  $\chi$  gets identified with  $e$ , and we may write  $\chi + \mathfrak{m}^\perp \subset \mathfrak{g}$ .

Now, let  $x \in \chi + \mathfrak{m}^\perp$  be an arbitrary element and put  $O := \text{Ad } G(x)$ . We are going to reduce the statement (i) for  $O$  to the special case of the orbit  $\mathbb{O}$  using the  $\bullet$ -action as follows.

Observe that the tangent space to  $O$  at the point  $x$  equals  $T_x O = [\mathfrak{g}, x] = \text{ad } x(\mathfrak{g})$ , a vector subspace of  $\mathfrak{g}$ . Proving part (i) amounts to showing that the composite  $\text{pr} \circ \kappa \circ \text{ad } x : \mathfrak{g} \rightarrow \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^* \rightarrow \mathfrak{m}^*$  is a surjective linear map, for any  $x \in \chi + \mathfrak{m}^\perp$ . It is clear that, for any  $x \in \mathfrak{g}$  sufficiently close to  $e$ , the map  $\text{pr} \circ \kappa \circ \text{ad } x$  is surjective, by continuity. Since the  $\bullet$ -action on  $\chi + \mathfrak{m}^\perp$  is a contraction, we deduce by  $\mathbb{C}^\times$ -equivariance that the surjectivity holds for any  $x \in e + \mathfrak{m}^\perp$ . Part (i) is proved.

The isomorphism of part (iii) follows from Proposition 1.2.1, by restricting isomorphism (1.4.2) to  $\mathcal{N}$ . All the other claims of part (iii) then follow from the isomorphism.

To prove (ii), we observe first, that since  $\chi$  vanishes on  $[\mathfrak{m}, \mathfrak{m}]$ , the vector space  $\text{ad } \mathfrak{m}(\chi)$  is an isotropic subspace of the tangent space  $T_\chi \mathbb{O}$ . This implies, by  $M$ -equivariance, that  $\text{Ad } M(\chi)$  is an isotropic submanifold of  $\mathbb{O}$ . This  $\text{Ad } M$ -orbit is closed since  $M$  is a unipotent group. Finally, the  $M$ -action being free we find  $\dim \text{Ad } M(\chi) = \dim M = \frac{1}{2} \dim \mathbb{O}$ , by formula (1.3.2). It follows that  $\text{Ad } M(\chi)$  is a Lagrangian submanifold of  $\mathbb{O}$ .  $\square$

## 2. SPRINGER RESOLUTION OF SLODOWY SLICES

**2.1. The Slodowy variety.** Let  $\mathcal{B}$  be the flag variety, i.e., the variety of all Borel subalgebras in  $\mathfrak{g}$ . Let  $T^*\mathcal{B}$  be the total space of the cotangent bundle on  $\mathcal{B}$ , equipped with the standard symplectic structure and the natural Hamiltonian  $G$ -action. An associated moment map is given by the first projection

$$(2.1.1) \quad \pi : T^*\mathcal{B} = \{(\lambda, \mathfrak{b}) \in \mathfrak{g}^* \times \mathcal{B} \mid \lambda \in \mathfrak{b}^\perp\} \longrightarrow \mathfrak{g}^*, \quad (\lambda, \mathfrak{b}) \mapsto \lambda.$$

The map  $\pi$ , called Springer resolution, is a *symplectic resolution* of  $\mathcal{N}$ . This means that, one has  $\mathcal{N} = \pi(T^*\mathcal{B})$  and, moreover,  $\pi$  is a resolution of singularities

of  $\mathcal{N}$  such that the pull-back morphism  $\pi^* : \mathcal{O}_{\mathfrak{g}^*} \rightarrow \mathcal{O}_{T^*\mathcal{B}}$  intertwines the Kirillov-Kostant Poisson bracket on  $\mathcal{O}_{\mathfrak{g}^*}$  with the Poisson bracket on  $\mathcal{O}_{T^*\mathcal{B}}$  coming from the symplectic structure on  $T^*\mathcal{B}$ .

The *Slodowy variety* is defined as  $\tilde{\mathcal{S}} := \pi^{-1}(\mathcal{S}) = \pi^{-1}(\mathbb{S})$ , a scheme theoretic preimage of the Slodowy slice under the Springer resolution. Also, let  $\mathcal{B}_\chi := \pi^{-1}(\chi)_{\text{red}}$  be the Springer fiber over  $\chi$ , equipped with *reduced* scheme structure. Clearly, we have  $\mathcal{B}_\chi \subset \tilde{\mathcal{S}}$ .

**Proposition 2.1.2.** (i) *The map  $\pi : \tilde{\mathcal{S}} \rightarrow \mathcal{S}$  is a symplectic resolution, in particular,  $\tilde{\mathcal{S}}$  is a smooth and connected symplectic submanifold in  $T^*\mathcal{B}$  of dimension  $\dim \tilde{\mathcal{S}} = 2 \dim \mathcal{B}_\chi$ .*

(ii) *The Springer fiber  $\mathcal{B}_\chi$  is a (not necessarily irreducible) Lagrangian subvariety of  $\tilde{\mathcal{S}}$ .*

Our next goal is to find a Hamiltonian reduction construction for the variety  $\tilde{\mathcal{S}}$ . Specifically, we would like to get an analogue of formula (1.3.5) where the variety  $\mathcal{S}$  is replaced by  $\tilde{\mathcal{S}}$  and where the symplectic manifold  $T^*\mathcal{B}$  plays the role of the Poisson variety  $\mathcal{N}$ . To do so, it is natural to try to replace, in formula (1.3.5), the space  $\chi + \mathfrak{m}^\perp$  by  $\pi^{-1}(\chi + \mathfrak{m}^\perp)$ . Thus, we are led to introducing the scheme

$$(2.1.3) \quad \Sigma := \pi^{-1}(\chi + \mathfrak{m}^\perp) = (\varpi \circ \pi)^{-1}(\chi_{\mathfrak{m}}) \subset T^*\mathcal{B}.$$

**Proposition 2.1.4.** (i) *The scheme  $\Sigma$  is a (reduced) smooth connected manifold, and we have  $\dim \Sigma = \dim \mathcal{B} + \dim \mathcal{B}_\chi = \dim T^*\mathcal{B} - \dim \mathfrak{m}$ .*

(ii) *The scheme  $\Sigma$  is  $M$ -stable, and the action-map induces an  $M$ -equivariant isomorphism*

$$(2.1.5) \quad M \times \tilde{\mathcal{S}} \xrightarrow{\sim} \Sigma.$$

(iii) *The scheme  $\Sigma$  is a coisotropic submanifold in  $T^*\mathcal{B}$ ; the nil-foliation on  $\Sigma$  coincides with the fibration by  $M$ -orbits.*

**2.2. Proof of Propositions 2.1.2 and 2.1.4.** There is a natural  $\mathbb{C}^\times$ -action on  $T^*\mathcal{B}$  along the fibers of the cotangent bundle projection  $T^*\mathcal{B} \rightarrow \mathcal{B}$ . Thus, the group  $G \times \mathbb{C}^\times$  acts on  $T^*\mathcal{B}$  by  $(g, z) : x \mapsto z \cdot g(x)$ . Further, we define a  $\bullet$ -action of the torus  $\mathbb{C}^\times$  on  $T^*\mathcal{B}$  by the formula  $\mathbb{C}^\times \ni t : x \mapsto t \bullet x := t^2 \cdot \gamma_{t^{-1}}(x)$ ; cf. (1.4.3).

The Springer resolution (2.1.1) is a  $G \times \mathbb{C}^\times$ -equivariant morphism. Hence, the map  $\pi$  commutes with the  $\bullet$ -action as well. It follows, in particular, that  $\mathcal{B}_\chi$ ,  $\tilde{\mathcal{S}}$ , and  $\Sigma$ , are all  $\bullet$ -stable subschemes of  $T^*\mathcal{B}$ . The  $\bullet$ -action retracts  $\chi + \mathfrak{m}^\perp$  to  $\chi$ , hence, provides a retraction of  $\Sigma = \pi^{-1}(\chi + \mathfrak{m}^\perp)$  to  $\mathcal{B}_\chi = \pi^{-1}(\chi)$ .

Further, a result of Spaltenstein [Spa] says that the Springer fiber is a connected variety and, moreover, all irreducible components of  $\mathcal{B}_\chi$  have the same dimension (cf. also [CG, Corollaries 7.6.16 and 3.3.24]), which is equal to

$$(2.2.1) \quad \dim \mathcal{B}_\chi = \dim \mathcal{B} - \frac{1}{2} \cdot \dim \mathbb{O}.$$

*Proof of Proposition 2.1.4.* Corollary 1.3.8(i) insures that the point  $\chi_{\mathfrak{m}} \in \mathfrak{m}^*$  is a regular value of the map  $\varpi \circ \pi : T^*\mathcal{B} \rightarrow \mathfrak{m}^*$ . It follows, in particular, that  $\Sigma$  is a (reduced) smooth subscheme of  $T^*\mathcal{B}$  and that  $\dim \Sigma = \dim T^*\mathcal{B} - \dim \mathfrak{m}$ . Furthermore, (2.1.5) holds as a scheme theoretic isomorphism and we have  $\dim \Sigma = \dim \tilde{\mathcal{S}} + \dim \mathfrak{m}$ .

Since  $\mathcal{B}_\chi$  is connected and the  $\bullet$ -action contracts  $\tilde{\mathcal{S}}$ , resp.  $\Sigma$ , to  $\mathcal{B}_\chi$ , we deduce that  $\tilde{\mathcal{S}}$ , resp.  $\Sigma$ , is a connected manifold. This completes the proof of Proposition 2.1.4(i).

Part (ii) of the proposition is immediate from the isomorphism of Proposition 1.3.3(i). Part (iii) is a general property of the fiber of a moment map over a regular value; see e.g. [GuS].  $\square$

*Proof of Proposition 2.1.2.* First of all, we observe that the smoothness of  $\Sigma$ , combined with (2.1.5), implies that  $\tilde{\mathcal{S}}$  is a smooth scheme. Furthermore, from part (iii) of Proposition 2.1.4 and the isomorphism  $\tilde{\mathcal{S}} \cong \Sigma/M$  we deduce that the symplectic 2-form on  $T^*\mathcal{B}$  restricts to a nondegenerate 2-form on  $\tilde{\mathcal{S}}$ . This yields Proposition 2.1.2(i).

Further, we know that  $\pi : \tilde{\mathcal{S}} \rightarrow \mathcal{S}$  is a projective and dominant morphism which is an isomorphism over the open dense subset of  $\mathcal{S}$  formed by regular nilpotent elements. It follows that the map  $\pi : \tilde{\mathcal{S}} \rightarrow \mathcal{S}$  is a symplectic resolution. The scheme  $\mathcal{S}$  being irreducible, we deduce that  $\tilde{\mathcal{S}}$  is connected (we have already proved this fact differently in the course of the proof of Proposition 2.1.4).

By Proposition 1.2.1, we get  $\dim \tilde{\mathcal{S}} = \dim \mathcal{S} = \dim \mathfrak{g} - \text{rk } \mathfrak{g} - \dim \mathbb{O}$ . Hence, using (2.2.1) and the equality  $2 \dim \mathcal{B} + \text{rk } \mathfrak{g} = \dim \mathfrak{g}$ , we find  $\dim \tilde{\mathcal{S}} = (2 \dim \mathcal{B} + \text{rk } \mathfrak{g}) - \text{rk } \mathfrak{g} - 2(\dim \mathcal{B} - \dim \mathcal{B}_\chi) = 2 \dim \mathcal{B}_\chi$ . This completes the proof of part (i) of Proposition 2.1.2.

To prove part (ii), let  $y = (\mathfrak{b}, \chi) \in \mathcal{B}_\chi$ . Recall that any tangent vector to  $T^*\mathcal{B}$  at  $y$  can be written in the form  $\text{ad}^* a(y) + \alpha$ , for some  $a \in \mathfrak{g}$  and some vertical vector  $\alpha \in \mathfrak{b}^\perp$  (i.e. a vector tangent to the fiber of the cotangent bundle). It is clear that such a vector  $\text{ad}^* a(y) + \alpha$  is tangent to  $\mathcal{B}_\chi$  if and only if one has  $\alpha + \text{ad}^* a(\chi) = 0$ . Now, let  $\text{ad}^* b(y) + \beta$  be a second tangent vector at  $y$  which is tangent to  $\mathcal{B}_\chi$  at the point  $y$ . Thus, we have  $\text{ad}^* a(\chi) = -\alpha$  and  $\text{ad}^* b(\chi) = -\beta$ .

Using an explicit formula for the symplectic 2-form  $\omega_{T^*\mathcal{B}}$  on  $T^*\mathcal{B}$  (see e.g. [CG, Proposition 1.4.11]), we find

$$\begin{aligned} \omega_{T^*\mathcal{B}}(\text{ad}^* a(y) + \alpha, \text{ad}^* b(y) + \beta) &= \chi([a, b]) + \alpha(b) - \beta(a) \\ &= \chi([a, b]) - (\text{ad}^* a(\chi))(b) + (\text{ad}^* b(\chi))(a) \\ &= \chi([a, b]) - \chi([a, b]) - \chi([a, b]) = -\chi([a, b]). \end{aligned}$$

Next, let  $\omega_{\mathbb{O}}$  denote the Kirillov-Kostant 2-form on the orbit  $\mathbb{O}$ . By definition, we have  $\omega_{\mathbb{O}}(\text{ad}^* a(\chi), \text{ad}^* b(\chi)) = \chi([a, b])$ . Further, it is known that  $\mathbb{O} \cap \mathfrak{b}^\perp$  is an isotropic subvariety in  $\mathbb{O}$ ; cf. [CG, Theorem 3.3.7]. The vectors  $\text{ad}^* a(\chi) = -\alpha$ ,  $\text{ad}^* b(\chi) = -\beta \in \mathfrak{b}^\perp$  are clearly tangent to that subvariety. We deduce  $0 = \omega_{\mathbb{O}}(\text{ad}^* a(\chi), \text{ad}^* b(\chi)) = \chi([a, b])$ .  $\square$

*Remark 2.2.2.* The smoothness statement in Proposition 2.1.2(i) is a special case of the following elementary general result.

Let  $X$  be a  $G$ -scheme, and let  $F : X \rightarrow \mathfrak{g}^*$  be a  $G$ -equivariant morphism such that  $\chi \in F(X)$ . Set  $\tilde{\mathbb{S}} := F^{-1}(\mathbb{S})$ , a scheme theoretic preimage of  $\mathbb{S}$ , and write  $F_{\tilde{\mathbb{S}}} = F|_{\tilde{\mathbb{S}}} : \tilde{\mathbb{S}} \rightarrow \mathbb{S}$ .

Then, for any  $x \in F^{-1}(\chi)$ , there are local isomorphisms  $(X, x) \xrightarrow{\sim} (\mathbb{O}, \chi) \times (\tilde{\mathbb{S}}, x)$ , resp.  $(\mathfrak{g}^*, \chi) \xrightarrow{\sim} (\mathbb{O}, \chi) \times (\mathbb{S}, \chi)$ , in *étale* topology, such that the map  $F$  goes, under the isomorphisms, to the map  $\text{Id}_{\mathbb{O}} \times F_{\tilde{\mathbb{S}}} : \mathbb{O} \times \tilde{\mathbb{S}} \rightarrow \mathbb{O} \times \mathbb{S}$ .

2.3. Given a manifold  $Y$ , we write  $p : T^*Y \rightarrow Y$  for the cotangent bundle projection.

Let  $X \subset Y$  be a submanifold. Below, we will use the following:

**Definition 2.3.1.** A submanifold  $\Lambda \subset p^{-1}(X)$  is said to be a **twisted conormal bundle** on  $X$  if  $\Lambda$  is a Lagrangian submanifold of  $T^*Y$  and, moreover, the map  $p$  makes the projection  $\Lambda \rightarrow X$  an affine bundle.

Let  $\sigma$  be the restriction to  $\Sigma \subset T^*\mathcal{B}$  of the projection  $p : T^*\mathcal{B} \rightarrow \mathcal{B}$ . It is clear that  $\sigma(\Sigma)$  is an  $M$ -stable subset of  $\mathcal{B}$ , and one has the following diagram of  $M$ -equivariant maps

$$(2.3.2) \quad \begin{array}{ccc} & \Sigma & \\ \sigma \swarrow & & \searrow \pi \\ \sigma(\Sigma) & & \mathcal{N} \cap (\chi + \mathfrak{m}^\perp) \end{array}$$

**Proposition 2.3.3.** (i) For a Borel subalgebra  $\mathfrak{b}$ , we have:  $\mathfrak{b} \in \sigma(\Sigma) \Leftrightarrow \chi|_{\mathfrak{m} \cap \mathfrak{b}} = 0$ .  
 (ii) For any  $M$ -orbit  $X \subset \sigma(\Sigma)$ , the map  $\sigma$  makes the projection  $\sigma^{-1}(X) \rightarrow X$  a twisted conormal bundle on the submanifold  $X \subset \mathcal{B}$ .

*Remark 2.3.4.* Let  $\mathcal{B}^\circ$  be the set of all Borel subalgebras  $\mathfrak{b}$  such that  $\mathfrak{b} \cap \mathfrak{m} = 0$ . It is easy to see that  $\mathcal{B}^\circ$  is an  $M$ -stable Zariski open and dense subset of  $\mathcal{B}$ . Part (i) of Proposition 2.3.3 implies that we have  $\mathcal{B}^\circ \subset \sigma(\Sigma)$ .

*Proof.* Clearly,  $\mathfrak{b} \in \sigma(\Sigma)$  if and only if the fiber  $\sigma^{-1}(\mathfrak{b})$  is nonempty. By definition, we have

$$(2.3.5) \quad \sigma^{-1}(\mathfrak{b}) = \{(\lambda, \mathfrak{b}) \in \mathfrak{g}^* \times \mathcal{B} \mid \lambda \in \mathfrak{b}^\perp \text{ and } \lambda \in \chi + \mathfrak{m}^\perp\} \cong \mathfrak{b}^\perp \cap (\chi + \mathfrak{m}^\perp).$$

Note that  $\chi|_{\mathfrak{m} \cap \mathfrak{b}} = 0$  says that  $\chi \in (\mathfrak{b} \cap \mathfrak{m})^\perp$ . Thus, the equivalences below yield part (i),

$$\mathfrak{b} \in \sigma(\Sigma) \iff \sigma^{-1}(\mathfrak{b}) \neq \emptyset \iff \mathfrak{b}^\perp \cap (\chi + \mathfrak{m}^\perp) \neq \emptyset \iff \chi \in \mathfrak{b}^\perp + \mathfrak{m}^\perp = (\mathfrak{b} \cap \mathfrak{m})^\perp.$$

To prove (ii), fix a point  $\mathfrak{b} \in \sigma(\Sigma)$ , and let  $X := M \cdot \mathfrak{b} \subset \mathcal{B}$  be the  $M$ -orbit of the point  $\mathfrak{b}$ . Thus, writing  $B$  for the Borel subgroup corresponding to  $\mathfrak{b}$ , we have  $X \cong M/M \cap B$ . Further, we may use the last displayed formula above and choose  $\lambda_{\mathfrak{b}} \in \mathfrak{b}^\perp$  and  $\mu_{\mathfrak{b}} \in \mathfrak{m}^\perp$  such that  $\chi = \lambda_{\mathfrak{b}} + \mu_{\mathfrak{b}}$ .

*Claim 2.3.6.* The map  $\sigma : \sigma^{-1}(X) \rightarrow X$  is  $M$ -equivariantly isomorphic to the fibration  $\phi : M \times_{M \cap B} (\lambda_{\mathfrak{b}} + (\mathfrak{m}^\perp \cap \mathfrak{b}^\perp)) \rightarrow M/M \cap B$ .

The above claim implies part (ii) since the fibration  $\phi$ , of the claim, is well known to be a twisted conormal bundle on the submanifold  $X \subset \mathcal{B} = G/B$ ; cf. [CG, Proposition 1.4.14].

To prove Claim 2.3.6, we write  $\chi + \mathfrak{m}^\perp = \lambda_{\mathfrak{b}} + \mu_{\mathfrak{b}} + \mathfrak{m}^\perp = \lambda_{\mathfrak{b}} + \mathfrak{m}^\perp$ . Hence, using (2.3.5), we may write

$$\sigma^{-1}(\mathfrak{b}) = \mathfrak{b}^\perp \cap (\chi + \mathfrak{m}^\perp) = \mathfrak{b}^\perp \cap (\lambda_{\mathfrak{b}} + \mathfrak{m}^\perp) = \{\lambda \in \mathfrak{b}^\perp \mid \lambda - \lambda_{\mathfrak{b}} \in \mathfrak{m}^\perp\} = \lambda_{\mathfrak{b}} + (\mathfrak{b}^\perp \cap \mathfrak{m}^\perp),$$

which is an affine-linear subspace in  $\mathfrak{b}^\perp$  obtained from the vector space  $\mathfrak{b}^\perp \cap \mathfrak{m}^\perp \subset \mathfrak{b}^\perp$  by translation. It follows that  $\sigma^{-1}(M \cdot \mathfrak{b}) \cong M \times_{M \cap B} \sigma^{-1}(\mathfrak{b}) \cong M \times_{M \cap B} (\lambda_{\mathfrak{b}} + (\mathfrak{m}^\perp \cap \mathfrak{b}^\perp))$ .

Now, we identify the tangent space to  $\mathcal{B}$ , resp. to the  $M$ -orbit  $X$ , at the point  $\mathfrak{b}$  with  $T_{\mathfrak{b}}\mathcal{B} = \mathfrak{g}/\mathfrak{b}$ , resp. with  $T_{\mathfrak{b}}X = (\mathfrak{m} + \mathfrak{b})/\mathfrak{b} \subset \mathfrak{g}/\mathfrak{b}$ . We see that the vector space

$\mathfrak{b}^\perp \cap \mathfrak{m}^\perp = (\mathfrak{m} + \mathfrak{b})^\perp$  equals the annihilator of  $T_{\mathfrak{b}}X$  in  $T_{\mathfrak{b}}\mathcal{B}$ . This proves Claim 2.3.6, and part (ii) follows.  $\square$

### 3. QUANTIZATION

**3.1. The Premet algebra.** Given a Lie algebra  $\mathfrak{k}$ , we write  $\text{Sym } \mathfrak{k}$  for the Symmetric, resp.  $\mathcal{U}\mathfrak{k}$  for the enveloping, algebra of  $\mathfrak{k}$ . We keep the notation of the previous section, fix  $\chi \in \mathcal{N}$ , and define a linear map  $\mathfrak{m} \rightarrow \text{Sym } \mathfrak{g}$ , resp.  $\mathfrak{m} \rightarrow \mathcal{U}\mathfrak{g}$ , by the assignment  $m \mapsto m - \chi(m)$ . Let  $\mathfrak{m}_\chi$  denote the image. Using the canonical isomorphism  $\mathbb{C}[\mathfrak{g}^*] = \text{Sym } \mathfrak{g}$ , one can identify  $\mathfrak{m}_\chi$  with the space of degree  $\leq 1$  polynomials on  $\mathfrak{g}^*$  vanishing on  $\chi + \mathfrak{m}^\perp$ .

Let  $\mathfrak{Z}\mathfrak{g}$  denote the center of  $\mathcal{U}\mathfrak{g}$  and write  $\text{Specm } \mathfrak{Z}\mathfrak{g}$  for the set of maximal ideals of the algebra  $\mathfrak{Z}\mathfrak{g}$ . Given  $c \in \text{Specm } \mathfrak{Z}\mathfrak{g}$ , let  $\mathfrak{J}_c \subset \mathfrak{Z}\mathfrak{g}$  denote the corresponding maximal ideal, and put  $\mathcal{U}_c := \mathcal{U}\mathfrak{g}/\mathfrak{U}\mathfrak{g} \cdot \mathfrak{J}_c$ . Let  $\mathcal{U}_c \cdot \mathfrak{m}_\chi$  denote the left ideal of the algebra  $\mathcal{U}_c$  generated by the image of the composite  $\mathfrak{m}_\chi \hookrightarrow \mathcal{U}\mathfrak{g} \twoheadrightarrow \mathcal{U}_c$ .

Similarly, let  $I_o$  denote the augmentation ideal of the algebra  $(\text{Sym } \mathfrak{g})^{\text{Ad } G}$ , and put  $S_o := \text{Sym } \mathfrak{g}/\text{Sym } \mathfrak{g} \cdot I_o$ . Thus,  $S_o = \mathbb{C}[\mathcal{N}]$ . Let  $S_o \cdot \mathfrak{m}_\chi$  denote an ideal of the algebra  $S_o$  generated by the image of the composite  $\mathfrak{m}_\chi \hookrightarrow \text{Sym } \mathfrak{g} \twoheadrightarrow S_o$ . It is clear that we have

$$(3.1.1) \quad \mathbb{C}[\mathcal{N} \cap (\chi + \mathfrak{m}^\perp)] = S_o/S_o \cdot \mathfrak{m}_\chi; \quad \text{by analogy, we put } Q_c := \mathcal{U}_c/\mathcal{U}_c \cdot \mathfrak{m}_\chi.$$

The left ideal  $\mathcal{U}_c \cdot \mathfrak{m}_\chi \subset \mathcal{U}_c$ , resp. the ideal  $S_o \cdot \mathfrak{m}_\chi \subset S_o$ , is  $\text{ad } \mathfrak{m}$ -stable. It is clear that the scheme isomorphism (1.3.5) translates into the following algebra isomorphism

$$(3.1.2) \quad \mathbb{C}[\mathcal{S}] \cong (S_o/S_o \cdot \mathfrak{m}_\chi)^{\text{ad } \mathfrak{m}}; \quad \text{by analogy, we put } A_c := (\mathcal{U}_c/\mathcal{U}_c \cdot \mathfrak{m}_\chi)^{\text{ad } \mathfrak{m}}.$$

It is easy to see that multiplication in  $\mathcal{U}\mathfrak{g}$  gives rise to a well-defined (not necessarily commutative) associative algebra structure on  $A_c$ . Furthermore, the above formulas show that the algebra  $\mathbb{C}[\mathcal{S}]$  is obtained from  $\mathbb{C}[\mathcal{N}]$  by a *classical Hamiltonian reduction*, resp. the algebra  $A_c$  is obtained from  $\mathcal{U}_c$  by a *quantum Hamiltonian reduction*. In particular,  $\mathbb{C}[\mathcal{S}]$  has a natural Poisson algebra structure. The family of algebras  $\{A_c, c \in \text{Specm } \mathfrak{Z}\mathfrak{g}\}$  may be viewed as “quantizations” of the Poisson algebra  $\mathbb{C}[\mathcal{S}]$ .

*Remark 3.1.3.* Each of the algebras  $A_c$  is a quotient of a single algebra  $A := (\mathcal{U}\mathfrak{g}/\mathfrak{U}\mathfrak{g} \cdot \mathfrak{m}_\chi)^{\text{ad } \mathfrak{m}}$ . This algebra  $A$ , that has been introduced and studied by Premet in [P1], is a Hamiltonian reduction of the algebra  $\mathcal{U}\mathfrak{g}$ . The natural imbedding  $\mathfrak{Z}\mathfrak{g} \hookrightarrow \mathcal{U}\mathfrak{g}$  descends to a well-defined algebra map  $j: \mathfrak{Z}\mathfrak{g} \rightarrow A$  with central image. It is easy to see that, for any central character  $c$ , one has  $A/A \cdot j(\mathfrak{J}_c) = A_c$ .

It is immediate to check that, for any right  $\mathcal{U}_c$ -module  $N$ , the assignment  $u: n \mapsto nu$  induces a well-defined right  $A_c$ -action on the coinvariant space  $N/N \cdot \mathfrak{m}_\chi$ . This yields, in particular, a right  $A_c$ -action on  $Q_c$  (cf. (3.1.1)) that commutes with the natural left  $\mathcal{U}_c$ -action. In this way,  $Q_c$  becomes an  $(\mathcal{U}_c, A_c)$ -bimodule. Moreover, the right action of  $A_c$  gives an algebra isomorphism  $A_c^{\text{op}} \xrightarrow{\sim} \text{End}_{\mathcal{U}_c} Q_c$ .

**3.2. Kazhdan filtrations.** Given an algebra  $A$  with an ascending  $\mathbb{Z}$ -filtration  $F_\bullet A$ , one puts  $\text{gr}_F A := \bigoplus_{n \in \mathbb{Z}} F_n A / F_{n-1} A$ , an associated graded algebra, resp.  $\text{Rees}_F A := \bigoplus_{n \in \mathbb{Z}} F_n A$ , the Rees algebra. From now on, we assume that  $\text{gr}_F A$  is a finitely generated commutative algebra.

Let  $V$  be a left  $A$  module equipped with an ascending  $\mathbb{Z}$ -filtration  $F_\bullet V$  which is compatible with the one on  $A$ . Then,  $\text{Rees}_F V := \bigoplus_{n \in \mathbb{Z}} F_n V$  acquires the structure of a left  $\text{Rees}_F A$ -module. The filtration on  $V$  is called *good* if that module is finitely generated. In such a case,  $\text{gr}_F V := \bigoplus_{n \in \mathbb{Z}} F_n V / F_{n-1} V$  is a finitely generated  $\text{gr}_F A$ -module, and the *support* of  $\text{gr}_F V$  is a closed subset  $\text{Supp}(\text{gr}_F V) \subset \text{Spec}(\text{gr}_F A)$  (equipped with reduced scheme structure).

The following standard result is well known; cf. [ABO], Theorem 1.8 and Proposition 2.6.

**Lemma 3.2.1.** *Let  $F_\bullet A$  be a  $\mathbb{Z}$ -filtration on an algebra  $A$  such that  $\text{Rees}_F A$  is both left and right noetherian, and  $\text{gr}_F A$  is commutative. Then, for any left  $A$ -module  $V$ , we have*

(i) *All good filtrations on  $V$  are equivalent to each other and the set  $\text{Var} V := \text{Supp}(\text{gr}_F V)$  is independent of the choice of such a filtration.*

(ii) *For any good filtration  $F_\bullet V$  and any  $A$ -submodule  $N \subset V$ , the induced filtration  $F_\bullet N := N \cap F_\bullet V$  is a good filtration on  $N$ . □*

Given a vector space  $V$  equipped with an ascending  $\mathbb{Z}$ -filtration  $F_\bullet V$  and with a direct sum decomposition  $V = \bigoplus_{a \in \mathbb{C}} V\langle a \rangle$ , one defines an associated *Kazhdan filtration*, a new ascending  $\mathbb{Z}$ -filtration on  $V$ , as follows:

$$(3.2.2) \quad K_n V := \sum_{a+2j \leq n} (F_j V \cap V\langle a \rangle).$$

Let  $\mathcal{U}\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathcal{U}\mathfrak{g}\langle i \rangle$  be the  $\mathbb{Z}$ -grading induced by *some* Lie algebra grading  $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}\langle i \rangle$ . We may take  $V := \mathcal{U}\mathfrak{g}$  and let  $F_j V := \mathcal{U}^{\leq j} \mathfrak{g}$ ,  $j = 0, 1, \dots$ , be the canonical ascending PBW filtration on the enveloping algebra. Then, formula (3.2.2) gives an associated Kazhdan filtration  $K_\bullet \mathcal{U}\mathfrak{g}$ , on  $\mathcal{U}\mathfrak{g}$ . This filtration is multiplicative, i.e., one has  $K_i \mathcal{U}\mathfrak{g} \cdot K_j \mathcal{U}\mathfrak{g} \subset K_{i+j} \mathcal{U}\mathfrak{g}$ ,  $\forall i, j$ .

For any  $i \in \mathbb{Z}$ , we have  $\mathfrak{g}\langle i \rangle \in K_{i+2} \mathcal{U}\mathfrak{g}$ . We see as in [GG, §4.2] that the identity map  $\mathfrak{g} \rightarrow \mathfrak{g}$  extends to a Poisson algebra isomorphism  $\text{gr}_K \mathcal{U}\mathfrak{g} \cong \text{Sym } \mathfrak{g}$ . Further, it follows easily that the algebra  $\text{Rees}_K \mathcal{U}\mathfrak{g}$  is isomorphic to a quotient of  $T\mathfrak{g} \otimes \mathbb{C}[t]$  by the two-sided ideal generated by the elements  $x \otimes y - y \otimes x - [x, y] \otimes t^2$ ,  $x, y \in \mathfrak{g}$ . Thus,  $\text{Rees}_K \mathcal{U}\mathfrak{g}$ , is a finitely generated algebra. Moreover,  $\text{Rees}_K \mathcal{U}\mathfrak{g}$ , viewed as an algebra *without grading*, is *independent* of the grading on  $\mathfrak{g}$  used in (3.2.2) (for  $V = \mathcal{U}\mathfrak{g}$ ). For the trivial grading  $\mathfrak{g} = \mathfrak{g}\langle 0 \rangle$ , the filtration  $K_\bullet \mathcal{U}\mathfrak{g}$  is clearly nonnegative, and the corresponding algebra  $\text{Rees}_K \mathcal{U}\mathfrak{g}$  is easily seen to be both left and right noetherian. It follows that  $\text{Rees}_K \mathcal{U}\mathfrak{g}$  is both left and right noetherian for *any* Lie algebra  $\mathbb{Z}$ -grading on  $\mathfrak{g}$ . In particular, Lemma 3.2.1 applies.

From now on, we let  $K_\bullet \mathcal{U}\mathfrak{g}$  be the Kazhdan filtration associated with the grading (1.3.1) by formula (3.2.2). Given a left  $\mathcal{U}\mathfrak{g}$ -module  $V$ , one may consider ascending filtrations  $F_\bullet V$  which are compatible with the PBW filtration on  $\mathcal{U}\mathfrak{g}$ , in the sense that  $\mathcal{U}^{\leq i} \mathfrak{g} \cdot F_j V \subset F_{i+j} V$  holds for any  $i, j \in \mathbb{Z}$ . One may also consider filtrations  $K_\bullet V$  which are compatible with the above defined Kazhdan filtration on  $\mathcal{U}\mathfrak{g}$ , to be referred to as Kazhdan filtrations on  $V$ .

Assume, in addition, that the  $h$ -action on  $V$  is locally finite. Then there is a direct sum decomposition  $V = \bigoplus_{a \in \mathbb{C}} V\langle a \rangle$  where, for any  $a \in \mathbb{C}$ , one defines a generalized  $a$ -eigenspace by the formula  $V\langle a \rangle := \{v \in V \mid \exists N = N(v) : (h - a)^N v = 0\}$ . For any  $h$ -stable ascending filtration  $F_\bullet V$  which is compatible with the PBW filtration on  $\mathcal{U}\mathfrak{g}$ , formula (3.2.2) gives an associated Kazhdan filtration  $K_\bullet V$ , on  $V$ . Clearly, one has  $F_j V = \bigoplus_{a \in \mathbb{C}} (F_j V \cap V\langle a \rangle)$ . Using this, one obtains by an appropriate

“re-grading procedure”, a canonical isomorphism of  $\text{Sym } \mathfrak{g}$ -modules, resp.  $\text{Rees}_\kappa \mathcal{U}\mathfrak{g}$ -modules (that does *not* necessarily respect the natural gradings):

$$(3.2.3) \quad \text{gr}_F V \cong \text{gr}_\kappa V, \quad \text{resp.} \quad \text{Rees}_F V \cong \text{Rees}_\kappa V.$$

**Corollary 3.2.4.** *Let  $V$  be a finitely generated  $\mathcal{U}\mathfrak{g}$ -module such that the  $h$ -action on  $V$  is locally finite. Then, one has:*

(i) *For any good  $h$ -stable filtration  $F_\bullet V$ , on  $V$ , formula (3.2.2) gives a good and separated Kazhdan filtration on  $V$ .*

(ii) *Any good Kazhdan filtration  $\mathbf{K}'_\bullet V$ , on  $V$ , is equivalent to (3.2.2). Hence, it is a separated filtration and, in  $\mathfrak{g}^*$ , one has  $\text{Var}_{\mathbf{K}'} V = \text{Var}_F V$  (set theoretic equality).*

*Proof.* Let  $F_\bullet V$  be a good  $h$ -stable filtration. Clearly, it is bounded from below, hence, separated. It follows that  $\text{Rees}_F V$  is a finitely generated  $\text{Rees}_F \mathcal{U}\mathfrak{g}$ -module and, moreover, we have  $\bigcap_{j \in \mathbb{Z}} t^j \cdot \text{Rees}_F V = 0$ . Thus, from (3.2.3) we deduce that  $\text{Rees}_\kappa V$  is a finitely generated  $\text{Rees}_\kappa \mathcal{U}\mathfrak{g}$ -module and, moreover, we have that  $\bigcap_{j \in \mathbb{Z}} t^j \cdot \text{Rees}_\kappa V = 0$ . This yields (i). Part (ii) is now a consequence of Lemma 3.2.1. □

From now on, in the setting of Corollary 3.2.4, we will use simplified notation  $\text{Var} V$  for  $\text{Var}_{\mathbf{K}'} V = \text{Var}_F V$ .

**3.3. Whittaker functors.** Recall the space  $Q_c = \mathcal{U}_c / \mathcal{U}_c \mathfrak{m}_\chi$ , which is an  $(\mathcal{U}_c, A_c)$ -bimodule.

Associated with any left  $\mathcal{U}\mathfrak{g}$ -module, resp.  $\mathcal{U}_c$ -module,  $V$ , is its *Whittaker subspace*:

$$(3.3.1) \quad \text{Hom}_{\mathcal{U}_c}(Q_c, V) = \text{Wh}^{\mathfrak{m}}(V) := \{v \in V \mid mv = \chi(m)v, \forall m \in \mathfrak{m}\}.$$

The right  $A_c$ -action on  $Q_c$  gives the space  $\text{Wh}^{\mathfrak{m}} V$  a structure of a *left*  $A_c$ -module. Similarly, for any *right*  $\mathcal{U}_c$ -module  $N$ , the right  $A_c$ -action on  $Q_c$  gives the coinvariant space  $N/N\mathfrak{m}_\chi = N \otimes_{\mathcal{U}_c} Q_c$  a structure of *right*  $A_c$ -module.

Below, we are also going to use a version of the Whittaker functor for bimodules. Given an  $(\mathcal{U}_{c'}, \mathcal{U}_c)$ -bimodule  $K$ , the subspace  $K\mathfrak{m}_\chi \subset K$  is stable under the adjoint  $\mathfrak{m}$ -action on  $K$ . Hence, the latter action descends to a well-defined  $\text{ad } \mathfrak{m}$ -action on  $K/K\mathfrak{m}_\chi$ . It is clear that, for any  $x \in K/K\mathfrak{m}_\chi$  and  $m \in \mathfrak{m}$ , one has  $mx = \text{ad } m(x) + \chi(m) \cdot x$ . We see, in particular, that the adjoint  $\mathfrak{m}$ -action and the above defined right  $A_c$ -action on  $K/K\mathfrak{m}_\chi$  commute.

Finally, we put

$$(3.3.2) \quad \text{Wh}_{\mathfrak{m}}^{\mathfrak{m}}(K) := \text{Hom}_{\mathcal{U}_{c'}}(Q_{c'}, K \otimes_{\mathcal{U}_c} Q_c) = (K/K\mathfrak{m}_\chi)^{\text{ad } \mathfrak{m}}.$$

The right  $A_{c'}$ -action on  $Q_{c'}$  and the right  $A_c$ -action on  $Q_c$  make  $\text{Wh}_{\mathfrak{m}}^{\mathfrak{m}}(K)$  an  $(A_{c'}, A_c)$ -bimodule.

**Definition 3.3.3.** Let  $(\mathcal{U}_c, \mathfrak{m}_\chi)$ -mod be the abelian category of finitely generated  $\mathcal{U}_c$ -modules  $V$  satisfying the following condition: *for any  $v \in V$  there exists an integer  $n = n(v) \gg 0$  such that we have  $(m_1 \cdot m_2 \cdot \dots \cdot m_n)v = 0$ ,  $\forall m_1, \dots, m_n \in \mathfrak{m}_\chi$ .*

Objects of the category  $(\mathcal{U}_c, \mathfrak{m}_\chi)$ -mod are called Whittaker modules. It is clear that  $Q_c$  is a Whittaker module. We let  $\mathbf{K} \cdot \mathcal{U}_c$ , resp.  $\mathbf{K} \cdot Q_c$ , be the quotient filtration induced by the Kazhdan filtration on  $\mathcal{U}\mathfrak{g}$ . The Kazhdan filtration on  $Q_c$  induces, by restriction, an algebra filtration  $\mathbf{K} \cdot A_c$ , on  $A_c \subset Q_c$ . Note that, unlike the case of  $\mathbf{K} \cdot \mathcal{U}_c$ , the filtration  $\mathbf{K} \cdot Q_c$ , hence also  $\mathbf{K} \cdot A_c$ , is *nonnegative*.

The proof of the following result repeats the proof of [GG, Proposition 5.2].

**Proposition 3.3.4.** *For any  $c \in \text{Specm } \mathfrak{J}\mathfrak{g}$ , one has a graded Poisson algebra isomorphism  $\text{gr}_{\mathbb{K}} A_c \cong \mathbb{C}[\mathcal{S}]$ ; cf. (3.1.2). Further, one has a graded  $(\text{gr}_{\mathbb{K}} \mathcal{U}_c, \text{gr}_{\mathbb{K}} A_c)$ -bimodule isomorphism  $\text{gr}_{\mathbb{K}} Q_c \cong S_o/S_o\mathfrak{m}_{\chi} =$ ; cf. (3.1.1).  $\square$*

From this proposition, using Proposition 1.2.1 and a result of Bjork [Bj], we deduce:

**Corollary 3.3.5.** *The algebra  $A_c$  is Cohen-Macaulay and Auslander-Gorenstein.  $\square$*

Let  $\mathbf{K}, V$  be a good Kazhdan filtration on an object  $V \in (\mathcal{U}_c, \mathfrak{m}_{\chi})\text{-mod}$ . Using that the Kazhdan filtration on  $Q_c$  is nonnegative, one proves that the filtration  $\mathbf{K}, V$  is bounded from below. If, in addition, the filtration  $\mathbf{K}, V$  is  $\mathfrak{m}$ -stable, then  $\text{gr}_{\mathbb{K}} V$  is an  $M$ -equivariant  $\mathbb{C}[\mathcal{N}]$ -module such that the  $\mathbb{C}[\mathcal{N}]$ -action factors through an action of the algebra  $\mathbb{C}[\mathcal{N} \cap (\chi + \mathfrak{m}^{\perp})]$ . In particular, we have  $\text{Var } V \subset \mathcal{N} \cap (\chi + \mathfrak{m}^{\perp})$ .

One has a graded algebra isomorphism  $\mathbb{C}[\mathcal{N} \cap (\chi + \mathfrak{m}^{\perp})] \cong \mathbb{C}[M] \otimes \mathbb{C}[\mathcal{S}]$  that results from the scheme isomorphism of Proposition 1.3.3(iii). Here, the grading on  $\mathbb{C}[M]$  is the weight grading with respect to the adjoint action of the 1-parameter subgroup  $t \mapsto \gamma_{t^{-1}}$ . Hence, we get  $\text{gr}_{\mathbb{K}} Q_c = \mathbb{C}[M] \otimes \mathbb{C}[\mathcal{S}]$ , by Proposition 3.3.4.

For any noetherian algebra  $B$ , let  $\text{mod } B$  denote the category of finitely generated left  $B$ -modules. Also, in part (ii) of the proposition below, given a filtration on an  $A_c$ -module  $N$ , we equip  $Q_c \otimes_{A_c} N$  with the tensor product filtration using the Kazhdan filtration on  $Q_c$ .

**Proposition 3.3.6.** (i) *The functor  $\text{Wh}^{\mathfrak{m}} : (\mathcal{U}_c, \mathfrak{m}_{\chi})\text{-mod} \rightarrow \text{mod } A_c$  is an equivalence.*

(ii) *For any good filtration on a  $A_c$ -module  $N$ , the natural map  $\mathbb{C}[M] \otimes \text{gr } N \xrightarrow{\sim} \text{gr}_{\mathbb{K}}(Q_c \otimes_{A_c} N)$  yields a graded  $\mathbb{C}[\mathcal{N}]$ -module isomorphism. Furthermore, the functor  $N \mapsto Q_c \otimes_{A_c} N$  provides a quasi-inverse to the equivalence in (i).*

(iii) *For any good  $\mathfrak{m}$ -stable Kazhdan filtration  $\mathbf{K}, V$ , on  $V \in (\mathcal{U}_c, \mathfrak{m}_{\chi})\text{-mod}$ , there are canonical isomorphisms  $\text{gr}(\text{Wh}^{\mathfrak{m}} V) \cong (\text{gr}_{\mathbb{K}} V)^M \cong (\text{gr}_{\mathbb{K}} V)|_{\mathcal{S}}$ , of graded  $\mathbb{C}[\mathcal{S}]$ -modules.  $\square$*

The equivalences in parts (i) and (ii) of the above proposition are due to Skryabin, [Sk], and the graded isomorphisms in parts (ii) and (iii) are immediate consequences of the results of [GG].

#### 4. WEAK HARISH-CHANDRA BIMODULES

4.1. Let  $B$  and  $B'$  be an arbitrary pair of nonnegatively filtered algebras such that  $\text{gr } B$  and  $\text{gr } B'$ , the corresponding associated graded algebras, are finitely generated commutative algebras *isomorphic to each other*. Thus, there is a well-defined subset  $\Delta \subset \text{Spec}(\text{gr } B) \times \text{Spec}(\text{gr } B')$ , the diagonal.

Associated with any finitely generated  $(B, B')$ -bimodule  $K$ , viewed as a left  $B \otimes (B')^{\text{op}}$ -module, is its characteristic variety  $\text{Var } K \subset \text{Spec}(\text{gr } B) \times \text{Spec}(\text{gr } B')$ .

**Definition 4.1.1.** A finitely generated  $(B, B')$ -bimodule  $K$  is called a *weak Harish-Chandra (wHC) bimodule* if, set theoretically, one has  $\text{Var } K \subset \Delta$ .

It is straightforward to show the following:

**Proposition 4.1.2.** (i) *Any wHC  $(B, B')$ -bimodule, viewed either as a left  $B$ -module, or as a right  $B'$ -module, is finitely generated.*

(ii) *The category  $\mathcal{WHC}(B, B')$ , of wHC bimodules, is an abelian category.  $\square$*

Given a closed subset  $Z \subset \text{Spec}(\text{gr } B) = \text{Spec}(\text{gr } B')$ , let  $\text{mod}_Z B$ , resp.  $\text{mod}_Z B'$  be the category of finitely generated left  $B$ -modules, resp.  $B'$ -modules,  $K$  such that  $\text{Var } K \subset Z$ . One similarly defines  $\mathcal{WHC}_Z(B, B')$  to be the Serre subcategory of  $\mathcal{WHC}(B, B')$  formed by the wHC bimodules  $K$  such that  $\text{Var } K \subset Z$ .

A tensor product over  $B'$  gives the bi-functor

$$(4.1.3) \quad \mathcal{WHC}(B, B') \times \text{mod}_Z B \longrightarrow \text{mod}_Z B', \quad K \times V \mapsto K \otimes_{B'} V.$$

We have, in particular, the category  $\mathcal{WHC}(\mathfrak{U}\mathfrak{g}, \mathfrak{U}\mathfrak{g})$ , where the enveloping algebra  $\mathfrak{U}\mathfrak{g}$  is equipped with the PBW filtration, not with Kazhdan filtration. Similarly, for any  $c', c \in \text{Specm } \mathfrak{Z}\mathfrak{g}$ , one has the category  $\mathcal{WHC}(\mathcal{U}_{c'}, \mathcal{U}_c)$ , resp.  $\mathcal{WHC}(A_{c'}, A_c)$ .

A finitely generated  $(\mathfrak{U}\mathfrak{g}, \mathfrak{U}\mathfrak{g})$ -bimodule, resp.  $(\mathcal{U}_{c'}, \mathcal{U}_c)$ -bimodule,  $K$  such that the adjoint  $\mathfrak{g}$ -action,  $\text{ad } a : v \mapsto av - va$  on  $K$  is locally finite, is called a Harish-Chandra bimodule. It is clear that any Harish-Chandra bimodule is a weak Harish-Chandra bimodule. However, the converse is not true, in general; cf. section 6.6. We write  $\mathcal{HC}(\mathcal{U}_{c'}, \mathcal{U}_c)$  for the abelian category of Harish-Chandra  $(\mathcal{U}_{c'}, \mathcal{U}_c)$ -bimodules.

Let  $K$  be a Harish-Chandra  $(\mathcal{U}_{c'}, \mathcal{U}_c)$ -bimodule and let  $F \bullet K$  be a good ad  $\mathfrak{g}$ -stable filtration compatible with the tensor product of PBW filtrations on  $\mathcal{U}_{c'}$  and on  $\mathcal{U}_c^{\text{op}}$ . The ad  $h$ -action on  $K$  being locally finite, one can use formula (3.2.2) to define an associated Kazhdan filtration  $\mathbf{K} \bullet K$  on  $K$ . The latter induces a quotient filtration on  $K/K\mathfrak{m}_\chi$  which gives, by restriction, a filtration  $\mathbf{K} \bullet (\text{Wh}_m^m K)$  on  $\text{Wh}_m^m K$ . It is easy to see that this filtration is compatible with the algebra filtration on  $A_{c'} \otimes A_c^{\text{op}}$ .

Recall the notation  $\mathbb{O} = \text{Ad } G(\chi)$ . Let  $\mathcal{N}^\diamond$  be the union of all nilpotent Ad  $G$ -orbits  $\mathbb{O}' \subset \mathfrak{g}^*$  such that  $\mathbb{O} \not\subset \overline{\mathbb{O}'}$ . Thus,  $\mathcal{N}^\diamond \subset \mathcal{N}$  is a closed Ad  $G$ -stable subset such that  $\mathcal{N}^\diamond \cap \mathcal{S} = \emptyset$ .

The first main result of the paper is the following:

**Theorem 4.1.4.** *Let  $K \in \mathcal{HC}(\mathcal{U}_{c'}, \mathcal{U}_c)$ . Then, the following holds:*

(i) *For any good ad  $\mathfrak{g}$ -stable filtration on  $K$ , the associated filtration  $\mathbf{K} \bullet (\text{Wh}_m^m K)$  is good; moreover, one has a graded isomorphism of  $\text{gr } A_{c'}$ - and  $\text{gr } A_c$ -modules,  $\text{gr}(\text{Wh}_m^m K) \cong (\text{gr } K)|_{\mathcal{S}}$ ; in particular,  $\text{Var}(\text{Wh}_m^m K) = \mathcal{S} \cap \text{Var } K$ .*

(ii) *The functor  $K \otimes_{\mathcal{U}_c} (-)$  takes Whittaker modules to Whittaker modules, and there is a natural isomorphism of functors that makes the following diagram commute:*

$$(4.1.5) \quad \begin{array}{ccc} (\mathcal{U}_c, \mathfrak{m}_\chi)\text{-mod} & \xrightarrow{\text{Wh}^m} & \text{mod } A_c \\ K \otimes_{\mathcal{U}_c} (-) \downarrow & & \downarrow \text{Wh}_m^m K \otimes_{A_c} (-) \\ (\mathcal{U}_{c'}, \mathfrak{m}_\chi)\text{-mod} & \xrightarrow{\text{Wh}^m} & \text{mod } A_{c'} \end{array}$$

(iii) *The assignment  $K \mapsto \text{Wh}_m^m K$  induces a faithful exact functor*

$$\mathcal{HC}(\mathcal{U}_{c'}, \mathcal{U}_c) / \mathcal{HC}_{\mathcal{N}^\diamond}(\mathcal{U}_{c'}, \mathcal{U}_c) \rightarrow \mathcal{WHC}(A_{c'}, A_c).$$

The proof of Theorem 4.1.4 occupies subsections 4.3 and 4.4. From part (i) of the theorem, one immediately obtains

**Corollary 4.1.6.** *For  $K \in \mathcal{HC}(\mathcal{U}_{c'}, \mathcal{U}_c)$ , we have*

$$\begin{aligned} \mathbb{O} \subset \text{Var } K &\iff \text{Wh}_m^m K \neq 0, \\ \overline{\mathbb{O}} = \text{Var } K &\iff \dim(\text{Wh}_m^m K) < \infty. \end{aligned}$$

The following direct consequence of Theorem 4.1.4(ii) says that  $\text{Wh}_m^m$  is a monoidal functor.

**Corollary 4.1.7.** *There is a functorial isomorphism of  $(A_{c'}, A_{c''})$ -bimodules*

$$\begin{aligned} \text{Wh}_m^m(K \otimes_{\mathcal{U}_c} K') &\xrightarrow{\sim} (\text{Wh}_m^m K) \otimes_{A_c} (\text{Wh}_m^m K'), \\ K \in \mathcal{HC}(\mathcal{U}_{c'}, \mathcal{U}_c), K' \in \mathcal{HC}(\mathcal{U}_c, \mathcal{U}_{c''}). &\quad \square \end{aligned}$$

4.2. We are going to construct a functor from  $(A_{c'}, A_c)$ -bimodules to  $(\mathcal{U}_{c'}, \mathcal{U}_c)$ -bimodules as follows. Let  $N$  be an  $(A_{c'}, A_c)$ -bimodule. Then, the space  $Q_{c'} \otimes_{A_{c'}} N$  has an obvious structure of  $(\mathcal{U}_{c'}, A_c)$ -bimodule. Let  $\tilde{I}(N) := \text{Hom}_{A_c}(Q_c, Q_{c'} \otimes_{A_{c'}} N)$  be the space of linear maps  $Q_c \rightarrow Q_{c'} \otimes_{A_{c'}} N$  which commute with the right  $A_c$ -action. The left  $\mathcal{U}_c$ -action on  $Q_c$  and the left  $\mathcal{U}_{c'}$ -action on  $Q_{c'} \otimes_{A_{c'}} N$  make this space an  $(\mathcal{U}_{c'}, \mathcal{U}_c)$ -bimodule. We put

$$(4.2.1) \quad I(N) := \text{Hom}_{A_c}^{\text{fin}}(Q_c, Q_{c'} \otimes_{A_{c'}} N) \subset \tilde{I}(N),$$

the subspace of  $\tilde{I}(N)$  formed by ad  $\mathfrak{g}$ -locally finite elements.

Part (i) of the theorem below will be proved later, in section 6.5. It is stated here for reference purposes.

**Theorem 4.2.2.** (i) *Any object of  $\mathcal{WHC}(A_{c'}, A_c)$  has finite length.*

(ii) *The assignment  $N \mapsto I(N)$  gives a functor  $I : \mathcal{WHC}(A_{c'}, A_c) \rightarrow \mathcal{HC}(\mathcal{U}_{c'}, \mathcal{U}_c)$ , which is a right adjoint of  $\text{Wh}_m^m$ .*

*Proof.* We begin the proof of (ii) by showing the adjunction property. The latter says that, for any  $K \in \mathcal{HC}(\mathcal{U}_{c'}, \mathcal{U}_c)$ ,  $N \in \mathcal{WHC}(A_{c'}, A_c)$ , there is a bifunctorial isomorphism

$$(4.2.3) \quad \text{Hom}_{(A_{c'}, A_c)\text{-bimod}}(\text{Wh}_m^m K, N) \cong \text{Hom}_{(\mathcal{U}_{c'}, \mathcal{U}_c)\text{-bimod}}(K, I(N)).$$

To prove this, using the definition of  $\tilde{I}(N)$ , we compute

$$\begin{aligned} \text{Hom}_{(\mathcal{U}_{c'}, \mathcal{U}_c)\text{-bimod}}(K, \tilde{I}(N)) &= \text{Hom}_{(\mathcal{U}_{c'}, \mathcal{U}_c)\text{-bimod}}(K, \text{Hom}_{\text{mod-}A_c}(Q_c, Q_{c'} \otimes_{A_{c'}} N)) \\ &= \text{Hom}_{(\mathcal{U}_{c'}, \mathcal{U}_c \otimes A_c)\text{-bimod}}(K \otimes Q_c, Q_{c'} \otimes_{A_{c'}} N) \\ (4.2.4) \quad &= \text{Hom}_{(\mathcal{U}_{c'}, A_c)\text{-bimod}}(K \otimes_{\mathcal{U}_c} Q_c, Q_{c'} \otimes_{A_{c'}} N) \\ &= \text{Hom}_{(\mathcal{U}_{c'}, A_c)\text{-bimod}}(K/K\mathfrak{m}_\chi, Q_{c'} \otimes_{A_{c'}} N). \end{aligned}$$

Now, both  $K/K\mathfrak{m}_\chi$  and  $Q_{c'} \otimes_{A_{c'}} N$ , viewed as left  $\mathcal{U}_{c'}$ -modules, are objects of  $(\mathcal{U}_c, \mathfrak{m}_\chi)$ -mod. For these objects, we have

$$\text{Wh}_m^m(K/K\mathfrak{m}_\chi) = \text{Wh}_m^m K \quad \text{and} \quad \text{Wh}_m^m(Q_{c'} \otimes_{A_{c'}} N) = N.$$

Hence, by the Skryabin equivalence, we get  $\text{Hom}_{\mathcal{U}_{c'}}(K/K\mathfrak{m}_\chi, Q_{c'} \otimes_{A_{c'}} N) = \text{Hom}_{A_{c'}}(\text{Wh}_m^m K, N)$ .

We deduce the isomorphism

$$\text{Hom}_{(\mathcal{U}_{c'}, A_c)\text{-bimod}}(K/K\mathfrak{m}_\chi, Q_{c'} \otimes_{A_{c'}} N) = \text{Hom}_{(A_{c'}, A_c)\text{-bimod}}(\text{Wh}_m^m K, N).$$

Thus, from (4.2.4), we obtain

$$(4.2.5) \quad \text{Hom}_{(\mathcal{U}_{c'}, \mathcal{U}_c)\text{-bimod}}(K, \tilde{I}(N)) = \text{Hom}_{(A_{c'}, A_c)\text{-bimod}}(\text{Wh}_m^m K, N).$$

Observe next that since the ad  $\mathfrak{g}$ -action on  $K$  is locally finite, the imbedding  $I(N) \hookrightarrow \tilde{I}(N)$  induces a bijection

$$\text{Hom}_{(\mathcal{U}_{c'}, \mathcal{U}_c)\text{-bimod}}(K, I(N)) \xrightarrow{\sim} \text{Hom}_{(\mathcal{U}_{c'}, \mathcal{U}_c)\text{-bimod}}(K, \tilde{I}(N)).$$

Hence, in the Hom-space on the left of (4.2.5), one may replace  $\tilde{I}(N)$  by  $I(N)$ . The resulting isomorphism yields (4.2.3).

To complete the proof of part (ii) we must show that  $I(N)$  is a finitely generated  $(\mathcal{U}_{c'}, \mathcal{U}_{\mathfrak{g}})$ -bimodule. To this end, we need to enlarge the category  $\mathcal{HC}(\mathcal{U}_{c'}, \mathcal{U}_c)$  as follows. Recall first that  $\mathcal{J}_c \subset \mathfrak{Z}_{\mathfrak{g}}$  denotes the maximal ideal in the center of the enveloping algebra  $\mathcal{U}_{\mathfrak{g}}$ . Let  $K$  be a finitely generated  $(\mathcal{U}_{c'}, \mathcal{U}_{\mathfrak{g}})$ -bimodule. We say that  $K$  is a Harish-Chandra  $(\mathcal{U}_{c'}, \widehat{\mathcal{U}}_c)$ -bimodule if the adjoint  $\mathfrak{g}$ -action on  $K$  is locally finite and, moreover, there exists a large enough integer  $\ell = \ell(K) \gg 0$  such that  $K$  is annihilated by the right action of the ideal  $(\mathcal{J}_c)^\ell$ , that is, we have  $K \cdot (\mathcal{J}_c)^\ell = 0$ . Let  $\mathcal{HC}(\mathcal{U}_{c'}, \widehat{\mathcal{U}}_c)$  be the full subcategory of  $(\mathcal{U}_{c'}, \mathcal{U}_{\mathfrak{g}})$ -bimod whose objects are Harish-Chandra  $(\mathcal{U}_{c'}, \widehat{\mathcal{U}}_c)$ -bimodules. The structure of the category  $\mathcal{HC}(\mathcal{U}_{c'}, \widehat{\mathcal{U}}_c)$  has been analyzed by Bernstein and Gelfand [BGe]. It turns out that any Harish-Chandra  $(\mathcal{U}_{c'}, \widehat{\mathcal{U}}_c)$ -bimodule has finite length. Furthermore, it was shown in *loc cit* that the category  $\mathcal{HC}(\mathcal{U}_{c'}, \widehat{\mathcal{U}}_c)$  has enough projectives and there are only finitely many nonisomorphic indecomposable projectives  $P_j$ ,  $j = 1, \dots, m$ , say.

Next, we observe that the algebra  $A$ , introduced in Remark 3.1.3, comes equipped with a natural ascending filtration such that one has  $\text{gr } A = \mathbb{C}[\mathbb{S}]$ . A finitely generated  $(A_{c'}, A)$ -bimodule  $N$  will be called a weak Harish-Chandra  $(A_{c'}, A)$ -bimodule if one has  $\text{Var } N \subset \mathcal{S} \subset \mathcal{S} \times \mathbb{S}$ , where  $\mathcal{S} \hookrightarrow \mathcal{S} \times \mathbb{S}$  is the diagonal imbedding. We let  $\mathcal{WHC}(A_{c'}, \widehat{A}_c)$  denote the full subcategory of the category  $(A_{c'}, A)$ -bimod whose objects are weak Harish-Chandra  $(A_{c'}, A)$ -bimodules  $N$  such that  $N \cdot (\mathcal{J}_c)^\ell = 0$  holds for a large enough integer  $\ell = \ell(N) \gg 0$ . As we will see later, the arguments of §6.5 can be used to show that any object of the category  $\mathcal{WHC}(A_{c'}, \widehat{A}_c)$  has finite length.

Clearly, the category  $\mathcal{HC}(\mathcal{U}_{c'}, \mathcal{U}_c)$  may be viewed as a full subcategory in  $\mathcal{HC}(\mathcal{U}_{c'}, \widehat{\mathcal{U}}_c)$ , resp. the category  $\mathcal{WHC}(A_{c'}, A_c)$  may be viewed as a full subcategory in  $\mathcal{WHC}(A_{c'}, \widehat{A}_c)$ . It is straightforward to extend our earlier definitions and introduce a functor  $\text{Wh}_m^m : \mathcal{HC}(\mathcal{U}_{c'}, \widehat{\mathcal{U}}_c) \rightarrow \mathcal{WHC}(A_{c'}, \widehat{A}_c)$ . One also defines a functor  $I$  in the opposite direction such that an analogue of formula (4.2.3) holds.

We are now ready to complete the proof of Theorem 4.2.2(ii) by showing that  $I(N)$  is a finitely generated  $(\mathcal{U}_{c'}, \mathcal{U}_{\mathfrak{g}})$ -bimodule, for any  $N \in \mathcal{WHC}(A_{c'}, \widehat{A}_c)$ . It suffices to show, in view of the results of [BGe] cited above, that, for each  $j = 1, \dots, m$ , the vector space  $\text{Hom}_{(\mathcal{U}_{c'}, \widehat{\mathcal{U}}_c)\text{-bimod}}(P_j, I(N))$  has finite dimension. To see this, we use the analogue of formula (4.2.3), which yields

$$(4.2.6) \quad \dim \text{Hom}_{(\mathcal{U}_{c'}, \widehat{\mathcal{U}}_c)\text{-bimod}}(P_j, I(N)) = \dim \text{Hom}_{(A_{c'}, \widehat{A}_c)\text{-bimod}}(\text{Wh}_m^m P_j, N).$$

However, we know that  $\text{Wh}_m^m P_j \in \mathcal{WHC}(A_{c'}, \widehat{A}_c)$  and that the object  $N \in \mathcal{WHC}(A_{c'}, \widehat{A}_c)$  has finite length. It follows that the dimension on the right of formula (4.2.6) is finite, and we are done.  $\square$

**4.3. Homology vanishing.** Recall the Lie subalgebra  $\mathfrak{m}_\chi \subset \mathcal{U}_{\mathfrak{g}}$ . The Kazhdan filtration on  $\mathcal{U}_{\mathfrak{g}}$  restricts to a filtration on  $\mathfrak{m}_\chi$ . The latter induces an ascending filtration  $\mathbf{K}_j(\wedge^\bullet \mathfrak{m}_\chi)$ ,  $j \geq 0$ , on the exterior algebra of  $\mathfrak{m}_\chi$ .

Given a *right*  $\mathcal{U}_{\mathfrak{g}}$ -module  $V$  equipped with a Kazhdan filtration  $\mathbf{K}_\bullet V$ , we form a tensor product  $C := V \otimes \wedge^\bullet \mathfrak{m}_\chi$ , and let  $\mathbf{K}_n C = \sum_{n=i+j} \mathbf{K}_i V \otimes \mathbf{K}_j(\wedge^\bullet \mathfrak{m}_\chi)$  be the tensor product filtration.

We view  $V$  as a  $\mathfrak{m}_\chi$ -module, and write  $H_\bullet(\mathfrak{m}_\chi, V)$  for the corresponding Lie algebra homology with coefficients in  $V$ . The latter may be computed by means of the complex  $(C, \partial)$ , where  $\partial : C \rightarrow C$  is the standard Chevalley-Eilenberg differential, of degree  $(-1)$ . It is immediate to check that the differential respects the filtration  $\mathbf{K}\cdot C$ , making  $C$  a filtered complex.

Write  $B \subset Z \subset C$  for the subspaces of boundaries and cycles of the complex  $C$ , respectively. Thus, we have  $H_\bullet(\mathfrak{m}_\chi, V) = H_\bullet(C) = Z/B$ . The filtration on  $C$  induces, by restriction, a filtration  $\mathbf{K}_p Z := Z \cap \mathbf{K}_p C$ , on the space of cycles. The latter filtration induces a quotient filtration on homology. Explicitly, the induced filtration on homology is given by

$$(4.3.1) \quad \mathbf{K}_p H(C) := (B + \mathbf{K}_p Z)/B = \text{Im}[H(\mathbf{K}_p C) \rightarrow H(C)], \quad p \in \mathbb{Z}.$$

There is an associated standard spectral sequence with the 0-th term (cf. [CE], Ch. 15),

$$(4.3.2) \quad E_{p,q}^0 = \wedge^{p+q} \mathfrak{m}_\chi \otimes \text{gr}_{-q} V.$$

Recall the local algebra  $\mathcal{O}_\chi$  introduced above Corollary 1.3.8. The lemma below is a slight generalization of a result due to Holland; cf. [Ho], §2.4.

**Lemma 4.3.3.** *Let  $V$  be a right  $\mathcal{U}\mathfrak{g}$ -module equipped with a good Kazhdan filtration  $\mathbf{K}\cdot V$ . Assume that the localization of  $\text{gr}_\mathbf{K} V$  at  $\chi + \mathfrak{m}^\perp$  is a flat  $\varpi^\bullet \mathcal{O}_\chi$ -module and, moreover, the induced filtration on  $C$ , the Chevalley-Eilenberg complex, is convergent in the sense of [CE], p. 321.*

*Then, for any  $j > 0$ , we have  $H_j(\mathfrak{m}_\chi, \text{gr}_\mathbf{K} V) = 0$  and  $H_j(\mathfrak{m}_\chi, V) = 0$ . Moreover, the natural projection yields a canonical graded  $\mathbb{C}[\mathfrak{g}^*]$ -module isomorphism*

$$\text{gr}_\mathbf{K} V / (\text{gr}_\mathbf{K} V) \mathfrak{m}_\chi \xrightarrow{\sim} \text{gr}_\mathbf{K} (V / V \mathfrak{m}_\chi) = \text{gr}_\mathbf{K} H_0(\mathfrak{m}_\chi, V).$$

*Proof.* We recall various standard objects associated with the spectral sequence of a filtered complex. We follow [CE], Ch. 15, §1 and §2 closely, except that we use homological, rather than cohomological notation.

First of all, for each  $p \in \mathbb{Z}$ , one defines a pair of vector spaces  $Z_p^\infty$  and  $B_p^\infty$ , in such a way that one has

$$(4.3.4) \quad \text{gr}_p H_\bullet(\mathfrak{m}_\chi, V) = \text{gr}_p H_\bullet(C) \cong Z_p^\infty / B_p^\infty.$$

Further, one defines a chain of vector spaces (cf. [CE], p. 317),

$$\dots \subset B_p^r \subset B_p^{r+1} \subset \dots \subset B_p^\infty \subset Z_p^\infty \subset \dots \subset Z_p^{r+1} \subset Z_p^r \subset \dots$$

Precise definitions of these objects are not important for our purposes, they are given e.g. in [CE], Ch. 15, §1. What is important for us is that the definitions imply  $B_p^\infty = \bigcup_{r \geq 0} B_p^r$ . The assumption of the lemma that the filtration  $\mathbf{K}\cdot C$  be convergent means that, in addition, one has (see [CE], Ch. 15, §2),

$$(4.3.5) \quad Z_p^\infty = \bigcap_{r \geq 0} Z_p^r \quad \text{and} \quad \bigcap_{r \geq 0} \mathbf{K}_r H_p(C) = 0, \quad \forall p \in \mathbb{Z}.$$

Now, to prove the lemma, we observe that the zeroth page  $(E^0, d^0)$  of the spectral sequence (4.3.2) may be identified with the Koszul complex associated with the subscheme  $\varpi^{-1}(\chi_\mathfrak{m}) \subset \mathfrak{g}^*$ . Thus, the assumption of the lemma that the localization of  $\text{gr}_\mathbf{K} V$  at  $\chi + \mathfrak{m}^\perp$  be a flat  $\varpi^\bullet \mathcal{O}_\chi$ -module forces the Koszul complex to be acyclic in positive degrees. Therefore, the spectral sequence degenerates at  $E^1$ .

The degeneration implies that, for any  $p \in \mathbb{Z}$  and  $r \geq 1$ , one has  $Z_p^r = Z_p^1$  and  $B_p^r = B_p^1$ . Therefore, we have  $B_p^1 = B_p^\infty$ , and using the first equation in (4.3.5), we get  $Z_p^1 = Z_p^\infty$ . Hence,  $H(E^0, d^0) = Z^1/B^1 = Z_p^\infty/B_p^\infty = \text{gr}_p H(\mathfrak{m}_\chi, V)$ , by (4.3.4). We conclude that one has  $\text{gr}_K H_0(\mathfrak{m}_\chi, V) = (\text{gr}_k V)/(\text{gr}_k V)\mathfrak{m}_\chi$  and, moreover,  $\text{gr}_K H_j(\mathfrak{m}_\chi, V) = 0$  for any  $j > 0$ . Finally, thanks to the second equation in (4.3.5), we have  $\text{gr}_K H_j(\mathfrak{m}_\chi, V) = 0 \Rightarrow H_j(\mathfrak{m}_\chi, V) = 0$ , and the lemma is proved.  $\square$

It is important to observe that the filtration  $K_\bullet C$ , on  $C$ , gives rise to *two* natural filtrations on the subspace  $B = \partial C$ , of the boundaries. These two filtrations are defined as follows:

$$(4.3.6) \quad K_\bullet B := \partial(K_\bullet C), \quad \text{resp.} \quad K'_\bullet B := B \cap K_\bullet C.$$

It is clear that one has  $K_\bullet B \subset K'_\bullet B$ , but this inclusion need not be an equality, in general. We have the following result:

**Lemma 4.3.7.** *Assume the above filtrations  $K_\bullet B$  and  $K'_\bullet B$  are equivalent, i.e. there exists an integer  $\ell \geq 1$  such that, for all  $p \in \mathbb{Z}$ , one has  $K'_{p-\ell} B \subset K_{p-1} B$ .*

*Then, we have  $Z_p^\infty = \bigcap_{r \geq 0} Z_p^r$ ,  $\forall p \in \mathbb{Z}$ , i.e. the first equation in (4.3.5), holds.*

*Proof.* For any  $\ell \geq 1$ , we have  $K_{\bullet-\ell} C \subset K_{\bullet-1} C$ , hence, one gets an obvious imbedding  $B \cap K_{\bullet-\ell} C \hookrightarrow Z \cap K_{\bullet-1} C$ . Using the definitions of various filtrations introduced above, this imbedding may be rewritten as  $K'_{\bullet-\ell} B \hookrightarrow K_{\bullet-1} Z$ . We may further compose the imbedding with a projection to homology to obtain the following composite:

$$(4.3.8) \quad \delta : K'_{\bullet-\ell} B \hookrightarrow K_{\bullet-1} Z \twoheadrightarrow K_{\bullet-1} Z / K_{\bullet-1} B = K_{\bullet-1} Z / \partial(K_{\bullet-1} C) = H(K_{\bullet-1} C).$$

Next, we fix  $p \in \mathbb{Z}$  and consider the complex  $K_p C / K_{p-\ell} C$ . By definition, we have

$$H(K_p C / K_{p-\ell} C) = \frac{\{z \in K_p C \mid \partial(z) \in K_{p-\ell} C\}}{K_{p-\ell} Z + \partial(K_p C)} = \frac{\{z \in K_p C \mid \partial(z) \in K'_{p-\ell} B\}}{K_{p-\ell} Z + \partial(K_p C)}.$$

The differential  $\partial$  clearly annihilates the space  $K_{p-\ell} Z + \partial(K_p C)$ , in the denominator of the rightmost term. Therefore, we see that applying the differential  $\partial$  to the numerator of that term yields a well-defined map

$$\partial : H(K_p C / K_{p-\ell} C) \rightarrow K'_{p-\ell} B, \quad z \mapsto \partial(z).$$

Thus, we have constructed the following diagram:

$$H(K_\bullet C / K_{\bullet-\ell} C) \xrightarrow{\partial} K'_{\bullet-\ell} B \xrightarrow{\delta} H(K_{\bullet-1} C).$$

Now, let  $\ell$  be such that the assumption of the lemma holds, so that we have  $K'_{\bullet-\ell} B \subset K_{\bullet-1} B$ . Then the corresponding map  $\delta$ , in (4.3.8), clearly vanishes. It follows that the composite map  $\delta \circ \partial$  vanishes as well, and we have

$$\text{Im} [\delta \circ \partial : H(K_\bullet C / K_{\bullet-\ell} C) \rightarrow H(K_{\bullet-1} C)] = 0.$$

The last equation insures that we are in a position to apply a criterion given in [CE, Ch. 15, Proposition 2.1]. Applying that criterion yields the statement of the lemma.  $\square$

4.4. **Proof of Theorem 4.1.4.** In this subsection, we fix  $K \in \mathcal{HC}(\mathcal{U}_{c'}, \mathcal{U}_c)$ .

Choose a finite dimensional  $\text{ad } \mathfrak{g}$ -stable subspace  $K_0 \subset K$  that generates  $K$  as a bimodule. For any  $\ell \geq 0$ , let

$$F_\ell K := \sum_{i+j \leq \ell} \mathcal{U}^{\leq i} \mathfrak{g} \cdot K_0 \cdot \mathcal{U}^{\leq j} \mathfrak{g} = \mathcal{U}^{\leq \ell} \mathfrak{g} \cdot K_0 = K_0 \cdot \mathcal{U}^{\leq \ell} \mathfrak{g}.$$

In this way, one may define a good  $\text{ad } \mathfrak{g}$ -stable filtration on  $K$ .

Now, let  $F_\bullet K$  be an arbitrary good  $\text{ad } \mathfrak{g}$ -stable filtration on  $K$ . Let  $\mathbf{K}_\bullet K$  be the Kazhdan filtration associated with the filtration  $F_\bullet K$  via formula (3.2.2).

We first view  $K$  as a right  $\mathfrak{m}_\chi$ -module. A key step in the proof of Theorem 4.1.4 is played by the following:

**Lemma 4.4.1.** *For all  $j > 0$ , we have  $H_j(\mathfrak{m}_\chi, \text{gr}_K K) = 0$  and  $H_j(\mathfrak{m}_\chi, K) = 0$ .*

*Furthermore, the canonical projection  $\text{gr}_K K / (\text{gr}_K K) \mathfrak{m}_\chi \xrightarrow{\sim} \text{gr}_K (K / K \mathfrak{m}_\chi)$  is an isomorphism of  $M$ -equivariant  $\mathbb{C}[\mathcal{N}]$ -modules.*

*Proof.* The result is clearly a consequence of Lemma 4.3.3, provided we show that all the assumptions of that lemma hold in our present setup.

First, we verify the assumption that, for the above defined Kazhdan filtration on  $K$ , the  $\varpi^\bullet \mathcal{O}_\chi$ -module  $\varpi^* \mathcal{O}_\chi \otimes_{\mathbb{C}[\mathcal{N}]} \text{gr}_K K$  is flat. To this end, we note that the construction of the filtration  $F_\bullet K$  insures that  $\text{gr}_F K$  is an  $\text{Ad } G$ -equivariant finitely generated  $\mathbb{C}[\mathcal{N}]$ -module, where  $\mathcal{N} \subset \mathcal{N} \times \mathcal{N}$  is the diagonal copy of the nilpotent variety. By Corollary 1.3.8(ii), we conclude that the localization of  $\text{gr}_F K$  at  $\chi + \mathfrak{m}^\perp$  is a flat  $\varpi^\bullet \mathcal{O}_\chi$ -module.

Further, by (3.2.3), we have a  $\mathbb{C}[\mathcal{N}]$ -module isomorphism  $\text{gr}_F K \cong \text{gr}_K K$ . Moreover, it is immediate from definitions that this isomorphism is compatible with  $\text{ad } \mathfrak{m}$ -actions on each side. It follows, in particular, that  $\varpi^* \mathcal{O}_\chi \otimes_{\mathbb{C}[\mathcal{N}]} \text{gr}_K K$  is a flat  $\varpi^\bullet \mathcal{O}_\chi$ -module, and we have an  $\text{Ad } M$ -equivariant  $\mathbb{C}[\mathcal{N}]$ -module isomorphism  $\text{gr}_K K / (\text{gr}_K K) \mathfrak{m}_\chi \cong \text{gr}_F K / (\text{gr}_F K) \mathfrak{m}_\chi$ .

To complete the proof, we show that our filtration  $\mathbf{K}_\bullet K$  is convergent, i.e., both equations in (4.3.5) hold in the situation at hand.

To see this, we observe that the filtration on  $K$  being  $\text{ad } \mathfrak{g}$ -stable, it is good as a filtration on  $K$  viewed as a left  $\mathcal{U}_{c'}$ -module. Further, the differential in the Chevalley-Eilenberg complex  $C := \wedge^\bullet \mathfrak{m}_\chi \otimes K$ , involved in Lemma 4.3.3, is clearly a morphism of left  $\mathcal{U}_{c'}$ -modules. Thus, the subspace  $B = \partial C \subset C$ , of the boundaries of the Chevalley-Eilenberg complex, as well as each homology group  $H_\bullet(\mathfrak{m}_\chi, K)$ , acquires a natural structure of left  $\mathcal{U}_{c'}$ -module. Being a subquotient of  $C$ , any of these  $\mathcal{U}_{c'}$ -modules is finitely generated. Hence, each of the two Kazhdan filtrations on  $B$  defined in (4.3.6), as well as the Kazhdan filtration on  $H_\bullet(\mathfrak{m}_\chi, K)$  defined in (4.3.1), is a good filtration on the corresponding left  $\mathcal{U}_{c'}$ -module. It follows, in particular, that the two filtrations on  $B$  are equivalent; cf. Lemma 3.2.1. Thus, the first equation in (4.3.5) holds by Lemma 4.3.7.

It remains to prove that the Kazhdan filtration on  $H_\bullet(\mathfrak{m}_\chi, K)$  defined in (4.3.1) is separated. Recall that any good Kazhdan filtration on an object of the category  $(\mathcal{U}_c, \mathfrak{m}_\chi)\text{-mod}$  is bounded below, hence, separated. Thus, it suffices to show that, for any  $j \geq 0$ ,  $H_j(\mathfrak{m}_\chi, K)$  viewed as a left  $\mathcal{U}_c$ -module, is an object of  $(\mathcal{U}_c, \mathfrak{m}_\chi)\text{-mod}$ .

To this end, we recall that the Chevalley-Eilenberg complex of a right module over a Lie algebra has a natural action of that Lie algebra, by the ‘‘Lie derivative’’. It is well known that the Lie derivative action on the Chevalley-Eilenberg complex

induces the zero action on each homology group. Applying this to our Harish-Chandra  $\mathcal{U}\mathfrak{g}$ -bimodule  $K$ , we see that the complex  $C = K \otimes \wedge^\bullet \mathfrak{m}_\chi$  has a left  $\mathfrak{g}$ -action, defined as a tensor product of the left  $\mathfrak{g}$ -action on  $K$  and the zero  $\mathfrak{g}$ -action on  $\wedge^\bullet \mathfrak{m}_\chi$ . There is also an  $\mathfrak{m}_\chi$ -action, by the ‘‘Lie derivative’’. The left  $\mathfrak{g}$ -action and the  $\mathfrak{m}_\chi$ -action on  $C$  commute, and the difference of the left  $\mathfrak{m}_\chi$ -action and the Lie derivative  $\mathfrak{m}_\chi$ -action gives a well-defined  $\mathfrak{m}_\chi$ -action on  $C$ , to be called the adjoint action. The adjoint  $\mathfrak{g}$ -action on  $K$  being locally finite and the Lie algebra  $\mathfrak{m}_\chi$  being nilpotent, it follows easily that the adjoint  $\mathfrak{m}_\chi$ -action on  $C$  is locally nilpotent. We conclude that the left  $\mathfrak{m}_\chi$ -action on  $H_j(\mathfrak{m}_\chi, K)$ , induced by the left  $\mathfrak{m}_\chi$ -action on  $C$ , may be written as a sum of a locally nilpotent adjoint action and of the Lie derivative action, the latter being known to be the zero action. Thus, we have shown that  $H_j(\mathfrak{m}_\chi, K) \in (\mathcal{U}_c, \mathfrak{m}_\chi)$ -mod. This completes the proof.  $\square$

*Proof of Theorem 4.1.4.* It follows from the preceding paragraph that we have  $K/K\mathfrak{m}_\chi = H_0(\mathfrak{m}_\chi, K) \in (\mathcal{U}_c, \mathfrak{m}_\chi)$ -mod; cf. (3.3.2). Thus, we get a functor  $\text{Wh}_m : \mathcal{HC}(\mathcal{U}_{c'}, \mathcal{U}_c) \rightarrow (\mathcal{U}_c, \mathfrak{m}_\chi)$ -mod,  $K \mapsto K/K\mathfrak{m}_\chi$ . The homology vanishing of Lemma 4.4.1 implies that this functor is *exact*. Since the functor  $\text{Wh}^m : (\mathcal{U}_c, \mathfrak{m}_\chi)$ -mod  $\rightarrow A_c$ -mod is an equivalence (cf. Proposition 3.3.6(i)), we deduce the exactness of the composite functor  $\text{Wh}^m \circ \text{Wh}_m$ . The exactness statement of part (iii) of the theorem now follows by writing  $\text{Wh}_m^m = \text{Wh}^m \circ \text{Wh}_m$ .

Next, fix an ad  $\mathfrak{g}$ -stable good filtration on  $K$ , and write  $\text{Wh}_m^m K = \text{Wh}^m(K/K\mathfrak{m}_\chi)$ . It follows, in particular, that the induced filtration on  $K/K\mathfrak{m}_\chi$  is  $\mathfrak{m}$ -stable. Further, by Lemma 4.4.1, we get (below,  $(-)|_{\mathcal{S}}$  stands for a restriction of a  $\mathbb{C}[\mathcal{N}]$ -module to the subvariety  $\mathcal{S} \subset \mathfrak{g}^*$ )

$$\begin{aligned} \text{gr Wh}_m^m K &= \text{gr Wh}^m(K/K\mathfrak{m}_\chi) = [\text{gr}(K/K\mathfrak{m}_\chi)]|_{\mathcal{S}} \\ &= [\text{gr}_K K / (\text{gr}_K K)\mathfrak{m}_\chi]|_{\mathcal{S}} = (\text{gr}_K K)|_{\mathcal{S}}. \end{aligned}$$

where the second equality is due to Proposition 3.3.6(ii) applied to the object  $V = K/K\mathfrak{m}_\chi \in (\mathcal{U}_c, \mathfrak{m}_\chi)$ -mod. This proves part (i) of the theorem.

Observe that proving commutativity of the diagram of part (ii) is equivalent, thanks to Skryabin’s equivalences (cf. Proposition 3.3.6(i) and (ii)), to showing commutativity of the following diagram:

$$(4.4.2) \quad \begin{array}{ccc} (\mathcal{U}_c, \mathfrak{m}_\chi)\text{-mod} & \xleftarrow{Q_c \otimes_{A_c} (-)} & \text{mod } A_c \\ K \otimes_{\mathcal{U}_c} (-) \downarrow & & \downarrow \text{Wh}_m^m K \otimes_{A_c} (-) \\ (\mathcal{U}_{c'}, \mathfrak{m}_\chi)\text{-mod} & \xleftarrow{Q_{c'} \otimes_{A_{c'}} (-)} & \text{mod } A_{c'}. \end{array}$$

To prove (4.4.2), write  $\text{Wh}_m^m K = \text{Wh}^m \circ \text{Wh}_m K$ . Thus, one has a canonical map

$$(4.4.3) \quad Q_{c'} \otimes_{A_{c'}} \text{Wh}^m(\text{Wh}_m K) \xrightarrow{\Phi} \text{Wh}_m K = K/K\mathfrak{m}_\chi = K \otimes_{\mathcal{U}_c} Q_c.$$

Since  $\text{Wh}_m K \in (\mathcal{U}_{c'}, \mathfrak{m}_\chi)$ -mod, the map  $\Phi$  is actually an isomorphism, by Skryabin’s equivalence. Hence, tensoring diagram (4.4.3) with a left  $A_c$ -module  $N$ , we get a chain of isomorphisms

$$Q_{c'} \otimes_{A_{c'}} \text{Wh}_m^m K \otimes_{A_c} N = (Q_{c'} \otimes_{A_{c'}} \text{Wh}_m^m K) \otimes_{A_c} N \xrightarrow{(4.4.3)} (K \otimes_{\mathcal{U}_c} Q_c) \otimes_{A_c} N.$$

The composite isomorphism above provides the isomorphism of functors that makes diagram (4.4.2) commute, and Theorem 4.1.4(ii) follows.

We now complete the proof of part (iii) of the theorem. To this end, pick a good  $\text{ad } \mathfrak{g}$  stable filtration on our  $w\mathcal{H}\mathcal{C}$  bimodule  $K$ . We know by part (i) that  $\text{gr } \text{Wh}_m^m K = (\text{gr}_K K)|_{\mathcal{S}}$ . This is clearly a finitely generated  $\mathbb{C}[\mathcal{S} \times \mathcal{S}]$ -module supported on the diagonal in  $\mathcal{S} \times \mathcal{S}$ . It follows that  $\text{Wh}_m^m K$  is itself finitely generated and we have  $\text{Wh}_m^m K \in \mathcal{WH}\mathcal{C}(A_{c'}, A_c)$ .

Further, it is clear that if  $\text{Supp}(\text{gr } \text{Wh}_m^m K) \subset \mathcal{N}^\diamond$ , then we have  $\text{gr } \text{Wh}_m^m K = (\text{gr}_K K)|_{\mathcal{S}} = 0$  since  $\mathcal{N}^\diamond \cap \mathcal{S} = \emptyset$ . The filtration on  $\text{Wh}_m^m K$  is bounded below, hence, separated. Hence, the equation  $\text{gr } \text{Wh}_m^m K = 0$  implies  $\text{Wh}_m^m K = 0$ . Thus, we have shown that the functor  $\text{Wh}_m^m$  factors through the quotient category  $\mathcal{H}\mathcal{C}(\mathcal{U}_{c'}, \mathcal{U}_c) / \mathcal{H}\mathcal{C}_{\mathcal{N}^\diamond}(\mathcal{U}_{c'}, \mathcal{U}_c)$ .

It remains to show that the resulting functor  $\text{Wh}_m^m$  is faithful. To prove this, observe that for  $K \in \mathcal{H}\mathcal{C}(\mathcal{U}_{c'}, \mathcal{U}_c) / \mathcal{H}\mathcal{C}_{\mathcal{N}^\diamond}(\mathcal{U}_{c'}, \mathcal{U}_c)$  the equation  $\text{Wh}_m^m K = 0$  implies  $K = 0$ , by the first equivalence of Corollary 4.1.6. The faithfulness of  $\text{Wh}_m^m$  is now a consequence of the following general “abstract nonsense” result: *Let  $F : \mathcal{C} \rightarrow \mathcal{C}'$  be an exact functor between abelian categories such that  $F(M) = 0$  implies  $M = 0$ . Then  $F$  is faithful.*  $\square$

**4.5. Some applications.** Given an algebra  $B$  and a left, resp. right,  $B$ -module  $N$ , let  $\text{Ann}_B N$  denote the annihilator of  $N$ , a two-sided ideal in  $B$ .

Corollary 4.1.6 easily implies the main result of Matumoto, [Ma]. Matumoto’s result says that, for any finitely generated right  $\mathcal{U}_c$ -module  $M$ , one has

$$(4.5.1) \quad \text{Wh}_m M \neq 0 \implies \mathbb{O} \subset \text{Var}(\mathcal{U}_c / \text{Ann}_{\mathcal{U}_c} M).$$

To see this, put  $I := \text{Ann}_{\mathcal{U}_c} M$ . Since  $M$  is finitely generated, one can find an integer  $n \geq 1$  and an  $\mathcal{U}_c$ -module surjection  $(\mathcal{U}_c/I)^{\oplus n} \twoheadrightarrow M$ . The functor of coinvariants being right exact, we get a surjection  $\text{Wh}_m(\mathcal{U}_c/I)^{\oplus n} \twoheadrightarrow \text{Wh}_m M$ . Hence,  $\text{Wh}_m M \neq 0$  implies  $\text{Wh}_m(\mathcal{U}_c/I) \neq 0$ . We conclude that  $\text{Wh}_m^m(\mathcal{U}_c/I) \neq 0$ , by Proposition 3.3.6(i). Now, (4.5.1) follows from the first equivalence of Corollary 4.1.6 applied to  $K = \mathcal{U}_c/I$ .  $\square$

As a second application of our technique, we provide a new proof of a result (Theorem 4.5.2 below) conjectured by Premet [P2, Conjecture 3.2]. In the special case of rational central characters, the conjecture was first proved (using characteristic  $p$  methods) by Premet in [P3], and shortly afterwards by Losev [Lo1] in full generality. An alternate proof of the general case was later obtained in [P4]. Our approach is totally different from those used by Losev and Premet.

**Theorem 4.5.2.** *For any primitive ideal  $I \subset \mathcal{U}_c$  such that  $\text{Var}(\mathcal{U}_c/I) = \overline{\mathbb{O}}$ , there exists a simple finite dimensional  $A_c$ -module  $N$  such that one has  $\text{Ann}_{\mathcal{U}_c}(Q_c \otimes_{A_c} N) = I$ .*

*Proof.* Let  $I \subset \mathcal{U}_c$  be a primitive ideal such that  $\text{Var}(\mathcal{U}_c/I) = \overline{\mathbb{O}}$ . Corollary 4.1.6 implies that  $\text{Wh}_m^m(\mathcal{U}_c/I)$  is a finite dimensional  $(A_c, A_c)$ -bimodule.

Let  $N$  be a simple left  $A_c$ -submodule of  $\text{Wh}_m^m(\mathcal{U}_c/I)$ , the latter being viewed as a left  $A_c$ -module. By Skryabin’s Proposition 3.3.6, we have a diagram of  $\mathcal{U}_c$ -modules,

$$Q_c \otimes_{A_c} N \hookrightarrow Q_c \otimes_{A_c} \text{Wh}_m^m(\mathcal{U}_c/I) = \text{Wh}_m(\mathcal{U}_c/I) = \mathcal{U}_c / (I + \mathcal{U}_c \mathfrak{m}_\chi) \leftarrow \mathcal{U}_c / I.$$

Let  $J = \text{Ann}_{\mathcal{U}_c}(Q_c \otimes_{A_c} N)$ . From the above diagram, we get  $J \supset \text{Ann}_{\mathcal{U}_c} \mathcal{U}_c / (I + \mathcal{U}_c \mathfrak{m}_\chi) \supset I$ . Moreover,  $\text{Var}(\mathcal{U}_c/I)$  is an  $\text{Ad } G$ -stable subset of  $\mathcal{N}$ , and we have  $\{\chi\} \subset \text{Var}(Q_c \otimes_{A_c} N) \subset \text{Var}(\mathcal{U}_c/J)$ . Hence,  $\mathbb{O} \subset \text{Var}(\mathcal{U}_c/J)$  and we get  $\dim \text{Var}(\mathcal{U}_c/I) =$

$\dim \mathbb{O} \leq \dim \text{Var}(\mathcal{U}_c/J)$ . Since,  $I \subset J$ , we conclude that  $\dim \text{Var}(\mathcal{U}_c/I) = \dim \text{Var}(\mathcal{U}_c/J)$ .

Now, since  $N$  is a simple finite dimensional left  $A_c$ -module, we deduce using Skryabin’s equivalence that  $Q_c \otimes_{A_c} N$  is a simple  $\mathcal{U}_c$ -module. Thus,  $J$  is a primitive ideal in  $\mathcal{U}_c$ . The equation  $\dim \text{Var}(\mathcal{U}_c/I) = \dim \text{Var}(\mathcal{U}_c/J)$  combined with the inclusion  $I \subset J$  forces  $I = J$ , due to a result by Borho and Kraft [BoK], Korollar 3.6.  $\square$

*Remark 4.5.3.* Set  $n := \dim \mathcal{S}$ . The assignment  $N \mapsto \text{Ext}_{A_c\text{-mod}}^n(N, A_c)$  is an *exact* functor on the category of finite dimensional left  $A_c$ -modules, thanks to Corollary 3.3.5. That functor gives a contravariant duality

$$\mathcal{WHC}_{\{\chi\}}(A_c, A_c) \xrightarrow{\sim} \mathcal{WHC}_{\{\chi\}}(A_c, A_c)^{\text{op}}.$$

5.  $\mathcal{D}$ -MODULES

**5.1. Whittaker  $\mathcal{D}$ -modules.** Let  $\mathfrak{h}$  denote the abstract Cartan algebra for the Lie algebra  $\mathfrak{g}$ , and let  $\mathbb{X}^+ \subset \mathfrak{h}^*$  be the semigroup of dominant integral weights.

For any integral weight  $\lambda \in \mathfrak{h}^*$  one has a  $G$ -equivariant line bundle  $\mathcal{O}(\lambda)$  on  $\mathcal{B}$ . For any  $\nu \in \mathfrak{h}^*$ , let  $\mathcal{D}_\nu$  denote the sheaf of  $\nu$ -twisted algebraic differential operators on  $\mathcal{B}$ ; see [BB1]. In the case where  $\nu$  is integral, we have  $\mathcal{D}_\nu = \mathcal{D}(\mathcal{O}(\nu))$  is the sheaf of differential operators acting in the sections of the line bundle  $\mathcal{O}(\nu)$ . There is a canonical algebra isomorphism  $\mathcal{U}_\nu \cong \Gamma(\mathcal{B}, \mathcal{D}_\nu)$ ; see [BB1].

Let  $\mathcal{V}$  be a coherent  $\mathcal{D}_\nu$ -module on  $\mathcal{B}$  that has, viewed as a quasi-coherent  $\mathcal{O}_\mathcal{B}$ -module, an additional  $M$ -equivariant structure. Given an element  $x \in \mathfrak{m}$ , we write  $x_\mathcal{D}$  for the action on  $\mathcal{V}$  of the vector field corresponding to  $x$  via the  $\mathcal{D}_\nu$ -module structure, and  $x_M$  for the action on  $\mathcal{V}$  obtained by differentiating the  $M$ -action arising from the equivariant structure.

We say that an  $M$ -equivariant  $\mathcal{D}_\nu$ -module  $\mathcal{V}$  is an  $(\mathfrak{m}, \chi)$ -Whittaker  $\mathcal{D}_\nu$ -module if, for any  $x \in \mathfrak{m}$  and  $v \in \mathcal{V}$ , we have  $(x_\mathcal{D} - x_M)v = \chi(x) \cdot v$ . Write  $(\mathcal{D}_\nu, \mathfrak{m}_\chi)\text{-mod}$  for the abelian category of  $(\mathfrak{m}, \chi)$ -Whittaker coherent  $\mathcal{D}_\nu$ -modules on  $\mathcal{B}$ .

We put  $\mathcal{Q}_\nu := \mathcal{D}_\nu / \mathcal{D}_\nu \cdot \mathfrak{m}_\chi$ . This  $\mathcal{D}_\nu$ -module is clearly an object of  $(\mathcal{D}_\nu, \mathfrak{m}_\chi)\text{-mod}$ . For any  $\mathcal{D}_\nu$ -module  $\mathcal{V}$  from the definitions one finds  $\text{Hom}_{\mathcal{D}_\nu}(\mathcal{Q}_\nu, \mathcal{V}) = \text{Wh}^{\mathfrak{m}}(\Gamma(\mathcal{B}, \mathcal{V}))$ . This space has a right  $A_\nu$ -module structure. In particular, taking  $\mathcal{V} = \mathcal{Q}_\nu$ , one obtains an algebra homomorphism  $A_\nu^{\text{op}} \rightarrow \text{Hom}_{\mathcal{D}_\nu}(\mathcal{Q}_\nu, \mathcal{Q}_\nu)$  that makes  $\mathcal{Q}_\nu$  a  $(\mathcal{D}_\nu, A_\nu)$ -bimodule.

*Remark 5.1.1.* The natural  $\mathcal{D}$ -module projection  $\mathcal{D}_\nu \rightarrow \mathcal{Q}_\nu$  induces a  $\mathcal{U}_\nu$ -module map  $Q_\nu = \mathcal{U}_\nu / \mathcal{U}_\nu \mathfrak{m}_\chi \rightarrow \Gamma(\mathcal{B}, \mathcal{Q}_\nu)$ . The latter map turns out to be an isomorphism, according to a special case ( $\lambda = 0$ ) of Corollary 5.4.2(ii). If  $\nu$  is a dominant weight, then the same isomorphism follows from the Beilinson-Bernstein theorem [BB1]. In any case, one concludes that the canonical algebra map  $A_\nu^{\text{op}} \rightarrow \text{Hom}_{\mathcal{D}_\nu}(\mathcal{Q}_\nu, \mathcal{Q}_\nu)$  is an isomorphism as well.

For any regular and dominant  $\nu \in \mathfrak{h}^*$ , the localization theorem of Beilinson and Bernstein [BB1] yields a category equivalence

$$\Gamma(\mathcal{B}, -) : (\mathcal{D}_\nu, \mathfrak{m}_\chi)\text{-mod} \xrightarrow{\sim} (\mathcal{U}_\nu, \mathfrak{m}_\chi)\text{-mod}.$$

Combining the Beilinson-Bernstein and the Skryabin equivalences, one obtains, see [GG, Proposition 6.2], the following result:

**Proposition 5.1.2.** *For any regular and dominant  $\nu \in \mathfrak{h}^*$ , the following functors provide mutually inverse equivalences, of abelian categories,*

$$(\mathcal{D}_\nu, \mathfrak{m}_\chi)\text{-mod} \begin{array}{c} \xrightarrow{\text{Hom}_{\mathcal{D}_\nu}(\mathcal{Q}_\nu, -)} \\ \xleftarrow{\mathcal{Q}_\nu \otimes_{A_\nu} (-)} \end{array} \text{mod } A_\nu . \quad \square$$

For any Borel subalgebra  $\mathfrak{b}$ , let  $\mathbb{C}_{\mathfrak{b}}$  denote the corresponding 1-dimensional  $\mathcal{O}_{\mathcal{B}}$ -module, a skyscraper sheaf at the point  $\{\mathfrak{b}\} \subset \mathcal{B}$ . Recall diagram (2.3.2). The following result is standard:

**Lemma 5.1.3.** *For any  $\mathfrak{b} \in \mathcal{B} \setminus \sigma(\Sigma)$  and any  $\mathcal{V} \in (\mathcal{D}_\nu, \mathfrak{m}_\chi)\text{-mod}$ , we have that  $\mathbb{C}_{\mathfrak{b}} \stackrel{L}{\otimes}_{\mathcal{O}_{\mathcal{B}}} \mathcal{V} = 0$ .*

*Proof.* Fix  $\mathfrak{b} \in \mathcal{B}$  and write  $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$  for the nil-radical of  $\mathfrak{b}$ . Given a  $\mathcal{D}_\nu$ -module  $\mathcal{V}$ , let  $V := \Gamma(\mathcal{B}, \mathcal{V})$  be the corresponding  $\mathcal{U}_\nu$ -module. The Beilinson-Bernstein theory yields, in particular, a canonical isomorphism

$$\text{Tor}_\bullet^{\mathcal{O}_{\mathcal{B}}}(\mathbb{C}_{\mathfrak{b}}, \mathcal{V}) = H_\bullet(\mathfrak{n}, V)_o,$$

where  $H_\bullet(\mathfrak{n}, -)$  denotes the Lie algebra homology functor, and the subscript “ $o$ ” stands for a certain particular weight space of the natural Cartan subalgebra action on homology.

Assume now that  $\mathcal{V} \in (\mathcal{D}_\nu, \mathfrak{m}_\chi)\text{-mod}$ . Then,  $V \in (\mathcal{U}_\nu, \mathfrak{m}_\chi)\text{-mod}$ . Therefore, for any element  $x \in \mathfrak{m} \cap \mathfrak{n}$ , the natural action of  $x$  on  $H_\bullet(\mathfrak{n}, V)$  is such that the operator  $x - \chi(x)$  is nilpotent. On the other hand, the action of any Lie algebra on its homology is trivial, hence, the element  $x$  induces the zero operator on  $H_\bullet(\mathfrak{n}, V)$ . Thus, we must have  $\chi(x) = 0$ . Proposition 2.3.3(i) completes the proof.  $\square$

**5.2. Translation bimodules.** Let  $\mathcal{O}_{T^*\mathcal{B}}(\lambda)$  be the pull-back of the line bundle  $\mathcal{O}(\lambda)$ , on  $\mathcal{B}$ , via the cotangent bundle projection  $T^*\mathcal{B} \rightarrow \mathcal{B}$  and let  $\mathcal{O}_{\tilde{\mathcal{S}}}(\lambda)$  denote the restriction of the line bundle  $\mathcal{O}_{T^*\mathcal{B}}(\lambda)$  to the Slodowy variety  $\tilde{\mathcal{S}} \subset T^*\mathcal{B}$ . The sheaf  $\mathcal{O}_{\tilde{\mathcal{S}}}(\lambda)$  is equivariant with respect to the  $\bullet$ -action on  $\tilde{\mathcal{S}}$ . This gives the space  $\Gamma(\tilde{\mathcal{S}}, \mathcal{O}_{\tilde{\mathcal{S}}}(\lambda))$  a  $\mathbb{Z}$ -grading, which is bounded from below since  $\Gamma(\tilde{\mathcal{S}}, \mathcal{O}_{\tilde{\mathcal{S}}}(\lambda))$  is a finitely generated  $\mathbb{C}[\mathcal{S}]$ -module.

For any  $\nu \in \mathfrak{h}^*$  and an integral weight  $\lambda$ , we put  $\mathcal{D}_\nu^{\nu+\lambda} := \mathcal{O}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{D}_\nu$ , resp.  $\mathcal{Q}_\nu^{\nu+\lambda} := \mathcal{O}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{Q}_\nu$ . Let  $U_\nu^{\nu+\lambda} := \Gamma(\mathcal{B}, \mathcal{D}_\nu^{\nu+\lambda})$ , resp.  $Q_\nu^{\nu+\lambda} := \Gamma(\mathcal{B}, \mathcal{Q}_\nu^{\nu+\lambda})$ . Further, we define

$$A_\nu^{\nu+\lambda} = (Q_\nu^{\nu+\lambda})^{\text{ad } \mathfrak{m}} = \text{Wh}^{\mathfrak{m}}(Q_\nu^{\nu+\lambda}) = \text{Wh}^{\mathfrak{m}}(\Gamma(\mathcal{B}, \mathcal{O}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{Q}_\nu)).$$

The space  $A_\nu^{\nu+\lambda}$  comes equipped with a natural  $(A_{\lambda+\nu}, A_\nu)$ -bimodule structure.

The standard filtration on  $\mathcal{D}_\nu$  by the order of differential operator gives rise to a tensor product filtration on  $\mathcal{O}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{D}_\nu$ , where the factor  $\mathcal{O}(\lambda)$  is assigned filtration degree 0. This induces a natural ascending filtration on the vector space  $Q_\nu^{\nu+\lambda}$ , resp.  $A_\nu^{\nu+\lambda}$ . Further, the  $\bullet$ -action on  $\mathcal{B}$  gives a  $\mathbb{Z}$ -grading on the above vector spaces; cf §5.3 below. Hence, formula (3.2.2) provides the vector space  $Q_\nu^{\nu+\lambda}$ , resp.  $A_\nu^{\nu+\lambda}$ , with a natural Kazhdan filtration  $\text{K}Q_\nu^{\nu+\lambda}$ , resp.  $\text{K}A_\nu^{\nu+\lambda}$ .

Main properties of the bimodules  $A_\nu^{\nu+\lambda}$  are summarized in the following result.

**Proposition 5.2.1.** *Let  $\nu$  be dominant regular weights. Then, for any  $\lambda, \mu \in \mathbb{X}^+$ , we have:*

- (i)  $A_\nu^{\nu+\lambda}$  is finitely generated and projective as a left  $A_{\nu+\lambda}$ -module, as well as a right  $A_\nu$ -module.
- (ii) The natural pairing  $A_{\nu+\lambda}^{\nu+\lambda+\mu} \otimes_{A_{\nu+\lambda}} A_\nu^{\nu+\lambda} \rightarrow A_\nu^{\nu+\lambda+\mu}$  is an isomorphism.
- (iii) If  $\lambda$  is sufficiently dominant, then there is a canonical graded space isomorphism

$$(5.2.2) \quad \mathrm{gr}_K A_\nu^{\nu+\lambda} = \Gamma(\tilde{\mathcal{S}}, \mathcal{O}_{\tilde{\mathcal{S}}}(\lambda)),$$

which is compatible with the pairings in (ii).

- (iv) The following translation functor is an equivalence:

$$\Gamma_\nu^{\nu+\lambda} : \mathrm{mod} A_\nu \rightarrow \mathrm{mod} A_{\nu+\lambda}, \quad M \mapsto A_\nu^{\nu+\lambda} \otimes_{A_\nu} M.$$

*Remark 5.2.3.* Applying  $\mathrm{gr}(-)$  to the pairing of part (ii) of the proposition, one obtains a pairing

$$\mathrm{gr}_K A_{\nu+\lambda}^{\nu+\lambda+\mu} \otimes_{\mathrm{gr}_K A_{\nu+\lambda}} \mathrm{gr}_K A_\nu^{\nu+\lambda} \rightarrow \mathrm{gr}_K A_\nu^{\nu+\lambda+\mu}.$$

The last statement of Proposition 5.2.1(iii) means that the latter pairing gets identified, via the the isomorphisms in (5.2.2), with the natural pairing

$$\Gamma(\tilde{\mathcal{S}}, \mathcal{O}_{\tilde{\mathcal{S}}}(\lambda)) \otimes \Gamma(\tilde{\mathcal{S}}, \mathcal{O}_{\tilde{\mathcal{S}}}(\mu)) \rightarrow \Gamma(\tilde{\mathcal{S}}, \mathcal{O}_{\tilde{\mathcal{S}}}(\lambda + \mu)),$$

induced by the sheaf morphism  $\mathcal{O}_{\tilde{\mathcal{S}}}(\lambda) \otimes \mathcal{O}_{\tilde{\mathcal{S}}}(\mu) \rightarrow \mathcal{O}_{\tilde{\mathcal{S}}}(\lambda + \mu)$ .

We begin the proof of Proposition 5.2.1 with the following result that relates “geometric” and “algebraic” translation functors.

**Lemma 5.2.4.** *For any dominant regular weight  $\nu$  and any  $\lambda \in \mathbb{X}^+$ , the following diagram commutes:*

$$\begin{array}{ccc} (\mathcal{D}_\nu, \mathfrak{m}_\chi)\text{-mod} & \xrightarrow{\mathrm{Wh}^m \Gamma(\mathcal{B}, -)} & \mathrm{mod} A_\nu \\ \mathcal{O}(\lambda) \otimes_{\mathcal{O}_B} (-) \downarrow & & \downarrow \Gamma_\nu^{\nu+\lambda} \\ (\mathcal{D}_{\nu+\lambda}, \mathfrak{m}_\chi)\text{-mod} & \xrightarrow{\mathrm{Wh}^m \Gamma(\mathcal{B}, -)} & \mathrm{mod} A_{\nu+\lambda} \end{array}$$

*Proof.* First of all, we claim that the projection  $\mathcal{D}_\nu^{\nu+\lambda} \twoheadrightarrow \mathcal{D}_{\nu+\lambda}^{\nu+\lambda}$  induces an isomorphism

$$(5.2.5) \quad A_\nu^{\nu+\lambda} = \mathrm{Wh}_m^m \Gamma(\mathcal{B}, \mathcal{D}_\nu^{\nu+\lambda}).$$

To prove this, we are going to use Corollary 5.4.2 from section 5.4 below, which is independent of the intervening material. We have

$$A_\nu^{\nu+\lambda} = \mathrm{Wh}^m Q_\nu^{\nu+\lambda} = \mathrm{Wh}^m \Gamma(\mathcal{B}, \mathcal{D}_\nu^{\nu+\lambda}) = \mathrm{Wh}^m [\Gamma(\mathcal{B}, \mathcal{D}_\nu^{\nu+\lambda}) / \Gamma(\mathcal{B}, \mathcal{D}_\nu^{\nu+\lambda}) \mathfrak{m}_\chi],$$

where the last equality holds thanks to part (ii) of Corollary 5.4.2. The rightmost term in the above chain of equalities is nothing but  $\mathrm{Wh}_m^m \Gamma(\mathcal{B}, \mathcal{D}_\nu^{\nu+\lambda})$ , and formula (5.2.5) is proved.

Now, let  $\mathcal{V} \in (\mathcal{D}_\nu, \mathfrak{m}_\chi)\text{-mod}$ . One clearly has  $\mathcal{O}(\lambda) \otimes_{\mathcal{O}_B} \mathcal{V} = \mathcal{D}_\nu^{\nu+\lambda} \otimes_{\mathcal{D}_\nu} \mathcal{V}$ . Therefore, applying the Beilinson-Bernstein equivalence, we obtain

$$(5.2.6) \quad \Gamma(\mathcal{B}, \mathcal{O}(\lambda) \otimes_{\mathcal{O}_B} \mathcal{V}) = \Gamma(\mathcal{B}, \mathcal{D}_\nu^{\nu+\lambda}) \otimes_{\mathcal{U}_\nu} \Gamma(\mathcal{B}, \mathcal{V}).$$

We put  $K := \Gamma(\mathcal{B}, \mathcal{D}_\nu^{\nu+\lambda})$  so that (5.2.5) reads  $A_\nu^{\nu+\lambda} = \text{Wh}_m^m K$ . By Theorem 4.1.4(ii), we get

$$\text{Wh}^m(K \otimes_{\mathcal{U}_\nu} \Gamma(\mathcal{B}, \mathcal{V})) = \text{Wh}_m^m K \otimes_{A_\nu} \text{Wh}^m \Gamma(\mathcal{B}, \mathcal{V}) = A_\nu^{\nu+\lambda} \otimes_{A_\nu} \text{Wh}^m \Gamma(\mathcal{B}, \mathcal{V}).$$

The leftmost term in the last formula equals  $\text{Wh}^m \Gamma(\mathcal{B}, \mathcal{O}(\lambda) \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{V})$ , by (5.2.6). Thus, we have proved that  $\text{Wh}^m \Gamma(\mathcal{B}, \mathcal{O}(\lambda) \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{V}) = \mathbb{T}_\nu^{\nu+\lambda}(\Gamma(\mathcal{B}, \mathcal{V}))$ , and the lemma follows.  $\square$

*Proof of Proposition 5.2.1.* For dominant and regular  $\nu, \lambda$ , the horizontal functors in each row of the diagram of Lemma 5.2.4 are the equivalences of Proposition 5.1.2. Further, the functor  $\mathcal{O}(\lambda) \otimes_{\mathcal{O}_\mathcal{B}} (-)$  given by the vertical arrow on the left of the diagram is clearly an equivalence. We conclude that the functor  $\mathbb{T}_\nu^{\nu+\lambda}$  given by the vertical arrow on the right of the diagram is an equivalence as well, and (iv) is proved.

Note that the equivalence of part (iv) yields, in particular, an algebra isomorphism

$$(5.2.7) \quad \text{End}_{A_{\nu+\lambda}}(A_\nu^{\nu+\lambda}) = \text{End}_{A_{\nu+\lambda}}(\mathbb{T}_\nu^{\nu+\lambda}(A_\nu)) = \text{End}_{A_\nu}(A_\nu) = A_\nu^{\text{op}}.$$

Thus, the  $(A_{\nu+\lambda}, A_\nu)$ -bimodule  $A_\nu^{\nu+\lambda}$  fits the standard Morita context. Hence, the general Morita theory implies all the statements from part (i) of the proposition.

We now prove (ii). To this end, we observe that the composite functor  $\mathbb{T}_\nu^{\nu+\lambda+\mu} \circ \mathbb{T}_\nu^{\nu+\lambda}$  is clearly given by tensoring with  $A_{\nu+\lambda}^{\nu+\lambda+\mu} \otimes_{A_{\nu+\lambda}} A_\nu^{\nu+\lambda}$ , an  $(A_{\nu+\lambda+\mu}, A_\nu)$ -bimodule. Therefore, the canonical pairing in (ii) induces a morphism of functors

$$(5.2.8) \quad \mathbb{T}_\nu^{\nu+\lambda+\mu} \circ \mathbb{T}_\nu^{\nu+\lambda} \rightarrow \mathbb{T}_\nu^{\nu+\lambda+\mu}.$$

We may transport the latter morphism via the equivalences provided by Proposition 5.1.2. In this way, we get a morphism  $\mathcal{O}(\mu) \otimes_{\mathcal{O}_\mathcal{B}} (-) \circ \mathcal{O}(\lambda) \otimes_{\mathcal{O}_\mathcal{B}} (-) \rightarrow \mathcal{O}(\mu + \lambda) \otimes_{\mathcal{O}_\mathcal{B}} (-)$ , of functors  $(\mathcal{D}_\nu, \mathfrak{m}_\chi)\text{-mod} \rightarrow (\mathcal{D}_{\nu+\lambda+\mu}, \mathfrak{m}_\chi)\text{-mod}$ . It is clear from commutativity of diagram (4.1.5) that the latter morphism of functors is the one induced by the canonical morphism of sheaves  $\psi : \mathcal{O}(\mu) \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{O}(\lambda) \rightarrow \mathcal{O}(\mu + \lambda)$ .

Now, the morphism of sheaves  $\psi$  is clearly an isomorphism. It follows that the associated morphism of functors is an isomorphism as well. Thus, we conclude that the morphism in (5.2.8) is an isomorphism. For the corresponding bimodules, this implies that the pairing in (ii) must be an isomorphism.

The proof of part (iii) of the proposition will be given in §5.4.  $\square$

**5.3. Characteristic varieties.** We are going to define a *Kazhdan filtration* on  $\mathcal{D}_\nu$ . To this end, view  $\mathcal{B}$  as a  $\mathbb{C}^\times$ -variety via the action  $\mathbb{C}^\times \ni t : \mathfrak{b} \mapsto \text{Ad } \gamma_t(\mathfrak{b})$ . Any  $\mathbb{C}^\times$ -orbit in an arbitrary quasi-projective  $\mathbb{C}^\times$ -variety is known to be contained in an *affine* Zariski-open  $\mathbb{C}^\times$ -stable subset. Thus, we may view  $\mathcal{D}_\nu$  as a sheaf in the topology formed by Zariski-open  $\mathbb{C}^\times$ -stable subsets of  $\mathcal{B}$ .

For any Zariski-open subset  $U \subset \mathcal{B}$ , the order filtration on differential operators gives a filtration on the vector space  $\Gamma(U, \mathcal{D}_\nu)$ . If, in addition,  $U$  is  $\mathbb{C}^\times$ -stable, then the  $\mathbb{C}^\times$ -action gives a weight decomposition  $\Gamma(U, \mathcal{D}_\nu) = \bigoplus_{i \in \mathbb{Z}} \Gamma(U, \mathcal{D}_\nu)\langle i \rangle$ . We are therefore in a position to define an associated Kazhdan filtration on  $\Gamma(U, \mathcal{D}_\nu)$  by formula (3.2.2). For the associated graded sheaf, one has a canonical isomorphism  $\text{gr}_K \mathcal{D}_\nu = p_* \mathcal{O}_{T^*\mathcal{B}}$ .

Let  $K, \mathcal{V}$  be a good  $M$ -stable Kazhdan filtration on a  $\mathcal{D}$ -module  $\mathcal{V} \in (\mathcal{D}_\nu, \mathfrak{m}_\chi)\text{-mod}$ . Write  $\tilde{\text{gr}}_K \mathcal{V}$  for the  $M$ -equivariant coherent sheaf on  $T^*\mathcal{B}$  such that  $p_* \tilde{\text{gr}}_K \mathcal{V} = \text{gr}_K \mathcal{V}$ .

For any weight  $\lambda$ , the filtration on  $\mathcal{V}$  induces one on  $\mathcal{O}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{V}$ , hence, also on the vector space  $\Gamma(\mathcal{B}, \mathcal{O}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{V})$  and on  $\mathrm{Wh}^m \Gamma(\mathcal{B}, \mathcal{O}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{V})$ , by restriction.

**Lemma 5.3.1.** *In the above setting, for all sufficiently dominant  $\lambda \in \mathbb{X}^+$ , there is a canonical isomorphism*

$$\Gamma(\tilde{\mathcal{S}}, \mathcal{O}_{\tilde{\mathcal{S}}}(\lambda) \otimes_{\mathcal{O}_{\tilde{\mathcal{S}}}} (\tilde{\mathrm{gr}}_{\mathcal{K}} \mathcal{V}|_{\tilde{\mathcal{S}}})) \xrightarrow{\sim} \mathrm{gr}_{\mathcal{K}} \mathrm{Wh}^m (\Gamma(\mathcal{B}, \mathcal{O}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{V})).$$

*Proof.* Since  $\mathcal{V} \in (\mathcal{D}_{\nu}, \mathfrak{m}_{\chi})$ -mod, any good Kazhdan filtration  $\mathcal{K}_{\bullet} \mathcal{V}$  on  $\mathcal{V}$  is bounded below and we have  $\mathrm{Supp} \tilde{\mathrm{gr}}_{\mathcal{K}} \mathcal{V} \subset \Sigma$ . For any integral  $\lambda$ , put  $\mathcal{V}(\lambda) = \mathcal{O}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{V}$ . It is clear that  $\mathcal{K}_{\bullet} \mathcal{V}(\lambda) = \mathcal{O}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{K}_{\bullet} \mathcal{V}$ . Thus,  $\tilde{\mathrm{gr}}_{\mathcal{K}} \mathcal{V}(\lambda) = \mathcal{O}_{\Sigma}(\lambda) \otimes_{\mathcal{O}_{\Sigma}} \tilde{\mathrm{gr}}_{\mathcal{K}} \mathcal{V}$  is a coherent sheaf supported on  $\Sigma$ .

We apply the functor  $\mathrm{Wh}^m \Gamma(\mathcal{B}, -)$  to  $\mathcal{V}(\lambda)$ , a filtered sheaf. Thus, there is a standard convergent spectral sequence involving  $R^{\bullet} \mathrm{Wh}^m \Gamma(\mathcal{B}, -)$ , the right derived functors of the composite functor  $\mathrm{Wh}^m \Gamma(\mathcal{B}, -)$ . The spectral sequence reads

$$(5.3.2) \quad E_1 = (R^{\bullet} \mathrm{Wh}^m \Gamma(\mathcal{B}, \mathrm{gr}_{\mathcal{K}} \mathcal{V}(\lambda))) \Rightarrow \mathrm{gr}_{\mathcal{K}} (R^{\bullet} \mathrm{Wh}^m \Gamma(\mathcal{B}, \mathcal{V}(\lambda))),$$

where  $R^{\bullet} \mathrm{Wh}^m \Gamma(\mathcal{B}, -)$  stand for the derived functors of  $\mathrm{Wh}^m \Gamma(\mathcal{B}, -)$ , a left exact functor.

We claim that, for all  $\lambda$  dominant enough, one has  $R^i \mathrm{Wh}^m \Gamma(\mathcal{B}, \mathrm{gr}_{\mathcal{K}} \mathcal{V}(\lambda)) = 0$  for any  $i > 0$ . That would yield the collapse of the above spectral sequence which would give, in turn, the required canonical isomorphism

$$\mathrm{Wh}^m \Gamma(\mathcal{B}, \mathrm{gr}_{\mathcal{K}} \mathcal{V}(\lambda)) \xrightarrow{\sim} \mathrm{gr}_{\mathcal{K}} \mathrm{Wh}^m \Gamma(\mathcal{B}, \mathcal{V}(\lambda)).$$

To complete the proof we must check the above claim that  $R^i \mathrm{Wh}^m \Gamma(\mathcal{B}, \mathrm{gr}_{\mathcal{K}} \mathcal{V}(\lambda)) = 0$  for any  $i > 0$ . To this end, write  $q : \Sigma \rightarrow \Sigma/M = \tilde{\mathcal{S}}$  for the projection; cf. (2.1.5). Observe further that applying the functor  $\mathrm{Wh}^m$  to the vector space  $\Gamma(\Sigma, \tilde{\mathrm{gr}} \mathcal{V}(\lambda))$  amounts to taking  $M$ -invariants. Thus, writing  $\mathrm{Inv}^M$  for the functor of  $M$ -invariants, we get

$$\begin{aligned} \mathrm{Wh}^m \Gamma(\mathcal{B}, \mathrm{gr}_{\mathcal{K}} \mathcal{V}(\lambda)) &= \mathrm{Inv}^M \Gamma(\Sigma, \tilde{\mathrm{gr}} \mathcal{V}(\lambda)) = \mathrm{Inv}^M \Gamma(\tilde{\mathcal{S}}, q_{\bullet} \tilde{\mathrm{gr}} \mathcal{V}(\lambda)) \\ &= \Gamma(\tilde{\mathcal{S}}, \mathrm{Inv}^M q_{\bullet} \tilde{\mathrm{gr}} \mathcal{V}(\lambda)). \end{aligned}$$

Since the  $M$ -action on  $\Sigma$  is free, one can repeat the argument in the proof of [GG], formula (6.1) and Proposition 5.2, to show that the functor  $\mathrm{Inv}^M q_{\bullet}$  is exact (and is isomorphic to the functor of restriction to the closed submanifold  $\tilde{\mathcal{S}} \subset \Sigma$ ). Further, using the Springer resolution  $\pi : \tilde{\mathcal{S}} \rightarrow \mathcal{S}$ , we can write  $\Gamma(\tilde{\mathcal{S}}, -) = \Gamma(\mathcal{S}, \pi_{\bullet}(-))$ . Therefore, for any  $i \geq 0$ , we have isomorphisms of derived functors

$$R^i \mathrm{Wh}^m \Gamma(\mathcal{B}, -) \cong R^i \Gamma(\tilde{\mathcal{S}}, \mathrm{Inv}^M q_{\bullet}(-)) = \Gamma(\mathcal{S}, R^i \pi_{\bullet}(\mathrm{Inv}^M q_{\bullet}(-))),$$

where in the last isomorphism we have used that  $\Gamma(\mathcal{S}, -)$  is an exact functor since  $\mathcal{S}$  is affine.

Recall next that, for a dominant regular weight  $\lambda$  the sheaf  $\mathcal{O}_{\Sigma}(\lambda)$  is relatively ample with respect to the Springer resolution  $\pi$ . Therefore, for  $\lambda$  dominant enough, one has  $R^i \pi_{\bullet}(\mathrm{Inv}^M q_{\bullet} \tilde{\mathrm{gr}}_{\mathcal{K}} \mathcal{V}(\lambda)) = 0$ , for any  $i > 0$ . Hence, for such  $\lambda$ , and  $i > 0$ , we obtain  $R^i \mathrm{Wh}^m \Gamma(\mathcal{B}, \mathrm{gr}_{\mathcal{K}} \mathcal{V}(\lambda)) = 0$ , and we are done.  $\square$

**5.4. Harish-Chandra  $\mathcal{D}$ -modules.** For any pair  $\mu, \nu \in \mathfrak{h}^*$ , we put  $\mathcal{D}_{\mu, \nu} := \mathcal{D}_{\mu} \boxtimes \mathcal{D}_{\nu}$ , a sheaf of twisted differential operators on  $\mathcal{B} \times \mathcal{B}$ . Let  $G^{sc}$  denote a simply-connected cover of the semisimple group  $G$ . The group  $G^{sc}$  acts diagonally on  $\mathcal{B} \times \mathcal{B}$ .

We may consider the restriction of this action to the 1-parameter subgroup  $\mathbb{C}^\times$  and corresponding Kazhdan filtrations on the sheaf  $\mathcal{D}_{\mu,\nu}$  as well as on  $\mathcal{D}_{\mu,\nu}$ -modules.

Recall that a  $\mathcal{D}_{\mu,\nu}$ -module is called  $G$ -monodromic if it is  $G^{sc}$ -equivariant (with respect to the diagonal action on  $\mathcal{B} \times \mathcal{B}$ ). Let  $U \subset \mathcal{B} \times \mathcal{B}$  a Zariski open subset stable under the  $\mathbb{C}^\times$ -diagonal action. Then, for any  $G$ -monodromic  $\mathcal{D}_{\mu,\nu}$ -module  $\mathcal{V}$ , the induced  $\mathbb{C}^\times$ -action on  $\Gamma(U, \mathcal{V})$  is locally finite. Hence, any  $G$ -stable filtration on  $\mathcal{V}$  gives an associated Kazhdan filtration (3.2.2).

We define a set  $\mathcal{Z} := \{u \times v \in T^*\mathcal{B} \times T^*\mathcal{B} \mid \pi(u) = -\pi(v)\}$ , where  $\pi$  is the Springer resolution (2.1.1). We will view  $\mathcal{Z}$  as a closed *reduced* subscheme in  $T^*\mathcal{B} \times T^*\mathcal{B} = T^*(\mathcal{B} \times \mathcal{B})$ , called *Steinberg variety*. The assignment  $u \times v \mapsto \pi(u)$  gives a proper morphism  $\pi : \mathcal{Z} \rightarrow \mathcal{N}$ . For any  $G$ -monodromic  $\mathcal{D}_{\mu,\nu}$ -module  $\mathcal{V}$ , one has  $\text{Var } \mathcal{V} \subset \mathcal{Z}$ .

Let  $\mathcal{V}$  be a coherent  $\mathcal{D}_{\mu,\nu}$ -module and let  $pr : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$  denote the first projection. We abuse the notation and write  $\mathcal{V}/\mathcal{V}\mathfrak{m}_\chi := pr_*[\mathcal{V}/(1 \boxtimes \mathfrak{m}_\chi)\mathcal{V}]$  for a sheaf-theoretic direct image of the sheaf of coinvariants with respect to the action of the Lie algebra  $1 \boxtimes \mathfrak{m}_\chi \subset \mathcal{D}_\mu \boxtimes \mathcal{D}_\nu$ . A filtration on  $\mathcal{V}$  induces one on  $\mathcal{V}/\mathcal{V}\mathfrak{m}_\chi$ .

**Proposition 5.4.1.** *Let  $\mathcal{V}$  be a  $G$ -monodromic  $\mathcal{D}_{\mu,\nu}$ -module equipped with a  $G$ -stable good filtration. Then, the induced filtration on the  $\mathcal{D}_\mu$ -module  $\mathcal{V}/\mathcal{V}\mathfrak{m}_\chi$  is good and we have  $\mathcal{V}/\mathcal{V}\mathfrak{m}_\chi \in (\mathcal{D}_\nu, \mathfrak{m}_\chi)$ -mod. Furthermore, the corresponding Kazhdan filtration on  $\mathcal{V}/\mathcal{V}\mathfrak{m}_\chi$  is good as well.*

*In addition, we have  $H_j(\mathfrak{m}_\chi, \text{gr}_K \mathcal{V}) = 0$  and  $H_j(\mathfrak{m}_\chi, \mathcal{V}) = 0$ , for any  $j > 0$ ; moreover, the canonical map  $\text{gr}_K \mathcal{V}/(\text{gr}_K \mathcal{V})\mathfrak{m}_\chi \rightarrow \text{gr}_K(\mathcal{V}/\mathcal{V}\mathfrak{m}_\chi)$  is an isomorphism.*

*Proof.* Given a  $G$ -stable good filtration  $F \cdot \mathcal{V}$ , write  $\tilde{\text{gr}}_F \mathcal{V}$  for the coherent  $\mathcal{O}_{T^*\mathcal{B} \times T^*\mathcal{B}}$ -module such that  $p_* \tilde{\text{gr}}_F \mathcal{V} = \text{gr}_F \mathcal{V}$ . Then, we have that  $\tilde{\text{gr}}_F \mathcal{V}$  is a  $G^{sc}$ -equivariant coherent sheaf on  $\mathcal{Z}$ . The composite  $\tilde{pr} : \mathcal{Z} \hookrightarrow T^*\mathcal{B} \times T^*\mathcal{B} \rightarrow T^*\mathcal{B}$ , of the closed imbedding and the first projection, is a proper morphism. Hence, the push-forward  $\tilde{pr}_*(\tilde{\text{gr}}_F \mathcal{V})$  is a coherent sheaf on  $T^*\mathcal{B}$ . It follows that the induced filtration on  $\mathcal{V}/\mathcal{V}\mathfrak{m}_\chi$ , viewed as a left  $\mathcal{D}_\mu$ -module, is good and that  $\mathcal{V}/\mathcal{V}\mathfrak{m}_\chi$  is a coherent  $\mathcal{D}_\mu$ -module.

Now, applying Corollary 1.3.8(ii) to the morphism  $\pi : \mathcal{Z} \rightarrow \mathcal{N}$  we deduce that the stalk of the sheaf  $\tilde{\text{gr}}_F \mathcal{V}$  at any point of  $\pi^{-1}(\chi + \mathfrak{m}^\perp)$  is a flat  $(\varpi \circ \pi)^* \mathcal{O}_\chi$ -module. At this point, the proof of Lemma 4.4.1 (cf. the proof of Theorem 4.1.4 given in §4.4) goes through verbatim in our present situation.  $\square$

The Kazhdan filtration on  $\mathcal{D}_\nu$  induces a good Kazhdan filtration on each of the objects  $\mathcal{D}_\nu^{\nu+\lambda}$ ,  $\mathcal{D}_\nu$ , and also on  $\mathcal{Q}_\nu^{\nu+\lambda}$ . One may consider  $\mathcal{D}_\nu$  as a  $\mathcal{D}_\nu$ -bimodule, resp.  $\mathcal{D}_\nu^{\nu+\lambda}$  as a  $(\mathcal{D}_{\nu+\lambda}, \mathcal{D}_\nu)$ -bimodule. Such a bimodule may be viewed as a  $G$ -monodromic left  $\mathcal{D}_{\nu,\mu}$ -module, resp.  $\mathcal{D}_{\nu+\lambda,\mu}$ -module, supported on the diagonal  $\mathcal{B} \subset \mathcal{B} \times \mathcal{B}$ . Here, the weight  $\mu$  is given by the formula  $\mu = -w_0(\lambda) + 2\rho$ , where  $w_0$  stands for the longest element in the Weyl group.

Recall that the group  $\mathbb{C}^\times$  acts on  $\tilde{\mathcal{S}}$  via the  $\bullet$ -action.

**Corollary 5.4.2.** *For an  $\nu, \lambda \in \mathfrak{h}^*$ , we have*

- (i)  $\mathcal{Q}_\nu^{\nu+\lambda} \in (\mathcal{D}_\nu, \mathfrak{m}_\chi)$ -mod is a Cohen-Macaulay left  $\mathcal{D}_{\nu+\lambda}$ -module and there is an  $M \times \mathbb{C}^\times$ -equivariant isomorphism  $\text{gr}_K \mathcal{Q}_\nu^{\nu+\lambda} = p_* \mathcal{O}_\Sigma(\lambda)$ .
- (ii) If  $\lambda \in \mathbb{X}^+$ , then we have  $H^j(\mathcal{B}, \mathcal{Q}_\nu^{\nu+\lambda}) = 0$  for any  $j > 0$ ; moreover, the projection  $\mathcal{Q}_\nu^{\nu+\lambda} \rightarrow \mathcal{Q}_\nu^{\nu+\lambda}$  induces an isomorphism

$$\Gamma(\mathcal{B}, \mathcal{Q}_\nu^{\nu+\lambda})/\Gamma(\mathcal{B}, \mathcal{Q}_\nu^{\nu+\lambda})\mathfrak{m}_\chi \xrightarrow{\sim} \Gamma(\mathcal{B}, \mathcal{Q}_\nu^{\nu+\lambda}) = Q_\nu^{\nu+\lambda}.$$

Recall that the Cohen-Macaulay property claimed in part (i) of the corollary says that the sheaves  $\mathcal{E}xt_{\mathcal{D}_{\nu+\lambda}}^{\bullet}(\mathcal{Q}_{\nu}^{\nu+\lambda}, \mathcal{D}_{\nu+\lambda})$  vanish in all degrees but one.

*Proof.* We view  $\mathcal{V} := \mathcal{D}_{\nu}^{\nu+\lambda}$  as a  $\mathcal{D}_{\nu+\lambda, \mu}$ -module as explained above. Applying Proposition 5.4.1, we deduce the isomorphisms

$$\mathrm{gr}_{\mathbb{K}} \mathcal{D}_{\nu}^{\nu+\lambda} \cong \mathrm{gr}_{\mathbb{K}} \mathcal{D}_{\nu}^{\nu+\lambda} / (\mathrm{gr}_{\mathbb{K}} \mathcal{D}_{\nu}^{\nu+\lambda}) \mathfrak{m}_{\chi} \cong p \cdot \mathcal{O}_{\Sigma}(\lambda).$$

In addition, we deduce that the Chevalley-Eilenberg complex  $\mathcal{D}_{\nu}^{\nu+\lambda} \otimes \wedge^{\bullet} \mathfrak{m}_{\chi}$  is acyclic in positive degrees. Thus, forgetting the right action, we see that the Chevalley-Eilenberg complex provides a resolution of  $\mathcal{D}_{\nu}^{\nu+\lambda}$  by locally free left  $\mathcal{D}_{\nu+\lambda}$ -modules.

To prove the Cohen-Macaulay property, we use the Chevalley-Eilenberg complex as a resolution of  $\mathcal{D}_{\nu}^{\nu+\lambda}$ . Thus, the sheaves  $\mathcal{E}xt_{\mathcal{D}_{\nu+\lambda}}^{\bullet}(\mathcal{D}_{\nu}^{\nu+\lambda}, \mathcal{D}_{\nu+\lambda})$  may be obtained as cohomology groups of the complex

$$(5.4.3) \quad \mathcal{H}om_{\mathcal{D}_{\nu+\lambda}}(\mathcal{D}_{\nu}^{\nu+\lambda} \otimes \wedge^{\bullet} \mathfrak{m}_{\chi}, \mathcal{D}_{\nu+\lambda}) = \mathcal{H}om_{\mathcal{D}_{\nu+\lambda}}(\mathcal{O}(\lambda) \otimes \mathcal{D}_{\nu}, \mathcal{D}_{\nu+\lambda}) \otimes (\wedge^{\bullet} \mathfrak{m}_{\chi})^*.$$

Recall that the functor  $\mathcal{O}(\lambda) \otimes (-)$  is an equivalence and  $\mathcal{D}_{\nu+\lambda} = \mathcal{O}(\lambda) \otimes \mathcal{D}_{\nu} \otimes \mathcal{O}(-\lambda)$ . Hence, we get an isomorphism

$$\mathcal{H}om_{\mathcal{D}_{\nu+\lambda}}(\mathcal{O}(\lambda) \otimes \mathcal{D}_{\nu}, \mathcal{D}_{\nu+\lambda}) = \mathcal{H}om_{\mathcal{D}_{\nu}}(\mathcal{D}_{\nu}, \mathcal{D}_{\nu} \otimes \mathcal{O}(-\lambda)) = \mathcal{O}(-\lambda) \otimes (\mathcal{D}_{\nu})^{op}.$$

Put  $m := \dim \mathfrak{m}_{\chi}$  and recall that one has a canonical isomorphism  $(\mathcal{D}_{\nu})^{op} = \mathcal{D}_{\mu}$  where  $\mu = -w_0(\nu) + 2\rho$ . For any  $j \geq 0$ , we deduce

$$\begin{aligned} \mathcal{H}om_{\mathcal{D}_{\nu+\lambda}}(\mathcal{D}_{\nu}^{\nu+\lambda} \otimes \wedge^j \mathfrak{m}_{\chi}, \mathcal{D}_{\nu+\lambda}) &= \mathcal{O}(-\lambda) \otimes (\mathcal{D}_{\nu})^{op} \otimes \wedge^j \mathfrak{m}_{\chi}^* \\ &= [\mathcal{D}_{\mu}^{\mu-\lambda} \otimes \wedge^{m-j} \mathfrak{m}_{\chi}] \otimes \wedge^m \mathfrak{m}_{\chi}^*. \end{aligned}$$

Since the Lie algebra  $\mathfrak{m}_{\chi}$  is nilpotent, one has an isomorphism  $\wedge^m \mathfrak{m}_{\chi}^* \cong \mathbb{C}$  of  $\mathfrak{m}_{\chi}$ -modules. We see that the complex on the right of (5.4.3) may be identified, up to degree shift, with the Chevalley-Eilenberg complex  $\mathcal{D}_{\mu}^{\mu-\lambda} \otimes \wedge^{\bullet} \mathfrak{m}_{\chi}$ . But the latter complex is a resolution of the  $\mathcal{D}_{\mu-\lambda}$ -module  $\mathcal{D}_{\mu-\lambda}$ , by the first paragraph of the proof. Therefore, the complex (5.4.3) has a single nonvanishing cohomology group which is isomorphic to  $\mathcal{D}_{\mu-\lambda}$ . This completes the proof of part (i).

To prove part (ii) we recall that, for  $\lambda \in \mathbb{X}^+$  and any  $j > 0$ , one has the cohomology vanishing  $H^j(T^*\mathcal{B}, \mathcal{O}_{T^*\mathcal{B}}(\lambda)) = 0$ ; cf. [Br]. Using that  $\tilde{\mathrm{gr}} \mathcal{D}_{\nu}^{\nu+\lambda} = \mathcal{O}_{T^*\mathcal{B}}(\lambda)$ , by a standard spectral sequence argument, we deduce  $H^j(\mathcal{B}, \mathcal{D}_{\nu}^{\nu+\lambda}) = 0$  for all  $j > 0$ . Thus, the Chevalley-Eilenberg complex  $\mathcal{D}_{\nu}^{\nu+\lambda} \otimes \wedge^{\bullet} \mathfrak{m}_{\chi}$  yields a  $\Gamma$ -acyclic resolution of  $\mathcal{D}_{\nu}^{\nu+\lambda}$ , the latter being viewed as a sheaf on  $\mathcal{B}$ . We conclude that the sheaf cohomology groups  $H^j(\mathcal{B}, \mathcal{D}_{\nu}^{\nu+\lambda})$  may be computed as the cohomology groups of the complex

$$(5.4.4) \quad \begin{aligned} \cdots \rightarrow \Gamma(\mathcal{B}, \mathcal{D}_{\nu}^{\nu+\lambda} \otimes \wedge^3 \mathfrak{m}_{\chi}) &\rightarrow \Gamma(\mathcal{B}, \mathcal{D}_{\nu}^{\nu+\lambda} \otimes \wedge^2 \mathfrak{m}_{\chi}) \\ &\rightarrow \Gamma(\mathcal{B}, \mathcal{D}_{\nu}^{\nu+\lambda} \otimes \mathfrak{m}_{\chi}) \rightarrow \Gamma(\mathcal{B}, \mathcal{D}_{\nu}^{\nu+\lambda}). \end{aligned}$$

Next we apply Lemma 4.4.1 to  $K := \Gamma(\mathcal{B}, \mathcal{D}_{\nu}^{\nu+\lambda})$ , a Harish-Chandra  $(\mathcal{U}_{\nu+\lambda}, \mathcal{U}_{\mu})$ -bimodule. We conclude that the complex (5.4.4) is acyclic in positive degrees and, in degree 0, we have  $H^0(\mathcal{B}, \mathcal{D}_{\nu}^{\nu+\lambda}) = K/K\mathfrak{m}_{\chi}$ . This proves part (ii) of the corollary.  $\square$

*Proof of Proposition 5.2.1(iii).* For any weights  $\nu, \lambda$ , we have

$$(5.4.5) \quad \mathcal{D}_{\nu}^{\nu+\lambda} = \mathcal{O}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{D}_{\nu} / (\mathcal{O}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{D}_{\nu}) \mathfrak{m}_{\chi} = \mathcal{O}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} (\mathcal{D}_{\nu} / \mathcal{D}_{\nu} \mathfrak{m}_{\chi}).$$

We apply Lemma 5.3.1 to the  $\mathcal{D}_\nu$ -module  $\mathcal{V} = \text{Wh}_m \mathcal{D}_\nu := \mathcal{D}_\nu / \mathcal{D}_\nu \mathfrak{m}_\chi$ . The lemma says that, for all sufficiently dominant  $\lambda \in \mathbb{X}^+$ , one has

$$(5.4.6) \quad \text{gr}_K \text{Wh}^m \Gamma(\mathcal{B}, \mathcal{O}(\lambda) \otimes_{\mathcal{O}_B} (\mathcal{D}_\nu / \mathcal{D}_\nu \mathfrak{m}_\chi)) = \Gamma(\tilde{\mathcal{S}}, \mathcal{O}_{\tilde{\mathcal{S}}}(\lambda) \otimes_{\mathcal{O}_{\tilde{\mathcal{S}}}} (\tilde{\text{gr}}_K \text{Wh}_m \mathcal{D}_\nu)|_{\tilde{\mathcal{S}}}).$$

On the other hand, Corollary 5.4.2 yields  $\tilde{\text{gr}}_K \text{Wh}_m \mathcal{D}_\nu = \mathcal{O}_\Sigma$ , hence

$$(\tilde{\text{gr}}_K \text{Wh}_m \mathcal{D}_\nu)|_{\tilde{\mathcal{S}}} = \mathcal{O}_{\tilde{\mathcal{S}}}.$$

Thus, using (5.4.5) and (5.4.6), we get

$$\text{gr}_K A_\nu^{\nu+\lambda} = \text{gr}_K \text{Wh}^m \Gamma(\mathcal{B}, \mathcal{D}_\nu^{\nu+\lambda}) = \Gamma(\tilde{\mathcal{S}}, \mathcal{O}_{\tilde{\mathcal{S}}}(\lambda) \otimes_{\mathcal{O}_{\tilde{\mathcal{S}}}} \mathcal{O}_{\tilde{\mathcal{S}}}) = \Gamma(\tilde{\mathcal{S}}, \mathcal{O}_{\tilde{\mathcal{S}}}(\lambda)).$$

The verification of compatibilities with the pairings is left to the reader. □

### 6. NONCOMMUTATIVE RESOLUTIONS OF SLODOWY SLICES

**6.1. Directed algebras.** Let  $\Lambda$  be a torsion free abelian group, and  $\Lambda^+ \subset \Lambda$  a subsemigroup such that  $\Lambda^+ \cap (-\Lambda^+) = \{0\}$ . Given a pair  $\mu, \nu \in \Lambda$ , we write  $\mu \succcurlyeq \nu$  whenever  $\mu - \nu \in \Lambda^+$ . This gives a partial order on  $\Lambda$ .

A *directed algebra* is a vector space  $B := \bigoplus_{\mu \succcurlyeq \nu} B_{\mu\nu}$ , graded by pairs  $\mu, \nu \in \Lambda$  such that  $\mu \succcurlyeq \nu$ , and equipped, for each triple  $\mu \succcurlyeq \nu \succcurlyeq \lambda$ , with a bilinear multiplication pairing  $B_{\mu\nu} \otimes B_{\nu\lambda} \rightarrow B_{\mu\lambda}$ . These pairings are required to satisfy, for each quadruple  $\mu \succcurlyeq \nu \succcurlyeq \lambda \succcurlyeq \rho$ , a natural associativity condition; cf. [Mu]. In the special case where the group  $\Lambda = \mathbb{Z}$  is equipped with the usual order, our definition reduces to the notion of  $\mathbb{Z}$ -algebra used in [GS], and [Bo].

In general, for any directed algebra  $B$ , the multiplication pairings give each of the spaces  $B_{\mu\mu}$  an associative algebra structure. Similarly, for each pair  $\mu \succcurlyeq \nu$ , the space  $B_{\mu\nu}$  acquires a  $(B_{\mu\mu}, B_{\nu\nu})$ -bimodule structure. Note that, for any  $\mu \succcurlyeq \nu \succcurlyeq \lambda$ , the multiplication pairing descends to a well-defined map  $B_{\mu\nu} \otimes_{B_{\nu\nu}} B_{\nu\lambda} \rightarrow B_{\mu\lambda}$ .

**Example 6.1.1.** Let  $\mathcal{B} = \bigoplus_{\lambda \in \Lambda} \mathcal{B}_\lambda$  be an ordinary  $\Lambda$ -graded associative algebra. For any pair  $\mu \succcurlyeq \nu$ , put  $B_{\mu\nu} := \mathcal{B}_{\mu-\nu}$ . Then, the bigraded space  ${}^\sharp\mathcal{B} = \bigoplus_{\mu \succcurlyeq \nu} B_{\mu\nu}$  has a natural structure of directed algebra. It is called the directed algebra associated with the graded algebra  $\mathcal{B}$ .

We say that a directed algebra  $B$  is *filtered* provided, for each pair  $\mu \succ \nu$ , one has an ascending  $\mathbb{Z}$ -filtration  $F_\bullet B_{\mu,\nu}$ , on the corresponding component  $B_{\mu,\nu}$ , and multiplication pairings  $B_{\mu,\nu} \otimes B_{\nu,\lambda} \rightarrow B_{\mu,\lambda}$  respect the filtrations. There is an associated graded directed algebra  $\text{gr } B$ , with components  $(\text{gr } B)_{\mu\nu} := \bigoplus_{i \in \mathbb{Z}} F_i B_{\mu\nu} / F_{i-1} B_{\mu\nu}$ .

Given a directed algebra  $B = \bigoplus_{\mu \succcurlyeq \nu} B_{\mu\nu}$ , one has the notion of a  $\Lambda$ -graded  $B$ -module. Such a module is, by definition, a  $\Lambda$ -graded vector space  $M = \bigoplus_{\nu \in \Lambda} M_\nu$  equipped, for each pair  $\mu \succcurlyeq \nu$ , with an ‘‘action map’’  $\text{act}_{\mu,\nu} : B_{\mu\nu} \otimes M_\nu \rightarrow M_\mu$ , that satisfies a natural associativity condition for each triple  $\mu \succcurlyeq \nu \succcurlyeq \lambda$ . Such a module  $M$  is said to be finitely generated if there exists a finite collection of elements  $m_1 \in M_{\nu_1}, \dots, m_p \in M_{\nu_p}$  such that one has

$$(6.1.2) \quad M_\mu = \sum_{i=1}^p \text{act}_{\mu,\nu_i}(B_{\mu,\nu_i}, m_i), \quad \forall \mu \in \Lambda.$$

We let  $\text{grmod}(B)$  denote the category of finitely generated  $\Lambda$ -graded left  $B$ -modules.

Given a  $\Lambda$ -graded  $B$ -module  $M$ , we put  $\text{Spec } M := \{\nu \in \Lambda \mid M_\nu \neq 0\}$ . It is clear that, for any  $M \in \text{grmod}(B)$ , there exists a finite subset  $S \subset \Lambda$  such that we have  $\text{Spec } M \subset S + \Lambda^+$ . We say that  $M$  is *negligible* if there exists  $\nu \in \Lambda$  such that

$(\nu + \Lambda^+) \cap \text{Spec } M = \emptyset$ . Let  $\text{tails}(B)$  be the full subcategory of  $\text{grmod}(B)$  whose objects are negligible  $B$ -modules.

A directed algebra  $B$  is said to be *noetherian* if, for each  $\lambda \in \Lambda$ , the algebra  $B_{\lambda\lambda}$  is left noetherian and, moreover,  $B_{\lambda\nu}$  is a finitely generated left  $B_{\lambda\lambda}$ -module, for any  $\lambda \succ \nu$ . It is known (see Boyarchenko [Bo, Theorem 4.4(1)]) that, for a noetherian directed algebra  $B$ , the category  $\text{grmod}(B)$  is an abelian category, and  $\text{tails}(B)$  is its Serre subcategory. Thus, one can define  $\text{Qgrmod}(B) := \text{grmod}(B)/\text{tails}(B)$ , a Serre quotient category.

Fix  $\alpha \in \Lambda$ . Given a directed algebra  $B = \bigoplus_{\mu \succ \nu} B_{\mu\nu}$ , resp. a  $\Lambda$ -graded  $B$ -module  $M = \bigoplus_{\mu} M_{\mu}$ , put  $B^{\succ\alpha} = \bigoplus_{\mu \succ \nu \succ \alpha} B_{\mu\nu}$ , resp.  $M^{\succ\alpha} = \bigoplus_{\mu \succ \alpha} M_{\mu}$ . Thus,  $B^{\succ\alpha}$  may be viewed as a directed subalgebra of  $B$  which has zero homogeneous components  $(B^{\succ\alpha})_{\mu\nu}$  unless  $\mu \succ \nu \succ \alpha$ . Similarly,  $M^{\succ\alpha}$  may be viewed as a  $B$ -submodule in  $M$ . If  $B$  is noetherian, then  $B^{\succ\alpha}$  is clearly noetherian as well.

One easily proves the following result

**Proposition 6.1.3.** *Assume that the pair  $(\Lambda, \Lambda^+)$  satisfies the following two conditions:*

- The semi-group  $\Lambda_+$  is finitely generated.
- For any  $\mu, \nu \in \Lambda$  the set  $(\mu + \Lambda^+) \cap (\nu + \Lambda^+)$  is nonempty.

*Then, for any noetherian directed algebra  $B$  and any  $\alpha \in \Lambda$ , restriction of scalars gives a well-defined functor  $\text{grmod}(B) \rightarrow \text{grmod}(B^{\succ\alpha})$ . Furthermore, this functor induces an equivalence  $\text{Qgrmod}(B) \xrightarrow{\sim} \text{Qgrmod}(B^{\succ\alpha})$ .  $\square$*

**6.2. Geometric example.** Let  $X$  be a quasi-projective algebraic variety, and write  $\text{Coh } X$  for the abelian category of coherent sheaves on  $X$ . Given an ample line bundle  $\mathcal{L}$  on  $X$ , one defines  $\mathcal{B}(X, \mathcal{L}) := \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$ , a homogeneous coordinate ring of  $X$ . For the corresponding Proj-scheme, we have  $\text{Proj } \mathcal{B}(X, \mathcal{L}) \cong X$ .

Following Example 6.1.1 in the special case where  $\Lambda = \mathbb{Z}$  and  $\Lambda^+ = \mathbb{Z}_{\geq 0}$ , we may form the directed algebra  $\sharp\mathcal{B}(X, \mathcal{L})$  associated with  $\mathcal{B}(X, \mathcal{L})$ , the latter being viewed as a  $\Lambda$ -graded algebra. Then, one can construct a natural equivalence of categories

$$(6.2.1) \quad \text{Coh } X \cong \text{Qgrmod}(\sharp\mathcal{B}(X, \mathcal{L})).$$

We return to the setting of §1.1. Let  $\mathfrak{h}$  be the Cartan subalgebra for the semisimple Lie algebra  $\mathfrak{g}$ , and fix  $\mathbb{X}^+ \subset \mathfrak{h}^*$ , the subsemigroup of integral dominant weights. For each  $\lambda \in \mathbb{X}^+$ , we have the line bundle  $\mathcal{O}_{\tilde{\mathcal{S}}}(\lambda)$  on the Slodowy variety  $\tilde{\mathcal{S}}$ . The direct sum  $\mathcal{A}(e) = \bigoplus_{\lambda \in \mathbb{X}^+} \Gamma(\tilde{\mathcal{S}}, \mathcal{O}_{\tilde{\mathcal{S}}}(\lambda))$  has a natural structure of  $\mathbb{X}^+$ -graded algebra. For the corresponding multi-homogeneous Proj-scheme, one has  $\tilde{\mathcal{S}} \cong \text{Proj } \mathcal{A}(e)$ .

Now, in the setting of §6.1, we put  $\Lambda := \mathfrak{h}^*$ , resp  $\Lambda^+ := \mathbb{X}^+$ , and let  $\sharp\mathcal{A}(e)$  be the directed algebra associated to the  $\mathbb{X}^+$ -graded algebra  $\mathcal{A}(e)$ . One can show that  $\sharp\mathcal{A}(e)$  is a noetherian directed algebra and the following multi-homogeneous analogue of the equivalence (6.2.1) holds:

$$(6.2.2) \quad \text{Coh } \tilde{\mathcal{S}} \cong \text{Qgrmod}(\sharp\mathcal{A}(e)).$$

**6.3.** We are going to produce a family of quantizations of the directed algebra  $\sharp\mathcal{A}(e)$ . To this end, we exploit translation bimodules  $A_{\nu}^{\nu+\lambda}$  introduced in §5.2.

Given  $\nu \in \mathfrak{h}^*$ , we associate to our nilpotent element  $e \in \mathfrak{g}$  a directed algebra  $\mathfrak{A}(e, \nu) := \bigoplus_{\mu, \lambda \in \mathbb{X}^+} A_{\lambda+\nu}^{\mu+\lambda+\nu}$ , i.e., using directed algebra notation, for any  $\alpha \succ \beta \succ 0$ , we put  $A_{\alpha\beta} := A_{\beta+\nu}^{\alpha+\nu}$ .

The resulting directed algebra  $\mathfrak{A}(e, \nu)$  has the following properties:

(i) Each homogeneous component  $A_{\lambda+\nu}^{\mu+\nu}$ , of  $\mathfrak{A}(e, \nu)$ , comes equipped with a Kazhdan filtration  $\mathbf{K} \cdot A_{\lambda+\nu}^{\mu+\nu}$ , by nonnegative integers, such that the multiplication pairings respect the filtrations.

(ii) For all  $\lambda \in \mathbb{X}^+$ , one has  $A_{\lambda\lambda} = A_{\lambda+\nu}$  is the Premet algebra associated to the central character that corresponds to the weight  $\lambda + \nu$  via the Harish-Chandra isomorphism.

In addition, for a regular dominant  $\nu$  and any  $\mu, \lambda \in \mathbb{X}^+$ , part (iv), resp. part (ii), of Proposition 5.2.1, implies the following:

(iii) The  $(A_{\lambda+\nu}, A_\nu)$ -bimodule  $A_\nu^{\lambda+\nu}$  yields an equivalence  $\text{mod } A_\nu \xrightarrow{\sim} \text{mod } A_{\lambda+\nu}$ .

(iv) The multiplication in  $\mathfrak{A}(e, \nu)$  induces an isomorphism  $A_{\lambda+\nu}^{\mu+\lambda+\nu} \otimes_{A_{\lambda+\nu}} A_\nu^{\lambda+\nu} \xrightarrow{\sim} A_\nu^{\mu+\lambda+\nu}$ .

We make  $\mathfrak{A}(e, \nu)$  a filtered directed algebra by using the Kazhdan filtrations on each homogeneous component  $A_{\lambda+\nu}^{\mu+\lambda+\nu}$ ; cf. (i) above.

Note that, by construction, for each  $\lambda \in \mathbb{X}^+$ , the space  $A_{\lambda 0} = A_\nu^{\nu+\lambda}$  has a structure of right  $A_\nu$ -module. Thus, one can introduce a functor

$$(6.3.1) \quad \text{Loc} : \text{mod } A_\nu \rightarrow \text{Qgrmod}(\mathfrak{A}(e, \nu)), \quad M \mapsto \text{Loc } M := \bigoplus_{\lambda \in \mathbb{X}^+} A_\nu^{\lambda+\nu} \otimes_{A_\nu} M.$$

**Theorem 6.3.2.** *For any  $\nu \in \mathfrak{h}^*$ , we have that  $\mathfrak{A}(e, \nu)$  is a noetherian, filtered directed algebra. If  $\nu$  is dominant and regular, then  $\text{gr}_{\mathbf{K}} \mathfrak{A}(e, \nu) \cong \# \mathcal{A}(e)^{\succ \nu}$ , and the functor (6.3.1) is an equivalence.*

The equivalence of the theorem may be thought of as some sort of the Beilinson-Bernstein localization theorem for Slodowy slices; cf. the Introduction to [GS] for more discussions concerning this analogy.

A different approach to a result closely related to Theorem 6.3.2 was also proposed by Losev in an unpublished manuscript.

*Proof of Theorem 6.3.2.* The noetherian property is immediate from Proposition 5.2.1, and the isomorphism  $\text{gr}_{\mathbf{K}} \mathfrak{A}(e, \nu) \cong \# \mathcal{A}(e)^{\succ \nu}$  follows from Proposition 5.2.1 (iii).

The equivalence statement in the theorem is a consequence of properties (iii) and (iv) above, thanks to Gordon and Stafford [GS, Lemma 5.5], and Boyarchenko [Bo, Theorem 4.4]. □

*Remark 6.3.3.* Theorem 6.3.2 applies also to enveloping algebras. This corresponds, at least formally, to the case of the zero nilpotent element although the case  $e = 0$  was not considered in this paper. Specifically, introduce a filtered directed algebra  $\mathfrak{U}(\nu) := \bigoplus_{\mu, \lambda \in \mathbb{X}^+} U_{\lambda+\nu}^{\mu+\lambda+\nu}$ , where  $U_\nu^{\nu+\lambda} := \Gamma(\mathcal{B}, \mathcal{O}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{D}_\nu)$ . Then, arguing as in the proof of Theorem 6.3.2, one obtains, for any dominant regular  $\nu$ , an equivalence  $\text{mod } \mathfrak{U}_\nu \xrightarrow{\sim} \text{Qgrmod}(\mathfrak{U}(\nu))$ .

*Remark 6.3.4.* One can show that for a dominant, but not necessarily regular  $\nu$ , the functor  $M \mapsto \text{Loc } M$  is exact and, moreover, one has  $M \neq 0 \Rightarrow \text{Loc } M \neq 0$ .

**6.4. Characteristic varieties for  $\mathfrak{A}(e, \nu)$ -modules.** Fix  $\Lambda$  as in §6.1 and let  $B$  be a filtered directed algebra. A  $\Lambda$ -graded  $B$ -module  $M = \bigoplus_{\nu \in \Lambda} M_\nu$  is said to be filtered provided each of the spaces  $M_\nu$  is equipped with an ascending  $\mathbb{Z}$ -filtration  $F \cdot M_\nu$  such that, for any  $i, j \in \mathbb{Z}$  and  $\mu, \nu \in \Lambda$ , we have  $F_i B_{\mu\nu} \cdot F_j M_\nu \subset F_{i+j} M_\mu$ . In such a case, there is a well-defined associated graded  $\Lambda$ -module  $\text{gr } M$ , over  $\text{gr } B$ . Thus  $\text{gr } M$  has an additional  $\mathbb{Z}$ -grading.

A filtration on  $M$  is called good if  $\text{gr } M$  is a finitely generated  $\text{gr } B$ -module in the sense of (6.1.2).

Now let  $(\Lambda, \Lambda^+) = (\mathfrak{h}^*, \mathbb{X}^+)$ . This pair satisfies the conditions of Proposition 6.1.3. Thus, for any  $\nu \in \mathfrak{h}^*$ , combining the proposition and (6.2.2), we get a category equivalence  $\Phi : \mathbf{Qgrmod}(\# \mathcal{A}(e)^{\geq \nu}) \xrightarrow{\sim} \text{Coh } \tilde{\mathcal{S}}$ .

Assume first that  $\nu \in \mathfrak{h}^*$  is a dominant and regular weight. Then, we have  $\text{gr}_\kappa \mathfrak{A}(e, \nu) \cong \# \mathcal{A}(e)^{\geq \nu}$ , by Theorem 6.3.2. Therefore, for any  $\Lambda$ -graded  $\mathfrak{A}(e, \nu)$ -module  $M$  equipped with a good filtration, we may view  $\text{gr } M$  as a finitely generated graded  $\# \mathcal{A}(e)^{\geq \nu}$ -module and put  $\text{qgr } M := \Phi(\text{gr } M)$ . This is a coherent sheaf on  $\tilde{\mathcal{S}}$ . The additional  $\mathbb{Z}$ -grading on  $\text{gr } M$  gives that sheaf a  $\mathbb{C}^\times$ -equivariant structure. Thus,  $\text{Var } M := \text{Supp}(\text{qgr } M)$  is a  $\bullet$ -stable closed algebraic subset in  $\tilde{\mathcal{S}}$ .

Now, let  $\nu$  be an arbitrary, not necessarily dominant and regular, weight. Then, we find a sufficiently dominant integral weight  $\mu$  such that  $\nu + \mu$  is a dominant and regular weight. Given a  $\Lambda$ -graded  $\mathfrak{A}(e, \nu)$ -module  $M$ , we may view  $M^{\geq \nu + \mu}$  as a  $\Lambda$ -graded  $\mathfrak{A}(e, \nu + \mu)$ -module. A good filtration on  $M$  induces one on  $M^{\geq \nu + \mu}$ , and we have  $\text{gr}(M^{\geq \nu + \mu}) = (\text{gr } M)^{\geq \nu + \mu}$ . Thus, one may apply all the above constructions to the  $\mathfrak{A}(e, \nu + \mu)$ -module  $M^{\geq \nu + \mu}$ . This way, one defines the set  $\text{Var } M$  in the general case where  $\nu$  is an arbitrary weight.

One has the following standard result

**Proposition 6.4.1.** (i) *The set  $\text{Var } M$  is a coisotropic subset in  $\tilde{\mathcal{S}}$  which is independent of the choice of a good filtration on  $M$ .*

(ii) *For any finitely generated  $A_\nu$ -module  $N$ , we have  $\pi(\text{Var } \mathcal{L}oc N) \subset \text{Var } N$ .*

(iii) *Let  $\nu$  be a dominant and regular weight and  $\mathcal{V} \in (\mathcal{D}_\nu, \mathfrak{m}_\chi)$ -mod. Then, a good filtration on  $\mathcal{V}$  induces a good filtration on  $\mathcal{L}oc \text{Wh}^m \Gamma(\mathcal{B}, \mathcal{V})$  such that one has*

$$\tilde{\text{gr}}_\kappa \mathcal{V}|_{\tilde{\mathcal{S}}} \cong \text{qgr}(\mathcal{L}oc \text{Wh}^m \Gamma(\mathcal{B}, \mathcal{V})) \quad (\text{isomorphism in } \text{Coh } \tilde{\mathcal{S}}).$$

*Sketch of Proof.* The coisotropicness statement in part (i) is a special case of Gabber’s “integrability of characteristics” result [Ga]. Part (ii) is an analogous to a well-known result of Borho-Brylinski (cf. [BoBr], proof of Proposition 4.3). A key point is that the map  $\pi : \tilde{\mathcal{S}} \rightarrow \mathcal{S}$  is proper.

To prove part (iii), let  $\lambda \in \mathbb{X}^+$ . Lemma 5.2.4 yields an isomorphism  $A_\nu^{\nu+\lambda} \otimes_{A_\nu} \text{Wh}^m \Gamma(\mathcal{B}, \mathcal{V}) = \text{Wh}^m \Gamma(\mathcal{B}, \mathcal{O}(\lambda) \otimes_{\mathcal{O}_B} \mathcal{V})$ . Thus, by definition of the functor  $\mathcal{L}oc$ , we obtain

$$(6.4.2) \quad \text{qgr}(\mathcal{L}oc \text{Wh}^m \Gamma(\mathcal{B}, \mathcal{V})) = \text{qgr} \left( \bigoplus_{\lambda \in \mathbb{X}^+} \text{Wh}^m \Gamma(\mathcal{B}, \mathcal{O}(\lambda) \otimes_{\mathcal{O}_B} \mathcal{V}) \right).$$

Recall the equivalence  $\Phi : \mathbf{Qgrmod}(\# \mathcal{A}(e)^{\geq \nu}) \xrightarrow{\sim} \text{Coh } \tilde{\mathcal{S}}$  considered earlier in this subsection. The object on the right-hand side of (6.4.2) equals

$$\Phi \left( \bigoplus_{\lambda \in \mathbb{X}^+} \tilde{\text{gr}} \text{Wh}^m \Gamma(\mathcal{B}, \mathcal{O}(\lambda) \otimes_{\mathcal{O}_B} \mathcal{V}) \right),$$

by definition of the functor  $\text{qgr}$ . Further, for  $\lambda$  sufficiently dominant, we have an isomorphism  $\tilde{\text{gr}} \text{Wh}^m \Gamma(\mathcal{B}, \mathcal{O}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{V}) = \Gamma(\tilde{\mathcal{S}}, \mathcal{O}_{\tilde{\mathcal{S}}}(\lambda) \otimes_{\mathcal{O}_{\tilde{\mathcal{S}}}} (\tilde{\text{gr}}_{\mathcal{K}} \mathcal{V}|_{\tilde{\mathcal{S}}}))$ , by Lemma 5.3.1. Therefore, we deduce an isomorphism

$$(6.4.3) \quad \Phi \left( \bigoplus_{\lambda \in \mathbb{X}^+} \tilde{\text{gr}} \text{Wh}^m \Gamma(\mathcal{B}, \mathcal{O}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{V}) \right) = \Phi \left( \bigoplus_{\lambda \in \mathbb{X}^+} \Gamma(\tilde{\mathcal{S}}, \mathcal{O}_{\tilde{\mathcal{S}}}(\lambda) \otimes_{\mathcal{O}_{\tilde{\mathcal{S}}}} (\tilde{\text{gr}}_{\mathcal{K}} \mathcal{V}|_{\tilde{\mathcal{S}}})) \right).$$

In general, let  $\mathcal{F}$  be an arbitrary coherent sheaf on  $\tilde{\mathcal{S}}$ . By definition of the equivalence (6.2.2), in  $\text{Coh } \tilde{\mathcal{S}}$ , one has an isomorphism  $\Phi \left( \bigoplus_{\lambda \in \mathbb{X}^+} \Gamma(\tilde{\mathcal{S}}, \mathcal{O}_{\tilde{\mathcal{S}}}(\lambda) \otimes_{\mathcal{O}_{\tilde{\mathcal{S}}}} \mathcal{F}) \right) = \mathcal{F}$ . Applying this observation to the sheaf  $\mathcal{F} := \tilde{\text{gr}}_{\mathcal{K}} \mathcal{V}|_{\tilde{\mathcal{S}}}$ , and using isomorphisms (6.4.2) and (6.4.3), we deduce part (iii) of Proposition 6.4.1.  $\square$

To proceed further, we introduce the following terminology. A coherent sheaf  $\mathcal{F}$ , on a reduced scheme  $X$ , is said to be reduced if the annihilator of  $\mathcal{F}$  is a radical ideal in  $\mathcal{O}_X$ .

**Definition 6.4.4.** An object  $V \in \text{Qgrmod}(\mathfrak{A}(e, \nu))$  is said to have regular singularities if there exists a representative  $M \in \text{grmod}(\mathfrak{A}(e, \nu))$ , of  $V$ , and a good filtration on  $M$  such that the corresponding sheaf  $\text{qgr } M \in \text{Coh } \tilde{\mathcal{S}}$  is reduced and, moreover,  $\text{Supp}(\text{qgr } M)$  is a Lagrangian subvariety in  $\tilde{\mathcal{S}}$ .

**6.5. Definition of Harish-Chandra  $(A_{c'}, A_c)$ -bimodules.** We fix a pair of weights  $\beta, \gamma \in \mathfrak{h}^*$  and specialize the general setting at the beginning of section 6.4 to  $B = \mathfrak{A}(e, \beta) \otimes \mathfrak{A}(e, \gamma)^{\text{op}}$ , a filtered  $\Lambda \times \Lambda$ -graded directed algebra. By Theorem 6.3.2, we have  $\text{gr}_{\mathcal{K}}(\mathfrak{A}(e, \beta) \otimes \mathfrak{A}(e, \gamma)^{\text{op}}) = \sharp(\mathcal{A}(e) \otimes \mathcal{A}(e))^{\succ \beta \times \gamma}$ . Hence, there is a category equivalence  $\text{Qgrmod}(\text{gr}_{\mathcal{K}}(\mathfrak{A}(e, \beta) \otimes \mathfrak{A}(e, \gamma)^{\text{op}})) \cong \text{Coh}(\tilde{\mathcal{S}} \times \tilde{\mathcal{S}})$ . Therefore, associated with any  $\Lambda \times \Lambda$ -graded  $(\mathfrak{A}(e, \beta), \mathfrak{A}(e, \gamma))$ -bimodule  $M$  with a good filtration, there is a  $\mathbb{C}^\times$ -equivariant coherent sheaf  $\text{qgr } M$ , on  $\tilde{\mathcal{S}} \times \tilde{\mathcal{S}}$ .

One proves the following analogue of Theorem 6.3.2 for bimodules.

**Proposition 6.5.1.** *For dominant regular  $\beta, \gamma \in \mathfrak{h}^*$ , there is a canonical equivalence*

$$M \longmapsto \mathcal{L}oc M := \bigoplus_{\lambda, \mu \in \mathbb{X}^+} (A_{\beta}^{\lambda+\beta} \otimes_{A_{\beta}} M \otimes_{A_{\gamma}} A_{\mu}^{\mu+\gamma}),$$

*between the category  $\text{mod}(A_{\beta} \otimes A_{\gamma}^{\text{op}})$ , of finitely generated  $(A_{\beta}, A_{\gamma})$ -bimodules, and category  $\text{Qgrmod}(\mathfrak{A}(e, \beta) \otimes \mathfrak{A}(e, \gamma)^{\text{op}})$ .  $\square$*

Next, recall the Steinberg variety  $\mathcal{Z}$  and observe that  $\mathcal{Z} \cap (\tilde{\mathcal{S}} \times \tilde{\mathcal{S}})$  is a Lagrangian subvariety in  $\tilde{\mathcal{S}} \times \tilde{\mathcal{S}}$ . An analogue of Proposition 6.4.1 for bimodules yields the following result.

**Proposition 6.5.2.** *Let  $\beta, \gamma \in \mathfrak{h}^*$  be dominant weights. Then:*

- (i) *For any  $\text{whc } (A_{\beta}, A_{\gamma})$ -bimodule  $N$ , we have  $\text{Var}(\mathcal{L}oc N) \subset \mathcal{Z} \cap (\tilde{\mathcal{S}} \times \tilde{\mathcal{S}})$ .*
- (ii) *Let  $\beta$  and  $\gamma$  be dominant, and let  $\mathcal{V}$  be a  $G$ -monodromic  $\mathcal{D}_{\beta, \gamma}$ -module. Then, a good filtration on  $\mathcal{V}$  induces a good filtration on  $\mathcal{L}oc \text{Wh}_{\mathfrak{m}}^m \Gamma(\mathcal{B} \times \mathcal{B}, \mathcal{V})$  such that one has*

$$\tilde{\text{gr}}_{\mathcal{K}} \mathcal{V}|_{\tilde{\mathcal{S}} \times \tilde{\mathcal{S}}} \cong \text{qgr } \mathcal{L}oc \text{Wh}_{\mathfrak{m}}^m \Gamma(\mathcal{B} \times \mathcal{B}, \mathcal{V}).$$

In part (ii) above, we have abused the notation  $\text{Wh}_{\mathfrak{m}}^m$  and, given a left  $(\mathcal{U}_{\beta} \otimes \mathcal{U}_{\gamma})$ -module  $N$ , write  $\text{Wh}_{\mathfrak{m}}^m N$  for  $(\mathfrak{m}_{\chi} \otimes 1)$ -invariants in  $N/(1 \otimes \mathfrak{m}_{\chi})N$ .

Transition from left  $\mathcal{D}_{\beta,\gamma}$ -modules to  $(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g})$ -bimodules can be carried out as explained e.g. in [BG, §5], esp. Lemma 5.4 and formula (5.5).

*Proof of Theorem 4.2.2(i).* Fix a wHC  $(A_{c'}, A_c)$ -bimodule  $N$  and choose a good filtration on  $\mathcal{L}oc N$ . We know that  $Var(\mathcal{L}oc N) \subset \mathcal{Z} \cap (\tilde{\mathcal{S}} \times \tilde{\mathcal{S}})$ , by the proposition above, and that  $\mathcal{Z} \cap (\tilde{\mathcal{S}} \times \tilde{\mathcal{S}})$  is a Lagrangian variety. Therefore, Gabber’s theorem (cf. Proposition 6.4.1(i)), implies that any irreducible component of the scheme  $Supp(\mathcal{L}oc N)$  is Lagrangian, moreover, it is an irreducible component of  $\mathcal{Z} \cap (\tilde{\mathcal{S}} \times \tilde{\mathcal{S}})$ .

We define  $CC(\mathcal{L}oc N)$ , the *characteristic cycle* of  $\mathcal{L}oc N$ , to be a Lagrangian cycle in  $\tilde{\mathcal{S}} \times \tilde{\mathcal{S}}$  equal to the linear combination of the irreducible components of the scheme  $Supp(\mathcal{L}oc N)$ , counted with multiplicities. A standard argument shows that the cycle  $CC(\mathcal{L}oc N)$  is independent of the choice of good filtration. Further, an analogue of Remark 6.3.4 for bimodules implies the exactness of the functor  $N \mapsto \mathcal{L}oc N$ . It follows that the assignment  $N \mapsto CC(\mathcal{L}oc N)$  is additive on short exact sequences.

Now, let  $N = N^0 \supset N^1 \supset N^2 \supset \dots$  be a descending chain of sub-bimodules of  $N$  such that  $N^i/N^{i+1} \neq 0$  for any  $i$ . Thus, we have  $\mathcal{L}oc N^i/\mathcal{L}oc N^{i+1} = \mathcal{L}oc N^i/N^{i+1} \neq 0$ ; cf. Remark 6.3.4. It follows that the total multiplicity in the cycle  $CC(\mathcal{L}oc N^i)$ ,  $i = 0, 1, \dots$ , gives a strictly decreasing sequence of natural numbers. Therefore, this sequence terminates. We conclude that  $N^i = 0$  for  $i \gg 0$ , hence  $N$  has finite length.  $\square$

Further, it is clear that Definition 6.4.4 of “regular singularities” may be adapted to objects of  $\mathbf{Qgrmod}(\mathfrak{A}(e, \beta) \otimes \mathfrak{A}(e, \gamma)^{op})$  in an obvious way. The following definition is motivated by Corollary 6.6.2, to be discussed in section 6.6 below.

**Definition 6.5.3.** A weak Harish-Chandra  $(A_\mu, A_\nu)$ -bimodule  $N$  is called a Harish-Chandra bimodule if  $\mathcal{L}oc N \in \mathbf{Qgrmod}(\mathfrak{A}(e, \mu) \otimes \mathfrak{A}(e, \nu)^{op})$ , the object corresponding to  $N$  via the equivalence of Proposition 6.5.1, has regular singularities.

Part (ii) of Proposition 6.5.2, combined with Corollary 1.3.8(iii) and with Proposition 6.6.1 below, yields the following result.

**Proposition 6.5.4.** *If  $K \in \mathcal{HC}(\mathcal{U}_c, \mathcal{U}_{c'})$ , then  $Wh_m^m K$  is a Harish-Chandra  $(A_\mu, A_\nu)$ -bimodule; furthermore, we have  $Var(\mathcal{L}oc Wh_m^m K) = Var K \cap (\tilde{\mathcal{S}} \times \tilde{\mathcal{S}})$ .  $\square$*

**6.6. Harish-Chandra  $\mathcal{U}\mathfrak{g}$ -bimodules vs. regular singularities.**

**Proposition 6.6.1.** *Let  $\mu, \nu \in \mathfrak{h}^*$  be dominant regular weights and let  $\mathcal{M}$  be a  $\mathcal{D}_{\mu,\nu}$ -module such that, set-theoretically, one has  $Var \mathcal{M} \subset \mathcal{Z}$ . Then, the following conditions are equivalent:*

- (i) *The  $\mathcal{D}$ -module  $\mathcal{M}$  has regular singularities in the sense of [KK];*
- (ii) *There exists a good filtration on  $\mathcal{M}$  such that  $\tilde{gr} \mathcal{M}$ , the associated graded sheaf, is reduced.*
- (iii) *There exists a good filtration on  $\Gamma(\mathcal{B} \times \mathcal{B}, \mathcal{M})$  such that  $gr \Gamma(\mathcal{B} \times \mathcal{B}, \mathcal{M})$  is a symmetric  $(\mathbb{C}[\mathfrak{g}^*], \mathbb{C}[\mathfrak{g}^*])$ -bimodule, i.e., such that one has  $a \cdot m = m \cdot a$ ,  $\forall a \in \mathbb{C}[\mathfrak{g}^*], m \in gr \Gamma(\mathcal{B} \times \mathcal{B}, \mathcal{M})$ .*
- (iv) *The  $\mathfrak{g}$ -diagonal action on  $\Gamma(\mathcal{B} \times \mathcal{B}, \mathcal{M})$  is locally finite.*

*Proof.* The implication (i)  $\Rightarrow$  (ii) is a special case of a general result of Kashiwara-Kawai; cf. also [Kas, Definition 5.2]. The corresponding filtration was defined in [KK, Theorem 5.1.6].

To prove (ii)  $\Rightarrow$  (iii) let  $F_\bullet \mathcal{M}$  be a good filtration on  $\mathcal{M}$  such that (ii) holds. Any element  $x \in \mathfrak{g}$  gives rise to a linear function  $\mathfrak{g}^* \times \mathfrak{g}^* \rightarrow \mathbb{C}$ ,  $(\alpha, \beta) \mapsto \langle \alpha + \beta, x \rangle$ . Let  $\tilde{x}$  denote the pull-back of that function via the map  $\pi \times (-\pi) : T^*\mathcal{B} \times T^*\mathcal{B} \rightarrow \mathcal{N} \times \mathcal{N}$ ,  $u \times v \mapsto \pi(u) \times (-\pi(v))$ . Thus, we have  $\mathcal{Z} = [\pi \times (-\pi)]^{-1}(\mathcal{N}_\Delta)$ , where  $\mathcal{N}_\Delta \subset \mathcal{N} \times \mathcal{N}$  is the diagonal. Clearly, the function  $\tilde{x}$  vanishes on  $\mathcal{Z}$ . Hence, we have  $\tilde{x} \cdot \tilde{\text{gr}}\mathcal{M} = 0$ .

Observe next that  $\tilde{x}$  is a homogeneous function, of degree 1, along the fibers of the cotangent bundle  $T^*(\mathcal{B} \times \mathcal{B})$ . Thus, the equation  $\tilde{x} \cdot \tilde{\text{gr}}\mathcal{M} = 0$  implies that, for any  $j \in \mathbb{Z}$ , the good filtration on  $\mathcal{M}$  satisfies  $x(F_j\mathcal{M}) \subset F_j\mathcal{M}$ , where  $x(-)$  stands for the  $\mathfrak{g}$ -diagonal action of  $x$  on  $\mathcal{M}$ . It follows that the filtration on  $\Gamma(\mathcal{B} \times \mathcal{B}, \mathcal{M})$  defined, for any  $j \in \mathbb{Z}$ , by  $\Gamma_j(\mathcal{B} \times \mathcal{B}, \mathcal{M}) := \Gamma(\mathcal{B} \times \mathcal{B}, F_j\mathcal{M})$  makes  $\text{gr } \Gamma(\mathcal{B} \times \mathcal{B}, \mathcal{M})$  a symmetric  $(\mathbb{C}[\mathfrak{g}^*], \mathbb{C}[\mathfrak{g}^*])$ -bimodule.

To prove the implication (iii)  $\Rightarrow$  (iv), let  $\Gamma_\bullet(\mathcal{B} \times \mathcal{B}, \mathcal{M})$  be an arbitrary good filtration on the bimodule  $\Gamma(\mathcal{B} \times \mathcal{B}, \mathcal{M})$  that makes  $\text{gr } \Gamma(\mathcal{B} \times \mathcal{B}, \mathcal{M})$  a symmetric  $(\mathbb{C}[\mathfrak{g}^*], \mathbb{C}[\mathfrak{g}^*])$ -bimodule. The filtration being good, the vector space  $\Gamma_j(\mathcal{B} \times \mathcal{B}, \mathcal{M})$  is finite dimensional for each  $j \in \mathbb{Z}$ . The symmetry of the bimodule  $\text{gr } \Gamma(\mathcal{B} \times \mathcal{B}, \mathcal{M})$  implies, by an easy induction on  $j$ , that the vector space  $\Gamma_j(\mathcal{B} \times \mathcal{B}, \mathcal{M})$  is stable under the  $\mathfrak{g}$ -diagonal action. We conclude that the  $\mathfrak{g}$ -diagonal action on  $\Gamma(\mathcal{B} \times \mathcal{B}, \mathcal{M}) = \bigcup_j \Gamma_j(\mathcal{B} \times \mathcal{B}, \mathcal{M})$  is locally finite.

To complete the proof of the theorem, let  $\mathcal{M}$  be a  $\mathcal{D}$ -module such that (iv) holds and put  $M := \Gamma(\mathcal{B} \times \mathcal{B}, \mathcal{M})$ . It is clear that the  $\mathfrak{g}$ -diagonal action on  $M$  can be exponentiated to an algebraic  $G^{sc}$ -action. It follows that  $\mathcal{D}_{\mu, \nu} \otimes_{\mathcal{U}_\mu \otimes \mathcal{U}_\nu} M$  is a  $G^{sc}$ -equivariant  $\mathcal{D}$ -module. But,  $\mathcal{B} \times \mathcal{B}$  being a projective variety with finitely many  $G^{sc}$ -diagonal orbits, any  $G^{sc}$ -equivariant  $\mathcal{D}$ -module on  $\mathcal{B} \times \mathcal{B}$  has regular singularities, see e.g. [HTT, Theorem 11.6.1]. Finally, by the Beilinson-Bernstein theorem, we have  $\mathcal{M} = \mathcal{D}_{\mu, \nu} \otimes_{\mathcal{U}_\mu \otimes \mathcal{U}_\nu} M$ , and the implication (iv)  $\Rightarrow$  (i) follows.  $\square$

Using an analogue of the equivalence of Remark 6.3.3 for  $(\mathcal{U}_\mu, \mathcal{U}_\nu)$ -bimodules, one can reformulate Proposition 6.6.1 as follows:

**Corollary 6.6.2.** *Let  $\mu, \nu \in \mathfrak{h}^*$  be dominant regular weights and let  $M$  be a wHC  $(\mathcal{U}_\mu, \mathcal{U}_\nu)$ -bimodule. Then, the following conditions are equivalent:*

- (i) *The adjoint  $\mathfrak{g}$ -action on  $M$  is locally finite, i.e.,  $M \in \mathcal{HC}(\mathcal{U}_\mu, \mathcal{U}_\nu)$  is a Harish-Chandra bimodule in the sense of §4.1.*
- (ii) *The object  $\mathcal{L}oc M \in \mathbf{Qgrmod}(\mathfrak{U}(\mu) \otimes \mathfrak{U}(\nu)^{\text{op}})$  has regular singularities.*  $\square$

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