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CYCLOTOMIC q-SCHUR ALGEBRAS ASSOCIATED TO THE ARIKI-KOIKE ALGEBRA

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Abstract. Let $\mathcal{H}_{n,r}$ be the Ariki-Koike algebra associated to the complex reflection group $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$, and let $\mathcal{S}(\Lambda)$ be the cyclotomic q-Schur algebra associated to $\mathcal{H}_{n,r}$, introduced by Dipper, James and Mathas. For each $\mathbf{p} = (r_1, \dots, r_g) \in \mathbb{Z}^g_{>0}$ such that $r_1 + \cdots + r_g = r$, we define a subalgebra $\mathcal{S}^{\mathbf{p}}$ of $\mathcal{S}(\Lambda)$ and its quotient algebra $\overline{\mathcal{S}}^{\mathbf{p}}$. It is shown that $\mathcal{S}^{\mathbf{p}}$ is a standardly based algebra and $\overline{\mathcal{S}}^{\mathbf{p}}$ is a cellular algebra. By making use of these algebras, we prove a product formula for decomposition numbers of $\mathcal{S}(\Lambda)$, which asserts that certain decomposition numbers are expressed as a product of decomposition numbers for various cyclotomic q-Schur algebras associated to Ariki-Koike algebras $\mathcal{H}_{n_i,r_i}$ of smaller rank. This is a generalization of the result of N. Sawada. We also define a modified Ariki-Koike algebra $\overline{\mathcal{H}}^{\mathbf{p}}$ of type $\mathbf{p}$, and prove the Schur-Weyl duality between $\overline{\mathcal{H}}^{\mathbf{p}}$ and $\overline{\mathcal{S}}^{\mathbf{p}}$.

0. Introduction

Let $\mathcal{H} = \mathcal{H}_{n,r}$ be the Ariki-Koike algebra over an integral domain R associated to the complex reflection group $W_{n,r} = \mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ with parameters $q, Q_1, \dots, Q_r \in R$ such that q is a unit in R. Let $\widetilde{\mathcal{P}}_{n,r}$ (resp. $\mathcal{P}_{n,r}$) be the set of r-compositions (resp. r-partitions) of n. The cyclotomic q-Schur algebra $\mathcal{S}(\Lambda)$ associated to $\mathcal{H}$ was introduced by Dipper, James and Mathas [DJM], which is the endomorphism algebra of a certain $\mathcal{H}$-module $M = \bigoplus_{\mu \in \Lambda} M^\mu$, where Λ is a saturated subset of $\widetilde{\mathcal{P}}_{n,r}$. They showed that $\mathcal{S}(\Lambda)$ is a cellular algebra in the sense of Graham and Lehrer [GL], and Mathas [M] showed that the Schur-Weyl duality (i.e., the double centralizer property) holds between $\mathcal{H}$ and $\mathcal{S}(\Lambda)$ in the case where $\Lambda = \widetilde{\mathcal{P}}_{n,r}$ with a certain condition.

On the other hand, the modified Ariki-Koike algebra $\overline{\mathcal{H}}$ was introduced in [SawS], under the condition that (*) "$Q_i - Q_j$ are units in R for each $i \neq j$", based on the study of the Schur-Weyl duality between $\mathcal{H}$ and a certain subalgebra of the quantum group of type A ([SakS], [Sh]). By using the cellular structure of $\overline{\mathcal{H}}$, a cyclotomic q-Schur algebra associated to $\overline{\mathcal{H}}$ was constructed, in analogy to $\mathcal{S}(\Lambda)$. It was shown in [SawS] that this cyclotomic q-Schur algebra is isomorphic to the quotient algebra $\overline{\mathcal{S}}^0$ of a certain subalgebra $\mathcal{S}^0$ of $\mathcal{S}(\Lambda)$, and that the Schur-Weyl duality holds between $\overline{\mathcal{H}}$ and $\overline{\mathcal{S}}^0$. Moreover, the structure theorem for $\overline{\mathcal{S}}^0$ was proved, which asserts that $\overline{\mathcal{S}}^0$ is a direct sum of tensor products of various q-Schur algebras $\mathcal{S}(\widetilde{\mathcal{P}}_{n_i,1})$ associated to the Iwahori-Hecke algebra of type A_{n_i-1}.

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In [Sa], N. Sawada reconstructed the subalgebra $\mathcal{S}^0$ of $\mathcal{S}(\Lambda)$ and its quotient $\overline{\mathcal{S}}^0$ based on the cellular structure on $\mathcal{S}(\Lambda)$, which works without the assumption (*). He proved that $\mathcal{S}^0$ is a standardly based algebra in the sense of Du and Rui [DR], and showed, in the case where R is a field, that the decomposition number $d_{\lambda\mu}$ between the Weyl module W^λ and the irreducible module L^μ of $\mathcal{S}(\Lambda)$ ($\lambda, \mu \in \mathcal{P}_{n,r}$) coincides with the corresponding decomposition number for $\overline{\mathcal{S}}^0$ whenever $|\lambda^{(i)}| = |\mu^{(i)}|$ for $i = 1, \dots, r$. This implies in the case where $\Lambda = \widetilde{\mathcal{P}}_{n,r}$, under the condition (*), that $d_{\lambda\mu}$ can be written as a product of $d_{\lambda^{(i)}\mu^{(i)}}$ for $i = 1, \dots, r$, where $d_{\lambda^{(i)}\mu^{(i)}}$ is the decomposition number of the q-Schur algebra $\mathcal{S}(\widetilde{\mathcal{P}}_{n_i,1})$ with $|\lambda^{(i)}| = |\mu^{(i)}| = n_i$.

The subalgebra $\mathcal{S}^0$ is regarded, in some sense, as a Borel type subalgebra of $\mathcal{S}(\Lambda)$. For example, we have $\mathcal{S}(\Lambda) = \mathcal{S}^0 \cdot (\mathcal{S}^0)^*$, where $(\mathcal{S}^0)^*$ is the image of $\mathcal{S}^0$ under the involution $*$ of $\mathcal{S}(\Lambda)$, and $\overline{\mathcal{S}}^0$ is a quotient of both $\mathcal{S}^0$ and $(\mathcal{S}^0)^*$. Thus $\overline{\mathcal{S}}^0$ corresponds to a Cartan subalgebra. In this paper, we consider a parabolic analogue of $\mathcal{S}^0$ and $\overline{\mathcal{S}}^0$. We fix $\mathbf{p} = (r_1, \dots, r_g) \in \mathbb{Z}^g_{>0}$ such that $r_1 + \cdots r_g = r$. According to $\mathbf{p}$, we regard an r-partition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ as a g-tuple of multi-partitions $\lambda = (\lambda^{[1]}, \dots, \lambda^{[g]})$, where $\lambda^{[1]} = (\lambda^{(1)}, \dots, \lambda^{(r_1)}), \lambda^{[2]} = (\lambda^{(r_1+1)}, \dots, \lambda^{(r_1+r_2)})$, and so on. For example, $\lambda^{[i]} = \lambda^{(i)}$ for $i = 1, \dots, r$ if $\mathbf{p} = (1^r)$ with $g = r$, and $\lambda^{[1]} = \lambda$ if $\mathbf{p} = (r)$ with $g = 1$. For each $\mathbf{p}$, we define a subalgebra $\mathcal{S}^{\mathbf{p}}$ of $\mathcal{S}(\Lambda)$, and its quotient algebra $\overline{\mathcal{S}}^{\mathbf{p}}$. The algebra $\mathcal{S}^{\mathbf{p}}$ coincides with $\mathcal{S}^0$ if $\mathbf{p} = (1^r)$, and coincides with $\mathcal{S}(\Lambda)$ if $\mathbf{p} = (r)$. Thus $\mathcal{S}^{\mathbf{p}}$ is a generalization of $\mathcal{S}^0$, and is regarded as an intermediate object between $\mathcal{S}(\Lambda)$ and $\mathcal{S}^0$.

All of the results in [Sa] can be generalized to our cases; $\mathcal{S}^{\mathbf{p}}$ is a standardly based algebra and $\overline{\mathcal{S}}^{\mathbf{p}}$ is a cellular algebra. Assume that R is a field. For $\lambda = (\lambda^{[1]}, \dots, \lambda^{[g]}), \mu = (\mu^{[1]}, \dots, \mu^{[g]}) \in \mathcal{P}_{n,r}$ such that $|\lambda^{[i]}| = |\mu^{[i]}|$ for $i = 1, \dots, g$, one can show (Theorem 3.13) that the decomposition number $d_{\lambda\mu}$ coincides with the corresponding decomposition number in the algebra $\overline{\mathcal{S}}^{\mathbf{p}}$. In the case where $\Lambda = \widetilde{\mathcal{P}}_{n,r}$, we prove the structure theorem (Theorem 4.15) for $\overline{\mathcal{S}}^{\mathbf{p}}$, which asserts that $\overline{\mathcal{S}}^{\mathbf{p}}$ is a direct sum of tensor products of various $\mathcal{S}(\widetilde{\mathcal{P}}_{n_i,r_i})$. We remark, in contrast to the argument in [SawS], that no assumptions on parameters are required in this proof. Combining with the previous results, we obtain the product formula for decomposition numbers, namely, $d_{\lambda\mu}$ coincides with the product of $d_{\lambda^{[i]}\mu^{[i]}}$ for $i = 1, \dots, g$, where $d_{\lambda^{[i]}\mu^{[i]}}$ is the decomposition number for $\mathcal{S}(\widetilde{\mathcal{P}}_{n_i,r_i})$ with $|\lambda^{[i]}| = |\mu^{[i]}| = n_i$, which holds without any restriction on the parameters (Theorem 4.17).

By making use of the Schur functors on $\mathcal{S}(\Lambda)$, one can define a modified Ariki-Koike algebra $\overline{\mathcal{H}}^{\mathbf{p}}$ of type $\mathbf{p}$ as a certain subalgebra of $\mathcal{S}^{\mathbf{p}}$. The algebra $\overline{\mathcal{H}}^{\mathbf{p}}$ is isomorphic to $\overline{\mathcal{H}}$ if $\mathbf{p} = (1^r)$, and coincides with $\mathcal{H}$ if $\mathbf{p} = (r)$. Put $Q_i^{\mathbf{p}} = Q_{r_1+\cdots+r_i}$ for $i = 1, \dots, g$. Under the assumption (**) "$Q_i^{\mathbf{p}} - Q_j^{\mathbf{p}}$ are units in R for each $i \neq j$", we give a presentation of $\overline{\mathcal{H}}^{\mathbf{p}}$ which is a generalization of the presentation of $\overline{\mathcal{H}}$ given in [SawS]. We show that $\overline{\mathcal{S}}^{\mathbf{p}}$ is realized as an endomorphism algebra of a certain $\overline{\mathcal{H}}^{\mathbf{p}}$-module $\overline{M}^{\mathbf{p}} = \bigoplus_{\mu\in\Lambda} \overline{M}^\mu$, and prove the Schur-Weyl duality between $\overline{\mathcal{S}}^{\mathbf{p}}$ and $\overline{\mathcal{H}}^{\mathbf{p}}$. In the case where the parameters $q, Q_1, \dots, Q_r$ satisfy the separation condition in the sense of [A] (see (8.3.1)), it is shown that all the $\overline{\mathcal{H}}^{\mathbf{p}}$ are isomorphic to $\mathcal{H}$, and so the above results give new presentations of $\mathcal{H}$.

By using the Jantzen filtration, v-decomposition numbers $d_{\lambda\mu}(v)$ for $\mathcal{S}(\Lambda)$ can be defined, which is a polynomial analogue of $d_{\lambda\mu}$. The results in this paper concerning the decomposition numbers for $\mathcal{S}(\Lambda), \mathcal{S}^{\mathbf{p}}, \overline{\mathcal{S}}^{\mathbf{p}}$ are generalized to v-decomposition

numbers. In particular, the product formula for v-decomposition numbers is obtained, which is discussed in [W].

Notation. Let R be an integral domain and M a free R-module of finite rank. We denote by $\operatorname{End} M$ the endomorphism algebra of M, where the composition is defined by $(f \circ g)(m) = f(g(m))$ for $f, g \in \operatorname{End} M, m \in M$. Thus $\operatorname{End} M$ acts on M from the left by $(f, m) \mapsto f(m)$. We denote by $\operatorname{End}^0 M$ the opposite algebra of $\operatorname{End} M$. If an R-algebra A (resp. B) acts on M from the left (resp. from the right), then we have a natural homomorphism of R-algebras $A \to \operatorname{End} M$ (resp. $B \to \operatorname{End}^0 M$). If an R-algebra X acts on M from the right or left, we denote by $\operatorname{End}_X M$ the subalgebra of $\operatorname{End} M$ consisting of endomorphisms commuting with X. The subalgebra $\operatorname{End}^0_X M$ of $\operatorname{End}^0 M$ is defined similarly.

1. Recollection of cyclotomic q-Schur algebras

1.1. Let R be an integral domain, $q, Q_1, \dots, Q_r$ be elements in R with q invertible. The Ariki-Koike algebra $\mathcal{H} = \mathcal{H}_{n,r}$ associated to the complex reflection group $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ is an associative algebra over R with generators $T_0, T_1, \dots, T_{n-1}$ subject to the condition

$$\begin{aligned}
&(T_0 - Q_1) \cdots (T_0 - Q_r) = 0, && \\
&(T_i - q)(T_i + q^{-1}) = 0 && (i \geq 1), \\
&T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0, && \\
&T_i T_j = T_j T_i && (|i - j| \geq 2), \\
&T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} && (1 \leq i \leq n-2).
\end{aligned}$$

It is known that $\mathcal{H}$ is a free R-module with rank $n! r^n$. We denote by $\mathcal{H}_n$ the subalgebra of $\mathcal{H}$ generated by $T_1, \dots, T_{n-1}$, which is isomorphic to the Iwahori-Hecke algebra associated to the symmetric group $\mathfrak{S}_n$ of degree n.

1.2. It is known by [DJM] that $\mathcal{H}$ has a structure of the cellular algebra. In order to describe the cellular basis of $\mathcal{H}$, we prepare some notation. An element $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{Z}^m_{\geq 0}$ is called a composition of length $\leq m$, and $|\mu| = \sum \mu_i$ is called the size of μ. An r-composition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ is an r-tuple of compositions $\lambda^{(i)} = (\lambda^{(i)}_1, \dots, \lambda^{(i)}_{m_i})$. The size $|\lambda|$ of λ is defined by $|\lambda| = \sum_{i=1}^r |\lambda^{(i)}|$. We denote by λ by $\lambda = (\lambda^{(i)}_j)$. A composition $\mu = (\mu_1, \dots, \mu_m)$ is called a partition if $\mu_1 \geq \cdots \geq \mu_m \geq 0$. An r-composition λ is called an r-partition if $\lambda^{(i)}$ is a partition for all i. We fix $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r_{>0}$ once and for all, and denote by $\widetilde{\mathcal{P}}_{n,r} = \widetilde{\mathcal{P}}_{n,r}(\mathbf{m})$ the set of r-compositions $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ of size n such that $\lambda^{(i)}$ is a composition of length $\leq m_i$. Similarly, we define the set $\mathcal{P}_{n,r} = \mathcal{P}_{n,r}(\mathbf{m})$ of r-partitions. If $m_i \geq n$ for any i, $\mathcal{P}_{n,r}(\mathbf{m})$ are mutually identified for all $\mathbf{m}$. However, even in that case, $\widetilde{\mathcal{P}}_{n,r}(\mathbf{m})$ depends on the choice of $\mathbf{m}$.

For r-compositions $\lambda = (\lambda^{(i)}_j)$ and $\mu = (\mu^{(i)}_j)$, we define a dominance order $\lambda \trianglerighteq \mu$ by the condition

$$\sum_{c=1}^{k-1} |\lambda^{(c)}| + \sum_{j=1}^{i} |\lambda^{(k)}_j| \geq \sum_{c=1}^{k-1} |\mu^{(c)}| + \sum_{j=1}^{i} |\mu^{(k)}_j|$$

for any $1 \leq k \leq r$ and $1 \leq i \leq m_k$. If $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$, we write it as $\lambda \triangleright \mu$.

Let λ be an r-partition of n. We identify λ with the r-tuple of Young diagrams, and refer it as the Young diagram of λ. We denote by $\mathrm{Std}(\lambda)$ the set of standard tableau $\mathfrak{t} = (\mathfrak{t}^{(1)}, \dots, \mathfrak{t}^{(r)})$ of shape λ, i.e., $\mathfrak{t}$ is a Young diagram of λ with letters $1, \dots, n$ attached to the nodes of the diagram, under the condition that $\mathfrak{t}^{(i)}$ is a standard tableau in the usual sense for each i. We define $\mathfrak{t}^\lambda \in \mathrm{Std}(\lambda)$ by attaching the letters $1, 2, \dots, n$ to the nodes of the Young diagram λ in this order, from left to right, and from top to bottom for $\mathfrak{t}^{(1)}$, and then for $\mathfrak{t}^{(2)}$, and so on. $\mathfrak{S}_n$ acts naturally on the set of tableaux from the right, and we denote by $d(\mathfrak{t})$ the element in $\mathfrak{S}_n$ such that $\mathfrak{t} = \mathfrak{t}^\lambda d(\mathfrak{t})$ for each $\mathfrak{t} \in \mathrm{Std}(\lambda)$. More generally, the set r-$\mathrm{Std}(\mu)$ of row-standard tableaux of shape μ is defined for $\mu \in \widetilde{\mathcal{P}}_{n,r}$, by replacing a standard tableau $\mathfrak{t}^{(i)}$ by a row-standard tableau. Then $\mathfrak{t}^\mu$ is defined similarly, and $d(\mathfrak{t}) \in \mathfrak{S}_n$ is defined also for $\mathfrak{t} \in$ r-$\mathrm{Std}(\mu)$.

For $\mu \in \widetilde{\mathcal{P}}_{n,r}$, we define r-tuples of integers

$$\alpha(\mu) = (\alpha_1, \dots, \alpha_r), \qquad \mathbf{a}(\mu) = (a_1, \dots, a_r)$$

by $\alpha_i = |\mu^{(i)}|$, and $a_i = \sum_{j=1}^{i-1} |\mu^{(j)}|$ for $i = 1, \dots, r$. (Note that $a_1 = 0$.)

We define $L_k \in \mathcal{H}$ by $L_k = T_{k-1} \cdots T_1 T_0 T_1 \cdots T_{k-1}$ for $k = 1, \dots, n$. Then $L_1, \dots, L_n$ commute with each other. For $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{Z}^r_{\geq 0}$, we define $u^+_{\mathbf{a}} \in \mathcal{H}$ by $u^+_{\mathbf{a}} = u_{\mathbf{a},1} u_{\mathbf{a},2} \cdots u_{\mathbf{a},r}$, where

$$u_{\mathbf{a},k} = \prod_{i=1}^{a_k} (L_i - Q_k).$$

For $\mu \in \widetilde{\mathcal{P}}_{n,r}$, let $\mathfrak{S}_\mu = \mathfrak{S}_{\mu^{(1)}} \times \cdots \times \mathfrak{S}_{\mu^{(r)}}$ be the Young subgroup of $\mathfrak{S}_n$. We define $x_\mu \in \mathcal{H}_n$ by $x_\mu = \sum_{w \in \mathfrak{S}_\mu} q^{l(w)} T_w$, where $l(w)$ is the length of $w \in \mathfrak{S}_n$, and T_w is a basis element of $\mathcal{H}_n$ corresponding to $w \in \mathfrak{S}_n$. We define $m_\mu \in \mathcal{H}$ by $m_\mu = u^+_{\mathbf{a}} x_\mu = x_\mu u^+_{\mathbf{a}}$. For $\mathfrak{s}, \mathfrak{t} \in$ r-$\mathrm{Std}(\mu)$, we define $m_{\mathfrak{s}\mathfrak{t}} \in \mathcal{H}$ by $m_{\mathfrak{s}\mathfrak{t}} = T^*_{d(\mathfrak{s})} m_\mu T_{d(\mathfrak{t})}$, where $x \mapsto x^*$ is an anti-automorphism on $\mathcal{H}_n$ defined by $T_i^* = T_i$ for $i = 1, \dots, n-1$. Then it is known by [DJM, Theorem 3.26] that the set

$$\{m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ for some } \lambda \in \mathcal{P}_{n,r}\} \tag{1.2.1}$$

gives a cellular basis of $\mathcal{H}$ with respect to the dominance order on $\mathcal{P}_{n,r}$ in the sense of [GL]. In particular, if we denote by $h \mapsto h^*$ the anti-automorphism on $\mathcal{H}$ defined by $T_i^* = T_i$ for $i = 0, \dots, n-1$, we have $m^*_{\mathfrak{s}\mathfrak{t}} = m_{\mathfrak{t}\mathfrak{s}}$.

1.3. Here we recall the concept of semistandard tableau in the case of multi-partitions due to [DJM]. We consider the set X of pairs (i, k) with $1 \leq i \leq n, 1 \leq k \leq r$, and define a total order on this set by $(i_1, k_1) < (i_2, k_2)$ if $k_1 < k_2$, or if $k_1 = k_2$ and $i_1 < i_2$. For an r-partition λ of n, a Tableau T of shape λ is defined as a Young diagram λ with an element of X attached to each node of λ. For each $(i, k) \in X$, let $\mu_i^{(k)}$ be the number of entries of T containing (i, k). Then $\mu = (\mu_j^{(k)})$ is an r-composition of n. The Tableau T is called a λ-tableau of type μ. A Tableau $T = (T^{(1)}, \dots, T^{(r)})$ of shape λ is called a semistandard tableau if it satisfies the properties; the entries of $T^{(i)}$ are weakly increasing along the rows, strictly increasing along the columns with respect to X, and furthermore the entries of $T^{(k)}$ consist of (i, k') with $k' \geq k$. We denote by $\mathcal{T}_0(\lambda, \mu)$ the set of semistandard tableau of shape λ and type μ for $\lambda \in \mathcal{P}_{n,r}$ and $\mu \in \widetilde{\mathcal{P}}_{n,r}$. Note that $\mathcal{T}_0(\lambda, \mu)$ is empty unless $\lambda \trianglerighteq \mu$.

Let $\mathfrak{t}$ be a standard tableau of shape λ. For $\mu \in \widetilde{\mathcal{P}}_{n,r}$, we construct a tableau $\mu(\mathfrak{t})$ from $\mathfrak{t}$ as follows; replace the entry j in $\mathfrak{t}$ by (i,k) if j appears in the i-th row of the k-th component $(\mathfrak{t}^\mu)^{(k)}$ of $\mathfrak{t}^\mu$. $\mu(\mathfrak{t})$ is a λ-tableau of type μ, but it is not necessarily semistandard.

1.4. For each $\mu \in \widetilde{\mathcal{P}}_{n,r}$, we define a right $\mathcal{H}$-module M^μ by $M^\mu = m_\mu \mathcal{H}$. It is known by [DJM, Theorem 4.14] that M^μ is a free R-module with basis

$$\{m_{S\mathfrak{t}} \mid S \in \mathcal{T}_0(\lambda,\mu), \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ for some } \lambda \in \mathcal{P}_{n,r}\}, \tag{1.4.1}$$

where

$$m_{S\mathfrak{t}} = \sum_{\substack{\mathfrak{s}\in\mathrm{Std}(\lambda)\\ \mu(\mathfrak{s})=S}} q^{l(d(\mathfrak{s}))+l(d(\mathfrak{t}))} m_{\mathfrak{s}\mathfrak{t}}. \tag{1.4.2}$$

A subset Λ of $\widetilde{\mathcal{P}}_{n,r}(\mathbf{m})$ is called a saturated set if any partition λ such that $\lambda \trianglerighteq \mu$ for some $\mu \in \Lambda$ is contained in Λ. We denote by Λ^+ the set of r-partitions of n contained in Λ. Put $M = \bigoplus_{\mu\in\Lambda} M^\mu$. The cyclotomic q-Schur algebra $\mathcal{S}(\Lambda)$ associated to $\mathcal{H}$ (and to Λ) is defined by

$$\mathcal{S}(\Lambda) = \mathrm{End}_{\mathcal{H}}(M) = \bigoplus_{\nu,\mu\in\Lambda} \mathrm{Hom}_{\mathcal{H}}(M^\nu, M^\mu).$$

We consider the structure of $\mathrm{Hom}_{\mathcal{H}}(M^\nu, M^\mu)$ for $\mu,\nu \in \Lambda$. Let $M^{\nu *} = (M^\nu)^*$ be the image of M^ν under $*$. We have $M^{\nu *} = \mathcal{H} m_\nu$. It is easy to see that for any $m \in M^{\nu *} \cap M^\mu = \mathcal{H} m_\nu \cap m_\mu \mathcal{H}$, the map $m_\nu h \mapsto mh$ $(h \in \mathcal{H})$ gives rise to an $\mathcal{H}$-module homomorphism $\varphi_m : M^\nu \to M^\mu$. It is known by [DJM, Corollary 5.17] that the map $\varphi \mapsto \varphi(m_\nu)$ gives an isomorphism of R-modules

$$\mathrm{Hom}_{\mathcal{H}}(M^\nu, M^\mu) \to M^{\nu *} \cap M^\mu. \tag{1.4.3}$$

Suppose that $\mu,\nu \in \Lambda$, and $\lambda \in \Lambda^+$. We define for $S \in \mathcal{T}_0(\lambda,\mu), T \in \mathcal{T}_0(\lambda,\nu)$

$$m_{ST} = \sum_{\substack{\mathfrak{s},\mathfrak{t}\in\mathrm{Std}(\lambda)\\ \mu(\mathfrak{s})=S,\nu(\mathfrak{t})=T}} q^{l(d(\mathfrak{s}))+l(d(\mathfrak{t}))} m_{\mathfrak{s}\mathfrak{t}}. \tag{1.4.4}$$

Then it is known by [DJM, Proposition 6.3] that the set

$$\{m_{ST} \mid S \in \mathcal{T}_0(\lambda,\mu), T \in \mathcal{T}_0(\lambda,\nu) \text{ for some } \lambda \in \Lambda^+\}$$

gives rise to a basis of $M^{\nu *} \cap M^\mu$. We denote by φ_{ST} the element of $\mathrm{Hom}_{\mathcal{H}}(M^\nu, M^\mu)$ corresponding to m_{ST} via the isomorphism (1.4.3). Thus φ_{ST} is a map $M^\nu \to M^\mu$ defined by $\varphi_{ST}(m_\nu h) = m_{ST} h$ for any $h \in \mathcal{H}$. For each $\lambda \in \Lambda^+$, put $\mathcal{T}_0(\lambda) = \bigcup_{\mu\in\Lambda} \mathcal{T}_0(\lambda,\mu)$. The fundamental result of Dipper, James and Mathas is the following theorem.

Theorem 1.5 ([DJM]). *The cyclotomic q-Schur algebra $\mathcal{S}(\Lambda)$ is a cellular algebra with a cellular basis*

$$\mathcal{C}(\Lambda) = \{\varphi_{ST} \mid S,T \in \mathcal{T}_0(\lambda) \textit{ for some } \lambda \in \Lambda^+\}.$$

1.6. For $\lambda \in \Lambda^+$, let $T^\lambda = \lambda(\mathfrak{t}^\lambda)$. Then T^λ is a semistandard tableau obtained from $\mathfrak{t}^\lambda$ by replacing the entries j in $(\mathfrak{t}^\lambda)^{(k)}$ by (j,k). Then $\mathfrak{t} = \mathfrak{t}^\lambda$ is the unique standard tableau such that $\lambda(\mathfrak{t}) = T^\lambda$. It follows that $m_{T^\lambda T^\lambda} = m_{\mathfrak{t}^\lambda \mathfrak{t}^\lambda} = m_\lambda$, and $\varphi_{T^\lambda T^\lambda}$ is the identity element in $\mathrm{Hom}_{\mathcal{H}}(M^\lambda, M^\lambda)$. We put $\varphi_\lambda = \varphi_{T^\lambda T^\lambda}$.

For each $\lambda \in \Lambda^+$, we define $\mathcal{S}^{\vee\lambda}$ as the R-submodule of $\mathcal{S}(\Lambda)$ spanned by φ_{ST}, where $S, T \in \mathcal{T}_0(\lambda', \mu)$ for various $\lambda' \in \Lambda^+$ such that $\lambda' \triangleright \lambda$, and for various $\mu \in \Lambda$. Then $\mathcal{S}^{\vee\lambda}$ is a two-sided ideal of $\mathcal{S}(\Lambda)$, and we define the Weyl module W^λ as the right $\mathcal{S}(\Lambda)$-submodule of $\mathcal{S}(\Lambda)/\mathcal{S}^{\vee\lambda}$ generated by the image of $\varphi_\lambda \in \mathcal{S}(\Lambda)$. For each $T \in \mathcal{T}_0(\lambda)$, let φ_T be the image of $\varphi_{T^\lambda T}$ in W^λ. Then the following holds: W^λ is an R-free module with basis $\{\varphi_T \mid T \in \mathcal{T}_0(\lambda)\}$. There exists a canonical bilinear form $\langle\, , \rangle$ on W^λ determined by

$$\varphi_{T^\lambda S}\varphi_{TT^\lambda} \equiv \langle \varphi_S, \varphi_T\rangle \varphi_{T^\lambda T^\lambda} \mod \mathcal{S}^{\vee\lambda}.$$

Let $\operatorname{rad} W^\lambda = \{x \in W^\lambda \mid \langle x, y\rangle = 0 \text{ for any } y \in W^\lambda\}$. Then $\operatorname{rad} W^\lambda$ is an $\mathcal{S}(\Lambda)$-submodule of W^λ. Put $L^\lambda = W^\lambda/\operatorname{rad} W^\lambda$. Assume that R is a field. Then it is known by [DJM] that L^λ is a (non-zero) absolutely irreducible module, and that the set $\{L^\lambda \mid \lambda \in \Lambda^+\}$ gives a complete set of non-isomorphic irreducible $\mathcal{S}(\Lambda)$-modules.

2. Parabolic type subalgebras of $\mathcal{S}(\Lambda)$

2.1. In [Sa] Sawada constructed a subalgebra $\mathcal{S}^0$ of $\mathcal{S}(\Lambda)$ and showed that its quotient algebra $\overline{\mathcal{S}}^0$ coincides with the cyclotomic q-Schur algebra associated to the modified Ariki-Koike algebra discussed in [SawS] under some condition on parameters (see Introduction). $\mathcal{S}^0$ is regarded, in some sense, as a Borel type subalgebra of $\mathcal{S}(\Lambda)$. In this section, we extend his result to a more general situation, i.e, to the parabolic type subalgebras.

2.2. Let Λ and Λ^+ be as in Section 1. We fix $\mathbf{p} = (r_1, \dots, r_g) \in \mathbb{Z}^g_{>0}$ such that $r = r_1 + \cdots + r_g$ for some g, and put $p_k = \sum_{i=1}^{k-1} r_i$ for $k = 1, \dots, g$ with $p_1 = 0$. For each $\mu = (\mu^{(1)}, \dots, \mu^{(r)}) \in \Lambda$, put

$$\alpha_{\mathbf{p}}(\mu) = (n_1, \dots, n_g), \quad \mathbf{a}_{\mathbf{p}}(\mu) = (a_1, \dots, a_g),$$

where $n_k = \sum_{i=1}^{r_k} |\mu^{(p_k+i)}|$ and $a_k = \sum_{i=1}^{k-1} n_i$ for $k = 1, \dots, g$ with $a_1 = 0$. By making use of $\mathbf{p}$, we express the r-compositions as the g-tuples of multi-compositions as follows: let $\mu = (\mu^{(1)}, \dots, \mu^{(r)}) \in \widetilde{\mathcal{P}}_{n,r}$. We write μ as $(\mu^{[1]}, \dots, \mu^{[g]})$, where $\mu^{[k]} = (\mu^{(p_k+1)}, \dots, \mu^{(p_k+r_k)})$ is an r_k-composition of n_k. Note that $\mathbf{a}_{\mathbf{p}}(\mu)$ (resp. $\alpha_{\mathbf{p}}(\mu)$) coincides with $\mathbf{a}(\mu)$ (resp. $\alpha(\mu)$) in 1.2 in the special case where $\mathbf{p} = (1^r)$.

We define a partial order on $\mathbb{Z}^g_{\geq 0}$ by $\mathbf{a} = (a_1, \dots, a_g) \geq \mathbf{b} = (b_1, \dots, b_g)$ if $a_k \geq b_k$ for $k = 1, \dots, g$. We write $\mathbf{a} > \mathbf{b}$ if $\mathbf{a} \geq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$. The following properties are easily verified.

(2.2.1) Let $\mu, \nu \in \Lambda, \lambda \in \Lambda^+$. Then we have:

(i) $\mathbf{a}_{\mathbf{p}}(\mu) = \mathbf{a}_{\mathbf{p}}(\nu)$ if and only if $\alpha_{\mathbf{p}}(\mu) = \alpha_{\mathbf{p}}(\nu)$.
(ii) If $\nu \trianglerighteq \mu$, then $\mathbf{a}_{\mathbf{p}}(\nu) \geq \mathbf{a}_{\mathbf{p}}(\mu)$. In particular, if $\mathcal{T}_0(\lambda, \mu) \neq \emptyset$, then $\lambda \trianglerighteq \mu$ (cf. 1.3), and so $\mathbf{a}_{\mathbf{p}}(\lambda) \geq \mathbf{a}_{\mathbf{p}}(\mu)$.

For each $\lambda \in \Lambda^+, \mu \in \Lambda$, we define a set $\mathcal{T}_0^{\mathbf{p}}(\lambda, \mu)$ by $\mathcal{T}_0(\lambda, \mu)$ if $\mathbf{a}_{\mathbf{p}}(\lambda) = \mathbf{a}_{\mathbf{p}}(\mu)$ and by the empty set otherwise. Put $\mathcal{T}_0^{\mathbf{p}}(\lambda) = \bigcup_{\mu \in \Lambda} \mathcal{T}_0^{\mathbf{p}}(\lambda, \mu)$.

Example 2.3. Let $n = 20, r = 5$ and take $\mu = (21; 121; 32; 1^3; 41) \in \widetilde{\mathcal{P}}_{20,5}$. Let $\mathbf{p} = (2, 2, 1)$. Then $\alpha_{\mathbf{p}}(\mu) = (7, 8, 5)$ and $\alpha_{\mathbf{p}}(\mu) = (0, 7, 15)$. We have $\mu = (\mu^{[1]}, \mu^{[2]}, \mu^{[3]})$ with $\mu^{[1]} = (21; 121), \mu^{[2]} = (32; 1^3), \mu^{[3]} = (41)$.

2.4. Let $\mathcal{C}^{\mathbf{p}} = \mathcal{C}^{\mathbf{p}}(\Lambda)$ be the set of $\varphi_{ST} \in \mathcal{C}(\Lambda)$ for $S \in \mathcal{T}_0(\lambda, \mu), T \in \mathcal{T}_0(\lambda, \nu)$, where $\mu, \nu \in \Lambda, \lambda \in \Lambda^+$ are taken subject to the condition that $\mathbf{a}_{\mathbf{p}}(\lambda) > \mathbf{a}_{\mathbf{p}}(\mu)$ if $\alpha_{\mathbf{p}}(\mu) \neq \alpha_{\mathbf{p}}(\nu)$. We define an R-submodule $\mathcal{S}^{\mathbf{p}} = \mathcal{S}^{\mathbf{p}}(\Lambda)$ of $\mathcal{S}(\Lambda)$ as the R-span of $\mathcal{C}^{\mathbf{p}}$. We will see that $\mathcal{S}^{\mathbf{p}}$ is a subalgebra of $\mathcal{S}(\Lambda)$ and that $\mathcal{S}^{\mathbf{p}}$ turns out to be a standardly based algebra in the sense of Du and Rui [DR]. Note that in the case where $\mathbf{p} = (1^r)$, $\mathcal{S}^{\mathbf{p}}$ coincides with $\mathcal{S}^0$.

First we note that the identity element $1_{\mathcal{S}(\Lambda)}$ is contained in $\mathcal{S}^{\mathbf{p}}$. In fact, one can write $1_{\mathcal{S}(\Lambda)} = \sum_{\mu \in \Lambda} \varphi_\mu$, where $\varphi_\mu \in \mathcal{S}(\Lambda)$ is the identity map on M^μ. Since φ_μ is written as a linear combination of φ_{ST} with $S, T \in \mathcal{T}_0(\lambda, \mu)$, we see that $1_{\mathcal{S}(\Lambda)} \in \mathcal{S}^{\mathbf{p}}$.

In order to relate $\mathcal{S}^{\mathbf{p}}$ to the standardly based algebra, we introduce a different kind of labeling for $\mathcal{C}^{\mathbf{P}}$, following the idea of [Sa]. Let us define a subset $\Sigma^{\mathbf{p}}$ of $\Lambda^+ \times \{0, 1\}$ by

$$\Sigma^{\mathbf{p}} = (\Lambda^+ \times \{0,1\}) \backslash \{(\lambda, 1) \mid \mathcal{T}_0(\lambda, \mu) = \emptyset \\ \text{for any } \mu \in \Lambda \text{ such that } \mathbf{a}_{\mathbf{p}}(\lambda) > \mathbf{a}_{\mathbf{p}}(\mu)\}.$$

We define a partial order $\geq$ on $\Sigma^{\mathbf{p}}$ by $(\lambda_1, \varepsilon_1) > (\lambda_2, \varepsilon_2)$ if $\lambda_1 \rhd \lambda_2$ or $\lambda_1 = \lambda_2$ and $\varepsilon_1 > \varepsilon_2$.

For each $\eta = (\lambda, \varepsilon) \in \Sigma^{\mathbf{p}}$, put

$$I^{\mathbf{p}}(\eta) = \begin{cases} \mathcal{T}_0^{\mathbf{p}}(\lambda) & \text{if } \varepsilon = 0, \\ \displaystyle\bigcup_{\substack{\mu \in \Lambda \\ \mathbf{a}_{\mathbf{p}}(\lambda) > \mathbf{a}_{\mathbf{p}}(\mu)}} \mathcal{T}_0(\lambda, \mu) & \text{if } \varepsilon = 1, \end{cases}$$

$$J^{\mathbf{p}}(\eta) = \begin{cases} \mathcal{T}_0^{\mathbf{p}}(\lambda) & \text{if } \varepsilon = 0, \\ \mathcal{T}_0(\lambda) & \text{if } \varepsilon = 1. \end{cases}$$

Note that $I^{\mathbf{p}}(\eta)$ and $J^{\mathbf{p}}(\eta)$ are not empty. For $\eta \in \Sigma^{\mathbf{p}}$, if we put,

$$\mathcal{C}^{\mathbf{p}}(\eta) = \{\varphi_{ST} \mid S \in I^{\mathbf{p}}(\eta), T \in J^{\mathbf{p}}(\eta)\},$$

we see easily that

$$\mathcal{C}^{\mathbf{p}} = \coprod_{\eta \in \Sigma^{\mathbf{p}}} \mathcal{C}^{\mathbf{p}}(\eta).$$

For each $\eta \in \Sigma^{\mathbf{p}}$, we define a submodule $(\mathcal{S}^{\mathbf{p}})^{\vee \eta}$ of $\mathcal{S}^{\mathbf{p}}$ as the R-span of φ_{ST}, where $S \in I^{\mathbf{p}}(\eta'), T \in J^{\mathbf{p}}(\eta')$ for some $\eta' \in \Sigma^{\mathbf{p}}$ such that $\eta' > \eta$.

By using the cellular structure of $\mathcal{S}(\Lambda)$, the following result can be proved in a similar way as in [Sa, Lemma 2.4].

Lemma 2.5. *Take $\lambda_i \in \Lambda^+$, $\mu_i, \nu_i \in \Lambda$ for $i = 1, 2$ such that $\nu_1 = \mu_2$. Then for $\varphi_{S_iT_i} \in \mathcal{C}^{\mathbf{p}}$ with $S_i \in \mathcal{T}_0(\lambda_i, \mu_i), T_i \in \mathcal{T}_0(\lambda_i, \nu_i)$, the following hold:*

$$\varphi_{S_1T_1} \cdot \varphi_{S_2T_2} =$$
$$= \begin{cases} \sum_{\varphi_{ST} \in \mathcal{C}^{\mathbf{p}}(\lambda_1, 0)} r_{ST}\varphi_{ST} + \sum_{\lambda \rhd \lambda_1} \sum_{\varphi_{ST} \in \mathcal{C}^{\mathbf{p}}(\lambda)} r_{ST}\varphi_{ST} & \text{if } \varphi_{S_1T_1} \in \mathcal{C}^{\mathbf{p}}(\lambda_1, 0), \\ \sum_{\lambda \unrhd \lambda_1} \sum_{\varphi_{ST} \in \mathcal{C}^{\mathbf{p}}(\lambda, 1)} r_{ST}\varphi_{ST} & \text{if } \varphi_{S_1T_1} \in \mathcal{C}^{\mathbf{p}}(\lambda_1, 1), \\ \sum_{\lambda \unrhd \lambda_2} \sum_{\varphi_{ST} \in \mathcal{C}^{\mathbf{p}}(\lambda)} r_{ST}\varphi_{ST} & \text{if } \varphi_{S_2T_2} \in \mathcal{C}^{\mathbf{p}}(\lambda_2, 0), \\ \sum_{\lambda \unrhd \lambda_2} \sum_{\varphi_{ST} \in \mathcal{C}^{\mathbf{p}}(\lambda, 1)} r_{ST}\varphi_{ST} & \text{if } \varphi_{S_2T_2} \in \mathcal{C}^{\mathbf{p}}(\lambda_2, 1), \end{cases}$$

where $r_{ST} \in R$ and $\mathcal{C}^{\mathbf{p}}(\lambda) = \mathcal{C}^{\mathbf{p}}(\lambda, 0) \cup \mathcal{C}^{\mathbf{p}}(\lambda, 1)$.

The following theorem is a generalization of [Sa, Theorem 2.6]. The proof is done similarly by using Lemma 2.5.

Theorem 2.6. *$\mathcal{S}^{\mathbf{p}}$ is a subalgebra of $\mathcal{S}(\Lambda)$ containing the identity element of $\mathcal{S}(\Lambda)$. Moreover, $\mathcal{S}^{\mathbf{p}}$ turns out to be a standardly based algebra with the standard basis $\mathcal{C}^{\mathbf{p}}$ in the sense of* [DR]*, i.e., the following holds: for any $\varphi \in \mathcal{S}^{\mathbf{p}}, \varphi_{ST} \in \mathcal{C}^{\mathbf{p}}(\eta)$, we have*

$$\varphi \cdot \varphi_{ST} \equiv \sum_{S' \in I^{\mathbf{p}}(\eta)} f_{S'}\varphi_{S'T} \qquad \mathrm{mod}\ (\mathcal{S}^{\mathbf{p}})^{\vee\eta},$$
$$\varphi_{ST} \cdot \varphi \equiv \sum_{T' \in J^{\mathbf{p}}(\eta)} f'_{T'}\varphi_{ST'} \qquad \mathrm{mod}\ (\mathcal{S}^{\mathbf{p}})^{\vee\eta}$$

with $f_{S'}, f'_{T'} \in R$, where in the first formula $f_{S'}$ depends on (φ, S, S') but does not depend on T, and in the second formula $f'_{T'}$ depends on (φ, T, T') but does not depend on S.

2.7. For each $\eta \in \Sigma^{\mathbf{p}}$, let ${}^{\diamond}Z^{\eta}_{\mathbf{p}}$ be an R-module with a basis $\{\varphi^{\eta}_S \mid S \in I^{\mathbf{p}}(\eta)\}$, and let $Z^{\eta}_{\mathbf{p}}$ be an R-module with a basis $\{\varphi^{\eta}_T \mid T \in J^{\mathbf{p}}(\eta)\}$. In view of Theorem 2.6, one can define actions of $\mathcal{S}^{\mathbf{p}}$ on ${}^{\diamond}Z^{\eta}_{\mathbf{p}}$ and on $Z^{\eta}_{\mathbf{p}}$ by

$$\varphi \cdot \varphi^{\eta}_S = \sum_{S' \in I^{\mathbf{p}}(\eta)} f_{S'}\varphi^{\eta}_{S'} \qquad (S \in I^{\mathbf{p}}(\eta), \varphi \in \mathcal{S}^{\mathbf{p}}),$$
$$\varphi^{\eta}_T \cdot \varphi = \sum_{T' \in J^{\mathbf{p}}(\eta)} f'_{T'}\varphi^{\eta}_{T'} \qquad (T \in J^{\mathbf{p}}(\eta), \varphi \in \mathcal{S}^{\mathbf{p}}),$$

where $f_{S'}, f'_{T'}$ are as in the theorem. Then ${}^{\diamond}Z^{\eta}_{\mathbf{p}}$ (resp. $Z^{\eta}_{\mathbf{p}}$) has a structure of the left $\mathcal{S}^{\mathbf{p}}$-module (resp. the right $\mathcal{S}^{\mathbf{p}}$-module). Moreover, the theorem implies, for any $\varphi_{UT}, \varphi_{SV} \in \mathcal{C}^{\mathbf{p}}(\eta)$, that there exists $f_{TS} \in R$ (independent of the choice of U, V) such that

$$\varphi_{UT}\varphi_{SV} \equiv f_{TS}\varphi_{UV} \quad \mathrm{mod}\ (\mathcal{S}^{\mathbf{p}})^{\vee\eta}.$$

We define a bilinear form $\beta_\eta : {}^\diamond Z^\eta_{\mathbf{p}} \times Z^\eta_{\mathbf{p}} \to R$ by $\beta_\eta(\varphi^\eta_S, \varphi^\eta_T) = f_{TS}$ for $S \in I^{\mathbf{p}}(\eta), T \in J^{\mathbf{p}}(\eta)$. Put

$$\operatorname{rad} Z^\eta_{\mathbf{p}} = \{y \in Z^\eta_{\mathbf{p}} \mid \beta_\eta(x, y) = 0 \text{ for any } x \in {}^\diamond Z^\eta_{\mathbf{p}}\}.$$

Then $\operatorname{rad} Z^\eta_{\mathbf{p}}$ is an $\mathcal{S}^{\mathbf{p}}$-submodule of $Z^\eta_{\mathbf{p}}$ and we define the quotient module $L^\eta_{\mathbf{p}} = Z^\eta_{\mathbf{p}}/\operatorname{rad} Z^\eta_{\mathbf{p}}$. By the general theory of standardly based algebras (see [DR]), we obtain the following corollary, which is a strengthened form of [Sa, Proposition 3.7].

Corollary 2.8. *Assume that R is a field. Then:*

(i) *$L^\eta_{\mathbf{p}}$ is an absolutely irreducible $\mathcal{S}^{\mathbf{p}}$-module if it is non-zero.*
(ii) *The set $\{L^\eta_{\mathbf{p}} \neq 0 \mid \eta \in \Sigma^{\mathbf{p}}\}$ gives a complete set of non-isomorphic irreducible right $\mathcal{S}^{\mathbf{p}}$-modules.*

Remarks 2.9. (i) In [Sa], only the case $Z^{(\lambda,0)}_{\mathbf{p}}$ is discussed (for $\mathbf{p} = (1^r)$). In that case (for arbitrary $\mathbf{p}$), we have the following description on the basis of $Z^{(\lambda,0)}_{\mathbf{p}}$ as in the case of the Weyl module W^λ. For each $\lambda \in \Lambda^+$, $\varphi_\lambda = \varphi_{T^\lambda T^\lambda}$ is contained in $\mathcal{C}^{\mathbf{p}}(\lambda, 0)$. We consider the $\mathcal{S}^{\mathbf{p}}$-submodule $W^\lambda_{\mathbf{p}}$ of $\mathcal{S}^{\mathbf{p}}/(\mathcal{S}^{\mathbf{p}})^{\vee(\lambda,0)}$ generated by the image of φ_λ. Since $T^\lambda \in I^{\mathbf{p}}(\lambda, 0)$, we see that $\varphi_{T^\lambda T} \in \mathcal{C}^{\mathbf{p}}(\lambda, 0)$ for any $T \in J^{\mathbf{p}}(\lambda, 0)$. We denote by φ'_T the image of $\varphi_{T^\lambda T}$ on $\mathcal{S}^{\mathbf{p}}/(\mathcal{S}^{\mathbf{p}})^{\vee(\lambda,0)}$. Then one can check that $\varphi'_T \in W^\lambda_{\mathbf{p}}$ and that the map $\varphi_T \to \varphi'_T$ gives an isomorphism $Z^{(\lambda,0)}_{\mathbf{p}} \to W^\lambda_{\mathbf{p}}$ of $\mathcal{S}^{\mathbf{p}}$-modules. In particular, we see that $Z^{(\lambda,0)}_{\mathbf{p}}$ is generated by $\varphi^{(\lambda,0)}_{T^\lambda}$ as an $\mathcal{S}^{\mathbf{p}}$-module.

However, the above argument cannot be applied to $Z^{(\lambda,1)}_{\mathbf{p}}$ since $\varphi_{T^\lambda T} \notin \mathcal{S}^{\mathbf{p}}$ for $T \in J^{\mathbf{p}}(\lambda, 1) \backslash J^{\mathbf{p}}(\lambda, 0)$. It is not known whether $Z^{(\lambda,1)}_{\mathbf{p}}$ is generated by one element as an $\mathcal{S}^{\mathbf{p}}$-module.

(ii) For any $\lambda \in \Lambda^+$, we have $L^{(\lambda,0)}_{\mathbf{p}} \neq 0$. In fact, since $T^\lambda \in I^{\mathbf{p}}(\lambda, 0) \cap J^{\mathbf{p}}(\lambda, 0)$, we have $f_{T^\lambda T^\lambda} = 1$. This implies that $\beta_{(\lambda,0)}(\varphi^{(\lambda,0)}_{T^\lambda}, \varphi^{(\lambda,0)}_{T^\lambda}) = 1$ and we see that $\operatorname{rad} Z^{(\lambda,0)}_{\mathbf{p}} \neq Z^{(\lambda,0)}_{\mathbf{p}}$.

This argument cannot be applied to $Z^{(\lambda,1)}_{\mathbf{p}}$ since $T^\lambda \notin I^{\mathbf{p}}(\lambda, 1)$ and so $\varphi^{(\lambda,1)}_{T^\lambda} \notin {}^\diamond Z^{(\lambda,1)}_{\mathbf{p}}$. It is not known when $L^{(\lambda,1)}_{\mathbf{p}} \neq 0$.

2.10. Recall that $\varphi \mapsto \varphi^*$ is the anti-automorphism on $\mathcal{S}(\Lambda)$ defined by $\varphi_{ST} \mapsto \varphi_{TS}$, related to the cellular structure. Let $\mathcal{S}^{\mathbf{p}*} = (\mathcal{S}^{\mathbf{p}})^*$ be the image of $\mathcal{S}^{\mathbf{p}}$ under the map $*$. Then $\mathcal{S}^{\mathbf{p}*}$ is a subalgebra of $\mathcal{S}(\Lambda)$, and it is easy to check that $\mathcal{S}^{\mathbf{p}*}$ is a standardly based algebra with the standard basis $\mathcal{C}^{\mathbf{p}*} = \coprod_{\eta \in \Sigma^{\mathbf{p}}} \mathcal{C}^{\mathbf{p}}(\eta)^*$, where

$$\mathcal{C}^{\mathbf{p}}(\eta)^* = \{\varphi_{ST} \in \mathcal{C}(\Lambda) \mid S \in J^{\mathbf{p}}(\eta), T \in I^{\mathbf{p}}(\eta)\}.$$

In a similar way as in [Sa, Proposition 3.2], one can show the following result.

Proposition 2.11. *We have $\mathcal{S}(\Lambda) = \mathcal{S}^{\mathbf{p}} \cdot \mathcal{S}^{\mathbf{p}*}$.*

2.12. Let $\widehat{\mathcal{S}}^{\mathbf{p}}$ be the R-submodule of $\mathcal{S}^{\mathbf{p}}$ spanned by

$$\widehat{\mathcal{C}}^{\mathbf{p}} = \mathcal{C}^{\mathbf{p}} \backslash \{\varphi_{ST} \mid S, T \in \mathcal{T}^{\mathbf{p}}_0(\lambda) \text{ for some } \lambda \in \Lambda^+\}.$$

Then by the second and the fourth formulas in Lemma 2.5, $\widehat{\mathcal{S}}^{\mathbf{p}}$ turns out to be a two-sided ideal of $\mathcal{S}^{\mathbf{p}}$. We denote by $\overline{\mathcal{S}}^{\mathbf{p}} = \overline{\mathcal{S}}^{\mathbf{p}}(\Lambda)$ the quotient algebra $\mathcal{S}^{\mathbf{p}}/\widehat{\mathcal{S}}^{\mathbf{p}}$. Let

$\pi : \mathcal{S}^{\mathbf{p}} \to \overline{\mathcal{S}}^{\mathbf{p}}$ be the natural projection, and put $\overline{\varphi} = \pi(\varphi)$ for $\varphi \in \mathcal{S}^{\mathbf{p}}$. It is easy to see that $\overline{\mathcal{S}}^{\mathbf{p}}$ is an R-free module with the basis

$$\overline{\mathcal{C}}^{\mathbf{p}} = \{\overline{\varphi}_{ST} \mid S, T \in \mathcal{T}_0^{\mathbf{p}}(\lambda) \text{ for } \lambda \in \Lambda^+\}.$$

Similarly, one can define a quotient algebra $\overline{\mathcal{S}}^{\mathbf{p}*} = \mathcal{S}^{\mathbf{p}*}/\widehat{\mathcal{S}}^{\mathbf{p}*}$, where $\widehat{\mathcal{S}}^{\mathbf{p}*} = (\widehat{\mathcal{S}}^{\mathbf{p}})^*$ is a two-sided ideal of $\mathcal{S}^{\mathbf{p}*}$. Let π' be the natural projection $\mathcal{S}^{\mathbf{p}*} \to \overline{\mathcal{S}}^{\mathbf{p}*}$, and put $\overline{\varphi}' = \pi'(\varphi)$ for $\varphi \in \mathcal{S}^{\mathbf{p}*}$. Then $\overline{\mathcal{S}}^{\mathbf{p}*}$ has an R-free basis $\overline{\mathcal{C}}^{\mathbf{p}*} = \{\overline{\varphi}'_{ST} \mid S, T \in \mathcal{T}_0^{\mathbf{p}}(\lambda), \lambda \in \Lambda^+\}$. It is clear that $\overline{\varphi}_{ST} \mapsto \overline{\varphi}'_{ST}$ gives an isomorphism $\overline{\mathcal{S}}^{\mathbf{p}} \to \overline{\mathcal{S}}^{\mathbf{p}*}$ of R-algebras. On the other hand, the anti-algebra isomorphism $\mathcal{S}^{\mathbf{p}} \to \mathcal{S}^{\mathbf{p}*}$ induces an anti-algebra isomorphism $\overline{\mathcal{S}}^{\mathbf{p}} \to \overline{\mathcal{S}}^{\mathbf{p}*}, \overline{\varphi}_{ST} \mapsto \overline{\varphi}'_{TS}$. It follows that the map $\overline{\varphi}_{ST} \mapsto \overline{\varphi}_{TS}$ induces an anti-algebra automorphism $*$ on $\overline{\mathcal{S}}^{\mathbf{p}}$. Thus we have the following theorem (cf. [Sa, Theorem 4.8]). Note that the second assertion is obtained from the cellular structure of $\mathcal{S}(\Lambda)$.

Theorem 2.13. *$\overline{\mathcal{S}}^{\mathbf{p}}$ is a cellular algebra with a cellular basis $\overline{\mathcal{C}}^{\mathbf{p}}$, i.e., the following properties hold:*

(i) *$\overline{\varphi}_{ST} \mapsto (\overline{\varphi}_{ST})^* = \overline{\varphi}_{TS}$ gives an anti-algebra automorphism $*$ on $\overline{\mathcal{S}}^{\mathbf{p}}$.*

(ii) *Let $(\overline{\mathcal{S}}^{\mathbf{p}})^{\vee\lambda}$ be the R-submodule of $\overline{\mathcal{S}}^{\mathbf{p}}$ spanned by $\overline{\varphi}_{ST}$ such that $S, T \in \mathcal{T}_0^{\mathbf{p}}(\lambda')$ with $\lambda' \rhd \lambda$. Then for any $\lambda \in \Lambda^+, S, T \in \mathcal{T}_0^{\mathbf{p}}(\lambda), \overline{\varphi} \in \overline{\mathcal{S}}^{\mathbf{p}}$,*

$$\overline{\varphi}_{ST} \cdot \overline{\varphi} \equiv \sum_{T' \in \mathcal{T}_0^{\mathbf{p}}(\lambda)} r_{T'} \overline{\varphi}_{ST'} \qquad \mathrm{mod}\ (\overline{\mathcal{S}}^{\mathbf{p}})^{\vee\lambda},$$

where $r_{T'} \in R$ depends on $\lambda, T, \overline{\varphi}$, but does not depend on S.

2.14. We apply the general theory of cellular algebras to $\overline{\mathcal{S}}^{\mathbf{p}}$. For each $\lambda \in \Lambda^+$, $(\overline{\mathcal{S}}^{\mathbf{p}})^{\vee\lambda}$ is a two-sided ideal of $\overline{\mathcal{S}}^{\mathbf{p}}$, and we define the Weyl module $\overline{Z}_{\mathbf{p}}^{\lambda}$ as the $\overline{\mathcal{S}}^{\mathbf{p}}$-submodule of the right $\overline{\mathcal{S}}^{\mathbf{p}}$-module $\overline{\mathcal{S}}^{\mathbf{p}}/(\overline{\mathcal{S}}^{\mathbf{p}})^{\vee\lambda}$ generated by $\overline{\varphi}_{T^\lambda T^\lambda} + (\overline{\mathcal{S}}^{\mathbf{p}})^{\vee\lambda}$. Let $\overline{\varphi}_T$ be the image of $\overline{\varphi}_{T^\lambda T}$ on $\overline{\mathcal{S}}^{\mathbf{p}}/(\overline{\mathcal{S}}^{\mathbf{p}})^{\vee\lambda}$. Then the set $\{\overline{\varphi}_T \mid T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)\}$ gives a basis of $\overline{Z}_{\mathbf{p}}^{\lambda}$. The symmetric bilinear form $\langle\ ,\ \rangle_{\mathbf{p}} : \overline{Z}_{\mathbf{p}}^{\lambda} \times \overline{Z}_{\mathbf{p}}^{\lambda} \to R$ is defined by the equation

$$\langle \overline{\varphi}_S, \overline{\varphi}_T \rangle_{\mathbf{p}}\, \overline{\varphi}_{T^\lambda T^\lambda} \equiv \overline{\varphi}_{T^\lambda S} \overline{\varphi}_{T T^\lambda} \qquad \mathrm{mod}\ (\overline{\mathcal{S}}^{\mathbf{p}})^{\vee\lambda}.$$

Then the radical $\mathrm{rad}\, \overline{Z}_{\mathbf{p}}^{\lambda}$ of $\overline{Z}_{\mathbf{p}}^{\lambda}$ with respect to this form is an $\overline{\mathcal{S}}^{\mathbf{p}}$-submodule of $\overline{Z}_{\mathbf{p}}^{\lambda}$, and we define an $\overline{\mathcal{S}}^{\mathbf{p}}$-module $\overline{L}_{\mathbf{p}}^{\lambda}$ by $\overline{L}_{\mathbf{p}}^{\lambda} = \overline{Z}_{\mathbf{p}}^{\lambda}/\mathrm{rad}\, \overline{Z}_{\mathbf{p}}^{\lambda}$. Since $\langle \overline{\varphi}_{T^\lambda}, \overline{\varphi}_{T^\lambda} \rangle_{\mathbf{p}} = 1$, we see that $\overline{L}_{\mathbf{p}}^{\lambda} \neq 0$ for any $\lambda \in \Lambda^+$. By the general theory of cellular algebras, we have

Corollary 2.15. *Suppose that R is a field. Then, for any $\lambda \in \Lambda^+$, $\overline{L}_{\mathbf{p}}^{\lambda}$ is an absolutely irreducible $\overline{\mathcal{S}}^{\mathbf{p}}$-module, and the set $\{\overline{L}_{\mathbf{p}}^{\lambda} \mid \lambda \in \Lambda^+\}$ gives a complete set of non-isomorphic irreducible $\overline{\mathcal{S}}^{\mathbf{p}}$-modules.*

3. Decomposition numbers for $\mathcal{S}(\Lambda), \mathcal{S}^{\mathbf{p}}$ and $\overline{\mathcal{S}}^{\mathbf{p}}$

3.1. By the discussion in the previous section, we have the following diagram:

$$\begin{array}{ccc} \mathcal{S}^{\mathbf{p}} & \xrightarrow{\ \iota\ } & \mathcal{S}(\Lambda) \\ {\scriptstyle \pi}\big\downarrow & & \\ \overline{\mathcal{S}}^{\mathbf{p}} & & \end{array}$$

where ι is the inclusion map, and π is the natural surjective map. We have constructed the Weyl modules $W^\lambda, Z^\eta_{\mathbf{p}}$ and $\overline{Z}^\lambda_{\mathbf{p}}$ for $\mathcal{S}(\Lambda), \mathcal{S}^{\mathbf{p}}$ and $\overline{\mathcal{S}}^{\mathbf{p}}$, and assuming that R is a field, the irreducible modules $L^\mu, L^{\eta'}_{\mathbf{p}}, \overline{L}^\mu_{\mathbf{p}}$, respectively for $\lambda, \mu \in \Lambda^+, \eta, \eta' \in \Sigma^{\mathbf{p}}$. We consider the decomposition numbers

$$[W^\lambda : L^\mu]_{\mathcal{S}(\Lambda)}, \quad [Z^\eta_{\mathbf{p}} : L^{\eta'}_{\mathbf{p}}]_{\mathcal{S}^{\mathbf{p}}}, \quad [\overline{Z}^\lambda_{\mathbf{p}} : \overline{L}^\mu_{\mathbf{p}}]_{\overline{\mathcal{S}}^{\mathbf{p}}}$$

for $\mathcal{S}(\Lambda), \mathcal{S}^{\mathbf{p}}$ and $\overline{\mathcal{S}}^{\mathbf{p}}$. By using the above maps, we shall discuss the relationship among these decomposition numbers.

First we consider the relation between $\mathcal{S}^{\mathbf{p}}$ and $\overline{\mathcal{S}}^{\mathbf{p}}$. We regard an $\overline{\mathcal{S}}^{\mathbf{p}}$-module as an $\mathcal{S}^{\mathbf{p}}$-module through the map π. The following lemma is easily verified if we notice that $\pi((\mathcal{S}^{\mathbf{p}})^{\vee(\lambda,0)}) = \overline{\mathcal{S}}^{\vee\lambda}$ and that $\beta_{(\lambda,0)}(\varphi^{(\lambda,0)}_S, \varphi^{(\lambda,0)}_T) = \langle \overline{\varphi}_S, \overline{\varphi}_T \rangle_{\mathbf{p}}$ for $S, T \in \mathcal{T}^{\mathbf{p}}_0(\lambda)$.

Lemma 3.2. (i) *For any $\lambda \in \Lambda^+$, the map $\varphi^{(\lambda,0)}_T \mapsto \overline{\varphi}_T$ $(T \in \mathcal{T}^{\mathbf{p}}_0(\lambda))$ gives an isomorphism $Z^{(\lambda,0)}_{\mathbf{p}} \xrightarrow{\sim} \overline{Z}^\lambda_{\mathbf{p}}$ of $\mathcal{S}^{\mathbf{p}}$-modules.*

(ii) *Assume that R is a field. Then the above map induces an isomorphism $L^{(\lambda,0)}_{\mathbf{p}} \xrightarrow{\sim} \overline{L}^\lambda_{\mathbf{p}}$ of $\mathcal{S}^{\mathbf{p}}$-modules.*

The following proposition is proved in a similar way as in [Sa, Theorem 4.15] by taking the lemma into account.

Proposition 3.3. *Assume that R is a field. Then:*

(i) *The composition factors of $Z^{(\lambda,0)}_{\mathbf{p}}$ are isomorphic to $L^{(\mu,0)}_{\mathbf{p}}$ for some $\mu \in \Lambda^+$ such that $\lambda \rhd \mu$.*

(ii) *For any $\lambda, \mu \in \Lambda^+$, we have $[\overline{Z}^\lambda_{\mathbf{p}} : \overline{L}^\mu_{\mathbf{p}}]_{\overline{\mathcal{S}}^{\mathbf{p}}} = [Z^{(\lambda,0)}_{\mathbf{p}} : L^{(\mu,0)}_{\mathbf{p}}]_{\mathcal{S}^{\mathbf{p}}}$.*

(iii) *For $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) \neq \alpha_{\mathbf{p}}(\mu)$, we have $[\overline{Z}^\lambda_{\mathbf{p}} : \overline{L}^\mu_{\mathbf{p}}]_{\overline{\mathcal{S}}^{\mathbf{p}}} = 0$.*

3.4. Next we consider the relation between $\mathcal{S}(\Lambda)$ and $\mathcal{S}^{\mathbf{p}}$. Since $\mathcal{S}^{\mathbf{p}}$ is a subalgebra of $\mathcal{S}(\Lambda)$, we regard an $\mathcal{S}(\Lambda)$-module as an $\mathcal{S}^{\mathbf{p}}$-module by restriction. Recall that $J^{\mathbf{p}}(\lambda, 0) = \mathcal{T}^{\mathbf{p}}_0(\lambda), J^{\mathbf{p}}(\lambda, 1) = \mathcal{T}_0(\lambda)$ for $\lambda \in \Lambda^+$. Thus the basis of $Z^{(\lambda,0)}_{\mathbf{p}}$ is $\{\varphi^{(\lambda,0)}_T \mid T \in \mathcal{T}^{\mathbf{p}}_0(\lambda)\}$, the basis of $Z^{(\lambda,1)}_{\mathbf{p}}$ is $\{\varphi^{(\lambda,1)}_T \mid T \in \mathcal{T}_0(\lambda)\}$, and the basis of W^λ is $\{\varphi_T \mid T \in \mathcal{T}_0(\lambda)\}$, respectively. The following result is implicit in [Sa].

Lemma 3.5. *For each $\lambda \in \Lambda^+$, the following hold:*

(i) *The map $\varphi^{(\lambda,0)}_T \mapsto \varphi^{(\lambda,1)}_T$ $(T \in \mathcal{T}^{\mathbf{p}}_0(\lambda))$ gives an injective homomorphism $Z^{(\lambda,0)}_{\mathbf{p}} \to Z^{(\lambda,1)}_{\mathbf{p}}$ of $\mathcal{S}^{\mathbf{p}}$-modules.*

(ii) *The map $\varphi^{(\lambda,1)}_T \mapsto \varphi_T$ $(T \in \mathcal{T}_0(\lambda))$ gives an isomorphism $Z^{(\lambda,1)}_{\mathbf{p}} \xrightarrow{\sim} W^\lambda$ of $\mathcal{S}^{\mathbf{p}}$-modules.*

Proof. Take $\varphi_{ST} \in \mathcal{C}(\Lambda)$ $(S, T \in \mathcal{T}_0(\lambda))$, and $\varphi \in \mathcal{S}^{\mathbf{p}}$. By the property of the cellular algebra $\mathcal{S}(\Lambda)$, we have

$$\varphi_{ST} \cdot \varphi \equiv \sum_{T' \in \mathcal{T}_0(\lambda)} r_{T'} \varphi_{ST'} \qquad \mathrm{mod}\ \mathcal{S}(\Lambda)^{\vee\lambda}.$$

Since $\varphi_{ST} \in \mathcal{S}^{\mathbf{p}}$ and $\mathcal{S}(\Lambda)^{\vee\lambda} \cap \mathcal{S}^{\mathbf{p}} = (\mathcal{S}^{\mathbf{p}})^{\vee(\lambda,1)}$, the congruence relation by $\mathcal{S}(\Lambda)^{\vee\lambda}$ in the above formula can be replaced by $(\mathcal{S}^{\mathbf{p}})^{\vee(\lambda,1)}$. In particular, for $\varphi_{ST} \in$

$\mathcal{C}^{\mathbf{p}}(\lambda,1), \varphi \in \mathcal{S}^{\mathbf{p}}$, we have

$$\varphi_{ST} \cdot \varphi \equiv \sum_{T' \in \mathcal{T}_0(\lambda)} r_{T'} \varphi_{ST'} \qquad \mathrm{mod}\ (\mathcal{S}^{\mathbf{p}})^{\vee(\lambda,1)}. \tag{3.5.1}$$

On the other hand, by the second formula in Theorem 2.6 we have, for $\varphi_{ST} \in \mathcal{C}^{\mathbf{p}}(\lambda,0), \varphi \in \mathcal{S}^{\mathbf{p}}$,

$$\varphi_{ST} \cdot \varphi \equiv \sum_{T' \in \mathcal{T}_0^{\mathbf{p}}(\lambda)} f_{T'} \varphi_{ST'} \qquad \mathrm{mod}\ (\mathcal{S}^{\mathbf{p}})^{\vee(\lambda,0)};$$

but the first formula in Lemma 2.5 shows that $\varphi_{ST'} \in \mathcal{C}^{\mathbf{p}}(\lambda,1)$ does not appear in the expression of $\varphi_{ST} \cdot \varphi$ except $\varphi_{ST'} \in \mathcal{C}^{\mathbf{p}}(\lambda,0)$. It follows that the congruence relation $(\mathcal{S}^{\mathbf{p}})^{\vee(\lambda,0)}$ in the above formula can be replaced by $(\mathcal{S}^{\mathbf{p}})^{\vee(\lambda,1)}$. Thus we have, for $\varphi_{ST} \in \mathcal{C}^{\mathbf{p}}(\lambda,0), \varphi \in \mathcal{S}^{\mathbf{p}}$,

$$\varphi_{ST} \cdot \varphi \equiv \sum_{T' \in \mathcal{T}_0^{\mathbf{p}}(\lambda)} f_{T'} \varphi_{ST'} \qquad \mathrm{mod}\ (\mathcal{S}^{\mathbf{p}})^{\vee(\lambda,1)}. \tag{3.5.2}$$

We now prove (i). For $T \in \mathcal{T}_0^{\mathbf{p}}(\lambda), \varphi \in \mathcal{S}^{\mathbf{p}}$, one can write

$$\varphi_T^{(\lambda,0)} \cdot \varphi = \sum_{T' \in \mathcal{T}_0^{\mathbf{p}}(\lambda)} g_{T'} \varphi_{T'}^{(\lambda,0)},$$
$$\varphi_T^{(\lambda,1)} \cdot \varphi = \sum_{T' \in \mathcal{T}_0(\lambda)} g'_{T'} \varphi_{T'}^{(\lambda,1)}.$$

By the definition of Weyl modules, we see that $g_{T'} = f_{T'}$ and $g'_{T'} = r_{T'}$. Thus by comparing (3.5.1) and (3.5.2), we have

$$g'_{T'} = \begin{cases} g_{T'} & \text{if } T' \in \mathcal{T}_0^{\mathbf{p}}(\lambda), \\ 0 & \text{otherwise.} \end{cases}$$

This proves (i). The assertion (ii) is proved in a similar way. □

The following proposition can be proved in a similar way as in [Sa, Theorem 3.3].

Proposition 3.6. *For each $\lambda \in \Lambda^+$, there exists an isomorphism of $\mathcal{S}(\Lambda)$-modules*

$$Z_{\mathbf{p}}^{(\lambda,0)} \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}(\Lambda) \xrightarrow{\sim} W^\lambda$$

which maps $\varphi_{T^\lambda}^{(\lambda,0)} \psi \otimes \varphi$ to $\varphi_{T^\lambda} \psi \varphi$ for $\psi \in \mathcal{S}^{\mathbf{p}}, \varphi \in \mathcal{S}(\Lambda)$.

3.7. By Lemma 3.5, the map $\varphi_T^{(\lambda,0)} \mapsto \varphi_T$ gives an injective homomorphism $f_\lambda : Z_{\mathbf{p}}^{(\lambda,0)} \to W^\lambda$ of $\mathcal{S}^{\mathbf{p}}$-modules. By this map we regard $Z_{\mathbf{p}}^{(\lambda,0)}$ as an $\mathcal{S}^{\mathbf{p}}$-submodule of W^λ. We have the following lemma.

Lemma 3.8. *Assume that $\lambda \in \Lambda^+$.*

(i) *Let M be an $\mathcal{S}^{\mathbf{p}}$-submodule of $Z_{\mathbf{p}}^{(\lambda,0)}$, and let $\widetilde{M}$ be the $\mathcal{S}(\Lambda)$-submodule of W^λ generated by M. Then $\widetilde{M} \cap Z_{\mathbf{p}}^{(\lambda,0)} = M$.*

(ii) *Let $M_1 \subsetneq M_2$ be $\mathcal{S}^{\mathbf{p}}$-submodules of $Z_{\mathbf{p}}^{(\lambda,0)}$. Let ι_i be the inclusion map $M_i \to Z_{\mathbf{p}}^{(\lambda,0)}$, and $\iota_i \otimes \mathrm{Id}$ be the induced map $M_i \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}(\Lambda) \to Z_{\mathbf{p}}^{(\lambda,0)} \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}(\Lambda)$ for $i = 1, 2$. Then we have $\mathrm{Im}(\iota_1 \otimes \mathrm{Id}) \subsetneq \mathrm{Im}(\iota_2 \otimes \mathrm{Id})$.*

Proof. We prove (i). Take $x \in \widetilde{M} \cap Z_{\mathbf{p}}^{(\lambda,0)}$. We write $x = \sum_{T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)} r_T \varphi_T^{(\lambda,0)}$. Since $x \in \widetilde{M}$, one can write $x = \sum_i y_i \psi_i$ with $\psi_i \in \mathcal{S}(\Lambda)$, $y_i = \sum_{T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)} r_{T,i} \varphi_T^{(\lambda,0)} \in Z_{\mathbf{p}}^{(\lambda,0)}$. Hence we have a relation as elements in W^λ,

$$\sum_{T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)} r_T \varphi_T = \sum_i \sum_{T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)} r_{T,i} \varphi_T \psi_i.$$

This means that

$$\sum_{T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)} r_T \varphi_{T^\lambda T} \equiv \sum_i \sum_{T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)} r_{T,i} \varphi_{T^\lambda T} \psi_i \qquad \bmod \mathcal{S}(\Lambda)^{\vee\lambda}.$$

Put $\alpha = \alpha_{\mathbf{p}}(\lambda)$. Take $\nu \in \Lambda$ such that $\alpha_{\mathbf{p}}(\nu) = \alpha$ and multiply φ_ν on both sides of the above equation. Note that $\varphi_\nu \in \mathcal{S}^{\mathbf{p}}$ is a projection from M to M^ν, and we have $\mathcal{S}^{\mathbf{p}} \cap \mathcal{S}(\Lambda)^{\vee\lambda} = (\mathcal{S}^{\mathbf{p}})^{\vee(\lambda,1)} \subset (\mathcal{S}^{\mathbf{p}})^{\vee(\lambda,0)}$. It follows that

$$\sum_{T \in \mathcal{T}_0(\lambda,\nu)} r_T \varphi_{T^\lambda T} \equiv \sum_i \sum_{T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)} r_{T,i} \varphi_{T^\lambda T} \psi_i \varphi_\nu \qquad \bmod (\mathcal{S}^{\mathbf{p}})^{\vee(\lambda,0)}.$$

Since this holds for any $\nu \in \Lambda$ such that $\alpha_{\mathbf{p}}(\nu) = \alpha$, we have

$$\sum_{T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)} r_T \varphi_{T^\lambda T} \equiv \sum_{\substack{\nu \in \Lambda \\ \alpha_{\mathbf{p}}(\nu) = \alpha}} \sum_i \sum_{T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)} r_{T,i} \varphi_{T^\lambda T} \psi_i \varphi_\nu \qquad \bmod (\mathcal{S}^{\mathbf{p}})^{\vee(\lambda,0)}. \tag{3.8.1}$$

Put $\varphi_\alpha = \sum_\nu \varphi_\nu$, where ν runs over all the elements in Λ such that $\alpha_{\mathbf{p}}(\nu) = \alpha$. Since φ_α is the projection from M onto $M^\alpha = \bigoplus_\nu M^\nu$, we see that $\varphi_{T^\lambda T} \varphi_\alpha = \varphi_{T^\lambda T}$ for any $T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)$. Moreover, we note that $\varphi_\alpha \psi_i \varphi_\nu \in \mathcal{S}^{\mathbf{p}}$ since it is contained in $\mathrm{Hom}_{\mathcal{H}}(M^\nu, M^\alpha)$. It follows that

$$\varphi_{T^\lambda T} \psi_i \varphi_\nu = \varphi_{T^\lambda T}(\varphi_\alpha \psi_i \varphi_\nu) \in \mathcal{S}^{\mathbf{p}}$$

for $T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)$. Thus one can rewrite (3.8.1) as a relation on $Z_{\mathbf{p}}^{(\lambda,0)}$ as

$$x = \sum_{T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)} r_T \varphi_T^{(\lambda,0)} = \sum_{\substack{\nu \in \Lambda \\ \alpha_{\mathbf{p}}(\nu) = \alpha}} \sum_i \sum_{T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)} r_{T,i} \varphi_T^{(\lambda,0)} (\varphi_\alpha \psi_i \varphi_\nu).$$

This shows that $x = \sum_\nu \sum_i y_i (\varphi_\alpha \psi_i \varphi_\nu) \in M$ as asserted.

Next we prove (ii). Under the embedding $Z_{\mathbf{p}}^\lambda \hookrightarrow W^\lambda$, Proposition 3.6 implies that $\mathrm{Im}(\iota_i \otimes \mathrm{Id}) = \widetilde{M}_i$. Take $x \in M_2 \backslash M_1$. Suppose that $\widetilde{M}_1 = \widetilde{M}_2$. Then $x \in \widetilde{M}_1 \cap Z_{\mathbf{p}}^{(\lambda,0)} = M_1$ by (i). This is a contradiction. □

By making use of Lemma 3.8, we show the following lemma.

Lemma 3.9. *Assume that R is a field. Then for each $\lambda \in \Lambda^+$, there exists a unique maximal $\mathcal{S}(\Lambda)$-submodule N^λ of $L_{\mathbf{p}}^{(\lambda,0)} \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}(\Lambda)$ such that*

$$L_{\mathbf{p}}^{(\lambda,0)} \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}(\Lambda)/N^\lambda \simeq L^\lambda.$$

Proof. Since $L_{\mathbf{p}}^{(\lambda,0)} \simeq Z_{\mathbf{p}}^{(\lambda,0)}/\operatorname{rad} Z_{\mathbf{p}}^{(\lambda,0)}$, we have a surjective homomorphism

$$Z_{\mathbf{p}}^{(\lambda,0)} \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}(\Lambda) \to L_{\mathbf{p}}^{(\lambda,0)} \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}(\Lambda)$$

as $\mathcal{S}(\Lambda)$-modules. Since $L_{\mathbf{p}}^{(\lambda,0)} \neq 0$, we have $L_{\mathbf{p}}^{(\lambda,0)} \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}(\Lambda) \neq 0$ by Lemma 3.8, and the kernel of this map is a proper $\mathcal{S}(\Lambda)$-submodule of $Z_{\mathbf{p}}^{(\lambda,0)} \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}(\Lambda)$; but $Z_{\mathbf{p}}^{(\lambda,0)} \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}(\Lambda)$ is isomorphic to W^λ by Proposition 3.6, and $L^\lambda \simeq W^\lambda/\operatorname{rad} W^\lambda$.

Since $\operatorname{rad} W^\lambda$ is the unique maximal submodule of W^λ, we see that there exists a surjective homomorphism $L_{\mathbf{p}}^{(\lambda,0)} \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}(\Lambda) \to L^\lambda$ of $\mathcal{S}(\Lambda)$-modules. It is clear that N^λ is the unique maximal submodule since it is a quotient of $\operatorname{rad} W^\lambda$. □

Lemma 3.10. *Assume that R is a field. For each $\lambda \in \Lambda^+$, the $\mathcal{S}^{\mathbf{p}}$-module L^λ contains $L_{\mathbf{p}}^{(\lambda,0)}$ as a submodule.*

Proof. By definition, we have $\beta_\lambda(\varphi_S^{(\lambda,0)}, \varphi_T^{(\lambda,0)}) = \langle \varphi_S, \varphi_T \rangle$ for any $S, T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)$. Moreover, one can check that $\langle \varphi_S, \varphi_T \rangle = 0$ for $S \in \mathcal{T}_0(\lambda) \backslash \mathcal{T}_0^{\mathbf{p}}(\lambda)$, $T \in \mathcal{T}_0^{\mathbf{p}}(\lambda)$. It follows that $f_\lambda(\operatorname{rad} Z_{\mathbf{p}}^{(\lambda,0)}) \subset \operatorname{rad} W^\lambda$, where $f_\lambda : Z_{\mathbf{p}}^{(\lambda,0)} \hookrightarrow W^\lambda$ is the injective map given in 3.7. Then f_λ induces a homomorphism $\bar{f}_\lambda : L_{\mathbf{p}}^{(\lambda,0)} \to L^\lambda$ of $\mathcal{S}^{\mathbf{p}}$-modules. Since $f_\lambda(\varphi_{T^\lambda}^{(\lambda,0)}) = \varphi_{T^\lambda} \notin \operatorname{rad} W^\lambda$, $\bar{f}_\lambda$ is a non-zero map. Since $L_{\mathbf{p}}^{(\lambda,0)}$ is an irreducible $\mathcal{S}^{\mathbf{p}}$-module, $\bar{f}_\lambda$ is injective. This proves the lemma. □

The following two results are generalizations of [Sa, Theorem 5.6, Theorem 5.7].

Proposition 3.11. *Assume that R is a field. Then for $\lambda, \mu \in \Lambda^+$,*

$$[Z_{\mathbf{p}}^{(\lambda,0)} : L_{\mathbf{p}}^{(\mu,0)}]_{\mathcal{S}^{\mathbf{p}}} \leq [W^\lambda : L^\mu]_{\mathcal{S}(\Lambda)}.$$

Proof. We consider a composition series of $Z_{\mathbf{p}}^{(\lambda,0)}$ as an $\mathcal{S}^{\mathbf{p}}$-module

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k = Z_{\mathbf{p}}^{(\lambda,0)}$$

such that $M_j/M_{j-1} \simeq L_{\mathbf{p}}^{(\mu_j,0)}$. Let $i_j : M_j \hookrightarrow Z_{\mathbf{p}}^{(\lambda,0)}$ be the inclusion map and $\iota_j \otimes \mathrm{Id} : M_j \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}(\Lambda) \to Z_{\mathbf{p}}^{(\lambda,0)} \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}(\Lambda)$ be the induced map. Put $\mathcal{M}_j = \operatorname{Im}(\iota_j \otimes \mathrm{Id})$. We have a filtration

$$0 = \mathcal{M}_0 \subsetneq \mathcal{M}_1 \subsetneq \cdots \subsetneq \mathcal{M}_k = Z_{\mathbf{p}}^{(\lambda,0)} \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}(\Lambda) \simeq W^\lambda$$

of $\mathcal{S}(\Lambda)$-submodules of W^λ by Proposition 3.6 and Lemma 3.8. In order to prove the proposition, it is enough to show that L^{μ_j} occurs in the composition series of $\mathcal{M}_j/\mathcal{M}_{j-1}$ for each j. Since $M_j/M_{j-1} \simeq L_{\mathbf{p}}^{(\mu_j,0)}$, we have the following diagram of $\mathcal{S}(\Lambda)$-modules:

$$\begin{array}{ccccccccc}
 & & M_{j-1} \otimes \mathcal{S}(\Lambda) & \longrightarrow & M_j \otimes \mathcal{S}(\Lambda) & \longrightarrow & L_{\mathbf{p}}^{(\mu_j,0)} \otimes \mathcal{S}(\Lambda) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{M}_{j-1} & \longrightarrow & \mathcal{M}_j & \longrightarrow & \mathcal{M}_j/\mathcal{M}_{j-1} & \longrightarrow & 0
\end{array}$$

where the vertical maps are surjective. Thus we obtain a surjective homomorphism $L_{\mathbf{p}}^{(\lambda,0)} \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}(\Lambda) \to \mathcal{M}_j/\mathcal{M}_{j-1}$. On the other hand, by Lemma 3.9, we have a surjective homomorphism $L_{\mathbf{p}}^{(\lambda,0)} \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}(\Lambda) \to L^\mu$, whose kernel N^λ is the unique maximal submodule. This implies that we have a surjective homomorphism $\mathcal{M}_j/\mathcal{M}_{j-1} \to L^{\mu_j}$. Hence L^{μ_j} occurs in the composition series of $\mathcal{M}_j/\mathcal{M}_{j-1}$, and the proposition is proved. □

Proposition 3.12. *Assume that R is a field. Then for any $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, we have*

$$[Z_{\mathbf{p}}^{(\lambda,0)} : L_{\mathbf{p}}^{(\mu,0)}]_{\mathcal{S}^{\mathbf{p}}} \geq [W^\lambda : L^\mu]_{\mathcal{S}(\Lambda)}.$$

Proof. Consider a composition series of W^λ as an $\mathcal{S}(\Lambda)$-module

$$0 = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_k = W^\lambda$$

such that $W_j/W_{j-1} \simeq L^{\mu_j}$ for some $\mu_j \in \Lambda^+$. We consider this as a filtration of W^λ as $\mathcal{S}^{\mathbf{p}}$-modules. Since L^{μ_j} contains $L_{\mathbf{p}}^{(\mu_j,0)}$ as an $\mathcal{S}^{\mathbf{p}}$-submodule by Lemma 3.10, there exists an $\mathcal{S}^{\mathbf{p}}$-submodule W_j' of W_j containing W_{j-1} such that $W_j'/W_{j-1} \simeq L_{\mathbf{p}}^{(\mu_j,0)}$. By 3.7, we identify $Z_{\mathbf{p}}^{(\lambda.0)}$ as an $\mathcal{S}^{\mathbf{p}}$-submodule of W^λ, and put $M_j = Z_{\mathbf{p}}^{(\lambda,0)} \cap W_j$ and $M_j' = Z_{\mathbf{p}}^{(\lambda,0)} \cap W_j'$. We have a filtration of $Z_{\mathbf{p}}^{(\lambda,0)}$ by $\mathcal{S}^{\mathbf{p}}$-modules

$$0 = M_0 \subset M_1' \subset M_1 \subset \cdots \subset M_{k-1} \subset M_k' \subset M_k = Z_{\mathbf{p}}^{(\lambda,0)} \cap W^\lambda = Z_{\mathbf{p}}^{(\lambda,0)}.$$

We claim that

$$M_{j-1} \neq M_j' \quad \text{if} \quad \alpha_{\mathbf{p}}(\mu_j) = \alpha_{\mathbf{p}}(\lambda). \tag{3.12.1}$$

Note that (3.12.1) implies the proposition. In fact, $M_j'/M_{j-1} \simeq L_{\mathbf{p}}^{(\mu_j,0)}$ since it is isomorphic to a non-zero submodule of $L_{\mathbf{p}}^{(\mu_j,0)}$. It follows that $L_{\mathbf{p}}^{(\mu_j,0)}$ occurs in the composition series of M_j/M_{j-1} for each j, and the proposition follows.

We show (3.12.1). Assume that $\alpha_{\mathbf{p}}(\mu_j) = \alpha_{\mathbf{p}}(\lambda)$. Then the image of $\varphi_{T^{\mu_j}}^{(\mu_j,0)} \in Z_{\mathbf{p}}^{(\mu_j,0)}$ to $L_{\mathbf{p}}^{(\mu_j,0)}$ gives a non-zero element $\bar{\varphi}_j$in $L_{\mathbf{p}}^{(\mu_j,0)}$ by Remark 2.9 (ii). We choose $x_j \in W_j' \backslash W_{j-1}$ corresponding to $\bar{\varphi}_j$ under the isomorphism $W_j'/W_{j-1} \simeq L_{\mathbf{p}}^{(\mu_j,0)}$. Since $\bar{\varphi}_j \varphi_{\mu_j} = \bar{\varphi}_j$, we have $x_j \varphi_{\mu_j} \in W_j' \backslash W_{j-1}$. Since $x_j \in W_j' \subset W^\lambda$, one can write $x_j = \sum_{T \in \mathcal{T}_0(\lambda)} r_T \varphi_T$. Now φ_{μ_j} is a projection from M onto M^{μ_j}. Hence

$$x_j \varphi_{\mu_j} = \sum_{T \in \mathcal{T}_0^{\mathbf{p}}(\lambda,\mu_j)} r_T \varphi_T.$$

Here $\mathcal{T}_0^{\mathbf{p}}(\lambda,\mu_j) \subset \mathcal{T}_0^{\mathbf{p}}(\lambda) = J_{\mathbf{p}}(\lambda,0)$ since $\alpha_{\mathbf{p}}(\mu_j) = \alpha_{\mathbf{p}}(\lambda)$. It follows that the right-hand side of the above equation is contained in $Z_{\mathbf{p}}^{(\lambda,0)}$, and so $x_j \varphi_{\mu_j} \in M_j' \backslash M_{j-1}$. This proves (3.12.1), and the proposition follows. □

Combining Proposition 3.3, Proposition 3.11 and Proposition 3.12, we have the following theorem (cf. [Sa, Theorem 5.7]).

Theorem 3.13. *Assume that R is a field. For any $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, we have*

$$[\overline{Z}_{\mathbf{p}}^\lambda : \overline{L}_{\mathbf{p}}^\mu]_{\overline{\mathcal{S}}^{\mathbf{p}}} = [Z_{\mathbf{p}}^{(\lambda,0)} : L_{\mathbf{p}}^{(\mu,0)}]_{\mathcal{S}^{\mathbf{p}}} = [W^\lambda : L^\mu]_{\mathcal{S}(\Lambda)}.$$

4. Structure theorem for $\overline{\mathcal{S}}^{\mathbf{p}}$

4.1. In this section, we assume that $\Lambda = \widetilde{\mathcal{P}}_{n,r}(\mathbf{m})$. For each $\mu \in \Lambda$, let $\widehat{N}^{\mathbf{a_p}(\mu)}$ be the R-submodule of $\mathcal{H}$ spanned by $m_{\mathfrak{s}\mathfrak{t}}$ such that $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$ with $\mathbf{a_p}(\lambda) > \mathbf{a_p}(\mu)$. Since $\lambda \trianglerighteq \mu$ implies $\mathbf{a_p}(\lambda) \geq \mathbf{a_p}(\mu)$, $\widehat{N}^{\mathbf{a_p}(\mu)}$ is a two-sided ideal of $\mathcal{H}$. Put $\widehat{M}^\mu = M^\mu \cap \widehat{N}^{\mathbf{a_p}(\mu)}$. Then $\widehat{M}^\mu$ is an $\mathcal{H}$-module with the basis $\{m_{S\mathfrak{t}} \mid S \in \mathcal{T}_0(\lambda,\mu), \mathfrak{t} \in \mathrm{Std}(\lambda), \mathbf{a_p}(\lambda) > \mathbf{a_p}(\mu)\}$. We define an $\mathcal{H}$-module $\overline{M}^\mu$ by $\overline{M}^\mu = M^\mu/\widehat{M}^\mu$ and let $f : M^\mu \to \overline{M}^\mu$ be the natural surjection. Put $\overline{m}_{S\mathfrak{t}} = f(m_{S\mathfrak{t}})$ for a basis $m_{S\mathfrak{t}} \in M^\mu$. Then

$$\{\overline{m}_{S\mathfrak{t}} \mid S \in \mathcal{T}_0^{\mathbf{p}}(\lambda,\mu), \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ for } \lambda \in \Lambda^+\} \tag{4.1.1}$$

gives a basis of $\overline{M}^\mu$.

4.2. We write $\mathbf{m} = (m_1, \dots, m_r)$ in the form $\mathbf{m} = (\mathbf{m}^{[1]}, \dots, \mathbf{m}^{[g]})$ where $\mathbf{m}^{[k]} = (m_{p_k+1}, \dots, m_{p_k+r_k})$. For each $n_k \in \mathbb{Z}_{\geq 0}$, put $\Lambda_{n_k} = \widetilde{\mathcal{P}}_{n_k,r_k}(\mathbf{m}^{[k]})$ and $\Lambda^+_{n_k} = \mathcal{P}_{n_k,r_k}(\mathbf{m}^{[k]})$. ($\Lambda_{n_k}$ or $\Lambda^+_{n_k}$ is regarded as the empty set if $n_k = 0$.) Let $\mu = (\mu^{(1)}, \dots, \mu^{(r)}) \in \Lambda$ be an r-composition and write it as $\mu = (\mu^{[1]}, \dots, \mu^{[g]})$. Then a μ-tableau $\mathfrak{t} = (\mathfrak{t}^{(1)}, \dots, \mathfrak{t}^{(r)})$ can be expressed as $\mathfrak{t} = (\mathfrak{t}^{[1]}, \dots, \mathfrak{t}^{[g]})$ with $\mathfrak{t}^{[k]} = (\mathfrak{t}^{(p_k+1)}, \dots, \mathfrak{t}^{(p_k+r_k)})$, where $\mathfrak{t}^{[k]}$ is a $\mu^{[k]}$-tableau. Take $\lambda \in \Lambda^+$, $\mu \in \Lambda$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$. Then a λ-tableau $T = (T^{(1)}, \dots, T^{(r)})$ of type μ can be expressed as $T = (T^{[1]}, \dots, T^{[g]})$ with $T^{[k]} = (T^{(p_k+1)}, \dots, T^{(p_k+r_k)})$, where $T^{[k]}$ is a $\lambda^{[k]}$-tableau of type $\mu^{[k]}$.

The following lemma is easily verified.

Lemma 4.3. *Let $\alpha = (n_1, \dots, n_g) \in \mathbb{Z}^g_{>0}$ be such that $n_1 + \cdots + n_g = n$. Then:*

(i) *The map $\mu \mapsto (\mu^{[1]}, \dots, \mu^{[g]})$ gives a bijection between $\{\mu \in \Lambda \mid \alpha_{\mathbf{p}}(\mu) = \alpha\}$ and $\Lambda_{n_1} \times \cdots \times \Lambda_{n_g}$.*

(ii) *The map $\lambda \mapsto (\lambda^{[1]}, \dots, \lambda^{[g]})$ gives a bijection between $\{\lambda \in \Lambda^+ \mid \alpha_{\mathbf{p}}(\lambda) = \alpha\}$ and $\Lambda^+_{n_1} \times \cdots \times \Lambda^+_{n_g}$.*

(iii) *For each $\lambda \in \Lambda^+, \mu \in \Lambda$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, the map $T \mapsto (T^{[1]}, \dots, T^{[g]})$ gives a bijection $\mathcal{T}^{\mathbf{p}}_0(\lambda, \mu) \simeq \mathcal{T}_0(\lambda^{[1]}, \mu^{[1]}) \times \cdots \times \mathcal{T}_0(\lambda^{[g]}, \mu^{[g]})$.*

4.4. Let $\alpha = (n_1, \dots, n_g) \in \widetilde{\mathcal{P}}_{n,1}$. For each $\lambda \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha$, we define a subset $\mathrm{Std}(\lambda)_0$ of $\mathrm{Std}(\lambda)$ as the set of $\mathfrak{t} = (\mathfrak{t}^{[1]}, \dots, \mathfrak{t}^{[g]})$ such that the letters contained in the tableau $\mathfrak{t}^{[k]}$ are exactly $\{n_1 + \cdots + n_{k-1} + 1, \dots, n_1 + \cdots + n_k\}$. Then the set $\mathrm{Std}(\lambda)_0$ is in bijection with the set $\mathrm{Std}(\lambda^{[1]}) \times \cdots \times \mathrm{Std}(\lambda^{[g]})$ under the map $\mathfrak{t} \mapsto (\mathfrak{t}^{[1]}, \dots, \mathfrak{t}^{[g]})$. For each $\mu \in \Lambda$ such that $\alpha_{\mathbf{p}}(\mu) = \alpha$, we define an R-submodule $\overline{M}^{\mu}_0$ of $\overline{M}^{\mu}$ as the R-span of $\overline{m}_{S\mathfrak{t}}$ such that $S \in \mathcal{T}^{\mathbf{p}}_0(\lambda, \mu), \mathfrak{t} \in \mathrm{Std}(\lambda)_0$ for various $\lambda \in \Lambda^+$. We write $\mu = (\mu^{[1]}, \dots, \mu^{[g]})$ as before. Take $\mathfrak{s} \in \mathrm{Std}(\lambda)$ such that $\mu(\mathfrak{s}) = S$ for $S \in \mathcal{T}^{\mathbf{p}}_0(\lambda, \mu)$ with $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu) = \alpha$. Then $\mathfrak{s} \in \mathrm{Std}(\lambda)_0$ and $\mathfrak{s}^{[k]} \in \mathrm{Std}(\lambda^{[k]})$ has the property that $\mu^{[k]}(\mathfrak{s}^{[k]}) = S^{[k]}$. This gives a bijection between the set of $\mathfrak{s} \in \mathrm{Std}(\lambda)$ such that $\mu(\mathfrak{s}) = S$ and the set of $(\mathfrak{s}^{[1]}, \dots, \mathfrak{s}^{[g]}) \in \mathrm{Std}(\lambda^{[1]}) \times \cdots \times \mathrm{Std}(\lambda^{[g]})$ such that $\mu^{[k]}(\mathfrak{s}^{[k]}) = S^{[k]}$ for each k. Combined with (1.4.1) and (4.1.1), this implies that

(4.4.1) The map $\overline{m}_{S\mathfrak{t}} \mapsto m_{S^{[1]}\mathfrak{t}^{[1]}} \otimes \cdots \otimes m_{S^{[g]}\mathfrak{t}^{[g]}}$ gives an isomorphism of R-modules $\phi_\mu : \overline{M}^{\mu}_0 \xrightarrow{\sim} M^{\mu^{[1]}} \otimes \cdots \otimes M^{\mu^{[g]}}$.

Put $\mathcal{H}_\alpha = \mathcal{H}_{n_1,r_1} \otimes \cdots \otimes \mathcal{H}_{n_g,r_g}$. Since $M^{\mu^{[k]}}$ is an $\mathcal{H}_{n_k,r_k}$-module, $M^{\mu^{[1]}} \otimes \cdots \otimes M^{\mu^{[g]}}$ has a structure of an $\mathcal{H}_\alpha$-module. We denote by $T^{[k]}_0, \dots, T^{[k]}_{n_k-1}$ the generators of $\mathcal{H}_{n_k,r_k}$ corresponding to $T_0, \dots, T_{n-1}$ in the case of $\mathcal{H}_{n,r}$, and more generally we denote by $T^{[k]}_w$ for $w \in \mathfrak{S}_{n_k}$ the element corresponding to $T_w \in \mathcal{H}_n$. Then $T^{[k]}_i$ acts on $M^{\mu^{[1]}} \otimes \cdots \otimes M^{\mu^{[g]}}$ for $i = 0, \dots, n_k - 1$, through the action of $1^{\otimes(k-1)} \otimes T^{[k]}_i \otimes 1^{\otimes(g-k)} \in \mathcal{H}_\alpha$ on it.

Recall that $L_i = T_{i-1}T_{i-2} \cdots T_1T_0T_1 \cdots T_{i-2}T_{i-1} \in \mathcal{H}$ for $i = 0, \dots, n-1$. The following lemma is crucial for later discussions.

Lemma 4.5. *Let $\mu \in \Lambda$ be such that $\alpha_{\mathbf{p}}(\mu) = \alpha$. For $\mu \in \Lambda$, put $\mathbf{a}_{\mathbf{p}}(\mu) = (a_1, \dots, a_g)$. Then the action of L_{a_k+1} on $\overline{M}^{\mu}$ stabilizes the submodule $\overline{M}^{\mu}_0$, and it gives rise to the action of $T^{[k]}_0$ on $M^{\mu^{[1]}} \otimes \cdots \otimes M^{\mu^{[g]}}$ under the isomorphism ϕ_μ in (4.4.1).*

Proof. Take $\lambda \in \Lambda^+$ such that $\mathbf{a_p}(\lambda) = \mathbf{a_p}(\mu)$ and consider $\mathcal{T}_0^{\mathbf{P}}(\lambda, \mu)$. Let $\mathfrak{s} \in \mathrm{Std}(\lambda)$ such that $\mu(\mathfrak{s}) = S$ for $S = (S^{[1]}, \dots, S^{[g]}) \in \mathcal{T}_0^{\mathbf{P}}(\lambda, \mu)$. Then $\mathfrak{s} = (\mathfrak{s}^{[1]}, \dots, \mathfrak{s}^{[g]}) \in \mathrm{Std}(\lambda)_0$ and $\mu^{[k]}(\mathfrak{s}^{[k]}) = S^{[k]}$. Take $\mathfrak{s}$ as above, and take $\mathfrak{t} = (\mathfrak{t}^{[1]}, \dots, \mathfrak{t}^{[g]}) \in \mathrm{Std}(\lambda)_0$. We consider a basis $m_{\mathfrak{s}\mathfrak{t}} \in \mathcal{H}$ and $m_{\mathfrak{s}^{[k]}\mathfrak{t}^{[k]}} \in \mathcal{H}_{n_k, r_k}$. We show that

(4.5.1) $m_{\mathfrak{s}\mathfrak{t}} L_{a_k+1}$ is written as a linear combination of the basis elements $m_{\mathfrak{u}\mathfrak{v}}$ of $\mathcal{H}$, where $\mathfrak{u} = (\mathfrak{u}^{[1]}, \dots, \mathfrak{u}^{[g]})$ is obtained from $\mathfrak{s}$ by replacing $\mathfrak{s}^{[k]}$ by some $\mathfrak{u}^{[k]}$, and $\mathfrak{v}$ is obtained from $\mathfrak{t}$ similarly. Here $\mathfrak{u}^{[k]}$ and $\mathfrak{v}^{[k]}$ have the same shape. The coefficient of $m_{\mathfrak{u}\mathfrak{v}}$ in the expansion of $m_{\mathfrak{s}\mathfrak{t}} L_{a_k+1}$ coincides with the coefficient of $m_{\mathfrak{u}^{[k]}\mathfrak{v}^{[k]}}$ in the expansion of $m_{\mathfrak{s}^{[k]}\mathfrak{t}^{[k]}} T_0^{[k]}$ under the bijection $\mathfrak{u} \leftrightarrow \mathfrak{u}^{[k]}, \mathfrak{v} \leftrightarrow \mathfrak{v}^{[k]}$.

(4.5.1) implies the lemma since $\mathfrak{u}, \mathfrak{v}$ are standard tableau of shape ν with $\mathbf{a_p}(\nu) = \mathbf{a_p}(\mu)$ and $\mathfrak{v} \in \mathrm{Std}(\nu)_0$. We shall show (4.5.1). First we compute $m_{\mathfrak{s}^{[k]}\mathfrak{t}^{[k]}} T_0^{[k]}$ following the argument in the proof of [DJM, Proposition 3.20]. Recall that

$$\lambda^{[k]} = (\lambda^{(p_k+1)}, \dots, \lambda^{(p_k+r_k)}), \qquad \mathfrak{t}^{[k]} = (\mathfrak{t}^{(p_k+1)}, \dots, \mathfrak{t}^{(p_k+r_k)}),$$

and put

$$\beta = (|\lambda^{(p_k+1)}|, \dots, |\lambda^{(p_k+r_k)}|) = (\beta_1, \dots, \beta_{r_k}),$$
$$\mathbf{b} = (b_1, \dots, b_{r_k}) \text{ with } b_j = \sum_{i=1}^{j-1} \beta_i.$$

The letters contained in $\mathfrak{t}^{[k]}$ consist of $\{a_k+1, \dots, a_k+n_k\}$. By the shift by $-a_k$, we regard $\mathfrak{t}^{[k]}$ as the tableau consisting of letters $\{1, \dots, n_k\}$. Assume that the letter 1 is contained in $\mathfrak{t}^{(p_k+f)}$. One can write $d(\mathfrak{t}^{[k]}) = yc$, where $y \in \mathfrak{S}_\beta$ and c is a distinguished coset representative in $\mathfrak{S}_\beta \backslash \mathfrak{S}_{n_k}$. Then $\mathfrak{t}^{\lambda^{[k]}} y$ is a standard tableau, and c is a permutation which maps the letters $\{b_i+1, \dots, b_i+\beta_i\}$ to the letters contained in $\mathfrak{t}^{(p_k+i)}$ for $i = 1, \dots, r_k$. Thus y fixes the letter b_f+1, and c can be expressed as $c = (b_{f+1}, b_f)(b_f, b_{f-1}) \cdots (2,1)c'$, where $l(c) = b_f + l(c')$ and c' fixes the letter 1. It follows that $T_c^{[k]} = T_{b_f}^{[k]} T_{b_f-1}^{[k]} \cdots T_2^{[k]} T_1^{[k]} T_{c'}^{[k]}$ and $T_{c'}^{[k]} T_0^{[k]} = T_0^{[k]} T_{c'}^{[k]}$. Recall that $m_{\mathfrak{s}^{[k]}\mathfrak{t}^{[k]}} = T_{d(\mathfrak{s}^{[k]})}^{[k]*} m_{\lambda^{[k]}} T_{d(\mathfrak{t}^{[k]})}^{[k]}$ with $m_{\lambda^{[k]}} = u_{\mathbf{b}}^+ x_{\lambda^{[k]}} = x_{\lambda^{[k]}} u_{\mathbf{b}}^+$. Here $u_{\mathbf{b}}^+ = u_{\mathbf{b},1} \cdots u_{\mathbf{b},r_k}$, with

$$u_{\mathbf{b},j} = \prod_{i=1}^{b_j} (L_i^{[k]} - Q_j^{[k]}),$$

where $L_i^{[k]}$ is the element in $\mathcal{H}_{n_k,r_k}$ corresponding to $L_i \in \mathcal{H}$, and $Q_j^{[k]} = Q_{p_k+j}$. Then as in the computation in [DJM, Prop. 3.20, Lemma 3.4], by noticing that $T_y^{[k]}$ commutes with $L_{b_f+1}^{[k]}$, we have

$$\begin{aligned} u_{\mathbf{b}}^+ T_y^{[k]} T_c^{[k]} T_0^{[k]} &= u_{\mathbf{b}}^+ T_y^{[k]} T_{b_f}^{[k]} T_{b_f-1}^{[k]} \cdots T_1^{[k]} T_0^{[k]} T_{c'}^{[k]} \\ &= u_{\mathbf{b}}^+ L_{b_f+1}^{[k]} T_y^{[k]} (T_{b_f}^{[k]})^{-1} \cdots (T_1^{[k]})^{-1} T_{c'}^{[k]} \\ &= (Q_f^{[k]} u_{\mathbf{b}}^+ + u_{\mathbf{b}'}^+) T_y^{[k]} (T_{b_f}^{[k]})^{-1} \cdots (T_1^{[k]})^{-1} T_{c'}^{[k]} \end{aligned}$$

where $\mathbf{b}' = (b_1, \dots, b_{f-1}, b_f+1, b_{f+1}, \dots, b_{r_k})$. It follows that

$$m_{\mathfrak{s}^{[k]}\mathfrak{t}^{[k]}} T_0^{[k]} = T_{d(\mathfrak{s}^{[k]})}^{[k]*} x_{\lambda^{[k]}} (Q_f^{[k]} u_{\mathbf{b}}^+ + u_{\mathbf{b}'}^+) T_y^{[k]} h, \tag{4.5.2}$$

with

$$h = (T^{[k]}_{b_f})^{-1} \cdots (T^{[k]}_1)^{-1} T^{[k]}_{c'}.$$

Next we compute $m_{\mathfrak{s}\mathfrak{t}} L_{a_k+1}$ for $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)_0$. Recall that $m_{\mathfrak{s}\mathfrak{t}} = T^*_{d(\mathfrak{s})} m_\lambda T_{d(\mathfrak{t})}$ with $m_\lambda = x_\lambda u^+_{\mathbf{a}}$. Since $\mathfrak{t} \in \mathrm{Std}(\lambda)_0$, we have $d(\mathfrak{t}) = d(\mathfrak{t}^{[1]}) \cdots d(\mathfrak{t}^{[g]})$. (Note that the letters contained in $\mathfrak{t}^{[k]}$ consist of $\{a_k+1, \dots, a_k+n_k\}$, and we compute $d(\mathfrak{t}^{[k]})$ with respect to these letters.) We note that for $\mathfrak{t} = (\mathfrak{t}^{[1]}, \dots, \mathfrak{t}^{[g]})$, the letters contained in $\mathfrak{t}^{[1]}, \dots, \mathfrak{t}^{[k-1]}$ consist of $\{1, 2, \dots, a_k\}$, and the letters contained in $\mathfrak{t}^{[k]}$ consist of $\{a_k+1, \dots, a_k+n_k = a_{k+1}\}$, the letters contained in $\mathfrak{t}^{[k+1]}, \dots, \mathfrak{t}^{[g]}$ consist of $\{a_{k+1}+1, \dots, n\}$. It follows that

$$\begin{aligned} m_{\mathfrak{s}\mathfrak{t}} T_{a_k} T_{a_k-1} \cdots T_1 T_0 = T^*_{d(\mathfrak{s})} x_\lambda u^+_{\mathbf{a}} T_{d(\mathfrak{t}^{[1]})} \cdots T_{d(\mathfrak{t}^{[k]})} \\ \times T_{a_k} T_{a_k-1} \cdots T_1 T_0 T_{d(\mathfrak{t}^{[k+1]})} \cdots T_{d(\mathfrak{t}^{[g]})}. \end{aligned} \tag{4.5.3}$$

From the previous computation, we have $d(\mathfrak{t}^{[k]}) = yc$ with $y \in \mathfrak{S}_\beta$ and $c \in \mathfrak{S}_\beta \backslash \mathfrak{S}_{n_k}$. (Here we regard $\mathfrak{S}_{n_k}$ as the permutation group with respect to the letters $\{a_k + 1, \dots, a_k + n_k\}$. In particular, y fixes the letter $a_k + b_f + 1$.) Hence

$$T_{d(\mathfrak{t}^{[k]})} = T_y T_c = T_y T_{a_k+b_f} T_{a_k+b_f-1} \cdots T_{a_k+1} T_{c'}.$$

Let X be the left-hand side of (4.5.3). Since $T_{c'}$ commutes with $T_{a_k}, \dots, T_1, T_0$, we have

$$\begin{aligned} X = T^*_{d(\mathfrak{s})} x_\lambda u^+_{\mathbf{a}} T_{d(\mathfrak{t}^{[1]})} \cdots T_{d(\mathfrak{t}^{[k-1]})} T_y \\ \times T_{a_k+b_f} \cdots T_{a_k+1} T_{a_k} T_{a_k-1} \cdots T_1 T_0 T_{c'} T_{d(\mathfrak{t}^{[k+1]})} \cdots T_{d(\mathfrak{t}^{[g]})}. \end{aligned} \tag{4.5.4}$$

Recall that $\mathbf{a} = \mathbf{a}(\lambda) = (a'_1, \dots, a'_r)$ is defined by $a'_j = \sum_{i=1}^{j-1} |\lambda^{(i)}|$, and $u^+_{\mathbf{a}}$ is given by $u^+_{\mathbf{a}} = u_{\mathbf{a},1} u_{\mathbf{a},2} \cdots u_{\mathbf{a},r}$, where $u_{\mathbf{a},j} = \prod_{i=1}^{a'_j} (L_i - Q_j)$. Hence $\mathbf{a_p} = (a_1, \dots, a_g)$ is given by $a_i = a'_{p_i+1}$ for $i = 1, \dots, g$. Put

$$u_{\mathbf{a_p},i} = u_{\mathbf{a},p_i+1} \cdots u_{\mathbf{a},p_i+r_i} \tag{4.5.5}$$

for $i = 1, \dots, g$. Then we have $u^+_{\mathbf{a}} = u_{\mathbf{a_p},1} \cdots u_{\mathbf{a_p},g}$ and $u_{\mathbf{a_p},k}, \dots, u_{\mathbf{a_p},g}$ commutes with $T_{d(\mathfrak{t}^{[1]})}, \dots, T_{d(\mathfrak{t}^{[k-1]})}$, and $u_{\mathbf{a_p},k+1}, \dots, u_{\mathbf{a_p},g}$ commutes with T_y. It follows that

$$\begin{aligned} & u^+_{\mathbf{a}} T_{d(\mathfrak{t}^{[1]})} \cdots T_{d(\mathfrak{t}^{[k-1]})} T_y T_{a_k+b_f} \cdots T_1 T_0 \\ & \quad = u_{\mathbf{a_p},1} \cdots u_{\mathbf{a_p},k-1} T_{d(\mathfrak{t}^{[1]})} \cdots T_{d(\mathfrak{t}^{[k-1]})} u_{\mathbf{a_p},k} T_y u_{\mathbf{a_p},k+1} \cdots u_{\mathbf{a_p},g} T_{a_k+b_f} \cdots T_1 T_0. \end{aligned}$$

Since $u_{\mathbf{a_p},k+1}, \dots, u_{\mathbf{a_p},g}$ commutes with $T_{a_k+b_f}, \dots, T_1, T_0$, we have

$$u_{\mathbf{a_p},k} T_y u_{\mathbf{a_p},k+1} \cdots u_{\mathbf{a_p},g} T_{a_k+b_f} \cdots T_1 T_0 = u_{\mathbf{a_p},k} T_y T_{a_k+b_f} \cdots T_1 T_0 u_{\mathbf{a_p},k+1} \cdots u_{\mathbf{a_p},g}.$$

Since $T_{a_k+b_f} \cdots T_1 T_0 = L_{a_k+b_f+1} h'$ with $h' = T^{-1}_{a_k+b_f} \cdots T^{-1}_1$, and T_y commutes with $L_{a_k+b_f+1}$, we have by [DJM, Lemma 3.4],

$$\begin{aligned} & u_{\mathbf{a_p},k} T_y T_{a_k+b_f} \cdots T_1 T_0 u_{\mathbf{a_p},k+1} \cdots u_{\mathbf{a_p},g} \\ & \qquad = (Q_{p_k+f} u_{\mathbf{a_p},k} + u'_{\mathbf{a_p},k}) T_y h' u_{\mathbf{a_p},k+1} \cdots u_{\mathbf{a_p},g}, \end{aligned}$$

where $u'_{\mathbf{a_p},k}$ is defined as in (4.5.5) by replacing $\mathbf{a}$ by

$$\mathbf{a}' = (a'_1, \dots, a'_{p_k+f-1}, a'_{p_k+f} + 1, a'_{p_k+f+1}, \dots, a'_r).$$

Summing up the above computation, we have

$$\begin{aligned} X = T^*_{d(\mathfrak{s})} x_\lambda u_{\mathbf{a_p},1} \cdots u_{\mathbf{a_p},k-1} T_{d(\mathfrak{t}^{[1]})} \cdots T_{d(\mathfrak{t}^{[k-1]})} \\ \times (Q_{p_k+f} u_{\mathbf{a_p},k} + u'_{\mathbf{a_p},k}) T_y h' T_{c'} u_{\mathbf{a_p},k+1} \cdots u_{\mathbf{a_p},g} T_{d(\mathfrak{t}^{[k+1]})} \cdots T_{d(\mathfrak{t}^{[g]})} \end{aligned}$$

It follows that

(4.5.6)

$$m_{\mathfrak{s}\mathfrak{t}}L_{a_k+1} = XT_1\cdots T_{a_k} = T^*_{d(\mathfrak{s})}x_\lambda u_{\mathbf{a_p},1}\cdots u_{\mathbf{a_p},k-1}T_{d(\mathfrak{t}^{[1]})}\cdots T_{d(\mathfrak{t}^{[k-1]})}$$
$$\times (Q_{p_k+f}u_{\mathbf{a_p},k} + u'_{\mathbf{a_p},k})T_y h'' T_{c'} u_{\mathbf{a_p},k+1}\cdots u_{\mathbf{a_p},g}T_{d(\mathfrak{t}^{[k+1]})}\cdots T_{d(\mathfrak{t}^{[g]})}$$

where $h'' = T^{-1}_{a_k+b_f}\cdots T^{-1}_{a_k+1}$.

We now compare (4.5.2) and (4.5.6). The right-hand side of (4.5.2) is written as $X_1 + X_2$, where $X_1 = Q^{[k]}_f T^{[k]*}_{d(\mathfrak{s}^{[k]})}x_{\lambda^{[k]}}u^+_{\mathbf{b}}T^{[k]}_y h$ and $X_2 = T^{[k]*}_{d(\mathfrak{s}^{[k]})}x_{\lambda^{[k]}}u^+_{\mathbf{b}'}T^{[k]}_y h$. Since $x_{\lambda^{[k]}}u^+_{\mathbf{b}} = m_{\lambda^{[k]}}$, X_1 can be written, by Lemma 3.15 in [DJM], as a linear combination of the elements $m_{\mathfrak{s}^{[k]}\mathfrak{t}^{[k]}_i}$, where $\mathfrak{t}^{[k]}_i$ are row-standard tableaux. Then they are converted to a linear combination of the basis elements $m_{\mathfrak{u}^{[k]}\mathfrak{v}^{[k]}}$ in $\mathcal{H}_{n_k,r_k}$ by the procedure given in Proposition 3.18 in [loc. cit.], where $\mathfrak{u}^{[k]}, \mathfrak{v}^{[k]}$ are standard tableaux of shape $\mu^{[k]}$ for some r_k-partitions $\mu^{[k]}$. On the other hand, for X_2, first we convert $T^{[k]*}_{d(\mathfrak{s}^{[k]})}x_{\lambda^{[k]}}u^+_{\mathbf{b}'}$ to a linear combination of the elements $m_{\mathfrak{u}^{[k]}_1\mathfrak{v}^{[k]}_1}$ where $\mathfrak{u}^{[k]}_1, \mathfrak{v}^{[k]}_1$ are row-standard tableau of shape $\nu^{[k]}$ ($\nu^{[k]}$ is determined from $u_{\mathbf{b}'}$), and then we follow the argument in the case X_1. Note that in these computations, the parts $u^+_{\mathbf{b}}$ and $u^+_{\mathbf{b}'}$ remain unchanged.

Next we consider (4.5.6). Since $T_{d(\mathfrak{s})} = T_{d(\mathfrak{s}^{[1]})}\cdots T_{d(\mathfrak{s}^{[g]})}$ and $x_\lambda = x_{\lambda^{[1]}}\cdots x_{\lambda^{[g]}}$, one can write the formula (4.5.6) in the form

$$m_{\mathfrak{s}\mathfrak{t}}L_{a_k+1} = Z\cdot T^*_{d(\mathfrak{s}^{[k]})}x_{\lambda^{[k]}}(Q_{p_k+f}u_{\mathbf{a_p},k} + u'_{\mathbf{a_p},k})T_y h'' T_{c'}\cdot Z'$$

where

$$Z = T^*_{d(\mathfrak{s}^{[1]})}x_{\lambda^{[1]}}u_{\mathbf{a_p},1}T_{d(\mathfrak{t}^{[1]})}\cdots T^*_{d(\mathfrak{s}^{[k-1]})}x_{\lambda^{[k-1]}}u_{\mathbf{a_p},k-1}T_{d(\mathfrak{t}^{[k-1]})}$$
$$Z' = T^*_{d(\mathfrak{s}^{[k+1]})}x_{\lambda^{[k+1]}}u_{\mathbf{a_p},k+1}T_{d(\mathfrak{t}^{[k+1]})}\cdots T^*_{d(\mathfrak{s}^{[g]})}x_{\lambda^{[g]}}u_{\mathbf{a_p},g}T_{d(\mathfrak{t}^{[g]})}.$$

Put

$$Y_1 = Q_{p_k+f}T^*_{d(\mathfrak{s}^{[k]})}x_{\lambda^{[k]}}u_{\mathbf{a_p},k}T_y h'' T_{c'},$$
$$Y_2 = T^*_{d(\mathfrak{s}^{[k]})}x_{\lambda^{[k]}}u'_{\mathbf{a_p},k}T_y h'' T_{c'},$$

so that $m_{\mathfrak{s}\mathfrak{t}}L_{a_k+1} = Z(Y_1 + Y_2)Z'$. Let $\mathcal{H}'_{n_k}$ be the subalgebra of $\mathcal{H}_n$ generated by $T_{a_k+1},\cdots,T_{a_k+n_k-1}$. Then $T_y, T_{c'}, h''$ belong to $\mathcal{H}'_{n_k}$, and under the identification $\mathcal{H}'_{n_k} \simeq \mathcal{H}_{n_k}$, $T_y, T_{c'}$ coincide with $T^{[k]}_y, T^{[k]}_{c'}$, and $h''T_{c'}$ coincides with h. We also note that $Q_{p_k+f} = Q^{[k]}_f$. Now by applying Lemma 3.15 and Proposition 3.18 in [loc.cit], Y_1 can be expressed as a linear combination of the terms $T^*_{d(\mathfrak{u}^{[k]})}x_{\mu^{[k]}}u_{\mathbf{a_p},k}T_{d(\mathfrak{v}^{[k]})}$, where $\mathfrak{u}^{[k]}, \mathfrak{v}^{[k]}$ are standard tableaux of shape $\mu^{[k]}$ for some r_k-partitions $\mu^{[k]}$. Since this computation proceeds without referring $u_{\mathbf{a_p},k}$, the coefficients of these elements in the expansion of Y_1 are exactly the same as the coefficients of $m_{\mathfrak{u}^{[k]}\mathfrak{v}^{[k]}}$ in the expansion of X_1. For Y_2, we first convert $T^*_{d(\mathfrak{s}^{[k]})}x_{\lambda^{[k]}}u'_{\mathbf{a_p},k}$ to a linear combination of the terms $T^*_{d(\mathfrak{u}^{[k]}_1)}x_{\nu^{[k]}}u'_{\mathbf{a_p},k}T_{d(\mathfrak{v}^{[k]}_1)}$ by using Proposition 3.20 in [loc.cit]. By comparing $\mathbf{b}'$ and $\mathbf{a}'$, we see that the coefficients in this expansion are exactly the same as the coefficients of $m_{\mathfrak{u}^{[k]}_1\mathfrak{v}^{[k]}_1}$ in the expansion of $T^{[k]*}_{d(\mathfrak{s}^{[k]})}x_{\lambda^{[k]}}u^+_{\mathbf{b}'}$. Thus again by applying Lemma 3.15 and Proposition 3.18 in [loc.cit], we conclude that Y_2 can be written as a linear combination of the terms $T^*_{d(\mathfrak{u}^{[k]})}x_{\nu^{[k]}}u'_{\mathbf{a_p},k}T_{d(\mathfrak{v}^{[k]})}$, where $\mathfrak{u}^{[k]}, \mathfrak{v}^{[k]}$ are standard tableaux of shape $\nu^{[k]}$, and that their coefficients in the expansion of Y_2 is the same as the coefficients of $m_{\mathfrak{u}^{[k]}\mathfrak{v}^{[k]}}$ in the expansion of X_2.

Now one sees easily that $Z \cdot T^*_{d(\mathfrak{u}^{[k]})} x_{\mu^{[k]}} u_{\mathbf{a_p},k} T_{d(\mathfrak{v}^{[k]})} \cdot Z' = m_{\mathfrak{u}\mathfrak{v}}$, where μ is an r-partition obtained from λ by replacing $\lambda^{[k]}$ by $\mu^{[k]}$, and $\mathfrak{u}, \mathfrak{v}$ are standard tableau of shape μ obtained from $\mathfrak{s}, \mathfrak{t}$ by replacing $\mathfrak{s}^{[k]}, \mathfrak{t}^{[k]}$ by $\mathfrak{u}^{[k]}, \mathfrak{v}^{[k]}$. A similar result holds also for $Z \cdot T^*_{d(\mathfrak{u}^{[k]})} x_{\nu^{[k]}} u'_{\mathbf{a_p},k} T_{d(\mathfrak{v}^{[k]})} \cdot Z'$. Summing up the above arguments, we see that (4.5.1) holds. Hence the lemma is proved. □

The following lemma is easily verified by using a similar (but simpler) argument as in the proof of the previous lemma.

Lemma 4.6. *Let the notation be as in Lemma 4.5. Then, for $i = 1, \dots, n_k - 1$, the action of T_{a_k+i} on $\overline{M}^\mu$ stabilizes $\overline{M}^\mu_0$, and it gives rise to the action of $T_i^{[k]}$ on $M^{\mu^{[1]}} \otimes \cdots \otimes M^{\mu^{[g]}}$ under the identification ϕ_μ in (4.4.1).*

4.7. We fix $\alpha = (n_1, \dots, n_g) \in \widetilde{\mathcal{P}}_{n,1}$, and let $\mathcal{H}_\alpha$ be as in 4.4. Assume that $\alpha_{\mathbf{p}}(\mu) = \alpha$ for $\mu \in \Lambda$. Then $\mathcal{H}_\alpha$ acts naturally on $M^{\mu^{[1]}} \otimes \cdots \otimes M^{\mu^{[g]}}$. Let $\mathbf{a_p}(\mu) = (a_1, \dots, a_g)$ be as before, and let $\widetilde{\mathcal{H}}_\alpha$ be the subalgebra of $\mathcal{H}$ generated by $T_{a_k+1}, \dots, T_{a_k+r_k-1}, L_{a_k+1}$ for $k = 1, \dots, g$. As a corollary to Lemma 4.5 and Lemma 4.6, we have the following.

Corollary 4.8. *For each $\mu \in \Lambda$ such that $\alpha_{\mathbf{p}}(\mu) = \alpha$, $\overline{M}^\mu_0$ is $\widetilde{H}_\alpha$-stable. The action of $\widetilde{H}_\alpha$ on $\overline{M}^\mu_0$ coincides with the action of $\mathcal{H}_\alpha$ on $M^{\mu^{[1]}} \otimes \cdots \otimes M^{\mu^{[g]}}$.*

4.9. Recall that $\mathcal{S} = \bigoplus_{\mu,\nu\in\Lambda} \mathrm{Hom}_{\mathcal{H}}(M^\nu, M^\mu)$. It follows from the description of the basis of $\mathcal{S}^{\mathbf{p}}$ that

$$\mathcal{S}^{\mathbf{p}} = \bigoplus_{\mu,\nu\in\Lambda} H_{\mu\nu} \tag{4.9.1}$$

where $H_{\mu\nu} = \mathcal{S}^{\mathbf{p}} \cap \mathrm{Hom}_{\mathcal{H}}(M^\nu, M^\mu)$ is an R-submodule of $\mathrm{Hom}_{\mathcal{H}}(M^\nu, M^\mu)$ spanned by φ_{ST} with $S \in \mathcal{T}_0(\lambda, \mu)$, $T \in \mathcal{T}_0(\lambda, \nu)$ such that $\mathbf{a_p}(\lambda) > \mathbf{a_p}(\mu)$ if $\alpha_{\mathbf{p}}(\mu) \neq \alpha_{\mathbf{p}}(\nu)$. Then we have

$$\overline{\mathcal{S}}^{\mathbf{p}} = \bigoplus_{\substack{\mu,\nu\in\Lambda \\ \alpha_{\mathbf{p}}(\mu)=\alpha_{\mathbf{p}}(\nu)}} \overline{H}_{\mu\nu}, \tag{4.9.2}$$

where $\overline{H}_{\mu\nu} = \pi(H_{\mu\nu})$ is the R-span of the elements $\bar{\varphi}_{ST}$ such that $S \in \mathcal{T}_0^{\mathbf{p}}(\lambda, \mu)$, $T \in \mathcal{T}_0^{\mathbf{p}}(\lambda, \nu)$ for various $\lambda \in \Lambda^+$.

Assume that $\alpha_{\mathbf{p}}(\mu) = \alpha_{\mathbf{p}}(\nu)$. We claim that any $\varphi \in H_{\mu\nu}$ maps $\widehat{M}^\nu$ into $\widehat{M}^\mu$. In fact, take $\varphi \in H_{\mu\nu}$. Then by the property of φ_{ST}, there exists $h_\varphi \in \mathcal{H}$ such that $\varphi(m_\nu h) = h_\varphi m_\nu h$ for any $h \in \mathcal{H}$. Recall that $\widehat{M}^\nu$ is a linear combination of $m_{S\mathfrak{t}}$ with $S \in \mathcal{T}_0(\lambda, \nu), \mathfrak{t} \in \mathrm{Std}(\lambda)$ such that $\mathbf{a_p}(\lambda) > \mathbf{a_p}(\nu)$. Suppose that $m_{S\mathfrak{t}}$ is written as $m_{S\mathfrak{t}} = m_\nu h$ for some $h \in \mathcal{H}$. Then by the property of cellular basis, $\varphi(m_{St}) = h_\varphi m_{S\mathfrak{t}}$ is a linear combination of $m_{\mathfrak{s}'\mathfrak{t}'}$, where $\mathfrak{s}', \mathfrak{t}' \in \mathrm{Std}(\lambda')$ with $\lambda' \trianglerighteq \lambda$. Then we have $\mathbf{a_p}(\lambda') \geq \mathbf{a_p}(\lambda) > \mathbf{a_p}(\nu)$. Since $\mathbf{a_p}(\nu) = \mathbf{a_p}(\mu)$, we have $\mathbf{a_p}(\lambda') > \mathbf{a_p}(\mu)$, and so $\varphi(m_{S\mathfrak{t}}) \in \widehat{M}^\mu$. Thus the claim holds.

By the claim, φ induces a linear map $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{H}}(\overline{M}^\nu, \overline{M}^\mu)$ under the condition that $\alpha_{\mathbf{p}}(\mu) = \alpha_{\mathbf{p}}(\nu)$. We note that $\bar{\varphi} = 0$ if $\varphi \in \widehat{\mathcal{S}}^{\mathbf{p}}$. In fact, since $\mathbf{a_p}(\mu) = \mathbf{a_p}(\nu)$, we may consider the case where $\varphi = \varphi_{ST}$ for $S \in \mathcal{T}_0(\lambda, \mu), T \in \mathcal{T}_0(\lambda, \nu)$ with $\mathbf{a_p}(\lambda) \neq \mathbf{a_p}(\mu)$. Since $\lambda \trianglerighteq \mu$, we have $\mathbf{a_p}(\lambda) > \mathbf{a_p}(\mu)$. It follows that $\varphi_{ST}(m_\nu) = m_{ST} \in \widehat{M}^\mu$, and the image of φ is contained in $\widehat{M}^\mu$. Hence $\bar{\varphi} = 0$ as asserted.

The above discussion allows us to define a linear map $\theta : H_{\mu\nu} \to \mathrm{Hom}_{\mathcal{H}}(\overline{M}^{\nu}, \overline{M}^{\mu})$ by $\varphi \mapsto \bar{\varphi}$, which factors through the map $\bar{\theta} : \overline{H}_{\mu\nu} \to \mathrm{Hom}_{\mathcal{H}}(\overline{M}^{\nu}, \overline{M}^{\mu})$. We show the following lemma.

Lemma 4.10. (i) *For $\mu \in \Lambda$, let $\phi_\mu : \overline{M}_0^{\mu} \to M^{\mu^{[1]}} \otimes \cdots \otimes M^{\mu^{[g]}}$ be the isomorphism given in (4.4.1). Then we have*

$$\phi_\mu^{-1}(m_{\mu^{[1]}} \otimes \cdots \otimes m_{\mu^{[g]}}) = \overline{m}_\mu.$$

(ii) *Assume that $\alpha_{\mathbf{p}}(\mu) = \alpha_{\mathbf{p}}(\nu) = \alpha$. Then for any $\varphi \in \overline{H}_{\mu\nu}$, $\bar{\varphi} = \bar{\theta}(\varphi)$ maps $\overline{M}_0^{\nu}$ to $\overline{M}_0^{\mu}$. In particular, $\bar{\varphi} \in \mathrm{Hom}_{\widetilde{\mathcal{H}}_\alpha}(\overline{M}_0^{\nu}, \overline{M}_0^{\mu})$.*

Proof. First we prove(i). Put $\mathbf{a} = \mathbf{a}(\mu)$ and $\mathbf{a_p} = \mathbf{a_p}(\mu)$. Then $m_\mu = x_\mu u_{\mathbf{a}}^+$, and $x_\mu = x_{\mu^{[1]}} \cdots x_{\mu^{[g]}}$, $u_{\mathbf{a}}^+ = u_{\mathbf{a_p},1} \cdots u_{\mathbf{a_p},g}$, where $u_{\mathbf{a_p},i}$ is defined as in (4.5.5). One can write $m_\mu = x_1 x_2 \cdots x_g$ with $x_k = x_{\mu^{[k]}} u_{\mathbf{a_p},k}$. On the other hand, $m_{\mu^{[k]}} = x_{\mu^{[k]}} u_{\mathbf{b}}^+$, where $\mathbf{b} = \mathbf{a}(\mu^{[k]})$ is defined with respect to $\mu^{[k]} \in \widetilde{\mathcal{P}}_{n_k,r_k}(m^{[k]})$. Then by Proposition 3.18 in [DJM], $m_{\mu^{[k]}}$ is written as a linear combination of the basis elements $m_{\mathfrak{u}^{[k]}\mathfrak{v}^{[k]}}$ of $\mathcal{H}_{n_k,r_k}$, where $\mathfrak{u}^{[k]}, \mathfrak{v}^{[k]}$ are standard tableau of shape $\lambda^{[k]}$. By the same procedure, x_k is written as a linear combination of $x_{\mathfrak{u}^{[k]}\mathfrak{v}^{[k]}} = T^*_{d(\mathfrak{u}^{[k]})} x_{\lambda^{[k]}} u_{\mathbf{a_p},k} T_{d(\mathfrak{v}^{[k]})}$, and the corresponding coefficient coincides with each other. Note that in the latter case $\mathfrak{u}^{[k]}, d(\mathfrak{u}^{[k]})$, etc. are referred with respect to the letters $\{a_k+1, \ldots, a_k+n_k\}$ as in the proof of Lemma 4.5. We see that $x_{\mathfrak{u}^{[1]}\mathfrak{v}^{[1]}} \cdots x_{\mathfrak{u}^{[g]}\mathfrak{v}^{[g]}}$ gives rise to a basis element $m_{\mathfrak{u}\mathfrak{v}}$ of $\mathcal{H}$, where $\mathfrak{u} = (\mathfrak{u}^{[1]}, \ldots, \mathfrak{v}^{[g]})$ and $\mathfrak{v} = (\mathfrak{v}^{[1]}, \ldots, \mathfrak{v}^{[g]})$ are in $\mathrm{Std}(\lambda)_0$ with $\lambda = (\lambda^{[1]}, \ldots, \lambda^{[g]})$. The assertion (i) follows from this.

Next we prove (ii). Now we have $\overline{m}_\nu \in \overline{M}_0^{\nu}$. Since $M^{\nu^{[1]}} \otimes \cdots \otimes M^{\nu^{[g]}}$ is generated by $m_{\nu^{[1]}} \otimes \cdots \otimes m_{\nu^{[g]}}$ as an $\mathcal{H}_\alpha$-module, $\overline{M}_0^{\nu}$ is generated by $\overline{m}_\nu$ as an $\widetilde{H}_\alpha$-module. We take $\bar{\varphi}_{ST} \in \overline{H}_{\mu\nu}$. Then any element in $\overline{M}_0^{\nu}$ is written as $\overline{m}_\nu h$ with $h \in \widetilde{\mathcal{H}}_\alpha$, and $\varphi_{ST}(m_\nu h) = \varphi_{ST}(m_\nu)h = m_{ST}h$. Since $\overline{m}_{ST} \in \overline{M}_0^{\mu}$, we see that $\bar{\varphi}_{ST}(\overline{M}_0^{\nu}) \subseteq \overline{M}_0^{\mu}$. This proves (ii), and the lemma follows. □

4.11. We keep the previous setting. By Lemma 4.10, one can define an R-linear map $\Theta : \overline{H}_{\mu\nu} \to \mathrm{Hom}_{\widetilde{\mathcal{H}}_\alpha}(\overline{M}_0^{\nu}, \overline{M}_0^{\mu})$ induced from $\bar{\theta}$. On the other hand, in view of the isomorphisms ϕ_μ, ϕ_ν together with Corollary 4.8, we have a natural isomorphism of R-modules:

$$\begin{aligned} &\mathrm{Hom}_{\widetilde{\mathcal{H}}_\alpha}(\overline{M}_0^{\nu}, \overline{M}_0^{\mu}) \\ &\qquad \simeq \mathrm{Hom}_{\mathcal{H}_{n_1,r_1}}(M^{\nu^{[1]}}, M^{\mu^{[1]}}) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{H}_{n_g,r_g}}(M^{\nu^{[g]}}, M^{\mu^{[g]}}). \end{aligned} \tag{4.11.1}$$

We have the following lemma.

Lemma 4.12. *The map Θ gives an isomorphism*

$$\overline{H}_{\mu\nu} \simeq \mathrm{Hom}_{\widetilde{\mathcal{H}}_\alpha}(\overline{M}_0^{\nu}, \overline{M}_0^{\mu})$$

of R-modules. Let $\bar{\varphi}_{ST}$ be a basis element of $\overline{H}_{\mu\nu}$, where $S = (S^{[1]}, \ldots, S^{[g]}) \in \mathcal{T}_0^{\mathbf{P}}(\lambda, \mu)$ and $T = (T^{[1]}, \ldots, T^{[g]}) \in \mathcal{T}_0^{\mathbf{P}}(\lambda, \nu)$ for some $\lambda \in \Lambda^+$. Then under the identification in (4.11.1), Θ maps $\bar{\varphi}_{ST}$ to $\varphi_{S^{[1]}T^{[1]}} \otimes \cdots \otimes \varphi_{S^{[g]}T^{[g]}}$.

Proof. It is enough to show the second assertion since $\varphi_{S^{[1]}T^{[1]}} \otimes \cdots \otimes \varphi_{S^{[g]}T^{[g]}}$ gives a basis of $\mathrm{Hom}_{\widetilde{\mathcal{H}}_\alpha}(\overline{M}_0^{\nu}, \overline{M}_0^{\mu})$ under the identification in (4.11.1). Take $\bar{\varphi}_{ST} \in \overline{H}_{\mu\nu}$. Then $\bar{\varphi}_{ST}$ is defined by $\bar{\varphi}_{ST}(\overline{m}_\nu) = \overline{m}_{ST}$. By Lemma 4.10 (i), $\overline{m}_\nu$ is mapped to $m_{\nu^{[1]}} \otimes \cdots \otimes m_{\nu^{[g]}}$ via ϕ_ν. $\overline{m}_{ST}$ is also mapped to $m_{S^{[1]}T^{[1]}} \otimes \cdots \otimes m_{S^{[g]}T^{[g]}}$

via ϕ_μ. Hence via the isomorphism (4.11.1), $\bar{\varphi}_{ST}$ corresponds to the $\mathcal{H}_\alpha$-linear map sending $m_{\nu^{[1]}} \otimes \cdots \otimes m_{\nu^{[g]}}$ to $m_{S^{[1]}T^{[1]}} \otimes \cdots \otimes m_{S^{[g]}T^{[g]}}$, which coincides with $\varphi_{S^{[1]}T^{[1]}} \otimes \cdots \otimes \varphi_{S^{[g]}T^{[g]}}$. The lemma is proved. □

Remark 4.13. There exists an R-linear map $\psi : \bar{\theta}(\overline{H}_{\mu\nu}) \to \mathrm{Hom}_{\widetilde{\mathcal{H}}_\alpha}(\overline{M}_0^\nu, \overline{M}_0^\mu)$ such that $\psi \circ \bar{\theta} = \Theta$ by Lemma 4.10. Hence $\bar{\theta}$ is injective by Lemma 4.12. However, $\bar{\theta}$ is not necessarily surjective. In Section 7, we describe $\mathrm{Im}\,\bar{\theta}$ in terms of a modified Ariki-Koike algebra.

4.14. Let $\Delta_{n,g}$ be the set of $\alpha = (n_1, \dots, n_g) \in \mathbb{Z}^g_{\geq 0}$ such that $n_1 + \cdots + n_g = n$. For $\alpha \in \Delta_{n,g}$, put

$$M^\alpha = \bigoplus_{\substack{\mu \in \Lambda \\ \alpha_{\mathbf{p}}(\mu) = \alpha}} M^\mu, \qquad \overline{M}_0^\alpha = \bigoplus_{\substack{\mu \in \Lambda \\ \alpha_{\mathbf{p}}(\mu) = \alpha}} \overline{M}_0^\mu.$$

Then $\mathcal{S}^{\mathbf{p}}_\alpha = \mathrm{End}_{\mathcal{H}}(M^\alpha)$ is a subalgebra of $\mathcal{S}^{\mathbf{p}}$, and we have $\mathcal{S}^{\mathbf{p}}_\alpha = \bigoplus_{\mu,\nu} H_{\mu\nu}$, where the sum is taken over all $\mu, \nu \in \Lambda$ such that $\alpha_{\mathbf{p}}(\mu) = \alpha_{\mathbf{p}}(\nu) = \alpha$. Put $\overline{\mathcal{S}}^{\mathbf{p}}_\alpha = \pi(\mathcal{S}^{\mathbf{p}}_\alpha)$. Then $\overline{\mathcal{S}}^{\mathbf{p}}_\alpha$ is a subalgebra of $\overline{\mathcal{S}}^{\mathbf{P}}$ such that $\overline{\mathcal{S}}^{\mathbf{p}}_\alpha = \bigoplus_{\mu,\nu} \overline{H}_{\mu\nu}$. Hence we have

$$\overline{\mathcal{S}}^{\mathbf{p}} = \bigoplus_{\alpha \in \Delta_{n,g}} \overline{\mathcal{S}}^{\mathbf{p}}_\alpha. \tag{4.14.1}$$

On the other hand, Lemma 4.12 implies that

$$\overline{\mathcal{S}}^{\mathbf{p}}_\alpha \simeq \mathrm{End}_{\widetilde{\mathcal{H}}_\alpha}(\overline{M}_0^\alpha). \tag{4.14.2}$$

We define an $\mathcal{H}_{n_k,r_k}$-module $M^{[k]}$ by $M^{[k]} = \bigoplus_{\mu^{[k]} \in \Lambda_{n_k}} M^{\mu^{[k]}}$. Define a cyclotomic q-Schur algebra $\mathcal{S}(\Lambda_{n_k})$ associated to $\mathcal{H}_{n_k,r_k}$ by $\mathcal{S}(\Lambda_{n_k}) = \mathrm{End}_{\mathcal{H}_{n_k,r_k}} M^{[k]}$. Then we see that

$$\mathrm{End}_{\mathcal{H}_\alpha}\Big(\bigoplus_{\substack{\mu \in \Lambda \\ \alpha_{\mathbf{p}}(\mu) = \alpha}} M^{\mu^{[1]}} \otimes \cdots \otimes M^{\mu^{[g]}}\Big) \simeq \mathcal{S}(\Lambda_{n_1}) \otimes \cdots \otimes \mathcal{S}(\Lambda_{n_g}). \tag{4.14.3}$$

The following structure theorem follows from (4.14.1)–(4.14.3) together with (4.11.1). Note that in the special case where $\mathbf{p} = (1^r)$, this result was proved in [SawS, Theorem 5.5 (i)] under the assumption that $Q_i - Q_j$ are units in R for any $i \neq j$, and that $\Lambda = \widetilde{\mathcal{P}}_{n,r}(\mathbf{m})$ with $m_i \geq n$ for $i = 1, \dots, r$. In our case, we don't need any assumption for parameters Q_i nor $\mathbf{m}$.

Theorem 4.15. *Assume that $\Lambda = \widetilde{\mathcal{P}}_{n,r}(\mathbf{m})$. Then there exists an isomorphism of R-algebras*

$$\overline{\mathcal{S}}^{\mathbf{p}}(\Lambda) \simeq \bigoplus_{\substack{(n_1,\dots,n_g) \\ n_1 + \cdots + n_g = n}} \mathcal{S}(\Lambda_{n_1}) \otimes \cdots \otimes \mathcal{S}(\Lambda_{n_g}),$$

where $\bar{\varphi}_{ST}$ is mapped to $\varphi_{S^{[1]}T^{[1]}} \otimes \cdots \otimes \varphi_{S^{[g]}T^{[g]}}$.

For $\lambda^{[k]}, \mu^{[k]} \in \Lambda^+_{n_k}$, let $W^{\lambda^{[k]}}$ be the Weyl module, and let $L^{\mu^{[k]}}$ be the irreducible module with respect to $\mathcal{S}(\Lambda_{n_k})$. As a corollary to the previous theorem, we have

Corollary 4.16. *Assume that R is a field and Λ is as above. Let $\lambda, \mu \in \Lambda^+$. Then under the isomorphism in Theorem 4.15, we have the following:*

(i) $\overline{Z}^\lambda_{\mathbf{p}} \simeq W^{\lambda^{[1]}} \otimes \cdots \otimes W^{\lambda^{[g]}}$.

(ii) $\overline{L}^{\mu}_{\mathbf{p}} \simeq L^{\mu^{[1]}} \otimes \cdots \otimes L^{\mu^{[g]}}$.

(iii) $[\overline{Z}^{\lambda}_{\mathbf{p}} : \overline{L}^{\mu}_{\mathbf{p}}]_{\overline{\mathcal{S}}^{\mathbf{p}}} = \begin{cases} \prod_{k=1}^{g} [W^{\lambda^{[k]}} : L^{\mu^{[k]}}]_{\mathcal{S}(\Lambda_{n_k})} & \text{if } \alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu), \\ 0 & \text{otherwise.} \end{cases}$

Combining this with Theorem 3.13, we have the following product formula for the decomposition numbers of $\mathcal{S}(\Lambda)$, which is a generalization of [Sa, Corollary 5.10].

Theorem 4.17. *Assume that R is a field and that $\Lambda = \widetilde{\mathcal{P}}_{n,r}(\mathbf{m})$. For $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, we have*

$$[W^{\lambda} : L^{\mu}]_{\mathcal{S}(\Lambda)} = \prod_{k=1}^{g} [W^{\lambda^{[k]}} : L^{\mu[k]}]_{\mathcal{S}(\Lambda_{n_k})}.$$

5. Modified Ariki-Koike algebra of type $\mathbf{p}$

5.1. Throughout this section we assume the following property for $\mathbf{m} = (m_1, \dots, m_r)$.

$$m_i \geq n \text{ for } i = 1, \dots, r. \tag{5.1.1}$$

We keep the assumption that $\Lambda = \widetilde{\mathcal{P}}_{n,r}(\mathbf{m})$. Let $\Omega = \Omega^{\mathbf{p}}$ be a subset of Λ consisting of $\omega = (\omega_i^{(j)}) = (\omega^{[1]}, \dots, \omega^{[g]})$ satisfying the properties

(i) $\omega_i^{(j)} \in \{0, 1\}$,

(ii) $\sum_{j=1}^{r} \omega_i^{(j)} = 1$ for $1 \leq i \leq n, 1 \leq j \leq r$,

(iii) $\omega_i^{(j)} = 0$ unless $j = p_1 + r_1, \dots, p_g + r_g$. Hence $\omega^{[k]} = (-, \dots, -, \omega^{(p_k + r_k)})$ for $k = 1, \dots, g$.

Note that Ω coincides with Ω in [SawS, 7.1] in the case where $\mathbf{p} = (1^r)$ (i.e., the case $g = r$). While in the case where $\mathbf{p} = (r)$ (i.e., the case $g = 1$), $\Omega = \{\omega\}$, where ω is an r-partition $\omega = (-, \dots, -, (1^n))$ which coincides with ω in [M, §4].

Let $I = \{1, \dots, n\}$. For $\omega \in \Omega$, we denote by I_k the set of i such that $\omega_i^{(p_k + r_k)} = 1$ for $k = 1, \dots, g$. Then $I = \coprod_{k=1}^{g} I_k$ gives a partition of I into g parts, and thanks to (5.1.1), the set Ω is in bijection with the set of partitions of I into g parts. For $\mathfrak{t} = (\mathfrak{t}^{[1]}, \dots, \mathfrak{t}^{[t]}) \in \mathrm{Std}(\lambda)$, we denote by I_k the letters contained in the standard tableau $\mathfrak{t}^{[k]}$. Then $I = \coprod I_k$ determines $\omega = \omega_{\mathfrak{t}} \in \Omega$. We associate to $\mathfrak{t}$ a semi-standard tableau T of shape λ as follows: for each k ($1 \leq k \leq g$), the first terms of the entries of $T^{(p_k + i)}$ consist of the entries of $\mathfrak{t}^{(p_k + i)}$, and the second term of them has the common value $p_k + r_k$ for $i = 1, \dots, r_k$. Then $T \in \mathcal{T}_0^{\mathbf{p}}(\lambda, \omega)$, and any element of $\mathcal{T}_0^{\mathbf{p}}(\lambda, \omega)$ is obtained from $\mathfrak{t} \in \mathrm{Std}(\lambda)$ such that $\omega = \omega_{\mathfrak{t}}$ by the above procedure. The correspondence $\mathfrak{t} \mapsto T$ gives a bijective correspondence

$$\mathrm{Std}(\lambda) \simeq \bigcup_{\omega \in \Omega} \mathcal{T}_0^{\mathbf{p}}(\lambda, \omega). \tag{5.1.2}$$

We denote by $\mathrm{Std}(\lambda)_{\omega}$ the subset of $\mathrm{Std}(\lambda)$ corresponding to $\mathcal{T}_0^{\mathbf{p}}(\lambda, \omega)$ under the bijection (5.1.2), i.e., $\mathrm{Std}(\lambda)_{\omega} = \{\mathfrak{t} \in \mathrm{Std}(\lambda) \mid \omega_{\mathfrak{t}} = \omega\}$.

Assume that $\omega \in \Omega$ corresponds to the partition $I = \coprod_k I_k$, where $\mathbf{a}_{\mathbf{p}}(\omega) = (a_1, \dots, a_g)$. We write I_k as $I_k = \{i_{k1} < i_{k2} < \cdots < i_{kn_k}\}$. We define $d(\omega) \in \mathfrak{S}_n$ as

$$d(\omega) = \begin{pmatrix} \dots & a_k + 1 & a_k + 2 & \dots & a_k + n_k & \dots \\ \dots & i_{k1} & i_{k2} & \dots & i_{kn_k} & \dots \end{pmatrix}.$$

Suppose that $T \in \mathcal{T}_0^{\mathbf{p}}(\lambda,\omega)$ corresponds to $\mathfrak{t} \in \mathrm{Std}(\lambda)$ via (5.1.2). Let $\mathfrak{t}_1 \in \mathrm{Std}(\lambda)$ be such that $\mathfrak{t} = \mathfrak{t}_1 d(\omega)$. Then the letters contained in $\mathfrak{t}_1^{[k]}$ consist of $\{a_k+1,\dots,a_k+n_k\}$, and $\mathfrak{t}_1$ is the unique element in $\mathrm{Std}(\lambda)$ such that $\omega(\mathfrak{t}_1) = T$. In particular, assume that $S \in \mathcal{T}_0^{\mathbf{p}}(\lambda,\mu), T \in \mathcal{T}_0^{\mathbf{p}}(\lambda,\omega)$, for $\mu \in \Lambda, \omega \in \Omega$, and that $\mathfrak{t} \in \mathrm{Std}(\lambda)$ corresponds to T via (5.1.2). Then we have

$$m_{ST}T_{d(\omega)} = m_{S\mathfrak{t}}. \tag{5.1.3}$$

5.2. For each $\mu \in \Lambda$, let φ_μ be the identity map on M^μ. By 2.4, $\varphi_\mu \in H_{\mu\mu}$, and we put $\bar{\varphi}_\mu = \pi(\varphi_\mu) \in \overline{H}_{\mu\mu}$. If we put $\bar{\varphi}_\Omega = \sum_{\omega\in\Omega}\bar{\varphi}_\omega$, $\bar{\varphi}_\Omega$ is an idempotent in $\overline{\mathcal{S}}^{\mathbf{p}}$, and we define a subalgebra $\overline{\mathcal{H}}^{\mathbf{p}}$ of $\overline{\mathcal{S}}^{\mathbf{p}}$ by $\overline{\mathcal{H}}^{\mathbf{p}} = \bar{\varphi}_\Omega\overline{\mathcal{S}}^{\mathbf{p}}\bar{\varphi}_\Omega$. We call $\overline{\mathcal{H}}^{\mathbf{p}}$ the modified Ariki-Koike algebra of type $\mathbf{p}$. In the case where $\mathbf{p} = (1^r)$, $\overline{\mathcal{H}}^{\mathbf{p}}$ can be identified with the modified Ariki-Koike algebra given in [SawS] (see 7.1 in [loc. cit.]). One can write $\overline{\mathcal{H}}^{\mathbf{p}} = \bigoplus_{\omega,\omega'\in\Omega}\overline{H}_{\omega\omega'}$. In particular, $\overline{\mathcal{H}}^{\mathbf{p}}$ has an R-free basis

$$\mathcal{B}^{\mathbf{p}} = \{\overline{\varphi}_{ST} \mid S \in \mathcal{T}_0^{\mathbf{p}}(\lambda,\omega), T \in \mathcal{T}_0^{\mathbf{p}}(\lambda,\omega') \text{ for } \omega,\omega' \in \Omega, \lambda \in \Lambda^+\}. \tag{5.2.1}$$

Note that each $\overline{\varphi}_{ST} \in \mathcal{B}^{\mathbf{p}}$ determines uniquely the pair $\mathfrak{s},\mathfrak{t}$ of standard tableau of shape λ by (5.1.2). We denote $\overline{\varphi}_{ST}$ by $m^{\mathbf{p}}_{\mathfrak{s}\mathfrak{t}}$ if S,T correspond to $\mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\lambda)$. Thus we see that

$$\mathcal{B}^{\mathbf{p}} = \{m^{\mathbf{p}}_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\lambda) \text{ for some } \lambda \in \Lambda^+\}. \tag{5.2.2}$$

Note that $\overline{\mathcal{S}}^{\mathbf{p}}$ has a structure of the cellular algebra with the cellular basis $\overline{\mathcal{C}}^{\mathbf{p}}$. Since the involution $*$ on $\overline{\mathcal{S}}^{\mathbf{p}}$ stabilizes the set $\mathcal{B}^{\mathbf{p}}$, we see that

(5.2.3) $\overline{\mathcal{H}}^{\mathbf{p}}$ is a cellular algebra with the cellular basis $\mathcal{B}^{\mathbf{p}}$.

More generally, we consider for each $\mu \in \Lambda$ an R-submodule $\bar{\varphi}_\mu\overline{\mathcal{S}}^{\mathbf{p}}\bar{\varphi}_\Omega$ of $\overline{\mathcal{S}}^{\mathbf{p}}$. Then $\bar{\varphi}_\mu\overline{\mathcal{S}}^{\mathbf{p}}\bar{\varphi}_\Omega$ has an R-basis

$$\{\overline{\varphi}_{ST} \mid S \in \mathcal{T}_0^{\mathbf{p}}(\lambda,\mu), T \in \mathcal{T}_0^{\mathbf{p}}(\lambda,\omega) \text{ for } \omega \in \Omega, \lambda \in \Lambda^+\}.$$

Let $\overline{M}^\Omega = \bigoplus_{\omega\in\Omega}\overline{M}^\omega$, and put $\overline{m}_\Omega = \sum_{\omega\in\Omega}\overline{m}_\omega T_{d(\omega)} \in \overline{M}^\Omega$. Then for $S \in \mathcal{T}_0^{\mathbf{p}}(\lambda,\mu), T \in \mathcal{T}_0^{\mathbf{p}}(\lambda,\omega)$, we have

$$\overline{\varphi}_{ST}(\overline{m}_\Omega) = \overline{\varphi}_{ST}(\overline{m}_\omega T_{d(\omega)}) = \overline{m}_{ST}T_{d(\omega)} = \overline{m}_{S\mathfrak{t}}$$

by (5.1.3), where $\mathfrak{t} \in \mathrm{Std}(\lambda)$ corresponds to T via (5.1.2). Since $\{\overline{m}_{S\mathfrak{t}} \mid S \in \mathcal{T}_0^{\mathbf{p}}(\lambda,\mu), \mathfrak{t} \in \mathrm{Std}(\lambda)\}$ gives a basis of $\overline{M}^\mu$, we see that the map $\varphi \mapsto \varphi(\overline{m}_\Omega)$ gives an isomorphism of R-modules

$$\bar{\varphi}_\mu\overline{\mathcal{S}}^{\mathbf{p}}\bar{\varphi}_\Omega \simeq \overline{M}^\mu, \quad \overline{\varphi}_{ST} \leftrightarrow \overline{m}_{S\mathfrak{t}}. \tag{5.2.4}$$

Since $\bar{\varphi}_\Omega\overline{\mathcal{S}}^{\mathbf{p}}\bar{\varphi}_\Omega = \overline{\mathcal{H}}^{\mathbf{p}}$ acts naturally on $\bar{\varphi}_\mu\overline{\mathcal{S}}^{\mathbf{p}}\bar{\varphi}_\Omega$ from the right, one can define a right action of $\overline{\mathcal{H}}^{\mathbf{p}}$ on $\overline{M}^\mu$ through (5.2.4). Let $\mu,\nu \in \Lambda$ be such that $\alpha_{\mathbf{p}}(\mu) = \alpha_{\mathbf{p}}(\nu)$. By 4.9, we know that $\varphi \in \overline{H}_{\mu\nu}$ gives a map $\overline{\theta}(\varphi)$ from $\overline{M}^\nu$ to $\overline{M}^\mu$. It is clear by definition, that $\bar{\theta}(\varphi)$ commutes with the action of $\overline{\mathcal{H}}^{\mathbf{p}}$. Hence we have an R-linear map $\theta' : \overline{H}_{\mu\nu} \to \mathrm{Hom}_{\overline{\mathcal{H}}^{\mathbf{p}}}(\overline{M}^\nu,\overline{M}^\mu)$, which induces an R-algebra homomorphism $\theta' : \overline{\mathcal{S}}^{\mathbf{p}} \to \mathrm{End}_{\overline{\mathcal{H}}^{\mathbf{p}}}(\overline{M})$, where $\overline{M} = \bigoplus_{\mu\in\Lambda}\overline{M}^\mu$.

The following result is a generalization of Proposition 7.5 in [SawS].

Proposition 5.3. *For each $\alpha = (n_1, \dots, n_g) \in \Delta_{n,g}$, put $n_\alpha = n!/n_1! \cdots n_g!$. Then we have an isomorphism of R-algebras*

$$\overline{\mathcal{H}}^{\mathbf{p}} \simeq \bigoplus_{\alpha \in \Delta_{n,g}} M_{n_\alpha}(\mathcal{H}_{n_1,r_1} \otimes \cdots \otimes \mathcal{H}_{n_g,r_g}).$$

Proof. By (4.14.1), one can write

$$\overline{\mathcal{H}}^{\mathbf{p}} = \bar{\varphi}_\Omega \overline{\mathcal{S}}^{\mathbf{p}} \bar{\varphi}_\Omega = \bigoplus_{\alpha \in \Lambda_{n,g}} \bar{\varphi}_{\Omega,\alpha} \overline{\mathcal{S}}^{\mathbf{p}}_\alpha \bar{\varphi}_{\Omega,\alpha}.$$

Here $\bar{\varphi}_{\Omega,\alpha} = \sum_\omega \bar{\varphi}_\omega$ is an idempotent of $\overline{\mathcal{S}}^{\mathbf{p}}_\alpha$, where the sum is taken over all $\omega \in \Omega$ such that $\alpha_{\mathbf{p}}(\omega) = \alpha$. We define a subalgebra $\overline{\mathcal{H}}^{\mathbf{p}}_\alpha$ of $\overline{\mathcal{H}}^{\mathbf{p}}$ by $\overline{\mathcal{H}}^{\mathbf{p}}_\alpha = \bar{\varphi}_{\Omega,\alpha} \overline{\mathcal{S}}^{\mathbf{p}}_\alpha \bar{\varphi}_{\Omega,\alpha}$. Put $\overline{M}_0^{\Omega,\alpha} = \overline{M}^\Omega \cap \overline{M}_0^\alpha$. Then by (4.14.2) we have

$$\overline{\mathcal{H}}^{\mathbf{p}}_\alpha \simeq \mathrm{End}_{\widetilde{\mathcal{H}}_\alpha}(\overline{M}_0^{\Omega,\alpha}) = \bigoplus_{\substack{\omega,\omega' \in \Omega \\ \alpha_{\mathbf{p}}(\omega) = \alpha_{\mathbf{p}}(\omega') = \alpha}} \mathrm{Hom}_{\widetilde{\mathcal{H}}_\alpha}(\overline{M}_0^\omega, \overline{M}_0^{\omega'}). \tag{5.3.1}$$

Now the $\widetilde{\mathcal{H}}_\alpha$-module $\overline{M}_0^\omega$ is isomorphic to the $\mathcal{H}_\alpha$-module $M^{\omega^{[1]}} \otimes \cdots \otimes M^{\omega^{[g]}}$ by Corollary 4.8. In our case $M^{\omega^{[k]}} = \mathcal{H}_{n_k,r_k}$ (see 5.1). Hence for any $\omega, \omega' \in \Omega$ such that $\alpha_{\mathbf{p}}(\omega) = \alpha_{\mathbf{p}}(\omega') = \alpha$, we have

$$\begin{aligned} \mathrm{Hom}_{\widetilde{\mathcal{H}}_\alpha}(\overline{M}_0^\omega, \overline{M}_0^{\omega'}) &\simeq \mathrm{End}_{\mathcal{H}_\alpha}(\mathcal{H}_{n_1,r_1} \otimes \cdots \otimes \mathcal{H}_{n_g,r_g}) \\ &\simeq \mathcal{H}_{n_1,r_1} \otimes \cdots \otimes \mathcal{H}_{n_g,r_g}. \end{aligned} \tag{5.3.2}$$

The proposition follows from this by noticing that $\sharp\{\omega \in \Omega \mid \alpha_{\mathbf{p}}(\omega) = \alpha\} = n_\alpha$. $\square$

5.4. By $\bar{\theta}$, $\overline{\mathcal{S}}^{\mathbf{p}}$ acts on $\overline{M}$ from the left, and which commutes with the right action of $\mathcal{H}$. Hence we have a homomorphism $\rho : \mathcal{H} \to \mathrm{End}^0_{\mathcal{S}^{\mathbf{p}}} \overline{M}$ (see Notation). Since $\sum_{\mu \in \Lambda} \bar{\varphi}_\mu = \mathrm{Id}_{\overline{M}}$, we have $\overline{\mathcal{S}}^{\mathbf{p}} \bar{\varphi}_\Omega \simeq \overline{M}$ by (5.2.4). This implies a natural isomorphism of R-algebras

$$\mathrm{End}^0_{\mathcal{S}^{\mathbf{p}}} \overline{M} \simeq \mathrm{End}^0_{\mathcal{S}^{\mathbf{p}}}(\overline{\mathcal{S}}^{\mathbf{p}} \bar{\varphi}_\Omega) \simeq \bar{\varphi}_\Omega \overline{\mathcal{S}}^{\mathbf{p}} \bar{\varphi}_\Omega = \overline{\mathcal{H}}^{\mathbf{p}}, \tag{5.4.1}$$

where the second isomorphism is given by $f \mapsto f(\bar{\varphi}_\Omega)$ for $f \in \mathrm{End}_{\overline{\mathcal{S}}^{\mathbf{p}}}(\overline{\mathcal{S}}^{\mathbf{p}} \bar{\varphi}_\Omega)$. It follows that we have a homomorphism $\rho_0 : \mathcal{H} \to \overline{\mathcal{H}}^{\mathbf{p}}$ of R-algebras thorough $\mathcal{H} \to \mathrm{End}^0_{\mathcal{S}^{\mathbf{p}}} \overline{M}$. The homomorphism ρ_0 is explicitly given as follows: we have $\overline{\mathcal{H}}^{\mathbf{p}} = \bar{\varphi}_\Omega \overline{\mathcal{S}}^{\mathbf{p}} \bar{\varphi}_\Omega \simeq \overline{M}^\Omega$ via $\varphi \mapsto \varphi(\overline{m}_\Omega)$. Then for each $h \in \mathcal{H}$, there exists a unique $\varphi_h \in \overline{\mathcal{H}}^{\mathbf{p}}$ such that $\varphi_h(\overline{m}_\Omega) = \overline{m}_\Omega h \in \overline{M}^\Omega$. The map $h \mapsto \varphi_h$ gives ρ_0.

Now $\overline{\mathcal{H}}^{\mathbf{p}}$-module $\overline{M}$ is regarded as an $\mathcal{H}$-module via ρ_0, which coincides with the original $\mathcal{H}$-module $\overline{M}$. It follows that we have an injection

$$\mathrm{Hom}_{\overline{\mathcal{H}}^{\mathbf{p}}}(\overline{M}^\nu, \overline{M}^\mu) \hookrightarrow \mathrm{Hom}_{\mathcal{H}}(\overline{M}^\nu, \overline{M}^\mu),$$

and $\bar{\theta}$ factors through θ' via this injection. Since $\bar{\theta}$ is injective by Remark 4.13, we see that

(5.4.2) The map $\theta' : \overline{H}_{\mu\nu} \to \mathrm{Hom}_{\overline{\mathcal{H}}^{\mathbf{p}}}(\overline{M}^\nu, \overline{M}^\mu)$ is injective.

Since $\overline{M}^\mu$ is generated by $\overline{m}_\mu$ as an $\mathcal{H}$-module, it is generated by $\overline{m}_\mu$ as an $\overline{\mathcal{H}}^{\mathbf{p}}$-module, i.e., we have $\overline{M}^\mu = \overline{m}_\mu \overline{\mathcal{H}}^{\mathbf{p}}$. The following lemma is also clear from the fact that $\overline{\mathcal{H}}^{\mathbf{p}} \simeq \overline{M}^\Omega$ via $\varphi \mapsto \varphi(\overline{m}_\Omega)$ as noticed above.

Lemma 5.5. *We have $\overline{M}^{\Omega} = \overline{m}_{\Omega}\overline{\mathcal{H}}^{\mathbf{p}}$. The map $h \mapsto \overline{m}_{\Omega}h$ gives an isomorphism of R-modules $\overline{\mathcal{H}}^{\mathbf{p}} \to \overline{M}^{\Omega}$, namely $\overline{M}^{\Omega}$ is the regular representation of $\overline{\mathcal{H}}^{\mathbf{p}}$.*

6. Presentation for $\overline{\mathcal{H}}^{\mathbf{p}}$

6.1 We shall define several elements in $\overline{\mathcal{H}}^{\mathbf{p}}$, and show that they generate $\overline{\mathcal{H}}^{\mathbf{p}}$. For each $\omega \in \Omega$ let $I = \coprod I_k$ be the corresponding partition of I. Define a map $b_\omega : I \to \mathbb{Z}_{>0}$ by $b_\omega(i) = k$ if $i \in I_k$. We put $Q_k^{\mathbf{p}} = Q_{p_k+r_k}$ for $k = 1, \dots, g$. Under this notation, we define elements $\xi_i \in \overline{\mathcal{S}}^{\mathbf{p}}$, for $i = 1, \dots, n$, by

$$\xi_i = \sum_{\omega\in\Omega} Q^{\mathbf{p}}_{b_\omega(i)}\bar{\varphi}_\omega. \tag{6.1.1}$$

Clearly, $\bar{\varphi}_\Omega\xi_i\bar{\varphi}_\Omega = \xi_i$, and so $\xi_1, \dots, \xi_n$ are elements in $\overline{\mathcal{H}}^{\mathbf{p}}$. They commute each other. Moreover, they satisfy the relation

$$(\xi_j - Q_1^{\mathbf{p}})(\xi_j - Q_2^{\mathbf{p}}) \cdots (\xi_j - Q_g^{\mathbf{p}}) = 0 \tag{6.1.2}$$

for $j = 1, \dots, n$.

Under the isomorphism in (5.2.4), the action of ξ_i on the basis element $\overline{m}_{S\mathfrak{t}}$ in $\overline{M}^{\mu}$ is given as follows:

$$\overline{m}_{S\mathfrak{t}}\xi_i = Q^{\mathbf{p}}_{b_\omega(i)}\overline{m}_{S\mathfrak{t}} \quad \text{if } \mathfrak{t} \in \mathrm{Std}(\lambda)_\omega, \tag{6.1.3}$$

where $\mathrm{Std}(\lambda)_\omega$ is as in 5.1. Note that in this case $b_\omega(i)$ coincides with k such that the letter i is contained in $\mathfrak{t}^{[k]}$. By [DJM, Proposition 3.18], $\overline{m}_\mu$ is written, for $\mu \in \Lambda$, as a linear combination of $\overline{m}_{\mathfrak{s}\mathfrak{t}}$ such that the letters contained in the k component of $\mathfrak{t}$ is the same as that of $\mathfrak{t}^\mu$. It follows from this, by making use of (6.1.3), that

$$\overline{m}_\mu\xi_i = Q^{\mathbf{p}}_{b(i)}\overline{m}_\mu, \tag{6.1.4}$$

where $b(i) = k$ if $a_k + 1 \le i \le a_k + n_k$ under the notation $\mathbf{a}_{\mathbf{p}}(\mu) = (a_1, \dots, a_g)$ and $\alpha_{\mathbf{p}}(\mu) = (n_1, \dots, n_g)$.

Let $\rho_0 : \mathcal{H} \to \overline{\mathcal{H}}^{\mathbf{p}}$ be the homomorphism defined in 5.4. We note that

(6.1.5) The restriction of ρ_0 on $\mathcal{H}_n$ is injective.

In fact, it is enough to show that $\rho_0(T_w)$ ($w \in \mathfrak{S}_n$) are linearly independent as operators on $\overline{M}$. Now $\overline{M} = \bigoplus_{\alpha\in\Delta_{n,g}} \overline{M}^{\alpha}$, and T_w preserves the subspaces $\overline{M}^{\alpha}$. We choose α such that $\alpha = (n, 0, \dots, 0)$. Then $\mathcal{H}_n$ is contained in $\widetilde{\mathcal{H}}_\alpha = \mathcal{H}$, and $\rho_0(T_w)$ induces an operator on $\overline{M}_0^{\alpha}$. By our choice of α, Corollary 4.8 implies that $\overline{M}_0^{\alpha}$ can be identified with M', the $\mathcal{H}_{n,r_1}$-module corresponding to M for $\mathcal{H}$, and the action of $\widetilde{\mathcal{H}}_\alpha$ on $\overline{M}_0^{\alpha}$ coincides with the action of $\mathcal{H}_{n,r_1}$ on M'. In particular, the action of $\rho_0(T_w)$ on $\overline{M}_0^{\alpha}$ corresponds to the action of T_w on M' (we regard $T_w \in \mathcal{H}_n \subset \mathcal{H}_{n,r_1}$). Since T_w ($w \in \mathfrak{S}_n$) are linearly independent as operators on M', we see that $\rho_0(T_w)$ are linearly independent as asserted.

By (6.1.5), we regard $\mathcal{H}_n$ as a subalgebra of $\overline{\mathcal{H}}^{\mathbf{p}}$, and define the elements $T_1, \dots, T_{n-1} \in \overline{\mathcal{H}}^{\mathbf{p}}$ by the generators of $\mathcal{H}_n$.

6.2. We shall determine the commutation relations between T_j and ξ_k. In view of Lemma 5.5, we compare the elements $\overline{m}_\Omega T_j\xi_k$ and $\overline{m}_\Omega\xi_k T_j$. First we compute the element $\overline{m}_\Omega T_j$ for $T_j \in \mathcal{H}_n$. Since $\overline{m}_\Omega T_j = \sum_{\omega\in\Omega} \overline{m}_\omega T_{d(\omega)}T_j$, we compute

$m_\omega T_{d(\omega)} T_j$. Let $I = \coprod I_k$ be the partition corresponding to ω. Assume that $j \in I_k$ and $j+1 \in I_{k'}$. Then we see that

$$T_{d(\omega)} T_j = \begin{cases} T_{d(\omega)s_j} & \text{if } k \le k', \\ T_{d(\omega)s_j} + (q - q^{-1}) T_{d(\omega)} & \text{if } k > k', \end{cases}$$

where s_j is the element in $\mathfrak{S}_n$ corresponding to T_j. Note that $m_\omega = u_{\mathbf{a}}^+ = m_\lambda$, where λ is the multi-partition obtained from ω by rearranging the rows. Put $\mathfrak{t}_\omega = \mathfrak{t}^\lambda d(\omega) \in \mathrm{Std}(\lambda)$. Put $\mathfrak{v}_\omega = \mathfrak{t}_\omega s_j$. If $k \ne k'$, then $\mathfrak{v}_\omega \in \mathrm{Std}(\lambda)$ and it is expressed as $\mathfrak{t}_{\omega'}$, where $\omega' \in \Omega$ is obtained from ω by exchanging j and $j+1$ in I_k and $I_{k'}$. One can write $m_\omega T_{d(\omega)} = m_{S_\omega \mathfrak{t}_\omega}$ and $m_\omega T_{d(\omega)s_j} = m_{S_\omega \mathfrak{v}_\omega}$, where $S_\omega = \omega(\mathfrak{t}^\lambda) \in \mathcal{T}_0^{\mathbf{P}}(\lambda, \omega)$. Hence we have

$$\overline{m}_\omega T_{d(\omega)} T_j = \begin{cases} \overline{m}_{S_\omega \mathfrak{v}_\omega} & \text{if } k = k', \\ \overline{m}_{S_\omega \mathfrak{t}_{\omega'}} & \text{if } k < k', \\ \overline{m}_{S_\omega \mathfrak{t}_{\omega'}} + (q - q^{-1}) \overline{m}_{S_\omega \mathfrak{t}_\omega} & \text{if } k > k'. \end{cases} \tag{6.2.1}$$

Note that in the first case, by [DJM, Proposition 3.18], $\overline{m}_{S_\omega \mathfrak{v}_\omega}$ is expressed as a linear combination of basis elements $\overline{m}_{S'\mathfrak{v}}$ such that $\omega_{\mathfrak{v}} = \omega$. It follows from (6.2.1) that

$$\begin{aligned} \overline{m}_\Omega T_j = & \sum_{\substack{\omega \in \Omega \\ b_\omega(j) < b_\omega(j+1)}} \overline{m}_{S_\omega \mathfrak{t}_{\omega'}} \\ & + \sum_{\substack{\omega \in \Omega \\ b_\omega(j) = b_\omega(j+1)}} \overline{m}_{S_\omega \mathfrak{v}_\omega} + \sum_{\substack{\omega \in \Omega \\ b_\omega(j) > b_\omega(j+1)}} (\overline{m}_{S_\omega \mathfrak{t}_{\omega'}} + (q - q^{-1}) \overline{m}_{S_\omega \mathfrak{t}_\omega}), \end{aligned}$$

where $\omega' \in \Omega$ is obtained from ω by s_j as above, and $\mathfrak{v}_\omega = \mathfrak{t}_\omega s_j$. Thus by (6.1.3) and (6.1.4), we have

$$\begin{aligned} \overline{m}_\Omega T_j \xi_k = & \sum_{\substack{\omega \in \Omega \\ b_\omega(j) \ne b_\omega(j+1)}} Q^{\mathbf{P}}_{b_{\omega'}(k)} \overline{m}_{S_\omega \mathfrak{t}_{\omega'}} \\ & + \sum_{\substack{\omega \in \Omega \\ b_\omega(j) = b_\omega(j+1)}} Q^{\mathbf{P}}_{b_\omega(k)} \overline{m}_{S_\omega \mathfrak{v}_\omega} + \sum_{\substack{\omega \in \Omega \\ b_\omega(j) > b_\omega(j+1)}} Q^{\mathbf{P}}_{b_\omega(k)} (q - q^{-1}) \overline{m}_{S_\omega \mathfrak{t}_\omega}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \overline{m}_\Omega \xi_k T_j = & \sum_{\substack{\omega \in \Omega \\ b_\omega(j) \ne b_\omega(j+1)}} Q^{\mathbf{P}}_{b_\omega(k)} \overline{m}_{S_\omega \mathfrak{t}_{\omega'}} \\ & + \sum_{\substack{\omega \in \Omega \\ b_\omega(j) = b_\omega(j+1)}} Q^{\mathbf{P}}_{b_\omega(k)} \overline{m}_{S_\omega \mathfrak{v}_\omega} + \sum_{\substack{\omega \in \Omega \\ b_\omega(j) > b_\omega(j+1)}} Q^{\mathbf{P}}_{b_\omega(k)} (q - q^{-1}) \overline{m}_{S_\omega \mathfrak{t}_\omega}. \end{aligned}$$

It follows that

$$\overline{m}_\Omega (T_j \xi_k - \xi_k T_j) = \sum_{\substack{\omega \in \Omega \\ b_\omega(j) \ne b_\omega(j+1)}} (Q^{\mathbf{P}}_{b_{\omega'}(k)} - Q^{\mathbf{P}}_{b_\omega(k)}) \overline{m}_{S_\omega \mathfrak{t}_{\omega'}}. \tag{6.2.2}$$

Note that if $k \ne j, j+1$, then $b_\omega(k) = b_{\omega'}(k)$ for any ω. It follows that

$$T_j \xi_k = \xi_k T_j \text{ if } k \ne j, j+1. \tag{6.2.3}$$

6.3. Let A be a square matrix of degree g whose ij-entry is given by $(Q_j^{\mathbf{p}})^{i-1}$ for $1 \le i, j \le g$. Thus A is the Vandermonde matrix, and $\Delta = \det A = \prod_{i>j}(Q_i^{\mathbf{p}} - Q_j^{\mathbf{p}})$. We pose the following assumption so that $\Delta^{-1} \in R$.

$$Q_i^{\mathbf{p}} - Q_j^{\mathbf{p}} \text{ are units in } R \text{ for any } i \neq j. \tag{6.3.1}$$

We express $A^{-1} = \Delta^{-1}B$ with $B = (h_{ij})$ for $h_{ij} \in R$. We define a polynomial $F_i(X) \in R[X]$, for $1 \le i \le g$, by

$$F_i(X) = \sum_{j=1}^{g} h_{ij} X^{j-1}.$$

We denote by $\Omega_j^{[c]}$ the set of $\omega \in \Omega$ such that $b_\omega(j) = c$ for $1 \le j \le n, 1 \le c \le g$. As in 6.2, one can write $\overline{m}_\Omega = \sum_{\omega\in\Omega} \overline{m}_{S_\omega \mathfrak{t}_\omega}$, and so

$$\overline{m}_\Omega \xi_j^b = \sum_{\omega\in\Omega} (Q_{b_\omega(j)}^{\mathbf{p}})^b \overline{m}_{S_\omega \mathfrak{t}_\omega} = \sum_{c=1}^{g} (Q_c^{\mathbf{p}})^b \sum_{\omega\in\Omega_j^{[c]}} \overline{m}_{S_\omega \mathfrak{t}_\omega} \tag{6.3.2}$$

for $b = 0, \dots, g-1$. We regard (6.3.2) as a system of linear equations with unknown variables $\sum_{\omega\in\Omega_j^{[c]}} \overline{m}_{S_\omega \mathfrak{t}_\omega}$. Since $\Delta^{-1} \in R$, we see that

$$\sum_{\omega\in\Omega_j^{[c]}} \overline{m}_{S_\omega \mathfrak{t}_\omega} = \overline{m}_\Omega \cdot \Delta^{-1} \sum_{b=1}^{g} h_{cb} \xi_j^{b-1} = \overline{m}_\Omega \cdot \Delta^{-1} F_c(\xi_j).$$

Repeating a similar procedure, we have

$$\sum_{\omega\in\Omega_j^{[c_1]}\cap\Omega_{j+1}^{[c_2]}} \overline{m}_{S_\omega \mathfrak{t}_\omega} = \overline{m}_\Omega \cdot \Delta^{-2} F_{c_1}(\xi_j) F_{c_2}(\xi_{j+1}). \tag{6.3.3}$$

By applying T_j on both sides of (6.3.3), and by using (6.2.1), we have

$$\sum_{\omega\in\Omega_j^{[c_1]}\cap\Omega_{j+1}^{[c_2]}} \overline{m}_{S_\omega \mathfrak{t}_{\omega'}} = \begin{cases} \overline{m}_\Omega \cdot \Delta^{-2} F_{c_1}(\xi_j) F_{c_2}(\xi_{j+1}) T_j & \text{if } c_1 < c_2, \\ \overline{m}_\Omega \cdot \Delta^{-2} F_{c_1}(\xi_j) F_{c_2}(\xi_{j+1})(T_j - (q - q^{-1})) & \text{if } c_1 > c_2. \end{cases} \tag{6.3.4}$$

We show the following lemma, which is analogous to [Sh, Lemma 3.4].

Lemma 6.4. *For $j = 1, \dots, n-1$, we have*

$$\begin{aligned} T_j \xi_{j+1} &= \xi_j T_j + \Delta^{-2} \sum_{c_1>c_2} (Q_{c_2}^{\mathbf{p}} - Q_{c_1}^{\mathbf{p}})(q - q^{-1}) F_{c_1}(\xi_j) F_{c_2}(\xi_{j+1}), \\ T_j \xi_j &= \xi_{j+1} T_j - \Delta^{-2} \sum_{c_1>c_2} (Q_{c_2}^{\mathbf{p}} - Q_{c_1}^{\mathbf{p}})(q - q^{-1}) F_{c_1}(\xi_j) F_{c_2}(\xi_{j+1}), \\ T_j \xi_k &= \xi_k T_j \quad (k \neq j, j+1). \end{aligned}$$

Proof. The third formula is already shown in (6.2.3). So assume that $k = j$ or $j+1$. Substituting (6.3.4) into (6.2.2), and by using Lemma 5.5, we have

$$\begin{aligned} T_j \xi_k - \xi_k T_j = \varepsilon \Delta^{-2} \Big\{ &\sum_{c_1<c_2} (Q_{c_2}^{\mathbf{p}} - Q_{c_1}^{\mathbf{p}}) F_{c_1}(\xi_j) F_{c_2}(\xi_{j+1}) T_j \\ &+ \sum_{c_1>c_2} (Q_{c_2}^{\mathbf{p}} - Q_{c_1}^{\mathbf{p}}) F_{c_1}(\xi_j) F_{c_2}(\xi_{j+1})(T_j - (q - q^{-1})) \Big\}, \end{aligned} \tag{6.4.1}$$

where $\varepsilon = 1$ (resp. $\varepsilon = -1$) if $k = j$ (resp. $k = j+1$).

We note that the following formula holds.

$$\xi_{j+1} - \xi_j = \Delta^{-2} \sum_{c_1 < c_2} (Q^{\mathbf{p}}_{c_2} - Q^{\mathbf{p}}_{c_1}) \Big\{ F_{c_1}(\xi_j) F_{c_2}(\xi_{j+1}) - F_{c_2}(\xi_j) F_{c_1}(\xi_{j+1}) \Big\}. \tag{6.4.2}$$

In fact, it is enough to compare the values at $\overline{m}_{S_\omega \mathfrak{t}_\omega} \in \overline{M}^\Omega$. This is essentially the same as the case where $\mathbf{p} = (1^r)$, and in that case the formula is proved in [Sh, (3.4.2)].

Now (6.4.1) can be written, by making use of (6.4.2), as

$$\begin{aligned} T_j \xi_k - \xi_k T_j &= \varepsilon(\xi_{j+1} - \xi_j) T_j \\ &\quad - \varepsilon \sum_{c_1 > c2} (Q^{\mathbf{p}}_{c_2} - Q^{\mathbf{p}}_{c_1})(q - q^{-1}) F_{c_1}(\xi_j) F_{c_2}(\xi_{j+1}). \end{aligned}$$

The first and the second equalities in the lemma follow from this. □

6.5. For each $\alpha \in \Delta_{n,g}$ and for $k = 1, \dots, g$, we define $T^{[k]}_{\alpha,0} \in \overline{\mathcal{H}}^{\mathbf{p}}$ as follows. We regard $T^{[k]}_0 \in \mathcal{H}_{n_k, r_k}$ as an element in $\mathcal{H}_{n_1,r_1} \otimes \cdots \otimes \mathcal{H}_{n_g,r_g}$, and we denote by $T^{[k]}_{\alpha,0} \in \overline{\mathcal{H}}^{\mathbf{p}}_\alpha$ the diagonal matrix consisting of $T^{[k]}_0$ in the diagonal entries under the isomorphism in Proposition 5.3. In particular, one can write

$$\overline{m}_\Omega T^{[k]}_{\alpha,0} = \sum_{\omega \in \Omega^\alpha} \overline{m}_\omega T_{d(\omega)} L_{a_k+1},$$

where $\Omega^\alpha = \{\omega \in \Omega \mid \alpha_{\mathbf{p}}(\omega) = \alpha\}$. Thus we see that $T^{[k]}_{\alpha,0}$ acts on $\overline{M}^{\Omega,\alpha} = \overline{M}^\Omega \cap \overline{M}^\alpha$ as L_{a_k+1}, and annihilates $\overline{M}^{\Omega,\alpha'}$ for any $\alpha' \neq \alpha$.

For a given $\omega \in \Omega$, put $c_i = b_\omega(i)$ for $i = 1, \dots, n$. We define $F_\omega(\xi) \in \overline{\mathcal{H}}^{\mathbf{p}}$ by

$$F_\omega(\xi) = F_{c_1}(\xi_1) F_{c_2}(\xi_2) \cdots F_{c_n}(\xi_n). \tag{6.5.1}$$

We have the following lemma.

Lemma 6.6. *Under the assumption of (6.3.1), the elements*

$$\xi_i \ (1 \le i \le n), \quad T_j \ (1 \le j \le n-1), \quad T^{[k]}_{\alpha,0} \ (\alpha \in \Delta_{n,g}, 1 \le k \le g)$$

generate $\overline{\mathcal{H}}^{\mathbf{p}}$.

Proof. Let $\mathcal{K}$ be the subalgebra of $\overline{\mathcal{H}}^{\mathbf{p}}$ generated by elements in the lemma. In view of Lemma 5.5, it is enough to show that $\overline{M}^\Omega = \overline{m}_\Omega \mathcal{K}$. First we show that

$$\overline{m}_\omega \in \overline{m}_\Omega \mathcal{K} \tag{6.6.1}$$

for any $\omega \in \Omega$. In fact, we have $\bigcap_{i=1}^n \Omega_i^{[c_i]} = \{\omega\}$ with $c_i = b_{\omega(i)}$. Hence by repeating the argument used to prove (6.3.3), we see that

$$\overline{m}_\omega T_{d(\omega)} = \overline{m}_{S_\omega \mathfrak{t}_w} = \overline{m}_\Omega \cdot \Delta^{-n} F_\omega(\xi). \tag{6.6.2}$$

This implies that $\overline{m}_\omega T_{d(\omega)} \in \overline{m}_\Omega \mathcal{K}$. Since $T_{d(\omega)}$ is an invertible element in $\mathcal{K}$, we obtain (6.6.1).

Now take $\overline{m}_\omega$ and put $\alpha = \alpha_{\mathbf{p}}(\omega)$. We know that $\overline{m}_\omega \in \overline{M}_0^\omega$, and that $\overline{M}_0^\omega = \overline{m}_\omega \widetilde{\mathcal{H}}_\alpha$ (see the proof of Lemma 4.10). Note that $\widetilde{H}_\alpha$ is generated by L_{a_k+1} and $\widetilde{H}_\alpha \cap \mathcal{H}_n$, and the action of L_{a_k+1} on $\overline{M}_0^\alpha$ coincides with that of $T_{\alpha,0}^{[k]}$. It follows that $\overline{M}_0^\omega = \overline{m}_\omega \widetilde{\mathcal{H}}_\alpha \subset \overline{m}_\Omega \mathcal{K}$. Here $\overline{M}_0^\omega$ has the basis $\{\overline{m}_{S\mathfrak{t}}\}$ with $S \in \mathcal{T}_0^{\mathbf{p}}(\lambda, \omega)$ and $\mathfrak{t} \in \mathrm{Std}(\lambda)_0$. While the basis of $\overline{M}^\omega$ is given by $\{\overline{m}_{S\mathfrak{t}'}\}$ for $S \in \mathcal{T}^{\mathbf{p}}(\lambda, \omega)$ and $\mathfrak{t}' \in \mathrm{Std}(\lambda)$. If we take $\mathfrak{t} = \mathfrak{t}^\lambda \in \mathrm{Std}(\lambda)_0$, any $\mathfrak{t}'$ is obtained as $\mathfrak{t}' = \mathfrak{t}d(\mathfrak{t}')$, and we have $\overline{m}_{S\mathfrak{t}'} = \overline{m}_{S\mathfrak{t}} T_{d(\mathfrak{t}')}$. It follows that $\overline{M}^\omega \subseteq \overline{M}_0^\omega \mathcal{H}_n \subseteq \overline{m}_\Omega \mathcal{K}$ for any $\omega \in \Omega$, and so $\overline{M}^\Omega = \overline{m}_\Omega \mathcal{K}$. The lemma is proved. □

6.7. Recall that $\overline{\mathcal{H}}^{\mathbf{p}} = \bigoplus_{\alpha \in \Delta_{n,g}} \overline{\mathcal{H}}_\alpha^{\mathbf{p}}$. For each $\alpha \in \Delta_{n,g}$, we denote by $T_{\alpha,j}$ the projection of T_j onto $\overline{\mathcal{H}}_\alpha^{\mathbf{p}}$. Also we denote by $\xi_{\alpha,i}$ the projection of ξ_i onto $\overline{\mathcal{H}}_\alpha^{\mathbf{p}}$. Hence we have $T_j = \sum_\alpha T_{\alpha,j}$ and $\xi_i = \sum_\alpha \xi_{\alpha,i}$. It follows from the construction that under the isomorphism $\overline{M}_0^\omega \simeq M^{\omega^{[1]}} \otimes \cdots \otimes M^{\omega^{[g]}}$, the action of T_{α,a_k+i} corresponds to the action of $T_i^{[k]}$ on $M^{\omega^{[k]}}$.

We note the relation

$$\xi_{\alpha,i} T_{\alpha,0}^{[k]} = T_{\alpha,0}^{[k]} \xi_{\alpha,i} \tag{6.7.1}$$

for any i and any k. In fact, by (5.3.1), it is enough to show the formula regarding $\xi_{\alpha,i}$ and $T_{\alpha,0}^{[k]}$ as operators on $\overline{M}_0^{\Omega,\alpha}$. Under the isomorphism $\overline{M}_0^\omega \simeq M^{\omega^{[1]}} \otimes \cdots \otimes M^{\omega^{[g]}}$ for $\omega \in \Omega$ such that $\alpha_{\mathbf{p}}(\omega) = \alpha$, ξ_{α,a_h+i} corresponds to the operator $\xi_i^{[h]}$ on $M^{\omega^{[h]}}$, where $\xi_i^{[h]}$ is an element of $\mathcal{H}_{n_h,r_h}$ defined similar to ξ_i for $\overline{\mathcal{H}}^{\mathbf{p}}$ (i.e., the special case where $n = n_h, r = r_h, g = 1, \mathbf{p} = (r_h)$). But it is easy to see that in this case $\xi_i^{[h]}$ is a scalar multiplication on $M^{\omega^{[h]}}$ by $Q_h^{\mathbf{p}}$. Hence $\xi_{\alpha,i}$ is a scalars operator on $\overline{M}^\omega$, and so commutes with $T_{\alpha,0}^{[k]}$. Equation (6.7.1) follows from this.

For each $\omega \in \Omega$ and $\alpha \in \Delta_{n,g}$, let

$$F_\omega(\xi_\alpha) = F_{c_1}(\xi_{\alpha,1}) F_{c_2}(\xi_{\alpha,2}) \cdots F_{c_n}(\xi_{\alpha,n})$$

with $c_i = b_\omega(i)$. We claim that

$$F_\omega(\xi_\alpha) = 0 \quad \text{unless } \alpha_{\mathbf{p}}(\omega) = \alpha. \tag{6.7.2}$$

In fact, we have $\overline{m}_\Omega F_\omega(\xi_\alpha) \in \overline{M}^{\alpha'}$ by (6.6.2), where $\alpha' = \alpha_{\mathbf{p}}(\omega)$. But since $F_\omega(\xi_\alpha) \in \overline{\mathcal{H}}_\alpha^{\mathbf{p}} = \varphi_{\Omega,\alpha} \overline{\mathcal{S}}_\alpha^{\mathbf{p}} \varphi_{\Omega,\alpha}$, we have $\overline{m}_\Omega F_\omega(\xi_\alpha) \in \overline{M}^\alpha$. It follows that $\overline{m}_\Omega F_\omega(\xi_\alpha) = 0$ unless $\alpha_{\mathbf{p}}(\omega) = \alpha$, and the claim follows.

The following theorem gives a presentation of $\overline{\mathcal{H}}^{\mathbf{p}}$.

Theorem 6.8. *Assume that (6.3.1) holds. Recall that $Q_k^{\mathbf{p}} = Q_{p_k+r_k}$. Then for each $\alpha \in \Delta_{n,g}$, the algebra $\overline{\mathcal{H}}_\alpha^{\mathbf{p}}$ is generated by*

$$\xi_{\alpha,i} \ (1 \le i \le n), \quad T_{\alpha,j} \ (1 \le j \le n-1), \quad T_{\alpha,0}^{[k]} \ (1 \le k \le g)$$

with the relations

(A1) $(T_{\alpha,i}-q)(T_{\alpha,i}+q^{-1})=0 \quad (1\le i\le n-1)$,

(A2) $T_{\alpha,i}T_{\alpha,i+1}T_{\alpha,i}=T_{\alpha,i+1}T_{\alpha,i}T_{\alpha,i+1} \quad (1\le i\le n-2)$,

(A3) $T_{\alpha,i}T_{\alpha,j}=T_{\alpha,j}T_{\alpha,i} \quad (1\le i,j\le n-1, |i-j|\ge 2)$,

(A4) $(T^{[k]}_{\alpha,0}-Q_{p_k+1})\cdots(T^{[k]}_{\alpha,0}-Q_{p_k+r_k})=0 \quad (1\le k\le g)$,

(A5) $T^{[k]}_{\alpha,0}T_{\alpha,a_k+1}T^{[k]}_{\alpha,0}T_{\alpha,a_k+1}=T_{\alpha,a_k+1}T^{[k]}_{\alpha,0}T_{\alpha,a_k+1}T^{[k]}_{\alpha,0} \quad (1\le k\le g)$,

(A6) $T^{[k]}_{\alpha,0}=T_{\alpha,a_k}\cdots T_{\alpha,a_{k-1}+2}T_{\alpha,a_{k-1}+1}T^{[k-1]}_{\alpha,0}T_{\alpha,a_{k-1}+1}T_{\alpha,a_{k-1}+2}\cdots T_{\alpha,a_k}$,

(A7) $T^{[k]}_{\alpha,0}T_{\alpha,j}=T_{\alpha,j}T^{[k]}_{\alpha,0} \quad (j\ne a_k, a_k+1)$,

(A8) $(\xi_{\alpha,i}-Q^{\mathbf{p}}_1)(\xi_{\alpha,i}-Q^{\mathbf{p}}_2)\cdots(\xi_{\alpha,i}-Q^{\mathbf{p}}_g)=0 \quad (1\le i\le n)$,

(A9) $\xi_{\alpha,i}\xi_{\alpha,j}=\xi_{\alpha,j}\xi_{\alpha,i} \quad (1\le i,j\le n)$,

(A10) $F_\omega(\xi_\alpha)=0 \quad$ *if* $\alpha_{\mathbf{p}}(\omega)\ne\alpha$,

(A11)
$$T_{\alpha,j}\xi_{\alpha,j+1}=\xi_{\alpha,j}T_{\alpha,j} + \Delta^{-2}\sum_{c_1<c_2}(Q^{\mathbf{p}}_{c_2}-Q^{\mathbf{p}}_{c_1})(q-q^{-1})F_{c_1}(\xi_{\alpha,j})F_{c_2}(\xi_{\alpha,j+1}),$$

(A12)
$$T_{\alpha,j}\xi_{\alpha,j}=\xi_{\alpha,j+1}T_{\alpha,j} - \Delta^{-2}\sum_{c_1<c_2}(Q^{\mathbf{p}}_{c_2}-Q^{\mathbf{p}}_{c_1})(q-q^{-1})F_{c_1}(\xi_{\alpha,j})F_{c_2}(\xi_{\alpha,j+1}),$$

(A13) $T_{\alpha,j}\xi_{\alpha,k}=\xi_{\alpha,k}T_{\alpha,j} \quad (k\ne j, j+1)$,

(A14) $T^{[k]}_{\alpha,0}\xi_i=\xi_iT^{[k]}_{\alpha,0} \quad (1\le i\le n, 1\le k\le g)$.

Proof. One sees that these elements generate $\overline{\mathcal{H}}^{\mathbf{p}}_\alpha$ by Lemma 6.6. We show that these generators satisfy the relations (A1)–(A14). (A1)–(A3) follow from the relations for $\mathcal{H}_n$. Relation (A8) follows from (6.1.2). Relation (A9) is also clear from 6.1. Relation (A10) follows from (6.7.2). (A11)–(A13) follows from Lemma 6.4. Relation (A14) is given in (6.7.1). We show the remaining relations (A4)–(A7). We may prove the formulas, by regarding $T^{[k]}_{\alpha,0}$ and $T_{\alpha,j}$ as operators on $\overline{M}^\omega_0$ for $\omega\in\Omega$ such that $\alpha_{\mathbf{p}}(\omega)=\alpha$ by (5.3.1). Since $T^{[k]}_{\alpha,0}$ corresponds to the action of $T^{[k]}_0\in\mathcal{H}_{n_k,r_k}$ on $M^{\omega^{[k]}}$, and T_{α,a_k+i} corresponds to the action of $T^{[k]}_i$, (A4), (A5) and (A7) follow from the relations for $\mathcal{H}_{n_k,r_k}$. While (A6) follows from the property that $T^{[k]}_{\alpha,0}$ is the restriction of L_{a_k+1} on $\overline{M}^{\Omega,\alpha}$. Thus those generators satisfy the relations (A1)–(A14).

Next we show that (A1)–(A14) gives a fundamental relation for $\overline{\mathcal{H}}^{\mathbf{p}}_\alpha$. Let $\widehat{\mathcal{H}}_\alpha$ be the algebra with generators $\widehat{\xi}_{\alpha,i}, \widehat{T}_{\alpha,j}$ and $\widehat{T}^{[k]}_{\alpha,0}$, and relations as in the theorem. (We denote by $\widehat{X}$ the generator in $\widehat{\mathcal{H}}_\alpha$ corresponding to the generator X in $\overline{\mathcal{H}}^{\mathbf{p}}_\alpha$.) Let $\widehat{\mathcal{H}}^0_\alpha$ be the subalgebra of $\widehat{\mathcal{H}}_\alpha$ generated by $\widehat{T}^{[k]}_{\alpha,i}$ for $k=1,\dots,g, i=0,\dots,n_k-1$. Recall that $\widehat{T}^{[k]}_{\alpha,i}=\widehat{T}_{\alpha,a_k+i}$ for $1\le i\le n_k-1$. Then by the relations in the theorem, $\widehat{\mathcal{H}}^0_\alpha$ is isomorphic to the quotient of the algebra $\mathcal{H}_{n_1,r_1}\otimes\cdots\otimes\mathcal{H}_{n_g,r_g}$. Also, we note that the subalgebra $\widehat{\mathcal{H}}_n$ of $\widehat{\mathcal{H}}_\alpha$ generated by $\widehat{T}_{\alpha,j}$ is the quotient of $\mathcal{H}_n$. We denote by $\widehat{T}_{\alpha,w}$ the image of $T_w\in\mathcal{H}_n$ to $\widehat{\mathcal{H}}_n$ for $w\in\mathfrak{S}_n$. Let $\mathfrak{S}_\alpha$ be the Young subgroup of $\mathfrak{S}_n$ corresponding to the composition α of n. Let $\widehat{\Xi}_\alpha$ be the subalgebra of $\widehat{\mathcal{H}}_\alpha$

generated by $\widehat{\xi}_{\alpha,1},\dots,\widehat{\xi}_{\alpha,n}$. For each $\omega\in\Omega^\alpha$, we define $F_\omega(\widehat{\xi}_\alpha)\in\widehat{\Xi}_\alpha$ in a similar way as $F_\omega(\xi_\alpha)$, but replacing $\xi_{\alpha,i}$ by $\widehat{\xi}_{\alpha,i}$. We show that

(6.8.1) Any element of $\widehat{\mathcal{H}}_\alpha$ can be written as a linear combination of elements in

$$\mathcal{C}=\{F_\omega(\widehat{\xi}_\alpha)\widehat{\mathcal{H}}^0_\alpha\widehat{T}_{\alpha,w}\mid\omega\in\Omega^\alpha,w\in\mathfrak{S}_\alpha\backslash\mathfrak{S}_n\}.$$

In fact, let $\widehat{\mathcal{H}}^\natural_\alpha$ be the subalgebra of $\widehat{\mathcal{H}}_\alpha$ generated by $\widehat{T}_{\alpha,j}$ and $\widehat{T}^{[k]}_{\alpha,0}$. Then by the commuting relations in the theorem, $\widehat{\mathcal{H}}_\alpha$ can be written as

$$\widehat{\mathcal{H}}_\alpha=\sum_{c_1,\dots,c_n}\widehat{\xi}^{c_1}_{\alpha,1}\widehat{\xi}^{c_2}_{\alpha,2}\cdots\widehat{\xi}^{c_n}_{\alpha,n}\widehat{\mathcal{H}}^\natural_\alpha,$$

where c_i are integers such that $0\le c_i\le g-1$. It is easy to see that any element in $\widehat{\Xi}_\alpha$ can be written as a linear combination of $F_\omega(\widehat{\xi}_\alpha)$ for various $\omega\in\Omega$. Thus by (A10), any element in $\widehat{\mathcal{H}}_\alpha$ is written as a linear combination of $F_\omega(\widehat{\xi}_\alpha)\widehat{\mathcal{H}}^\natural_\alpha$ with $\omega\in\Omega^\alpha$. We now concentrate on $\widehat{\mathcal{H}}^\natural_\alpha$. Define $\widehat{L}^{[k]}_i\in\widehat{\mathcal{H}}^0_\alpha$ by

$$\widehat{L}^{[k]}_i=\widehat{T}_{\alpha,a_k+i-1}\cdots\widehat{T}_{\alpha,a_k+2}\widehat{T}_{\alpha,a_k+1}\widehat{T}^{[k]}_{\alpha,0}\widehat{T}_{\alpha,a_k+1}\widehat{T}_{\alpha,a_k+2}\cdots\widehat{T}_{\alpha,a_k+i-1}$$

for $i=1,\dots,n_k$. Then $L^{[k]}_i$ commutes with $\widehat{T}_{\alpha,j}$ for $j\ne a_k+i-1,a_k+i$ and we have

$$\widehat{T}_{\alpha,a_k+i}\widehat{L}^{[k]}_i\widehat{T}_{\alpha,a_k+i}=\begin{cases}\widehat{L}^{[k]}_{i+1} & \text{if } i\ne n_k,\\ \widehat{L}^{[k+1]}_1 & \text{if } i=n_k.\end{cases}$$

by (A6). (Note that in the latter case, $\widehat{T}_{\alpha,a_k+n_k}\notin\widehat{\mathcal{H}}^0_\alpha$.) It follows that any element in $\widehat{\mathcal{H}}^\natural_\alpha$ can be written as a linear combination of the elements in $\widehat{L}\widehat{\mathcal{H}}_n$, where $\widehat{L}$ is the subalgebra of $\widehat{\mathcal{H}}^0_\alpha$ generated by $\widehat{L}^{[k]}_i$. Let $\mathcal{H}_{n,\alpha}$ be the subalgebra of $\mathcal{H}_n$ corresponding to the Young subgroup $\mathfrak{S}_\alpha$ of $\mathfrak{S}_n$, and let $\widehat{\mathcal{H}}_{n,\alpha}$ be the corresponding subalgebra of $\widehat{\mathcal{H}}_n$. Since $\widehat{\mathcal{H}}_n$ is the quotient of $\mathcal{H}_n$, it is written as a sum of $\widehat{\mathcal{H}}_{n,\alpha}\widehat{T}_{\alpha,w}$ with $w\in\mathfrak{S}_\alpha\backslash\mathfrak{S}_n$. Since $\widehat{\mathcal{H}}_{n,\alpha}\subset\widehat{\mathcal{H}}^0_\alpha$, one sees that $\widehat{\mathcal{H}}^\natural_\alpha=\sum_{w\in\mathfrak{S}_\alpha\backslash\mathfrak{S}_n}\widehat{\mathcal{H}}^0_\alpha\widehat{T}_w$. Hence (6.8.1) holds.

Since $\overline{\mathcal{H}}^{\mathbf{p}}_\alpha$ satisfies the same relations, we have a surjective homomorphism $\psi:\widehat{\mathcal{H}}_\alpha\to\overline{\mathcal{H}}^{\mathbf{p}}_\alpha$. In order to show that ψ is injective, it is enough to see that the set of elements in (6.8.1) gives an R-free basis of $\widehat{\mathcal{H}}_\alpha$ and that the image under ψ of this basis gives a basis of $\overline{\mathcal{H}}^{\mathbf{p}}_\alpha$. We denote by $\mathcal{C}'$ the image of $\mathcal{C}$ under ψ. By a similar argument as above, we see that $\mathcal{C}'$ spans $\overline{\mathcal{H}}^{\mathbf{p}}_\alpha$ as an R-module. We show that $\mathcal{C}'$ gives an R-free basis of $\overline{\mathcal{H}}^{\mathbf{p}}$. For this, it is enough to see that the elements in $\mathcal{C}'$ are linearly independent over R, or equivalently, they are linearly independent over K, where K is the quotient field of R. It is easy to see that the cardinality of the set $\mathcal{C}'$ is equal to

$$|\Omega^\alpha|\times\dim\mathcal{H}_\alpha\times n_\alpha=n_\alpha^2\times\dim\mathcal{H}_\alpha=\dim\overline{\mathcal{H}}^{\mathbf{p}}_\alpha$$

by Proposition 5.3. Hence the elements of $\mathcal{C}'$ are linearly independent, and $\mathcal{C}'$ gives an R-free basis of $\overline{\mathcal{H}}^{\mathbf{p}}_\alpha$. This shows that the elements in $\mathcal{C}$ are also linearly independent, and so $\mathcal{C}$ is an R-free basis of $\widehat{\mathcal{H}}_\alpha$. Therefore ψ is an isomorphism, and the theorem is proved. □

Remark 6.9. In the case where $\mathbf{p}=(r)$, $\overline{\mathcal{H}}^{\mathbf{p}}_\alpha=\overline{\mathcal{H}}^{\mathbf{p}}$ coincides with $\mathcal{H}$, and the fundamental relation (A1)–(A14) is reduced to the fundamental relation for $\mathcal{H}$. On the other hand, in the case where $\mathbf{p}=(1^r)$, $\mathcal{H}_\alpha$ is a subalgebra of $\mathcal{H}_n$ for each

$\alpha \in \Delta_{n,r}$. Then $T^{[k]}_{\alpha,0}$ turns out to be scalar operators, and the relations (A4)–(A7), (A14) can be ignored. The remaining relations give the fundamental relation for $\overline{\mathcal{H}}^{\mathbf{p}}_{\alpha}$. Note that a similar argument as in the proof shows that the relations (A1)–(A3), (A8), (A9), (A11)–(A13) gives a fundamental relation for $\overline{\mathcal{H}}^{\mathbf{p}}$, which is nothing but the fundamental relation for the modified Ariki-Koike algebra given in [SawS].

7. Schur-Weyl duality

7.1. It is known by [M, §5] that the Schur-Weyl duality i.e., the double centralizer property holds between $\mathcal{H}$ and $\mathcal{S} = \mathrm{End}_{\mathcal{H}} M$. A similar duality also holds by [SawS, Theorem 8.2] for the modified Ariki-Koike algebra $\overline{\mathcal{H}}$ on the action of the tensor space $V^{\otimes n}$. In our setting, $\overline{\mathcal{H}}$ coincides with $\overline{\mathcal{H}}^{\mathbf{p}}$ with $\mathbf{p} = (1^r)$, and $V^{\otimes n} \simeq \overline{M}$ as $\overline{\mathcal{H}}^{\mathbf{p}}$-modules. In what follows we shall give a generalization of this property for the arbitrary $\mathbf{p}$, i.e., we show the Schur-Weyl duality between $\overline{\mathcal{S}}^{\mathbf{p}}$ and $\overline{\mathcal{H}}^{\mathbf{p}}$ acting on $\overline{M}$. Although the proof is carried out for the action on $\overline{M}$, we formulate the theorem for $\overline{\mathcal{H}}^{\mathbf{p}}$-module $M_{\mathbf{p}} = \bigoplus M^{\mu}_{\mathbf{p}}$ which is isomorphic to $\overline{M}$, where $M^{\mu}_{\mathbf{p}}$ is a right ideal of $\overline{\mathcal{H}}^{\mathbf{p}}$, so that it fits to the situation above.

7.2. In order to give an expression of $\overline{M}^{\mu}$ as a right ideal of $\overline{\mathcal{H}}^{\mathbf{p}}$, we describe the cellular basis $m^{\mathbf{p}}_{\mathfrak{s}\mathfrak{t}}$ of $\overline{\mathcal{H}}^{\mathbf{p}}$ more explicitly. For each $\alpha = (n_1, \dots, n_g) \in \Delta_{n,g}$ we define $F_\alpha \in \overline{\mathcal{H}}^{\mathbf{p}}$ by

$$F_\alpha = \Delta^{-n} F_{c_1}(\xi_1) \cdots F_{c_n}(\xi_n), \tag{7.2.1}$$

where

$$(c_1, \dots, c_n) = (\underbrace{1, \dots, 1}_{n_1\text{-times}}, \underbrace{2, \dots, 2}_{n_2\text{-times}}, \dots, \underbrace{g, \dots, g}_{n_g\text{-times}}).$$

If we define $\omega = \omega_\alpha$ as the unique element in Ω^α such that $d(\omega) = 1$, we see that $F_\alpha = \Delta^{-n} F_\omega(\xi)$ in the notation of (6.5.1). It follows from (6.6.2) that

$$\overline{m}_\Omega F_\alpha = \overline{m}_\omega. \tag{7.2.2}$$

Take $\lambda \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha$. Then $\mathfrak{t}^\lambda \in \mathrm{Std}(\lambda)_\omega$. Let $S_\omega = \omega(\mathfrak{t}^\lambda) \in \mathcal{T}^{\mathbf{p}}_0(\lambda, \omega)$. Then $\overline{m}_{S_\omega \mathfrak{t}^\lambda} = \overline{m}_{\mathfrak{t}^\lambda \mathfrak{t}^\lambda} = \overline{m}_\lambda \in \overline{M}^\omega$. Since $\overline{M}^\omega = \overline{m}_\omega \overline{\mathcal{H}}^{\mathbf{p}}$, there exists $y_\lambda \in \overline{\mathcal{H}}^{\mathbf{p}}$ such that $\overline{m}_\lambda = \overline{m}_\omega y_\lambda$. One can choose y_λ in the following way. Let $\overline{\mathcal{H}}^0_\alpha$ be the subalgebra of $\overline{\mathcal{H}}^{\mathbf{p}}$ consisting of scalar matrices with entries in $\mathcal{H}_\alpha = \mathcal{H}_{n_1,r_1} \otimes \cdots \otimes \mathcal{H}_{n_g,r_g}$ under the isomorphism in Proposition 5.3. Thus $\overline{\mathcal{H}}^0_\alpha \simeq \mathcal{H}_\alpha$. We have $\overline{m}_\omega, \overline{m}_\lambda \in \overline{M}^\omega_0$, and under the isomorphism $\overline{M}^\omega_0 \simeq M^{\omega^{[1]}} \otimes \cdots \otimes M^{\omega^{[g]}} = \mathcal{H}_\alpha$, $\overline{m}_\omega$ corresponds to $1 \otimes \cdots \otimes 1$, and $\overline{m}_\lambda$ corresponds to $m_{\lambda^{[1]}} \otimes \cdots \otimes m_{\lambda^{[g]}}$ in $\mathcal{H}_\alpha$. Then we choose $y_\lambda \in \overline{\mathcal{H}}^0_\alpha$ as the scalar matrix consisting of $m_{\lambda^{[1]}} \otimes \cdots \otimes m_{\lambda^{[g]}}$ in $\mathcal{H}_\alpha$ under the isomorphism in Proposition 5.3.

Note that F_α commutes with any element in $\overline{\mathcal{H}}^0_\alpha$. In fact, by (7.2.1) $F_\alpha \in R[\xi_1, \dots, \xi_n]^{\mathfrak{S}_\alpha}$ with $\mathfrak{S}_\alpha = \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_g}$, and a similar argument as in [SawS, Lemma 2.8] can be applied. In particular, y_λ commutes with F_α. Let $* : \overline{\mathcal{H}}^{\mathbf{p}} \to \overline{\mathcal{H}}^{\mathbf{p}}$ be the anti-automorphism. Since ξ_i are fixed by $*$, F_α is fixed by $*$. Also, y_λ is fixed by $*$ since the corresponding elements in $\mathcal{H}_{n_k,r_k}$ are fixed by $*$. We have the following lemma.

Lemma 7.3. *For each* $\mathfrak{t}, \mathfrak{s} \in \mathrm{Std}(\lambda)$*, we have*

$$m^{\mathbf{p}}_{\mathfrak{s}\mathfrak{t}} = T^*_{d(\mathfrak{s})} F_\alpha y_\lambda T_{d(\mathfrak{t})}.$$

Proof. By the construction in 7.2, we see that $\overline{m}_\Omega F_\alpha y_\lambda = \overline{m}_{\mathfrak{t}^\lambda \mathfrak{t}^\lambda}$ for λ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha$. Thus $\overline{m}_\Omega F_\alpha y_\lambda T_{d(\mathfrak{t})} = \overline{m}_{\mathfrak{t}^\lambda \mathfrak{t}}$ for any $\mathfrak{t} \in \mathrm{Std}(\lambda)$. If $T \in \mathcal{T}^{\mathbf{p}}_0(\lambda, \omega')$ corresponds to $\mathfrak{t} \in \mathrm{Std}(\lambda)$ under (5.1.2), and $S_\omega \in \mathcal{T}^{\mathbf{p}}_0(\lambda, \omega)$ with $\omega = \omega_\alpha$, then we have $\varphi_{S_\omega T}(\overline{m}_\Omega) = \overline{m}_{\mathfrak{t}^\lambda \mathfrak{t}}$. It follows that $\varphi_{S_\omega T} = F_\alpha y_\lambda T_{d(\mathfrak{t})}$. This shows that $\varphi_{S S_\omega} = T^*_{d(\mathfrak{s})} F_\alpha y_\lambda$. Take $T \in \mathcal{T}^{\mathbf{p}}_0(\lambda, \omega')$ corresponding to $\mathfrak{t} \in \mathrm{Std}(\lambda)$. Since $\varphi_{S S_\omega}(\overline{m}_\Omega) = \overline{m}_{\mathfrak{s}\mathfrak{t}^\lambda}$, we have

$$\overline{m}_\Omega \cdot T^*_{d(\mathfrak{s})} F_\alpha y_\lambda T_{d(\mathfrak{t})} = \overline{m}_{\mathfrak{s}\mathfrak{t}^\lambda} \cdot T_{d(\mathfrak{t})} = \overline{m}_{\mathfrak{s}\mathfrak{t}} = \varphi_{ST}(\overline{m}_\Omega).$$

Thus we have $m^{\mathbf{p}}_{\mathfrak{s}\mathfrak{t}} = \varphi_{ST} = T^*_{d(\mathfrak{s})} F_\alpha y_\lambda T_{d(\mathfrak{t})}$. □

7.4. For each $\mu \in \Lambda$ such that $\alpha_{\mathbf{p}}(\mu) = \alpha$, we define $y_\mu \in \overline{\mathcal{H}}^{\mathbf{p}}_\alpha$ similarly as before, by extending the definition of y_λ for $\lambda \in \Lambda^+$. We define a right ideal $M^\mu_{\mathbf{p}}$ of $\overline{\mathcal{H}}^{\mathbf{p}}$ by $M^\mu_{\mathbf{p}} = F_\alpha y_\mu \overline{\mathcal{H}}^{\mathbf{p}}$ and put $M_{\mathbf{p}} = \bigoplus_{\mu \in \Lambda} M^\mu_{\mathbf{p}}$. By Lemma 4.10, we have $\overline{m}_\Omega F_\alpha y_\mu = \overline{m}_\mu$ and so $\overline{m}_\Omega F_\alpha y_\mu \overline{\mathcal{H}}^{\mathbf{p}} = \overline{m}_\mu \overline{\mathcal{H}}^{\mathbf{p}} = \overline{M}^\mu$. This shows that there exists an isomorphism $\phi : M^\mu_{\mathbf{p}} \to \overline{M}^\mu$ of $\overline{\mathcal{H}}^{\mathbf{p}}$-modules by $F_\alpha y_\mu h \mapsto \overline{m}_\Omega F_\alpha y_\mu h = \overline{m}_\mu h$.

Recall that $\{\overline{m}_{S\mathfrak{t}} \mid S \in \mathcal{T}^{\mathbf{p}}_0(\lambda, \mu), \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ for } \lambda \in \Lambda^+\}$ gives a basis of $\overline{M}^\mu$. In connection with this, we define, for each $S \in \mathcal{T}^{\mathbf{p}}_0(\lambda, \mu), \mathfrak{t} \in \mathrm{Std}(\lambda)$ with $\lambda \in \Lambda^+$,

$$m^{\mathbf{p}}_{S\mathfrak{t}} = \sum_{\substack{\mathfrak{s} \in \mathrm{Std}(\lambda) \\ \mu(\mathfrak{s}) = S}} q^{l(d(\mathfrak{s})) + l(d(\mathfrak{t}))} m^{\mathbf{p}}_{\mathfrak{s}\mathfrak{t}}.$$

The following lemma holds.

Lemma 7.5. *The set* $\{m^{\mathbf{p}}_{S\mathfrak{t}}\}$ *gives rise to a basis of* $M^\mu_{\mathbf{p}}$*, and we have* $\phi(m^{\mathbf{p}}_{S\mathfrak{t}}) = \overline{m}_{S\mathfrak{t}}$ *for each basis element.*

Proof. By the proof of Lemma 7.3, we know that $\overline{m}_\Omega m^{\mathbf{p}}_{\mathfrak{s}\mathfrak{t}} = \overline{m}_{\mathfrak{s}\mathfrak{t}}$ for any $\mathfrak{t}, \mathfrak{s} \in \mathrm{Std}(\lambda)$. It follows that $\overline{m}_\Omega m^{\mathbf{p}}_{S\mathfrak{t}} = \overline{m}_{S\mathfrak{t}} \in \overline{M}^\mu$ for any $S \in \mathcal{T}^{\mathbf{p}}_0(\lambda, \mu)$ and $\mathfrak{t} \in \mathrm{Std}(\lambda)$. In particular, we see that $m^{\mathbf{p}}_{S\mathfrak{t}} \in M^\mu_{\mathbf{p}}$, and the lemma follows. □

The following result gives the Schur-Weyl duality, i.e., the double centralizer property between $\overline{\mathcal{H}}^{\mathbf{p}}$ and $\overline{\mathcal{S}}^{\mathbf{p}}$.

Theorem 7.6. *Under the assumptions (5.1.1) and (6.3.1), there exist isomorphisms of R-algebras*

$$\overline{\mathcal{S}}^{\mathbf{p}} \simeq \mathrm{End}_{\overline{\mathcal{H}}^{\mathbf{p}}} M_{\mathbf{p}}, \qquad \overline{\mathcal{H}}^{\mathbf{p}} \simeq \mathrm{End}^0_{\overline{\mathcal{S}}^{\mathbf{p}}} M_{\mathbf{p}}.$$

Proof. We argue on $\overline{M}$ instead of $M_{\mathbf{p}}$. The second isomorphism is already shown in (5.4.1). So we prove the first isomorphism. Let $\mu, \nu \in \Lambda$ be such that $\alpha_{\mathbf{p}}(\mu) = \alpha_{\mathbf{p}}(\nu) = \alpha$, and take $\varphi \in \mathrm{Hom}_{\overline{\mathcal{H}}^{\mathbf{p}}}(\overline{M}^\nu, \overline{M}^\mu)$. Since $\overline{M}^\nu = \overline{m}_\nu \overline{\mathcal{H}}^{\mathbf{p}}$, the map φ is determined by $\varphi(\overline{m}_\nu)$. We show that

$$\varphi(\overline{m}_\nu) \in \overline{M}^\mu_0. \tag{7.6.1}$$

In fact, since $\overline{m}_{S\mathfrak{t}}$ $(S \in \mathcal{T}^{\mathbf{p}}_0(\lambda, \mu), \mathfrak{t} \in \mathrm{Std}(\lambda))$ gives a basis of $\overline{M}^\mu$, one can write

$$\varphi(\overline{m}_\nu) = \sum_{S, \mathfrak{t}} c_{S\mathfrak{t}} \overline{m}_{S\mathfrak{t}}$$

with $c_{S\mathfrak{t}} \in R$. By (6.1.4), we have

$$\varphi(\overline{m}_\nu \xi_i) = Q^{\mathbf{p}}_{b(i)} \sum_{S,\mathfrak{t}} c_{S\mathfrak{t}} \overline{m}_{S\mathfrak{t}} \tag{7.6.2}$$

for $i = 1, \dots, n$, where $b(i) = k$ if $a_k + 1 \le i \le a_k + n_k$. On the other hand, by (6.1.3), we have

$$\varphi(\overline{m}_\nu \xi_i) = \varphi(\overline{m}_\nu)\xi_i = \sum_{S,\mathfrak{t}} c_{S\mathfrak{t}} Q^{\mathbf{p}}_{\mathfrak{t}(i)} \overline{m}_{S\mathfrak{t}}, \tag{7.6.3}$$

where $\mathfrak{t}(i) = k$ if the letter i is contained in $\mathfrak{t}^{[k]}$ (see the remark after (6.1.3)). Comparing (7.6.2) and (7.6.3), we see that $\mathfrak{t} \in \mathrm{Std}(\lambda)_0$. Since $\overline{M}^\mu_0$ is spanned by $\overline{m}_{S\mathfrak{t}}$ with $\mathfrak{t} \in \mathrm{Std}(\lambda)_0$, we obtain (7.6.1).

Let $\overline{\mathcal{H}}^0_\alpha$ be the subalgebra of $\overline{\mathcal{H}}^{\mathbf{p}}$ generated by $T^{[k]}_{\alpha,i}$ for $k = 1, \dots, g, i = 0, \dots, n_k - 1$. Since $\overline{M}^\nu_0 = \overline{m}_\nu \overline{\mathcal{H}}^0_\alpha$, and similarly for $\overline{M}^\mu_0$, it follows from (7.6.1) that any $\varphi \in \mathrm{Hom}_{\overline{\mathcal{H}}^{\mathbf{p}}}(\overline{M}^\nu, \overline{M}^\mu)$ has the property that $\varphi(\overline{M}^\nu_0) \subset \overline{M}^\mu_0$. Thus we have a natural R-linear map

$$\theta'' : \mathrm{Hom}_{\overline{\mathcal{H}}^{\mathbf{p}}}(\overline{M}^\nu, \overline{M}^\mu) \to \mathrm{Hom}_{\overline{\mathcal{H}}^0_\alpha}(\overline{M}^\nu_0, \overline{M}^\mu_0),$$

which is clearly injective. Let $\overline{H}_{\mu\nu}$ be the $\mu\nu$-part of $\overline{S}^{\mathbf{p}}$ as in (4.9.2). Since the action of $\widetilde{H}_\alpha$ on $\overline{M}^\nu_0$ coincides with the action of $\overline{\mathcal{H}}^0_\alpha$, we see that there exists an R-linear isomorphism

$$\Theta : \overline{H}_{\mu\nu} \to \mathrm{Hom}_{\overline{\mathcal{H}}^0_\alpha}(\overline{M}^\nu_0, \overline{M}^\mu_0)$$

by Lemma 4.12. On the other hand, by (5.4.2), we know that there exists an injective map $\theta' : \overline{H}_{\mu\nu} \to \mathrm{Hom}_{\overline{\mathcal{H}}^{\mathbf{p}}}(\overline{M}^\nu, \overline{M}^\mu)$. It is clear that the composite of θ' and θ'' coincides with Θ. Hence θ' is an isomorphism. This shows that $\overline{\mathcal{S}}^{\mathbf{p}} \simeq \mathrm{End}_{\overline{\mathcal{H}}^{\mathbf{p}}} \overline{M}$, and the theorem follows. □

Remark 7.7. The assumption (6.3.1) is used to give an expression of $\overline{M}$ as an ideal of $\overline{\mathcal{H}}^{\mathbf{p}}$. But the Schur-Weyl duality holds for $\overline{M}$ without referring to the ideal $M_{\mathbf{p}}$. In that case, (6.3.1) can be replaced by a weaker assumption "the parameters $Q^{\mathbf{p}}_1, \dots, Q^{\mathbf{p}}_g$ are all distinct.".

By making use of Theorem 7.6, we obtain the following additional information on the space $\overline{H}_{\mu\nu} = \mathrm{Hom}_{\overline{\mathcal{H}}^{\mathbf{p}}}(M^\nu_{\mathbf{p}}, M^\mu_{\mathbf{p}})$. Put $m^{\mathbf{p}}_\nu = F_\alpha y_\nu$ so that $M^\nu_{\mathbf{p}} = m^{\mathbf{p}}_\nu \overline{\mathcal{H}}^{\mathbf{p}}$.

Proposition 7.8. *Let $\mu, \nu \in \Lambda$ such that $\alpha_{\mathbf{p}}(\mu) = \alpha_{\mathbf{p}}(\nu) = \alpha$.*

(i) *The map $\varphi \mapsto \varphi(m^{\mathbf{p}}_\nu)$ gives an isomorphism of R-modules,*

$$\mathrm{Hom}_{\overline{\mathcal{H}}^{\mathbf{p}}}(M^\nu_{\mathbf{p}}, M^\mu_{\mathbf{p}}) \to M^{\nu *}_{\mathbf{p}} \cap M^\mu_{\mathbf{p}},$$

where $M^{\nu *}_{\mathbf{p}} = \overline{\mathcal{H}}^{\mathbf{p}} m^{\mathbf{p}}_\nu$ is the image of $M^\nu_{\mathbf{p}}$ under the operation $*$.

(ii) *We have*

$$M^{\nu *}_{\mathbf{p}} \cap M^\mu_{\mathbf{p}} = F_\alpha(\overline{\mathcal{H}}^0_\alpha y_\nu \cap y_\mu \overline{\mathcal{H}}^0_\alpha).$$

Proof. For each $m \in M^{\nu *}_{\mathbf{p}} \cap M^\mu_{\mathbf{p}}$, the map $m^{\mathbf{p}}_\nu h \mapsto mh$ ($h \in \overline{\mathcal{H}}^{\mathbf{p}}$) gives a well-defined map $\varphi_m \in \mathrm{Hom}_{\overline{\mathcal{H}}^{\mathbf{p}}}(M^\nu_{\mathbf{p}}, M^\mu_{\mathbf{p}})$, and the map $m \mapsto \varphi_m$ gives an R-linear map $M^{\nu *}_{\mathbf{p}} \cap M^\mu_{\mathbf{p}} \to \mathrm{Hom}_{\overline{\mathcal{H}}^{\mathbf{p}}}(M^\nu_{\mathbf{p}}, M^\mu_{\mathbf{p}})$, which is clearly injective.

On the other hand, we have

$$\begin{aligned}\mathrm{Hom}_{\overline{\mathcal{H}}^{\mathbf{p}}}(M^{\nu}_{\mathbf{p}}, M^{\mu}_{\mathbf{p}}) &= \mathrm{Hom}_{\overline{\mathcal{H}}^{\mathbf{p}}_{\alpha}}(M^{\nu}_{\mathbf{p}}, M^{\mu}_{\mathbf{p}})\\ &\simeq \mathrm{Hom}_{\overline{\mathcal{H}}^{0}_{\alpha}}(m^{\mathbf{p}}_{\nu}\overline{\mathcal{H}}^{0}_{\alpha}, m^{\mathbf{p}}_{\mu}\mathcal{H}^{0}_{\alpha})\\ &\simeq \mathrm{Hom}_{\mathcal{H}_{\alpha}}(M^{\nu^{[1]}}\otimes\cdots\otimes M^{\nu^{[g]}}, M^{\mu^{[1]}}\otimes\cdots\otimes M^{\mu^{[g]}}),\end{aligned}$$

where $\mathcal{H}_{\alpha} = \mathcal{H}_{n_1,r_1}\otimes\cdots\otimes\mathcal{H}_{n_g,r_g}$. (Note that $m^{\mathbf{p}}_{\mu}\overline{\mathcal{H}}^{0}_{\alpha}$ corresponds to $\overline{M}^{\mu}_{0}$ under the isomorphism $M^{\mu}_{\mathbf{p}} \simeq \overline{M}^{\mu}$.) It is known, by [DJM] that the last set is isomorphic to $\overline{\mathcal{H}}^{0}_{\alpha}y_{\nu}\cap y_{\mu}\overline{\mathcal{H}}^{0}_{\alpha}$ as R-modules. Hence

$$\mathrm{Hom}_{\overline{\mathcal{H}}^{0}_{\alpha}}(m^{\mathbf{p}}_{\nu}\overline{\mathcal{H}}^{0}_{\alpha}, m^{\mathbf{p}}_{\mu}\mathcal{H}^{0}_{\alpha}) \simeq (m^{\mathbf{p}}_{\nu}\overline{\mathcal{H}}^{0}_{\alpha})^{*}\cap m^{\mathbf{p}}_{\mu}\overline{\mathcal{H}}^{0}_{\alpha}$$

via the map $\varphi\mapsto\varphi(m^{\mathbf{p}}_{\nu})$. But since $m^{\mathbf{p}}_{\nu}\overline{\mathcal{H}}^{0}_{\alpha} = F_{\alpha}y_{\nu}\overline{\mathcal{H}}^{0}_{\alpha}$ and F_{α} commutes with y_{ν} and $\overline{\mathcal{H}}^{0}_{\alpha}$, we see that $(m^{\mathbf{p}}_{\nu}\overline{\mathcal{H}}^{0}_{\alpha})^{*} = F_{a}\overline{\mathcal{H}}^{0}_{\alpha}y_{\nu}$. This shows that

$$\mathrm{Hom}_{\overline{\mathcal{H}}^{\mathbf{p}}}(M^{\mathbf{p}}_{\nu}, M^{\mathbf{p}}_{\mu}) \simeq F_{\alpha}(\overline{\mathcal{H}}^{0}_{\alpha}y_{\nu}\cap y_{\mu}\overline{\mathcal{H}}^{0}_{\alpha}) \subseteq M^{\nu *}_{\mathbf{p}}\cap M^{\mu}_{\mathbf{p}},$$

where the first isomorphism is given by the map $\varphi\mapsto\varphi(m^{\mathbf{p}}_{\nu})$. Both statements of the proposition follow from this, by being combined with the remark in the first paragraph. □

8. Comparison of $\overline{\mathcal{H}}^{\mathbf{p}}$ for various $\mathbf{p}$

8.1. We shall consider the relationship among $\mathcal{S}^{\mathbf{p}}$ and $\overline{\mathcal{H}}^{\mathbf{p}}$ for various types $\mathbf{p}$. First consider the case of $\mathcal{S}^{\mathbf{p}}$. Let $\mathbf{p} = (r_1,\dots,r_g)$ and let $\mathbf{p}' = (r'_1,\dots,r'_{g'})$ be two compositions of r. We define $(p'_1,\dots,p'_{g'})$ for $\mathbf{p}'$ similar to $\mathbf{p}$. We write $\mathbf{p}'\preceq\mathbf{p}$ if $\mathbf{p}'$ is obtained as a refinement of $\mathbf{p}$, namely if p_j coincides with some p'_{k_j} for each j. In particular, we have $(1^r)\preceq\mathbf{p}\preceq(r)$ for any $\mathbf{p}$. Assume that $\mathbf{p}'\preceq\mathbf{p}$. Then we see that $\mathbf{a}_{\mathbf{p}}(\lambda)\geq\mathbf{a}_{\mathbf{p}}(\mu)$ if $\mathbf{a}_{\mathbf{p}'}(\lambda)\geq\mathbf{a}_{\mathbf{p}'}(\mu)$ for $\lambda,\mu\in\Lambda$. Moreover, $\alpha_{\mathbf{p}}(\lambda)=\alpha_{\mathbf{p}}(\mu)$ if $\alpha_{\mathbf{p}'}(\lambda)=\alpha_{\mathbf{p}'}(\mu)$. This implies that

$$\mathcal{S}^{\mathbf{p}'}\subseteq\mathcal{S}^{\mathbf{p}}\quad\text{if } \mathbf{p}'\preceq\mathbf{p}. \tag{8.1.1}$$

Concerning the modified Ariki-Koike algebras $\overline{\mathcal{H}}^{\mathbf{p}}$, we have the following.

Proposition 8.2. *There exists an algebra homomorphism $\rho_{\mathbf{p}'\mathbf{p}}:\overline{\mathcal{H}}^{\mathbf{p}}\to\overline{\mathcal{H}}^{\mathbf{p}'}$ for any pair $\mathbf{p},\mathbf{p}'$ such that $\mathbf{p}'\preceq\mathbf{p}$ satisfying the following property: for $\mathbf{p}''\preceq\mathbf{p}'\preceq\mathbf{p}$, we have $\rho_{\mathbf{p}''\mathbf{p}}=\rho_{\mathbf{p}''\mathbf{p}'}\circ\rho_{\mathbf{p}'\mathbf{p}}$.*

Proof. Let $\overline{M}=\bigoplus_{\mu}\overline{M}^{\mu}$ be the $\mathcal{H}$-module defined by $\widehat{N}^{\mathbf{a}_{\mathbf{p}}(\mu)}$ as before. We denote $\overline{M}$ by $\overline{M}_{\mathbf{p}}$ to indicate its dependence on $\mathbf{p}$. Assume that $\mathbf{p}'\preceq\mathbf{p}$. Then we have a natural surjection $\overline{M}_{\mathbf{p}}\to\overline{M}_{\mathbf{p}'}$ of $\mathcal{H}$-modules. If we regard $\overline{M}_{\mathbf{p}}$ as a left $\mathcal{S}^{\mathbf{p}}$-module, and $\overline{M}_{\mathbf{p}'}$ as a left $\mathcal{S}^{\mathbf{p}'}$-module, then the map $\overline{M}_{\mathbf{p}}\to\overline{M}_{\mathbf{p}'}$ is compatible with the actions of $\mathcal{S}^{\mathbf{p}}$ and $\mathcal{S}^{\mathbf{p}'}$ via the inclusion $\mathcal{S}^{\mathbf{p}'}\hookrightarrow\mathcal{S}^{\mathbf{p}}$. Let $M_{\mathbf{p}}=\bigoplus_{\mu}M^{\mu}_{\mathbf{p}}$ be as in 7.4. By Theorem 7.6, $\overline{\mathcal{H}}^{\mathbf{p}}$ is realized as $\overline{\mathcal{H}}^{\mathbf{p}}=\mathrm{End}^{0}_{\mathcal{S}^{\mathbf{p}}}M_{\mathbf{p}}$. By using the property of the cellular structure of $\overline{\mathcal{H}}^{\mathbf{p}}$ described in the beginning of Section 7, together with Lemma 7.5, the above property of $\overline{M}_{\mathbf{p}}$ can be made more precise for $M_{\mathbf{p}}$ as follows (which is a generalization of the argument in 4.1). Let $\widehat{N}^{\mathbf{a}_{\mathbf{p}'}(\mu)}_{\mathbf{p}}$ be the R-submodule of $\overline{\mathcal{H}}^{\mathbf{p}}$ spanned by $m^{\mathbf{p}}_{\mathfrak{s}\mathfrak{t}}$ such that $\mathfrak{s},\mathfrak{t}\in\mathrm{Std}(\lambda)$ with $\mathbf{a}_{\mathbf{p}'}(\lambda)>\mathbf{a}_{\mathbf{p}'}(\mu)$. Then $\widehat{N}^{\mathbf{a}_{\mathbf{p}'}(\mu)}_{\mathbf{p}}$

is a two-sided ideal of $\overline{\mathcal{H}}^{\mathbf{p}}$. Put $\widehat{M}^{\mu}_{\mathbf{p}} = M^{\mu}_{\mathbf{p}} \cap N^{\mathbf{a}_{\mathbf{p}'}(\mu)}_{\mathbf{p}}$. Then $\widehat{M}^{\mu}_{\mathbf{p}}$ is an $\overline{\mathcal{H}}^{\mathbf{p}}$-submodule of $M^{\mu}_{\mathbf{p}}$ with the basis $\{m^{\mathbf{p}}_{S\mathfrak{t}} \mid S \in \mathcal{T}^{\mathbf{p}}_0(\lambda,\mu), \mathfrak{t} \in \mathrm{Std}(\lambda), \mathbf{a}_{\mathbf{p}'}(\lambda) > \mathbf{a}_{\mathbf{p}'}(\mu)\}$, and we have an isomorphism of R-modules

$$M^{\mu}_{\mathbf{p}}/\widehat{M}^{\mu}_{\mathbf{p}} \simeq M^{\mu}_{\mathbf{p}'}.$$

A similar argument as in 4.9 shows that any $\varphi \in \overline{H}_{\mu\nu}$ maps $\widehat{M}^{\nu}$ to $\widehat{M}^{\mu}$, and so $\varphi \in \overline{\mathcal{S}}^{\mathbf{p}} = \mathrm{End}_{\overline{\mathcal{H}}^{\mathbf{p}}} M_{\mathbf{p}}$ induces an action on $M_{\mathbf{p}}/\widehat{M}_{\mathbf{p}}$, where $\widehat{M}_{\mathbf{p}} = \bigoplus_{\mu} \widehat{M}^{\mu}_{\mathbf{p}}$. This gives an isomorphism $M_{\mathbf{p}}/\widehat{M}_{\mathbf{p}} \simeq M_{\mathbf{p}'}$ as $\overline{\mathcal{S}}^{\mathbf{p}'}$-modules.

Now the action of $\overline{\mathcal{H}}^{\mathbf{p}}$ on $M_{\mathbf{p}}$ induces an action on $M_{\mathbf{p}}/\widehat{M}_{\mathbf{p}}$, which is compatible with the action of $\overline{\mathcal{S}}^{\mathbf{p}}$. Hence this induces an action of $\overline{\mathcal{H}}^{\mathbf{p}}$ on $M_{\mathbf{p}'}$ compatible with the action of $\overline{\mathcal{S}}^{\mathbf{p}'}$. Thus we have an R-algebra homomorphism

$$\rho_{\mathbf{p}'\mathbf{p}} : \overline{\mathcal{H}}^{\mathbf{p}} \to \mathrm{End}^0_{\overline{\mathcal{S}}^{\mathbf{p}'}} M_{\mathbf{p}'} \simeq \overline{\mathcal{H}}^{\mathbf{p}'}.$$

It is clear that this map $\rho_{\mathbf{p}'\mathbf{p}}$ satisfies the required property. □

8.3. In the case where $\mathbf{p} = (r)$, we have $\overline{\mathcal{H}}^{\mathbf{p}} \simeq \mathcal{H}$, and in the case where $\mathbf{p}' = (1^r)$, we have $\overline{\mathcal{H}}^{\mathbf{p}'} \simeq \overline{\mathcal{H}}$, the modified Ariki-Koike algebra introduced in [SawS]. We have $\mathbf{p}' \preceq \mathbf{p}$, and the map $\rho_{\mathbf{p}'\mathbf{p}} : \mathcal{H} \to \overline{\mathcal{H}}$ coincides with the map ρ_0 given in [SawS, Lemma 1.5]. We consider the following separation condition on parameters of $\mathcal{H}$, which was first introduced in [A].

(8.3.1) $q^{2k}Q_i - Q_j \in R$ are invertible in R for $|k| < n, i \neq j$.

Note that the condition (8.3.1) is stronger than the condition (6.3.1) for any $\mathbf{p}$. It is shown by [SawS, 8.3.2], based on the result in [HS], that $\rho_0 : \mathcal{H} \to \overline{\mathcal{H}}$ gives an isomorphism if the separation condition (8.3.1) holds. We have the following corollary.

Corollary 8.4. *Suppose that the condition (8.3.1) holds for $\mathcal{H}$. Then $\mathcal{H} \simeq \overline{\mathcal{H}}^{\mathbf{p}}$ for any $\mathbf{p}$. In particular, Theorem 6.8 gives a new presentation for the Ariki-Koike algebra $\mathcal{H}$.*

Proof. We have $(1^r) \preceq \mathbf{p} \preceq (r)$ for any $\mathbf{p}$. Since $\rho_0 : \mathcal{H} \to \overline{\mathcal{H}}$ is an isomorphism, the map $\rho_{\mathbf{p}'\mathbf{p}} : \overline{\mathcal{H}}^{\mathbf{p}} \to \overline{\mathcal{H}}^{\mathbf{p}'} \simeq \mathcal{H}$ is surjective by Proposition 8.2 for $\mathbf{p}' = (1^r)$. Since both $\mathcal{H}$ and $\overline{\mathcal{H}}^{\mathbf{p}}$ are free R-modules of the same rank, we obtain $\overline{\mathcal{H}}^{\mathbf{p}} \simeq \mathcal{H}$ as asserted. □

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