

A NEW CONSTRUCTION OF THE ASYMPTOTIC ALGEBRA ASSOCIATED TO THE q -SCHUR ALGEBRA

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ABSTRACT. We denote by A the ring of Laurent polynomials in the indeterminate v and by K its field of fractions. In this paper, we are interested in representation theory of the “generic” q -Schur algebra $\mathcal{S}_q(n, r)$ over A . We will associate to every symmetrising trace form τ on $K\mathcal{S}_q(n, r)$ a subalgebra \mathcal{J}_τ of $K\mathcal{S}_q(n, r)$ which is isomorphic to the “asymptotic” algebra $\mathcal{J}(n, r)_A$ defined by J. Du. As a consequence, we give a new hypothesis which implies James’ conjecture.

1. INTRODUCTION

This article is concerned with the representation theory of the “generic” q -Schur algebra $\mathcal{S}_q(n, r)$ over $A = \mathbb{Z}[v, v^{-1}]$. The q -Schur algebra was introduced by Dipper and James in [3] and [4]. There is an interest in studying the representations of this algebra, because they relate information about the modular representation theory of the finite general linear group $\mathrm{GL}_n(q)$ and of the quantum groups.

Using a new basis of $\mathcal{S}_q(n, r)$ constructed in [5] (which is analogous to the Kazhdan-Lusztig basis in Iwahori-Hecke algebras), J. Du introduced in [7] the asymptotic algebra $\mathcal{J}(n, r)_A$ over A and defined a homomorphism, $\Phi : \mathcal{S}_q(n, r) \rightarrow \mathcal{J}(n, r)_A$, the so-called Du-Lusztig homomorphism because its construction is similar to the Lusztig homomorphism for Iwahori-Hecke algebras.

There is a relevant open question in the representation theory of the q -Schur algebra, the so-called James’ conjecture. A precise formulation of this conjecture is recalled in Section 6. In [11] Meinolf Geck obtained a new formulation of this conjecture. More precisely, for k any field of characteristic ℓ and for R any integral domain with quotient field k , if $q \in R$ is invertible, we can define the corresponding q -Schur algebra $\mathcal{S}_q(n, r)_R$ over R and its extension of scalars $\mathcal{S}_q(n, r)_k$. Similarly, we can define $\mathcal{J}(n, r)_k$.

In [11, 1.2] M. Geck has shown that James’ conjecture holds if and only if, for $\ell > r$, the rank of the homomorphism $\Phi_k : \mathcal{S}_q(n, r)_k \rightarrow \mathcal{J}(n, r)_k$ only depends on the multiplicative order of q in k^\times , but not on ℓ .

Thus, in order to prove James’ conjecture, it is relevant to understand the rank of the Du-Lusztig homomorphism. The motivation of this paper is to develop new methods allowing us to study this rank. More precisely, we will give a new construction of the asymptotic algebra. Indeed, thanks to methods developed in [16] by the second author and adapted to our situation, we prove that $\mathcal{J}(n, r)_A$ is isomorphic to an algebra \mathcal{J}_τ , which only depends on the choice of a symmetrising

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trace form τ on the semisimple algebra $K\mathcal{S}_q(n, r)$ (here $K = \mathbb{Q}(v)$) such that

$$\mathcal{S}_q(n, r) \subseteq \mathcal{J}_\tau \subseteq K\mathcal{S}_q(n, r).$$

Note that we include non-degeneracy in the definition of “symmetrising trace form”. Our main tool is to use the structure of the left cell modules of $\mathcal{S}_q(n, r)$ to construct an explicit Wedderburn basis of $K\mathcal{S}_q(n, r)$ (see Theorem 4.12). The main result of this paper is Theorem 5.5.

The article is organised as follows. In Section 2, we recall the definition of the “generic” q -Schur algebra and of its analogue of the Kazhdan-Lusztig basis for Iwahori-Hecke algebras. In Section 3 we prove that the q -Schur algebra satisfies properties which are very similar to Lusztig’s conjectures **P1**, . . . , **P15** for Iwahori-Hecke algebras. In Section 4 we develop some tools to prove our main result in Section 5. Finally, in Section 6 we state a new hypothesis which implies James’ conjecture.

2. THE IWAHORI-HECKE ALGEBRA OF TYPE A AND THE q -SCHUR ALGEBRA

Let v be an indeterminate. We set $A = \mathbb{Z}[v, v^{-1}]$ to be the ring of Laurent polynomials in v and $K := \mathbb{Q}(v)$ its field of fractions. In order to introduce the q -Schur algebra over A , we have to recall some definitions and properties about Iwahori-Hecke algebras. We follow [15].

2.1. Iwahori-Hecke algebras and the Kazhdan-Lusztig basis. Let (W, S) be a Coxeter group (here S is the set of simple reflections). We define the corresponding Iwahori-Hecke algebra \mathcal{H} as the free A -module with basis $\{T_w\}_{w \in W}$ satisfying

$$\begin{aligned} T_w T_{w'} &= T_{ww'} && \text{if } l(ww') = l(w) + l(w'), \\ (T_s - v)(T_s + v^{-1}) &= 0 && \text{for } s \in S, \end{aligned}$$

where l is the length function on W . In [14, §1] Kazhdan and Lusztig define an A -basis $\{C_w \mid w \in W\}$ of \mathcal{H} which satisfies

$$\overline{C}_w = C_w \quad \text{and} \quad C_w = \sum_{y \leq w} p_{y,w} T_y \quad \text{for } w \in W,$$

where \leq is the Bruhat-Chevalley order on W , and $\overline{} : \mathcal{H} \rightarrow \mathcal{H}$ is the involutive automorphism of \mathcal{H} defined by $\overline{v} = v^{-1}$ and $\overline{\sum_{w \in W} a_w T_w} = \sum_{w \in W} \overline{a_w} T_w^{-1}$ and $p_{y,w} \in \langle v^k \mid k \leq 0 \rangle_{\mathbb{Z}}$ and $p_{w,w} = 1$.

Note that we use the more modern notation from [15], that is, our elements T_w here are the same as in [15] and were denoted by $v^{-l(w)} T_w$ in [14], and our elements C_w here were denoted by C'_w in [14] and by c_w in [15].

We denote by $g_{x,y,z}$ the structure constants of \mathcal{H} with respect to the basis $\{C_w \mid w \in W\}$, that is, we have

$$C_x C_y = \sum_{z \in W} g_{x,y,z} C_z \quad \text{for } x, y \in W.$$

We define a relation $y \preceq_L w$ on W by: either $y = w$ or there is an $s \in S$ such that $g_{s,w,y} \neq 0$. Let \leq_L be the transitive closure of the relation \preceq_L and denote by \sim_L the associated equivalence relation on W . The classes for this relation are the so-called left cells. Similarly, we define \leq_R and \sim_R , and we call the corresponding equivalence classes right cells. For $y, w \in W$, we write $y \leq_{LR} w$ if there is a sequence $y = y_0, y_1, \dots, y_n = w$ of elements of W such that, for $i \in \{0, \dots, n-1\}$,

we have $y_i \leq_L y_{i+1}$ or $y_i \leq_R y_{i+1}$. The classes of the equivalence relation \sim_{LR} on W corresponding to \leq_{LR} are the so-called two-sided cells.

In [15, §3.6], Lusztig shows that for $z \in W$, there is a unique integer $\mathbf{a}(z)$ such that for every $x, y \in W$, we have $g_{x,y,z} \in v^{\mathbf{a}(z)}\mathbb{Z}[v^{-1}]$ and $g_{x,y,z} \notin v^{\mathbf{a}(z)-1}\mathbb{Z}[v^{-1}]$. Moreover, for $z \in W$, we define $\Delta(z) = -\deg p_{1,z}$. For $x, y, z \in W$, we write $\gamma_{x,y,z^{-1}} \in \mathbb{Z}$ for the coefficient of $v^{\mathbf{a}(z)}$ in $g_{x,y,z}$ and we set

$$\mathcal{D} = \{d \in W \mid \mathbf{a}(d) = \Delta(d)\},$$

the set of distinguished involutions. In the case that W is a finite Weyl group, an affine Weyl group, or a dihedral group, Lusztig proved that the following conjectures hold (see [15, §§15–17]):

- P1** For any $z \in W$ we have $\mathbf{a}(z) \leq \Delta(z)$.
- P2** Let $x, y \in W$; if $\gamma_{x,y,d} \neq 0$ for some $d \in \mathcal{D}$, then we have $x = y^{-1}$.
- P3** If $y \in W$, there exists a unique $d \in \mathcal{D}$ such that $\gamma_{y^{-1},y,d} \neq 0$.
- P4** If $x \leq_{LR} y$, then $\mathbf{a}(x) \geq \mathbf{a}(y)$.
- P5** If $d \in \mathcal{D}$ and $y \in W$ are such that $\gamma_{y^{-1},y,d} \neq 0$, then $\gamma_{y^{-1},y,d} = \pm 1$.
- P6** For $d \in \mathcal{D}$, we have $d = d^{-1}$.
- P7** For every $x, y, z \in W$, we have $\gamma_{x,y,z} = \gamma_{y,z,x} = \gamma_{z,x,y}$.
- P8** Let $x, y, z \in W$ be such that $\gamma_{x,y,z} \neq 0$, then $x \sim_L y^{-1}$, $y \sim_L z^{-1}$ and $z \sim_L x^{-1}$.
- P9** If $x \leq_L y$ and $\mathbf{a}(x) = \mathbf{a}(y)$, then $x \sim_L y$.
- P10** If $x \leq_R y$ and $\mathbf{a}(x) = \mathbf{a}(y)$, then $x \sim_R y$.
- P11** If $x \leq_{LR} y$ and $\mathbf{a}(x) = \mathbf{a}(y)$, then $x \sim_{LR} y$.
- P13** Every left cell contains a unique element $d \in \mathcal{D}$ and $\gamma_{y^{-1},y,d} \neq 0$ for every $y \sim_L d$.
- P14** For every $x \in W$, we have $x \sim_{LR} x^{-1}$.
- P15** Let v' be a second indeterminate and let $g'_{x,y,z} \in \mathbb{Z}[v', v'^{-1}]$ be obtained from $g_{x,y,z}$ by the substitution $v \mapsto v'$. If $x, x', y, w \in W$ satisfy $\mathbf{a}(w) = \mathbf{a}(y)$, then

$$\sum_{y'} g'_{w,x',y'} g_{x',y',y} = \sum_{y'} g_{x,w,y'} g'_{y',x',y}.$$

Note that in this paper we only consider the case of type A, in which W is the symmetric group on $|S| + 1$ points.

2.2. The q -Schur algebra $\mathcal{S}_q(n, r)$. In the following, we denote by W the symmetric group of degree r , and by S the set of transpositions $s_i = (i, i+1)$ for $1 \leq i \leq r-1$ and \mathcal{H} is the associated Iwahori-Hecke algebra as in §2.1. Let $n, r \geq 1$, we denote by $\Lambda(n, r)$ the set of compositions of r into at most n parts. For $\lambda \in \Lambda(n, r)$, we denote by $W_\lambda \subseteq W$ the corresponding Young subgroup. For $\lambda, \mu \in \Lambda(n, r)$, we set $D_{\lambda,\mu}$ to be the set of distinguished double coset representatives of W with respect to W_λ and W_μ . We set

$$M(n, r) = \{(\lambda, w, \mu) \mid \lambda, \mu \in \Lambda(n, r), w \in D_{\lambda,\mu}\}.$$

For $\underline{a} = (\lambda, w, \mu) \in M(n, r)$, we write $ro(\underline{a}) = \lambda$ and $co(\underline{a}) = \mu$ and we set $\underline{a}^t = (\mu, w^{-1}, \lambda)$. For $\lambda, \mu \in \Lambda(n, r)$, we set $M_{\lambda,\mu} = \{\underline{a} \in M(n, r) \mid ro(\underline{a}) = \lambda, co(\underline{a}) = \mu\}$. We remark that if $w \in D_{\lambda,\mu}$, then the double coset $W_\lambda w W_\mu$ has a unique longest element. To prove this, we can proceed as follows: We denote by w_0 the longest element of W , then ${}^{w_0}W_\mu = W_{\tilde{\mu}}$. Here $\tilde{\mu} = (\mu_s, \mu_{s-1}, \dots, \mu_1)$, where $\mu = (\mu_1, \dots, \mu_s)$. Moreover, $r_{w_0} : W \rightarrow W$, $x \mapsto xw_0$ induces a bijection from the

double coset $W_\lambda w w_0 W_{\tilde{\mu}}$ to the double coset $W_\lambda w W_\mu$. Thanks to [15, 11.3], we deduce that r_{w_0} reverses the Bruhat order. Since the double coset $W_\lambda w w_0 W_{\tilde{\mu}}$ has a unique element of minimal length, the result follows. We write $D_{\lambda,\mu}^+$ for the set of double coset representatives of maximal length. We denote by $\ell_{\lambda,\mu}$ the bijection from $D_{\lambda,\mu}$ to $D_{\lambda,\mu}^+$ that associates to the representative of minimal length w of the double coset $W_\lambda w W_\mu$ the representative of maximal length. We remark that if $w \in D_{\lambda,\mu}$, then $w^{-1} \in D_{\mu,\lambda}$. Moreover, we have

$$\ell_{\lambda,\mu}(w)^{-1} = \ell_{\mu,\lambda}(w^{-1}).$$

In the following, we set $\sigma(\underline{a}) := \ell_{\lambda,\mu}(w)$ for $\underline{a} = (\lambda, w, \mu)$.

We now recall the definition of the q -Schur algebra $\mathcal{S}_q(n, r)$ introduced by Dipper and James in [3]. We set $q = v^2$, then the q -Schur algebra $\mathcal{S}_q(n, r)$ of degree (n, r) is the endomorphism algebra

$$\mathcal{S}_q(n, r) = \text{End}_{\mathcal{H}} \left(\bigoplus_{\lambda \in \Lambda(n, r)} x_\lambda \mathcal{H} \right),$$

where $x_\lambda = \sum_{w \in W_\lambda} v^{l(w)} T_w \in \mathcal{H}$. In [2, 3.4] Dipper and James prove that $\mathcal{S}_q(n, r)$ has a standard basis $\{\phi_{\lambda,\mu}^w \mid (\lambda, w, \mu) \in M(n, r)\}$ indexed by the set $M(n, r)$, which plays the same role as the basis $\{T_w \mid w \in W\}$ for the Iwahori-Hecke algebra \mathcal{H} . Moreover, in [5] Du proves that $\mathcal{S}_q(n, r)$ has another basis $\{\theta_{\underline{a}} \mid \underline{a} \in M(n, r)\}$ whose construction is analogous to the Kazhdan-Lusztig basis of \mathcal{H} . We denote by $f_{\underline{a}, \underline{b}, \underline{c}} \in A$ the structure constants with respect to this basis, that is, we have

$$\theta_{\underline{a}} \theta_{\underline{b}} = \sum_{\underline{c} \in M(n, r)} f_{\underline{a}, \underline{b}, \underline{c}} \theta_{\underline{c}} \quad \text{for all } \underline{a}, \underline{b} \in M(n, r).$$

We recall the following lemma:

Lemma 2.3. *We have $f_{\underline{a}, \underline{b}, \underline{c}} \neq 0$ only if $co(\underline{a}) = ro(\underline{b})$ and $(ro(\underline{a}), co(\underline{b})) = (ro(\underline{c}), co(\underline{c}))$. In this case, we have*

$$f_{\underline{a}, \underline{b}, \underline{c}} = h_\mu^{-1} g_{\sigma(\underline{a}), \sigma(\underline{b}), \sigma(\underline{c})}.$$

where $\mu = co(\underline{a}) = ro(\underline{b})$ and $h_\mu = \sum_{w \in W_\mu} v^{2l(w)-l(w_\mu)}$ (here w_μ denotes the longest element in W_μ) and $g_{\sigma(\underline{a}), \sigma(\underline{b}), \sigma(\underline{c})}$ is the structure constant of \mathcal{H} defined in Section 2.1.

Proof. See [5, Prop. 3.4]. We want to explain why we have a further hypothesis here than in [5, Prop. 3.4]: For $\underline{a} = (\lambda, w, \mu) \in M(n, r)$ the element $\theta_{\underline{a}}$ is by definition a linear combination of basis elements $\phi_{\lambda,\mu}^z$ for $z \in \mathcal{D}_{\lambda,\mu}$. Thus, viewed as endomorphism of $\bigoplus_{\lambda \in \Lambda(n, r)} x_\lambda \mathcal{H}$ it vanishes on all summands except on $x_\mu \mathcal{H}$ and maps into the summand $x_\lambda \mathcal{H}$. Thus, if either $co(\underline{a}) \neq ro(\underline{b})$ or $(ro(\underline{a}), co(\underline{b})) \neq (ro(\underline{c}), co(\underline{c}))$, the structure constant $f_{\underline{a}, \underline{b}, \underline{c}}$ vanishes also. If both equations hold, the proof in [5, Prop. 3.4] works using $g_{\sigma(\underline{a}), \sigma(\underline{b}), \sigma(\underline{c})}$.

We are not claiming that [5, Prop. 3.4] is wrong as stated there. However, the notation $g_{\underline{a}, \underline{b}, \underline{c}}$ there needs proper interpretation (see [5, Section 3.3]), a problem we avoid here. \square

Remark 2.4. To further explain the just mentioned change of notation, consider the following: Let $n = r = 3$, $\lambda := (2, 1, 0)$, $\mu := (1, 1, 1)$, and $\nu := (2, 1, 0)$. Then W is the symmetric group on 3 letters, generated by the two Coxeter generators $s_1 =$

(1, 2) and $s_2 = (2, 3)$. Thus $\mathcal{D}_{\lambda, \mu}^+ := \{s_1, s_1 s_2, s_1 s_2 s_1\}$, $\mathcal{D}_{\mu, \nu}^+ = \{s_1, s_2 s_1, s_1 s_2 s_1\}$ and $\mathcal{D}_{\lambda, \nu}^+ = \{s_1, s_1 s_2 s_1\}$.

By the relations, we have $T_{s_1} \cdot T_{s_2 s_1} = T_{s_1 s_2 s_1}$ and thus $g_{s_1, s_2 s_1, s_1 s_2 s_1} = 1$. We now set $\underline{a} := (\lambda, \text{id}, \mu)$, $\underline{b} := (\mu, s_2, \nu)$ and $\underline{c} := (\lambda, s_2, \nu)$. Thus, we get

$$f_{\underline{a}, \underline{b}, \underline{c}} = 1 \cdot g_{\sigma(\underline{a}), \sigma(\underline{b}), \sigma(\underline{c})} = g_{s_1, s_2 s_1, s_1 s_2 s_1} = 1,$$

since here $h_\mu = 1$.

However, if we set $\underline{a}' := (\mu, s_1, \mu)$, then $f_{\underline{a}', \underline{b}, \underline{c}} = 0$, because of $ro(\underline{a}') \neq ro(\underline{c})$ and the arguments in the proof of Lemma 2.3. On the other hand, we have $ro(\underline{a}') = co(\underline{b})$ and $g_{\sigma(\underline{a}'), \sigma(\underline{b}), \sigma(\underline{c})} = g_{s_1, s_2 s_1, s_1 s_2 s_1} = 1$. This shows that we indeed need all of the hypotheses in Lemma 2.3. The statement in [5, Prop. 3.4] is true if one interprets $g_{\underline{a}', \underline{b}, \underline{c}}$ to be zero.

Definition 2.5 (The \mathbf{a} -function and the distinguished elements). Following [7, Section 2], we extend the \mathbf{a} -function to $M(n, r)$ by setting $\mathbf{a}(\underline{a}) = \mathbf{a}(\sigma(\underline{a}))$ for every $\underline{a} \in M(n, r)$ and we extend the set \mathcal{D} to the set

$$\mathcal{D}(n, r) = \{\underline{d} \in M(n, r) \mid co(\underline{d}) = ro(\underline{d}), \sigma(\underline{d}) \in \mathcal{D}\}.$$

Moreover, for every $\underline{a}, \underline{b}, \underline{c} \in M(n, r)$, we define

$$\gamma_{\underline{a}, \underline{b}, \underline{c}^t} = \begin{cases} \gamma_{\sigma(\underline{a}), \sigma(\underline{b}), \sigma(\underline{c}^t)} = \gamma_{\sigma(\underline{a}), \sigma(\underline{b}), \sigma(\underline{c})}^{-1} & \text{if } f_{\underline{a}, \underline{b}, \underline{c}} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.6. Note that our definition for $\gamma_{\underline{a}, \underline{b}, \underline{c}}$ differs slightly from the one in [7, Section 2.2]. His $\gamma_{\underline{a}, \underline{b}, \underline{c}}$ is our $\gamma_{\underline{a}, \underline{b}, \underline{c}^t}$. With our definition we follow the setup in [15] more closely and get nicer cyclic symmetries in our formulas.

Remark 2.7. In comparison to [7, Section 2.1] we added the explicit hypothesis for the elements $\underline{d} \in \mathcal{D}(n, r)$ that $ro(\underline{d}) = co(\underline{d})$. However, this hypothesis is implicit in [7], since otherwise the statements in [7, 4.1,(a)–(d)] and some others would not be true.

Now, for $\underline{a}, \underline{b} \in M(n, r)$, if there is $\underline{c} \in M(n, r)$ such that $f_{\underline{c}, \underline{b}, \underline{a}} \neq 0$, then we write $\underline{a} \leq_L \underline{b}$. We define \leq_R by $\underline{a} \leq_R \underline{b}$ if and only if $\underline{a}^t \leq_L \underline{b}^t$. Moreover, we define \leq_{LR} as in the Iwahori-Hecke algebra case. These relations induce corresponding equivalence relations \sim_L , \sim_R and \sim_{LR} . We call the corresponding equivalence classes the left, right and two-sided cells of $M(n, r)$, respectively.

Let Γ be a left cell of $M(n, r)$. We set

$$\mathcal{S}_{\leq \Gamma} = \sum_{\underline{b} \leq_L \underline{a}} A\theta_{\underline{b}} \quad \text{and} \quad \mathcal{S}_{< \Gamma} = \sum_{\underline{b} \leq_L \underline{a}, \underline{b} \not\sim_L \underline{a}} A\theta_{\underline{b}},$$

for some $\underline{a} \in \Gamma$, both are clearly left ideals of $\mathcal{S}_q(n, r)$ by the definition of \leq_L . Then the left cell module $\text{LC}^{(\Gamma)}$ corresponding to Γ is defined as the quotient $\mathcal{S}_{\leq \Gamma} / \mathcal{S}_{< \Gamma}$.

We define the right cell module $\text{RC}^{(\Gamma)}$ corresponding to a right cell Γ of $M(n, r)$ similarly. To see that we get right ideals we have to use Lemma 2.3 and $g_{x, y, z} = g_{y^{-1}, x^{-1}, z^{-1}}$ for $x, y, z \in W$ (see [15, 13.2.(e)]) together with $\sigma(\underline{a}^t) = \sigma(\underline{a})^{-1}$. This implies $f_{\underline{a}, \underline{b}, \underline{c}} = 0$ if and only if $f_{\underline{b}^t, \underline{a}^t, \underline{c}^t} = 0$.

3. LUSZTIG'S CONJECTURES FOR THE q -SCHUR ALGEBRA

In this section, we prove that the q -Schur algebra satisfies properties very similar to **P1**, \dots , **P15** for the Iwahori-Hecke algebra. First, we give some preliminary results.

Lemma 3.1. *If $\underline{a} \leq_L \underline{b}$ (resp. \leq_R, \leq_{LR}), then $\sigma(\underline{a}) \leq_L \sigma(\underline{b})$ (resp. \leq_R, \leq_{LR}).*

Proof. Since $\underline{a} \leq_L \underline{b}$, there is $\underline{c} \in M(n, r)$ such that $f_{\underline{c}, \underline{b}, \underline{a}} \neq 0$; but we have $f_{\underline{c}, \underline{b}, \underline{a}} = h_{\text{co}(\underline{a})}^{-1} g_{\sigma(\underline{c}), \sigma(\underline{b}), \sigma(\underline{a})}$ with $h_{\text{co}(\underline{a})}^{-1} \neq 0$. Thus $g_{\sigma(\underline{c}), \sigma(\underline{b}), \sigma(\underline{a})} \neq 0$ and $\sigma(\underline{a}) \leq_L \sigma(\underline{b})$. \square

Lemma 3.2. *If $\underline{a} \leq_L \underline{b}$, then $\text{co}(\underline{a}) = \text{co}(\underline{b})$. If $\underline{a} \leq_R \underline{b}$, then $\text{ro}(\underline{a}) = \text{ro}(\underline{b})$.*

Proof. Since $\underline{a} \leq_L \underline{b}$ there is $\underline{c} \in M(n, r)$ such that $f_{\underline{c}, \underline{b}, \underline{a}} \neq 0$. From Lemma 2.3 it follows that $(\text{ro}(\underline{a}), \text{co}(\underline{a})) = (\text{ro}(\underline{c}), \text{co}(\underline{b}))$ and the result is proved. \square

Lemma 3.3. *Let $\lambda, \mu, \nu \in \Lambda(n, r)$, $x \in D_{\lambda, \mu}^+$ and $y \in D_{\mu, \nu}^+$. If $g_{x, y, z} \neq 0$ for some $z \in W$, then $z \in D_{\lambda, \nu}^+$.*

Proof. For $\lambda \in \Lambda(n, r)$ we set $S_\lambda := W_\lambda \cap S$, the set of Coxeter generators of the parabolic subgroup W_λ . Let $x \in D_{\lambda, \mu}^+$ and $y \in D_{\mu, \nu}^+$ and $g_{x, y, z} \neq 0$. On one hand, this means that $l(sx) < l(x)$ for all $s \in S_\lambda$ and $l(ys) < l(y)$ for all $s \in S_\nu$. On the other hand, we get $z \leq_L y$ and $z \leq_R x$ and thus $l(zs) < l(z)$ for all $s \in S$ with $l(ys) < l(y)$ and $l(sz) < l(z)$ for all $s \in S$ with $l(sx) < l(x)$ by [15, Lemma 8.6]. Thus we have, in particular, that $l(zs) < l(z)$ for all $s \in S_\nu$ and $l(sz) < l(z)$ for all $s \in S_\lambda$. Hence z is the longest element in its W_λ - W_ν -double coset in W . \square

Lemma 3.4. *We have $\underline{a} \leq_R \underline{b}$ if and only if there is a $\underline{c} \in M(n, r)$ with $f_{\underline{b}, \underline{c}, \underline{a}} \neq 0$.*

Proof. By definition, $\underline{a} \leq_R \underline{b}$ is equivalent to $\underline{a}^t \leq_L \underline{b}^t$. This in turn means that there is a $\underline{c} \in M(n, r)$ such that $f_{\underline{c}^t, \underline{b}^t, \underline{a}^t} \neq 0$. As mentioned at the end of Section 2.2 we have $f_{\underline{b}, \underline{c}, \underline{a}} = 0$ if and only if $f_{\underline{c}^t, \underline{b}^t, \underline{a}^t} = 0$ which directly implies the statement in the lemma. \square

We now recall the following result of Du [8, 2.2, 2.3].

Lemma 3.5. *For any $\underline{a}, \underline{b} \in M(n, r)$, we have*

- (1) $\underline{a} \sim_{LR} \underline{b} \iff \sigma(\underline{a}) \sim_{LR} \sigma(\underline{b})$.
- (2) $\underline{a} \sim_L \underline{b} \iff \text{co}(\underline{a}) = \text{co}(\underline{b}) \quad \text{and} \quad \sigma(\underline{a}) \sim_L \sigma(\underline{b})$.

Remark 3.6. Note that to prove this result, the set $M(n, r)$ is identified with the set of $n \times n$ matrices whose entry sum is r . If $(a_{ij})_{1 \leq i, j \leq n}$ is the matrix corresponding to $\underline{a} \in M(n, r)$, then $\text{co}(\underline{a}) = (\sum_i a_{i1}, \dots, \sum_i a_{in})$ and $\text{ro}(\underline{a}) = (\sum_i a_{1i}, \dots, \sum_i a_{ni})$; see [8, §2] for more details.

Proposition 3.7. *The following properties hold for the q -Schur algebra:*

- Q1** For any $\underline{a} \in M(n, r)$ we have $\mathbf{a}(\underline{a}) \leq \Delta(\sigma(\underline{a}))$.
- Q2** If $\gamma_{\underline{a}, \underline{b}, \underline{d}} \neq 0$ for some $\underline{d} \in \mathcal{D}(n, r)$, then we have $\underline{b} = \underline{a}^t$.
- Q3** For every $\underline{a} \in M(n, r)$, there is a unique $\underline{d} \in \mathcal{D}(n, r)$ with $\gamma_{\underline{a}^t, \underline{a}, \underline{d}} \neq 0$.
- Q4** If $\underline{a} \leq_{LR} \underline{b}$, then $\mathbf{a}(\underline{a}) \geq \mathbf{a}(\underline{b})$.
- Q5** If $\underline{d} \in \mathcal{D}(n, r)$ and $\underline{a} \in M(n, r)$ are such that $\gamma_{\underline{a}^t, \underline{a}, \underline{d}} \neq 0$, then $\gamma_{\underline{a}^t, \underline{a}, \underline{d}} = 1$.
- Q6** For $\underline{d} \in \mathcal{D}(n, r)$, we have $\underline{d} = \underline{d}^t$.
- Q7** For every $\underline{a}, \underline{b}, \underline{c} \in M(n, r)$, we have $\gamma_{\underline{a}, \underline{b}, \underline{c}} = \gamma_{\underline{b}, \underline{c}, \underline{a}} = \gamma_{\underline{c}, \underline{a}, \underline{b}}$.
- Q8** Let $\underline{a}, \underline{b}, \underline{c} \in M(n, r)$ be such that $\gamma_{\underline{a}, \underline{b}, \underline{c}} \neq 0$, then $\underline{a} \sim_L \underline{b}^t$, $\underline{b} \sim_L \underline{c}^t$ and $\underline{c} \sim_L \underline{a}^t$.
- Q9** If $\underline{a} \leq_L \underline{b}$ and $\mathbf{a}(\underline{a}) = \mathbf{a}(\underline{b})$, then $\underline{a} \sim_L \underline{b}$.
- Q10** If $\underline{a} \leq_R \underline{b}$ and $\mathbf{a}(\underline{a}) = \mathbf{a}(\underline{b})$, then $\underline{a} \sim_R \underline{b}$.
- Q11** If $\underline{a} \leq_{LR} \underline{b}$ and $\mathbf{a}(\underline{a}) = \mathbf{a}(\underline{b})$, then $\underline{a} \sim_{LR} \underline{b}$.
- Q13** Every left cell contains a unique element $\underline{d} \in \mathcal{D}(n, r)$ and $\gamma_{\underline{a}^t, \underline{a}, \underline{d}} \neq 0$ for every $\underline{a} \sim_L \underline{d}$.
- Q14** For every $\underline{a} \in M(n, r)$, we have $\underline{a} \sim_{LR} \underline{a}^t$.
- Q15** Let v' be a second indeterminate and let $f'_{x,y,z} \in \mathbb{Z}[v', v'^{-1}]$ be obtained from $f_{x,y,z}$ by the substitution $v \mapsto v'$. If $\underline{a}, \underline{a}', \underline{b}, \underline{c} \in W$ satisfy $\mathbf{a}(\underline{c}) = \mathbf{a}(\underline{b})$, then

$$\sum_{\underline{b}'} f'_{\underline{c}, \underline{a}', \underline{b}'} f_{\underline{a}, \underline{b}', \underline{b}} = \sum_{\underline{b}'} f_{\underline{a}, \underline{c}, \underline{b}'} f'_{\underline{b}', \underline{a}', \underline{b}}.$$

Proof. We note that **Q1** is a direct consequence of Property **P1**.

We now will prove Property **Q2**. We suppose that $\gamma_{\underline{a}, \underline{b}, \underline{d}} \neq 0$ for some $\underline{a}, \underline{b} \in M(n, r)$ and $\underline{d} \in \mathcal{D}(n, r)$. Since $\gamma_{\underline{a}, \underline{b}, \underline{d}} \neq 0$, it follows that $f_{\underline{a}, \underline{b}, \underline{d}} \neq 0$. Thus we have $co(\underline{a}) = ro(\underline{b})$, $ro(\underline{a}) = ro(\underline{d})$ and $co(\underline{b}) = co(\underline{d})$ by Lemma 2.3; but $co(\underline{d}) = ro(\underline{d})$ implies $ro(\underline{a}) = co(\underline{b})$. We now write $\underline{a} = (\lambda, w_a, \mu)$ and $\underline{b} = (\mu, w_b, \lambda)$. We have $\gamma_{\underline{a}, \underline{b}, \underline{d}} = \gamma_{\sigma(\underline{a}), \sigma(\underline{b}), \sigma(\underline{d})}$. From $\sigma(\underline{d}) \in \mathcal{D}$ we deduce using **P2** that $\sigma(\underline{a}) = \sigma(\underline{b})^{-1}$. It follows that $\ell_{\lambda, \mu}(w_a) = \ell_{\mu, \lambda}(w_b)^{-1} = \ell_{\lambda, \mu}(w_b^{-1})$, we get $w_a = w_b^{-1}$ and thus **Q2** holds.

Let $\underline{a} = (\lambda, w, \mu) \in M(n, r)$. Thanks to Property **P3**, there is a unique $d \in \mathcal{D}$ such that $\gamma_{\sigma(\underline{a})^{-1}, \sigma(\underline{a}), d} \neq 0$. Since $\sigma(\underline{a})^{-1} = \sigma(\underline{a}^t)$, we deduce that $g_{\sigma(\underline{a}^t), \sigma(\underline{a}), d} \neq 0$; but $\sigma(\underline{a}^t) \in D_{\mu, \lambda}^+$ and $\sigma(\underline{a}) \in D_{\lambda, \mu}^+$, then Lemma 3.3 gives $d \in D_{\mu, \mu}^+$. We denote by \tilde{d} the representative of minimal length of the coset $W_\mu d W_\mu$ and we set $\underline{d} := (\mu, \tilde{d}, \mu)$. Then $\underline{d} \in \mathcal{D}(n, r)$ and $\sigma(\underline{d}) = d$. It follows that $\gamma_{\underline{a}^t, \underline{a}, \underline{d}} \neq 0$ and thus **Q3** holds.

Property **Q4** follows from **P4** and Lemma 3.1. Property **Q5** directly follows from **P5**, since in our case W is of type A and thus all coefficients of all Kazhdan-Lusztig polynomials are non-negative by [15, 15.1].

Let $\underline{d} = (\lambda, w, \lambda) \in \mathcal{D}(n, r)$; we have $\sigma(\underline{d}) \in \mathcal{D}$, thus **P6** gives $\sigma(\underline{d})^{-1} = \sigma(\underline{d})$. Therefore, we have $\ell_{\lambda, \lambda}(w) = \sigma(\underline{d})^{-1} = \sigma(\underline{d}^t) = \ell_{\lambda, \lambda}(w^{-1})$, and it follows that $w = w^{-1}$; thus **Q6** holds. Property **Q7** follows directly from **P7**.

Suppose that $\gamma_{\underline{a}, \underline{b}, \underline{c}} \neq 0$ for some $\underline{a}, \underline{b}, \underline{c} \in M(n, r)$, then $f_{\underline{a}, \underline{b}, \underline{c}^t} \neq 0$ and it follows that $co(\underline{a}) = ro(\underline{b})$ and $(ro(\underline{a}), co(\underline{b})) = (ro(\underline{c}^t), co(\underline{c}^t))$. Then we have

$$\begin{aligned} f_{\underline{b}^t, \underline{a}^t, \underline{c}} &= h_{co(\underline{a})} g_{\sigma(\underline{b}^t), \sigma(\underline{a}^t), \sigma(\underline{c})} \\ &= h_{co(\underline{a})} g_{\sigma(\underline{b})^{-1}, \sigma(\underline{a})^{-1}, \sigma(\underline{c})} \\ &= h_{co(\underline{a})} g_{\sigma(\underline{a}), \sigma(\underline{b}), \sigma(\underline{c})^{-1}} \\ &= f_{\underline{a}, \underline{b}, \underline{c}^t}. \end{aligned}$$

It follows that $\underline{c}^t \leq_L \underline{b}$ and $\underline{c} \leq_L \underline{a}^t$. Using **Q7** and the same arguments applied to $\gamma_{\underline{b}, \underline{c}, \underline{a}} = \gamma_{\underline{c}, \underline{a}, \underline{b}} \neq 0$, we deduce that $\underline{a} \sim_L \underline{b}^t$, $\underline{b} \sim_L \underline{c}^t$ and $\underline{c} \sim_L \underline{a}^t$. Thus **Q8** holds.

Next we prove **Q13**. Let $\underline{a} \in M(n, r)$. By **Q3** there is a unique $\underline{d} \in \mathcal{D}(n, r)$ with $\gamma_{\underline{a}^t, \underline{a}, \underline{d}} \neq 0$ and for this \underline{d} holds $\underline{a} \sim_L \underline{d}$ by **Q8**. But for $\underline{d}, \underline{d}' \in \mathcal{D}(n, r)$ with $\underline{d} \sim_L \underline{d}'$ we conclude $ro(\underline{d}) = co(\underline{d}) = co(\underline{d}') = ro(\underline{d}')$ using Lemma 3.2 and $\sigma(\underline{d}) = \sigma(\underline{d}')$ using **P13** since $\sigma(\underline{d}) \sim_L \sigma(\underline{d}')$ because of Lemma 3.1. Thus we have proved **Q13**.

Now, we prove **Q9**. Let $\underline{a}, \underline{b} \in M(n, r)$ with $\underline{a} \leq_L \underline{b}$ and $\mathbf{a}(\underline{a}) = \mathbf{a}(\underline{b})$. By Lemma 3.1, we have $\sigma(\underline{a}) \leq_L \sigma(\underline{b})$. Hence, Property **P9** implies that $\sigma(\underline{a}) \sim_L \sigma(\underline{b})$. Moreover, by Lemma 3.2, $co(\underline{a}) = co(\underline{b})$. We then conclude with Lemma 3.5. The proof of Property **Q11** is similar. Property **Q10** follows from **Q9** by transposition since $\mathbf{a}(\underline{a}) = \mathbf{a}(\underline{a}^t)$ for all $\underline{a} \in M(n, r)$ (use [15, 13.9 (a)]).

Let $\underline{a} \in M(n, r)$ and $\underline{d} \in \mathcal{D}(n, r)$ be the unique element such that $\underline{a} \sim_L \underline{d}$ given by **Q13**. Then $\underline{a}^t \sim_R \underline{d}^t = \underline{d}$ and **Q14** holds.

Finally, we prove **Q15**. We first remark that $f'_{\underline{c}, \underline{a}', \underline{b}'} \neq 0$ if and only if $f_{\underline{a}, \underline{c}, \underline{b}'} \neq 0$, and $f_{\underline{a}, \underline{b}', \underline{b}} \neq 0$ if and only if $f'_{\underline{b}', \underline{a}', \underline{b}} \neq 0$. Moreover, if $f'_{\underline{c}, \underline{a}', \underline{b}'} \neq 0$, then $f'_{\underline{c}, \underline{a}', \underline{b}'} = h'_{ro(\underline{a}')} g_{\sigma(\underline{c}), \sigma(\underline{a}'), \sigma(\underline{b}')}$ and $f_{\underline{a}, \underline{c}, \underline{b}'} = h_{co(\underline{a})} g_{\sigma(\underline{a}), \sigma(\underline{c}), \sigma(\underline{b}')}$. If $f_{\underline{a}, \underline{c}, \underline{b}'} \neq 0$, then $f_{\underline{a}, \underline{c}, \underline{b}'} = h_{co(\underline{a})} g_{\sigma(\underline{a}), \sigma(\underline{c}), \sigma(\underline{b}')}$ and $f'_{\underline{b}', \underline{a}', \underline{b}} = h'_{ro(\underline{a}')} g_{\sigma(\underline{b}'), \sigma(\underline{a}'), \sigma(\underline{b})}$. Here h'_μ is obtained from h_μ by the substitution $v \mapsto v'$. We note that $h_{ro(\underline{a}')}$ and $h_{co(\underline{a})}$ do not depend on \underline{b}' . It follows from **P15** that

$$\begin{aligned} \sum_{\underline{b}'} f'_{\underline{c}, \underline{a}', \underline{b}'} f_{\underline{a}, \underline{b}', \underline{b}} &= h_{ro(\underline{a}')} h_{co(\underline{a})} \sum_{\underline{b}'} g'_{\sigma(\underline{c}), \sigma(\underline{a}'), \sigma(\underline{b}')} g_{\sigma(\underline{a}), \sigma(\underline{b}'), \sigma(\underline{b})} \\ &= h_{ro(\underline{a}')} h_{co(\underline{a})} \sum_{\underline{b}'} g_{\sigma(\underline{a}), \sigma(\underline{c}), \sigma(\underline{b}')} f'_{\sigma(\underline{b}'), \sigma(\underline{a}'), \sigma(\underline{b})} \\ &= \sum_{\underline{b}'} f_{\underline{a}, \underline{c}, \underline{b}'} f'_{\underline{b}', \underline{a}', \underline{b}} \quad \square \end{aligned}$$

Proposition 3.8. *If $\underline{a} \sim_L \underline{b}$ and $\underline{a} \sim_R \underline{b}$, then $\underline{a} = \underline{b}$.*

Proof. By [7, 4.2(a)–(e)] and [7, 4.4], the intersection of a left cell and a right cell both inside a two-sided cell contains exactly one element, as required. \square

4. IRREDUCIBLE CELL MODULES AND DUAL BASIS

In this section we view the extension of scalars $K\mathcal{S}_q(n, r)$ of the q -Schur algebra $\mathcal{S}_q(n, r)$ as a symmetric algebra in the sense of [12, 7.1.1]. This is possible, since it is semisimple (see [1, (9.8)]). We can take as symmetrising trace form any K -linear form $\tau : K\mathcal{S}_q(n, r) \rightarrow K$ that is a K -linear combination

$$(4.1) \quad \tau = \sum_{\chi \in \text{Irr}(K\mathcal{S}_q(n, r))} \frac{\chi}{c_\chi}$$

of the irreducible characters where the $c_\chi \in K$ are non-zero constants, the so-called Schur elements (see [12, 7.1.1 and 7.2.6]). Clearly, τ is non-degenerate. Note that in [12, 7.2] it is shown that a symmetrising trace form uniquely determines the Schur elements via equation (4.1) and vice versa. Therefore, we will talk about Schur elements whenever a symmetrising trace form is given in the rest of this paper.

Having fixed τ , we denote for any K -basis $(B_{\underline{a}})_{\underline{a} \in M(n, r)}$ of $K\mathcal{S}_q(n, r)$ its dual basis with respect to τ by $(B_{\underline{b}}^\vee)_{\underline{b} \in M(n, r)}$. That is, we have $\tau(B_{\underline{a}} \cdot B_{\underline{b}}^\vee) = \tau(B_{\underline{b}}^\vee \cdot B_{\underline{a}}) = \delta_{\underline{a}, \underline{b}}$ for all $\underline{a}, \underline{b} \in M(n, r)$. Note that this immediately implies that we can write every element $x \in K\mathcal{S}_q(n, r)$ in the following form:

$$(4.2) \quad x = \sum_{\underline{a} \in M(n, r)} \tau(x \cdot B_{\underline{a}}^\vee) B_{\underline{a}} = \sum_{\underline{a} \in M(n, r)} \tau(x \cdot B_{\underline{a}}) B_{\underline{a}}^\vee$$

(just write x as a linear combination of the $B_{\underline{a}}$, multiply by some $B_{\underline{b}}$ and apply τ).

Remark 4.1. We have $f_{\underline{a}, \underline{b}, \underline{c}} = \tau(\theta_{\underline{a}} \cdot \theta_{\underline{b}} \cdot \theta_{\underline{c}}^{\vee})$ for all $\underline{a}, \underline{b}, \underline{c} \in M(n, r)$. Moreover, we note that formula (4.2) immediately gives us nice formulas for the matrix representations coming from the left cell modules. For a left cell Γ and an element $h \in \mathcal{S}_q(n, r)$ the representing matrix of h on the left cell module $\text{LC}^{(\Gamma)}$ with respect to the basis $\{\theta_{\underline{a}} + \mathcal{S}_{<\Gamma} \mid \underline{a} \in \Gamma\}$ is $\left(\tau(\theta_{\underline{b}}^{\vee} \cdot h \cdot \theta_{\underline{a}})\right)_{\underline{b}, \underline{a} \in \Gamma}$ since $h \cdot \theta_{\underline{a}} = \sum_{\underline{b} \in M(n, r)} \tau(\theta_{\underline{b}}^{\vee} \cdot h \cdot \theta_{\underline{a}}) \cdot \theta_{\underline{b}}$ and it is enough to sum over those \underline{b} with $\underline{b} \leq_L \underline{a}$.

Lemma 4.2 (Characterisation of \leq_L and \leq_R). *We have $\underline{a} \leq_L \underline{b}$ if and only if $\theta_{\underline{b}} \theta_{\underline{a}}^{\vee} \neq 0$ and $\underline{a} \leq_R \underline{b}$ if and only if $\theta_{\underline{a}}^{\vee} \theta_{\underline{b}} \neq 0$.*

Proof. We only show the version with \leq_L , the other is completely analogous thanks to Lemma 3.4. If $\underline{a} \leq_L \underline{b}$ there exists a $\underline{c} \in M(n, r)$ with $f_{\underline{c}, \underline{b}, \underline{a}} = \tau(\theta_{\underline{c}} \theta_{\underline{b}} \theta_{\underline{a}}^{\vee}) \neq 0$ which implies $\theta_{\underline{b}} \theta_{\underline{a}}^{\vee} \neq 0$. If we assume the latter, then by the non-degeneracy of τ there is some $\underline{c} \in M(n, r)$ with $\tau(\theta_{\underline{c}} \theta_{\underline{b}} \theta_{\underline{a}}^{\vee}) \neq 0$ and $\underline{a} \leq_L \underline{b}$ follows. \square

The other major ingredient is the fact that cell modules are simple, more precisely:

Theorem 4.3 (Simple cell modules, see [6] or [7, 4.3]). *Let Γ be a left cell and recall $K = \mathbb{Q}(v)$. The extension of scalars $K \text{LC}^{(\Gamma)}$ of the left cell module $\text{LC}^{(\Gamma)}$ for a left cell Γ is a simple $K\mathcal{S}_q(n, r)$ -module.*

Proof. See [6] or [7, 4.3]. \square

Remark 4.4. This, in particular, implies that all simple $K\mathcal{S}_q(n, r)$ -modules can be realised over the ring A , since their corresponding representing matrices involve only structure constants of $\mathcal{S}_q(n, r)$.

We now directly obtain useful algebra elements by using the simple cell modules:

Theorem 4.5 (Basis of an isotypic component). *Let Γ be a left cell and χ the corresponding irreducible character of the left cell module $\text{LC}^{(\Gamma)}$, then the elements*

$$(c_{\chi}^{-1} \theta_{\underline{a}} \theta_{\underline{b}}^{\vee})_{\underline{a}, \underline{b} \in \Gamma}$$

are K -linearly independent and span the isotypic component of $K\mathcal{S}_q(n, r)$ belonging to the character χ . Furthermore, we have the relations

$$(c_{\chi}^{-1} \theta_{\underline{a}} \theta_{\underline{b}}^{\vee}) \cdot (c_{\chi}^{-1} \theta_{\underline{a}'} \theta_{\underline{b}'}^{\vee}) = \delta_{\underline{b}, \underline{a}'} \cdot c_{\chi}^{-1} \theta_{\underline{a}} \theta_{\underline{b}'}^{\vee}$$

for all $\underline{a}, \underline{b}, \underline{a}', \underline{b}' \in \Gamma$. That is, these elements form a matrix unit for the isotypic component of $K\mathcal{S}_q(n, r)$ corresponding to the simple module $K \text{LC}^{(\Gamma)}$.

Proof. By [12, 7.2.7] we get a matrix unit for the isotypic component of $K\mathcal{S}_q(n, r)$ corresponding to the simple module $K \text{LC}^{(\Gamma)}$ by the elements

$$\frac{1}{c_{\chi}} \sum_{\underline{c} \in M(n, r)} \tau(\theta_{\underline{b}}^{\vee} \cdot \theta_{\underline{c}} \cdot \theta_{\underline{a}}) \cdot \theta_{\underline{c}}^{\vee} = \frac{1}{c_{\chi}} \sum_{\underline{c} \in M(n, r)} \tau(\theta_{\underline{c}} \cdot \theta_{\underline{a}} \theta_{\underline{b}}^{\vee}) \cdot \theta_{\underline{c}}^{\vee}$$

for $\underline{a}, \underline{b} \in \Gamma$. But this is equal to $c_{\chi}^{-1} \theta_{\underline{a}} \theta_{\underline{b}}^{\vee}$ by formula (4.2). \square

Corollary 4.6. *Let Γ be a left cell and χ the corresponding irreducible character of the left cell module $\text{LC}^{(\Gamma)}$. Then the element*

$$e_\Gamma := \frac{1}{c_\chi} \sum_{\underline{a} \in \Gamma} \theta_{\underline{a}} \theta_{\underline{a}}^\vee$$

is the central primitive idempotent of $K\mathcal{S}_q(n, r)$ corresponding to the irreducible character χ .

Proof. By Theorem 4.5 e_Γ lies in the isotypic component corresponding to the character χ and is mapped to the identity matrix in the corresponding matrix representation. \square

Lemma 4.7 (Isomorphism of left cell modules and two-sided cells). *Let Γ and Γ' be left cells. If $K\text{LC}^{(\Gamma)}$ and $K\text{LC}^{(\Gamma')}$ are isomorphic $K\mathcal{S}_q(n, r)$ -modules, then Γ and Γ' lie in the same two-sided cell.*

Proof. Let χ be the irreducible character of the left cell module $\text{LC}^{(\Gamma)}$ and χ' that of $\text{LC}^{(\Gamma')}$. The modules $K\text{LC}^{(\Gamma)}$ and $K\text{LC}^{(\Gamma')}$ are isomorphic if and only if $e_\Gamma \cdot e_{\Gamma'} = e_{\Gamma'} \cdot e_\Gamma \neq 0$ (and in this case $e_\Gamma = e_{\Gamma'}$). Now assume this case. Then

$$0 \neq \frac{1}{c_\chi^2} \sum_{\underline{a} \in \Gamma} \sum_{\underline{b} \in \Gamma'} \theta_{\underline{a}} \theta_{\underline{a}}^\vee \theta_{\underline{b}} \theta_{\underline{b}}^\vee = \frac{1}{c_\chi^2} \sum_{\underline{a} \in \Gamma} \sum_{\underline{b} \in \Gamma'} \theta_{\underline{b}} \theta_{\underline{b}}^\vee \theta_{\underline{a}} \theta_{\underline{a}}^\vee$$

and thus there is at least one pair $(\underline{a}, \underline{b}) \in \Gamma \times \Gamma'$ such that $\theta_{\underline{a}} \theta_{\underline{b}}^\vee \neq 0$. By Lemma 4.2 this implies $\underline{a} \leq_R \underline{b}$. Since e_Γ and $e_{\Gamma'}$ commute, the same argument shows $\underline{b}' \leq_R \underline{a}'$ for some $\underline{a}' \in \Gamma$ and $\underline{b}' \in \Gamma'$. Thus, Γ and Γ' lie in the same two-sided cell in that case. \square

For what follows we need the following statement about Iwahori-Hecke algebras of type A:

Theorem 4.8 (Equal cell modules in the Iwahori-Hecke algebra). *Let \mathcal{H} be a generic Iwahori-Hecke algebra of type A as in Section 2. Let τ_0 be the standard symmetrising trace form on \mathcal{H} ,*

$$\tau_0 \left(\sum_{w \in W} a_w T_w \right) := a_1,$$

where $1 \in W$ denotes the identity. We denote by $\{D_{w^{-1}}\}_{w \in W}$ the dual basis of $\{C_w\}_{w \in W}$ with respect to τ_0 , that is, we have $\tau_0(C_w \cdot D_{x^{-1}}) = \delta_{w,x}$ for $w, x \in W$.

If $x \sim_L y$ and $z \sim_L w$ and $x \sim_R z$ and $y \sim_R w$, then $C_x D_{y^{-1}} = C_z D_{w^{-1}}$. In particular, we have

$$g_{u,x,y} = \tau_0(C_u C_x D_{y^{-1}}) = \tau_0(C_u C_z D_{w^{-1}}) = g_{u,z,w}$$

for all $u \in W$.

Proof. This statement is already implicitly stated in [14]. Namely, it is shown there in the proof of Theorem 1.4 that the two left cell modules defined by the left cell containing x, y and the one containing z, w are isomorphic since all four lie in the same two-sided cell. The exact statement there is that two W -graphs are isomorphic, which means, in particular, that not only the two left cell modules

are isomorphic, but that even the matrix representations with respect to the bases $\{C_v \mid v \sim_L x\}$ and $\{C_w \mid w \sim_L z\}$ are equal. But this exactly means, that

$$\tau_0(C_u C_x D_{y^{-1}}) = \tau_0(D_{y^{-1}} C_u C_x) = \tau_0(D_{w^{-1}} C_u C_z) = \tau_0(C_u C_z D_{w^{-1}})$$

for all $u \in W$ which we claim. For more details see [17, 1.7.3]. \square

Remark 4.9. Note that Theorem 4.8 implies that the bases for a Specht module labelled by a two-sided cell constructed in [9, Lemma 6.1] are all the same.

Now we begin to use statements **Q1** to **Q14**:

Theorem 4.10 (Equality of different left cell modules). *Let Γ, Γ' be left cells such that $K\text{LC}^{(\Gamma)}$ and $K\text{LC}^{(\Gamma')}$ are isomorphic $KS_q(n, r)$ -modules. Let \underline{d} be the unique element in $\Gamma' \cap \mathcal{D}(n, r)$ (use **Q13**) and $\underline{c} \sim_L \underline{d}$ that is $\underline{c} \in \Gamma'$. Then there are unique $\underline{a}, \underline{b} \in \Gamma$ with $\underline{a} \sim_R \underline{c}$ and $\underline{b} \sim_R \underline{d}$ and we have $\theta_{\underline{a}} \theta_{\underline{b}}^\vee = \theta_{\underline{c}} \theta_{\underline{d}}^\vee$.*

Proof. Let χ be the irreducible character of the left cell module $\text{LC}^{\Gamma'}$. We denote by c_χ the corresponding Schur element. Since $\underline{c} \sim_L \underline{d}$, it follows from Theorem 4.5 that

$$\theta_{\underline{d}} \theta_{\underline{c}}^\vee \theta_{\underline{c}} \theta_{\underline{d}}^\vee = c_\chi \theta_{\underline{d}} \theta_{\underline{d}}^\vee.$$

Therefore, we have $\tau(\theta_{\underline{c}}^\vee \theta_{\underline{c}} \theta_{\underline{d}}^\vee \theta_{\underline{d}}) \neq 0$ and hence $\theta_{\underline{c}} \theta_{\underline{d}}^\vee$ acts non-trivially on the module $\text{LC}^{(\Gamma')}$ (see Remark 4.1) and thus also on the isomorphic module $\text{LC}^{(\Gamma)}$.

This means that there is at least one pair $(\underline{a}, \underline{b}) \in \Gamma \times \Gamma$ such that

$$\tau(\theta_{\underline{b}} \theta_{\underline{a}}^\vee \cdot \theta_{\underline{c}} \theta_{\underline{d}}^\vee) = \tau(\theta_{\underline{a}}^\vee \cdot \theta_{\underline{c}} \theta_{\underline{d}}^\vee \cdot \theta_{\underline{b}}) = \tau(\theta_{\underline{c}} \theta_{\underline{d}}^\vee \cdot \theta_{\underline{b}} \theta_{\underline{a}}^\vee) \neq 0.$$

But then, in particular, $\theta_{\underline{a}}^\vee \theta_{\underline{c}} \neq 0$ and thus $\underline{a} \leq_R \underline{c}$ by Lemma 4.2. Since Γ and Γ' lie in the same two-sided cell by Lemma 4.7, we conclude $\underline{a} \sim_{LR} \underline{c}$ and thus by **Q4** and **Q10** $\underline{a} \sim_R \underline{c}$. Analogously, we show $\underline{b} \sim_R \underline{d}$. By Proposition 3.8 we conclude that there is only one such pair $(\underline{a}, \underline{b})$ since both are uniquely defined by their membership in a left and a right cell.

We now show that $f_{\underline{e}, \underline{a}, \underline{b}} = f_{\underline{e}, \underline{c}, \underline{d}}$ for all $\underline{e} \in M(n, r)$ and thus $\theta_{\underline{a}} \theta_{\underline{b}}^\vee = \theta_{\underline{c}} \theta_{\underline{d}}^\vee$. We have $co(\underline{a}) = co(\underline{b})$ and $co(\underline{c}) = co(\underline{d}) = ro(\underline{d}) = ro(\underline{b})$ and $ro(\underline{a}) = ro(\underline{c})$ by Lemma 3.2 and the fact that $\underline{d} \in \mathcal{D}(n, r)$. Thus, if $ro(\underline{e}) \neq ro(\underline{b})$ or $co(\underline{e}) \neq ro(\underline{a})$, then both sides are zero by Lemma 2.3. Otherwise, we have

$$f_{\underline{e}, \underline{a}, \underline{b}} = h_{co(\underline{e})}^{-1} \cdot g_{\sigma(\underline{e}), \sigma(\underline{a}), \sigma(\underline{b})} \quad \text{and} \quad f_{\underline{e}, \underline{c}, \underline{d}} = h_{co(\underline{e})}^{-1} \cdot g_{\sigma(\underline{e}), \sigma(\underline{c}), \sigma(\underline{d})}$$

and thus the equality $f_{\underline{e}, \underline{a}, \underline{b}} = f_{\underline{e}, \underline{c}, \underline{d}}$ follows from

$$\sigma(\underline{a}) \sim_L \sigma(\underline{b}) \sim_R \sigma(\underline{d}) \sim_L \sigma(\underline{c}) \sim_R \sigma(\underline{a})$$

using Lemma 3.1 and Theorem 4.8. The non-degeneracy of τ now immediately implies $\theta_{\underline{a}} \theta_{\underline{b}}^\vee = \theta_{\underline{c}} \theta_{\underline{d}}^\vee$. \square

With this we get the following result, for which we first need one more piece of notation:

Definition 4.11 (Schur elements of characters of left cell modules). Let $\underline{d} \in \mathcal{D}(n, r)$ and Γ the unique left cell with $\underline{d} \in \Gamma$ (remember **Q13**). We denote the left cell module $\text{LC}^{(\Gamma)}$ by $\text{LC}^{(\underline{d})}$ and the Schur element corresponding to the irreducible character of $\text{LC}^{(\underline{d})}$ by $c_{\underline{d}}$. Note that $c_{\underline{d}} = c_{\underline{d}'}$ if the two left cell modules $\text{LC}^{(\underline{d})}$ and $\text{LC}^{(\underline{d}')}$ are isomorphic, because by [7, 4.3.1, 4.4.1], \underline{d} and \underline{d}' lie in the same two-sided cell if and only if $\text{LC}^{(\underline{d})}$ is isomorphic to $\text{LC}^{(\underline{d}')}$.

Theorem 4.12 (Wedderburn basis). *Let τ be an arbitrary symmetrising trace form (that is, non-degenerate) on $K\mathcal{S}_q(n, r)$. The set*

$$\mathcal{B} := \{c_{\underline{d}}^{-1}\theta_{\underline{c}}\theta_{\underline{d}}^{\vee} \mid \underline{c} \in M(n, r), \underline{d} \in \mathcal{D}(n, r), \underline{c} \sim_L \underline{d}\}$$

is a Wedderburn basis of $K\mathcal{S}_q(n, r)$. Two elements $c_{\underline{d}}^{-1}\theta_{\underline{c}}\theta_{\underline{d}}^{\vee}$ and $c_{\underline{d}'}^{-1}\theta_{\underline{c}'}\theta_{\underline{d}'}^{\vee}$ lie in the same isotypic component if and only if $\mathrm{LC}^{(\underline{d})} \cong \mathrm{LC}^{(\underline{d}')}$.

For $c_{\underline{d}}^{-1}\theta_{\underline{c}}\theta_{\underline{d}}^{\vee}, c_{\underline{d}'}^{-1}\theta_{\underline{c}'}\theta_{\underline{d}'}^{\vee} \in \mathcal{B}$ we have the following equation:

$$c_{\underline{d}}^{-1}\theta_{\underline{c}}\theta_{\underline{d}}^{\vee} \cdot c_{\underline{d}'}^{-1}\theta_{\underline{c}'}\theta_{\underline{d}'}^{\vee} = \begin{cases} 0 & \text{if } \mathrm{LC}^{(\underline{d})} \not\cong \mathrm{LC}^{(\underline{d}')}, \\ 0 & \text{if } \mathrm{LC}^{(\underline{d})} \cong \mathrm{LC}^{(\underline{d}')} \text{ and } \underline{d} \not\sim_R \underline{c}', \\ c_{\underline{d}'}^{-1}\theta_{\underline{c}''}\theta_{\underline{d}'}^{\vee} & \text{if } \mathrm{LC}^{(\underline{d})} \cong \mathrm{LC}^{(\underline{d}')} \text{ and } \underline{d} \sim_R \underline{c}'. \end{cases}$$

Here, \underline{c}'' in the last case is the unique element with $\underline{c}'' \sim_L \underline{d}'$ and $\underline{c}'' \sim_R \underline{c}$ and the statement contains the information that such a \underline{c}'' in fact exists.

Proof. By Theorem 4.5 the elements $c_{\underline{d}}^{-1}\theta_{\underline{c}}\theta_{\underline{d}}^{\vee}$ and $c_{\underline{d}'}^{-1}\theta_{\underline{c}'}\theta_{\underline{d}'}^{\vee}$ both lie in an isotypic component. Thus, if $\mathrm{LC}^{(\underline{d})} \not\cong \mathrm{LC}^{(\underline{d}')}$, then clearly their product is zero.

Now assume that the left cell modules are isomorphic. Let Γ be an arbitrary left cell, such that $K\mathrm{LC}^{(\Gamma)}$ is isomorphic to $K\mathrm{LC}^{(\underline{d})}$ and $K\mathrm{LC}^{(\underline{d}')}$ and denote the corresponding irreducible character by χ . By Theorem 4.10 there are unique $\underline{a}, \underline{b}, \underline{a}', \underline{b}' \in \Gamma$ with

$$\underline{a} \sim_R \underline{c} \text{ and } \underline{b} \sim_R \underline{d} \text{ and } \underline{a}' \sim_R \underline{c}' \text{ and } \underline{b}' \sim_R \underline{d}'$$

and we have $\theta_{\underline{a}}\theta_{\underline{b}}^{\vee} = \theta_{\underline{c}}\theta_{\underline{d}}^{\vee}$ and $\theta_{\underline{a}'}\theta_{\underline{b}'}^{\vee} = \theta_{\underline{c}'}\theta_{\underline{d}'}^{\vee}$. Thus, Theorem 4.5 implies that the product in the theorem is 0 if $\underline{b} \neq \underline{a}'$ and equal to $c_{\chi}^{-1}\theta_{\underline{a}}\theta_{\underline{b}'}^{\vee}$ otherwise. We remark that if $\underline{d} \sim_R \underline{c}'$, then $\underline{a}' \sim_R \underline{b}$ by transitivity. But using Proposition 3.8, $\underline{a}', \underline{b} \in \Gamma$ implies $\underline{b} = \underline{a}'$. Hence $\underline{b} = \underline{a}'$ if and only if $\underline{d} \sim_R \underline{c}'$ which proves case two in the equation.

Finally, we assume also that $\underline{d} \sim_R \underline{c}'$. Then, as \underline{c}'' runs through the left cell that contains \underline{d}' , we can apply Theorem 4.10 to each $\theta_{\underline{c}''}\theta_{\underline{d}'}^{\vee}$ and the left cell Γ . Since $\underline{b}' \in \Gamma$ and $\underline{b}' \sim_R \underline{d}'$ we get that

$$\{\theta_{\underline{c}''}\theta_{\underline{d}'}^{\vee} \mid \underline{c}'' \sim_L \underline{d}'\} = \{\theta_{\underline{a}''}\theta_{\underline{b}'}^{\vee} \mid \underline{a}'' \in \Gamma\}$$

and both sets have cardinality $|\Gamma|$. Thus, there is a unique \underline{c}'' with $\theta_{\underline{c}''}\theta_{\underline{d}'}^{\vee} = \theta_{\underline{a}}\theta_{\underline{b}'}^{\vee}$ characterised by $\underline{a} \sim_R \underline{c}'' \sim_L \underline{d}'$ and the theorem is proved. \square

Corollary 4.13 (Idempotents). *The elements $c_{\underline{d}}^{-1}\theta_{\underline{d}}\theta_{\underline{d}}^{\vee}$ with $\underline{d} \in \mathcal{D}(n, r)$ are pairwise orthogonal primitive idempotents whose sum is the identity $1 \in \mathcal{S}_q(n, r)$. The central primitive idempotent corresponding to an irreducible character χ of $K\mathcal{S}_q(n, r)$ is equal to*

$$\sum_{\substack{\underline{d} \in \mathcal{D}(n, r) \\ \mathrm{LC}^{(\underline{d})} \text{ has character } \chi}} c_{\underline{d}}^{-1}\theta_{\underline{d}}\theta_{\underline{d}}^{\vee}.$$

Proof. This follows directly from Theorems 4.12, 4.10 and 4.5. \square

Corollary 4.14 (Left cell modules as submodules). *Let $\underline{d} \in \mathcal{D}(n, r)$. Then the A -span*

$$\mathcal{L}_{\underline{d}} := \langle \theta_{\underline{c}}\theta_{\underline{d}}^{\vee} \mid \underline{c} \sim_L \underline{d} \rangle_A$$

is a left $\mathcal{S}_q(n, r)$ -module by the multiplication in $K\mathcal{S}_q(n, r)$ that is isomorphic to the left cell module $\mathrm{LC}^{(\underline{d})}$. In fact, the representing matrices with respect to the basis

$(\theta_{\underline{c}} \theta_{\underline{d}}^\vee)_{\underline{c} \sim_L \underline{d}}$ are equal to the representing matrices coming from the left cell module $\mathrm{LC}^{(\underline{d})}$ with respect to its standard basis.

Proof. Let Γ be the left cell that contains \underline{d} . Then by Formula (4.2) we have for every $h \in \mathcal{S}_q(n, r)$:

$$h\theta_{\underline{c}} = \sum_{\underline{c}' \in M(n, r)} \tau(\theta_{\underline{c}'}^\vee \cdot h\theta_{\underline{c}}) \cdot \theta_{\underline{c}'}.$$

Moreover, for $\underline{a} \in A$, there is $\alpha_{\underline{a}} \in A$ such that

$$h = \sum_{\underline{a} \in M(n, r)} \alpha_{\underline{a}} \theta_{\underline{a}}.$$

Hence, for $\underline{c}, \underline{c}' \in M(n, r)$, we have $\tau(\theta_{\underline{c}'}^\vee \cdot h\theta_{\underline{c}}) \in A$, because $\tau(\theta_{\underline{c}'}^\vee \cdot \theta_{\underline{a}} \theta_{\underline{c}}) \in A$ (see Remark 4.1). Multiplying this from the right with $\theta_{\underline{d}}^\vee$ we get

$$h\theta_{\underline{c}} \theta_{\underline{d}}^\vee = \sum_{\underline{c}' \in M(n, r)} \tau(h\theta_{\underline{c}} \theta_{\underline{c}'}^\vee) \cdot \theta_{\underline{c}'} \theta_{\underline{d}}^\vee,$$

where we only have to sum over $\underline{c}' \in \Gamma$, since all the summands are zero unless $\underline{d} \leq_L \underline{c}' \leq_L \underline{c}$ by Lemma 4.2, which is equivalent to $\underline{c}' \in \Gamma$. We then deduce that $\mathcal{L}_{\underline{d}}$ is a left $\mathcal{S}_q(n, r)$ -module. Moreover, comparing with Remark 4.1, this shows the statement of the representing matrices. \square

Corollary 4.15. *The Schur algebra $\mathcal{S}_q(n, r)$ is contained in the A -span of the Wedderburn basis \mathcal{B} :*

$$\mathcal{S}_q(n, r) \subseteq \langle \mathcal{B} \rangle_A.$$

Proof. Let $\Gamma_1, \dots, \Gamma_n$ be left cells, such that the corresponding left cell modules form a system of representatives for the isomorphism types of simple left $K\mathcal{S}_q(n, r)$ -modules. The mapping that maps $h \in K\mathcal{S}_q(n, r)$ to its tuple of representing matrices in the cell modules $\mathrm{LC}^{(\Gamma_1)}, \dots, \mathrm{LC}^{(\Gamma_n)}$ with respect to their standard basis is an explicit isomorphism to a direct sum of full matrix rings over K . In this isomorphism, the elements of \mathcal{B} are mapped to a matrix unit, that is, to tuples of matrices, in which exactly one matrix is non-zero, and this matrix contains exactly one non-zero coefficient equal to 1. The elements of $\mathcal{S}_q(n, r)$ are mapped to tuples of matrices with entries in A , since their representing matrices on the cell modules have entries in A (see the remark after Theorem 4.3). Therefore, $\mathcal{S}_q(n, r)$ lies in the A -span of \mathcal{B} . \square

Proposition 4.16. *Let τ be a symmetrising trace form on $K\mathcal{S}_q(n, r)$. We denote by \mathcal{B} the corresponding Wedderburn basis obtained in Theorem 4.12. Then, the dual basis of \mathcal{B} relative to τ is*

$$\mathcal{B}^\vee = \{\theta_{\underline{c}} \theta_{\underline{d}}^\vee \mid \underline{c} \in M(n, r), \underline{d} \in \mathcal{D}(n, r), \underline{c} \sim_L \underline{d}\}.$$

Proof. Note first, that since τ is non-degenerate and \mathcal{B} is a basis of $K\mathcal{S}_q(n, r)$, there must be at least one element $c_{\underline{d}'}^{-1} \theta_{\underline{c}} \theta_{\underline{d}'}^\vee \in \mathcal{B}$ such that $\tau(c_{\underline{d}}^{-1} \theta_{\underline{c}} \theta_{\underline{d}}^\vee \cdot c_{\underline{d}'}^{-1} \theta_{\underline{c}'} \theta_{\underline{d}'}^\vee)$ is non-zero. Since $c_{\underline{d}'} \neq 0$, we have, in particular, $\tau(c_{\underline{d}}^{-1} \theta_{\underline{c}} \theta_{\underline{d}}^\vee \theta_{\underline{c}'} \theta_{\underline{d}'}^\vee) \neq 0$. We try to find out, which element $\theta_{\underline{c}'} \theta_{\underline{d}'}^\vee$ this can be:

By Theorem 4.12, the value $\tau(c_{\underline{d}}^{-1} \theta_{\underline{c}} \theta_{\underline{d}}^\vee \theta_{\underline{c}'} \theta_{\underline{d}'}^\vee)$ is equal to zero, if $\mathrm{LC}^{(\underline{d})} \not\cong \mathrm{LC}^{(\underline{d}'')}$ or $\underline{d} \not\sim_R \underline{c}'$. If, however, $\mathrm{LC}^{(\underline{d})} \cong \mathrm{LC}^{(\underline{d}'')}$ and $\underline{d} \sim_R \underline{c}'$, then it is equal to $\tau(\theta_{\underline{c}'} \theta_{\underline{d}'}^\vee)$ where \underline{c}'' is uniquely defined by $\underline{c}'' \sim_L \underline{d}'$ and $\underline{c}'' \sim_R \underline{c}$. If $\underline{c}'' \neq \underline{d}'$, then this value

is also equal to 0 because of the original definition of $\{\theta_{\underline{a}}^\vee \mid \underline{a} \in M(n, r)\}$. If, however, $\underline{c}'' = \underline{d}'$, we can show that $\underline{c}' = \underline{c}^t$ using Proposition 3.8: Namely, we have $\underline{c}' \sim_L \underline{d}' = \underline{c}'' \sim_R \underline{c}$ and thus $\underline{c}' \sim_L \underline{c}^t$ by transposition. Further, we have $\underline{c}' \sim_R \underline{d} \sim_L \underline{c}$ and thus again by transposition $\underline{c}' \sim_R \underline{c}^t$. Thus, \underline{c}' and \underline{c}^t are both left and right equivalent and therefore equal.

Thus, we deduce that

$$\tau(c_{\underline{d}}^{-1} \theta_{\underline{c}} \theta_{\underline{d}}^\vee \cdot \theta_{\underline{c}'} \theta_{\underline{d}'}^\vee) = \delta_{\underline{c}', \underline{c}^t}$$

for all $\underline{c} \in M(n, r)$ and $\underline{d} \in \mathcal{D}(n, r)$ with $\underline{c} \sim_L \underline{d}$, and all $\underline{c}' \in M(n, r)$ and $\underline{d}' \in \mathcal{D}(n, r)$ with $\underline{c}' \sim_L \underline{d}'$. \square

Remark 4.17. Note that as a byproduct we have proved the following result: If $\underline{c} \in M(n, r)$ and $\underline{d} \in \mathcal{D}(n, r)$ with $\underline{c} \sim_L \underline{d}$, and $\underline{d}' \in \mathcal{D}(n, r)$ with $\underline{c}^t \sim_L \underline{d}'$, then $\text{LC}(\underline{d}) \cong \text{LC}(\underline{d}')$.

We now talk about A -sublattices of $K\mathcal{S}_q(n, r)$.

Definition/Proposition 4.18 (A -sublattices of $K\mathcal{S}_q(n, r)$ and their duals). By an A -lattice in $K\mathcal{S}_q(n, r)$ we mean an A -free A -submodule that contains a K -basis of $K\mathcal{S}_q(n, r)$. Let $L \subseteq K\mathcal{S}_q(n, r)$ be an A -lattice. Then we set

$$L^\vee := \{h \in K\mathcal{S}_q(n, r) \mid \tau(hx) \in A \text{ for all } x \in L\}$$

and call it the **dual lattice of L** . Since τ is non-degenerate, L^\vee is again an A -lattice in $K\mathcal{S}_q(n, r)$, namely, if $(b_{\underline{a}})_{\underline{a} \in M(n, r)}$ is an A -basis of L , then the dual basis $(b_{\underline{a}}^\vee)_{\underline{a} \in M(n, r)}$ is an A -basis of L^\vee . Clearly, if $L \subseteq N$ are two A -lattices in $K\mathcal{S}_q(n, r)$, then $N^\vee \subseteq L^\vee$.

Note that we do not require an A -lattice to be an A -algebra! \square

Proposition 4.19 (The dual is an $\mathcal{S}_q(n, r)$ -module). *We have $\mathcal{S}_q(n, r) \cdot \mathcal{S}_q(n, r)^\vee \subseteq \mathcal{S}_q(n, r)^\vee$.*

Proof. Fix $h \in \mathcal{S}_q(n, r)$ and $k \in \mathcal{S}_q(n, r)^\vee$. We have to show that $hk \in \mathcal{S}_q(n, r)^\vee$. However, for every $x \in \mathcal{S}_q(n, r)$ holds $\tau(hkx) = \tau(kxh)$. Since $xh \in \mathcal{S}_q(n, r)$ (because $\mathcal{S}_q(n, r)$ is an algebra), and $k \in \mathcal{S}_q(n, r)^\vee$ we get $\tau(kxh) \in A$. \square

For the rest of this section we let $\tau = \sum_{\chi \in \text{Irr}(K\mathcal{S}_q(n, r))} \chi$, that is, we choose τ such that all Schur elements are equal to 1.

Proposition 4.20 (The Wedderburn-basis is self-dual). *Let $\tau = \sum_{\chi \in \text{Irr}(K\mathcal{S}_q(n, r))} \chi$. Then*

$$\langle \mathcal{B} \rangle_A^\vee = \langle \mathcal{B} \rangle_A$$

for the Wedderburn basis \mathcal{B} from Theorem 4.12.

Proof. Since τ is the sum of the irreducible characters, all Schur elements c_χ are equal to one. It is then a direct consequence of Proposition 4.16. \square

Corollary 4.21 (The dual of $\mathcal{S}_q(n, r)$). *From Lemma 4.15 and Proposition 4.20 follows*

$$\langle \mathcal{B} \rangle_A \subseteq \mathcal{S}_q(n, r)^\vee.$$

Proof. Dualising reverses inclusion. \square

5. THE ASYMPTOTIC ALGEBRA AND THE DU-LUSZTIG HOMOMORPHISM

In this section we briefly recall the definition of the asymptotic algebra $\mathcal{J}(n, r)$ for the q -Schur algebra $\mathcal{S}_q(n, r)$ and of the Du-Lusztig homomorphism Φ from $\mathcal{S}_q(n, r)$ to $\mathcal{J}(n, r)$. We then show that this algebra is isomorphic to the algebra $\langle \mathcal{B} \rangle_A$ spanned by our Wedderburn basis \mathcal{B} and that the Du-Lusztig homomorphism can be interpreted as the inclusion of $\mathcal{S}_q(n, r)$ into $\langle \mathcal{B} \rangle_A$.

Definition 5.1 (The asymptotic algebra $\mathcal{J}(n, r)$). Let $\mathcal{J}(n, r)$ be the free abelian group with basis $\{t_{\underline{a}} \mid \underline{a} \in M(n, r)\}$. We define a multiplication on $\mathcal{J}(n, r)$ by setting

$$t_{\underline{a}} t_{\underline{b}} = \sum_{\underline{c} \in M(n, r)} \gamma_{\underline{a}, \underline{b}, \underline{c}} \cdot t_{\underline{c}}.$$

We set $\mathcal{D}(n, r)_{\lambda} := \mathcal{D}(n, r) \cap M_{\lambda, \lambda}$. Following Du, we denote the extension of scalars of $\mathcal{J}(n, r)$ to A by $\mathcal{J}(n, r)_A$.

Lemma 5.2 (See [7, (2.2.1)]). *The \mathbb{Z} -algebra $\mathcal{J}(n, r)$ is associative with the identity element*

$$\sum_{\underline{d} \in \mathcal{D}(n, r)} t_{\underline{d}}.$$

Theorem 5.3 (The Du-Lusztig homomorphism Φ , see [7, (2.3)]). *The A -linear map $\Phi : \mathcal{S}_q(n, r) \rightarrow \mathcal{J}(n, r)_A$ defined by*

$$\Phi(\theta_{\underline{a}}) := \sum_{\substack{\underline{b} \in M(n, r) \\ \underline{d} \in \mathcal{D}(n, r)_{\mu} \\ \mathbf{a}(\underline{d}) = \mathbf{a}(\underline{b})}} f_{\underline{a}, \underline{d}, \underline{b}} \cdot t_{\underline{b}} = \sum_{\substack{\underline{b} \in M(n, r) \\ \underline{d} \in \mathcal{D}(n, r) \\ \underline{d} \sim_L \underline{b}}} f_{\underline{a}, \underline{d}, \underline{b}} \cdot t_{\underline{b}}, \quad \text{where } \mu = \text{co}(\underline{a})$$

is an algebra homomorphism and becomes an isomorphism $K\mathcal{S}_q(n, r) \rightarrow \mathcal{J}(n, r)_K$ when tensored with the field of fractions K of A .

Proof. See [7, 2.3]. The latter equation holds, since $f_{\underline{a}, \underline{b}, \underline{d}} = 0$ unless $\underline{d} \leq_L \underline{b}$, and **Q9** implies $\underline{d} \sim_L \underline{b}$ in this case. Also, we can safely sum over all of $\mathcal{D}(n, r)$ neglecting the index μ , since all elements $\underline{d} \in \mathcal{D}(n, r)$ fulfill $\text{ro}(\underline{d}) = \text{co}(\underline{d})$ by definition (see Definition 2.5 and the remark there) and $f_{\underline{a}, \underline{d}, \underline{b}} = 0$ unless $\text{co}(\underline{a}) = \text{ro}(\underline{d})$ anyway. \square

We can now present our main theorem, which links our Wedderburn basis \mathcal{B} to the asymptotic algebra:

Theorem 5.4 (Preimage of the t -basis under the Du-Lusztig homomorphism). *Let τ be an arbitrary symmetrising trace form. All dual bases in the following are meant with respect to τ .*

With the above notation we have

$$\Phi(c_{\underline{d}}^{-1} \theta_{\underline{c}} \theta_{\underline{d}}^{\vee}) = t_{\underline{c}} \quad \text{for all } \underline{c} \in M(n, r).$$

Proof. The rightmost sum in Theorem 5.3 has the advantage that it provides a formula for the image of an arbitrary element $h \in K\mathcal{S}_q(n, r)$ under the Du-Lusztig homomorphism, since it is obviously K -linear in $\theta_{\underline{a}}$:

$$\Phi(h) = \sum_{\substack{\underline{b} \in M(n, r) \\ \underline{d}' \in \mathcal{D}(n, r) \\ \underline{d}' \sim_L \underline{b}}} \tau(h \cdot \theta_{\underline{d}'} \theta_{\underline{b}}^{\vee}) \cdot t_{\underline{b}}$$

(recall $\tau(\theta_{\underline{a}}\theta_{\underline{d}'}\theta_{\underline{b}}^\vee) = f_{\underline{a},\underline{d}',\underline{b}}$). But now we can immediately set $h := c_{\underline{d}}^{-1}\theta_{\underline{c}}\theta_{\underline{d}}^\vee$ for some $\underline{c} \in M(n, r)$ and $\underline{d} \in \mathcal{D}(n, r)$ with $\underline{c} \sim_L \underline{d}$. The value $\tau(c_{\underline{d}}^{-1}\theta_{\underline{c}}\theta_{\underline{d}}^\vee \cdot \theta_{\underline{d}'}\theta_{\underline{b}}^\vee)$ is zero (see Lemma 4.2) unless $\underline{b} \leq_R \underline{c} \sim_L \underline{d} \leq_R \underline{d}' \sim_L \underline{b}$ and this implies $\underline{b} \sim_R \underline{c}$ and $\underline{d}' \sim_R \underline{d}$ using **Q4** and **Q10**; but this means $\underline{d}' = \underline{d}$ by **Q13** and the definition of \sim_R and thus $\underline{b} = \underline{c}$ because of Lemma 3.8. Thus, in the sum there is only one non-zero summand, which is $\tau(c_{\underline{d}}^{-1}\theta_{\underline{c}}\theta_{\underline{d}}^\vee \cdot \theta_{\underline{d}}\theta_{\underline{c}}^\vee)t_{\underline{c}}$. Now everything is in a single left cell such that we can use Theorem 4.5 to get

$$\tau(c_{\underline{d}}^{-1}\theta_{\underline{c}}\theta_{\underline{d}}^\vee \cdot \theta_{\underline{d}}\theta_{\underline{c}}^\vee) \cdot t_{\underline{c}} = \tau(\theta_{\underline{c}}\theta_{\underline{c}}^\vee) \cdot t_{\underline{c}} = t_{\underline{c}}$$

as claimed. □

We can summarise our results in the following way:

Theorem 5.5 (New interpretation of the Du-Lusztig homomorphism). *Let τ be an arbitrary symmetrising trace form on $KS_q(n, r)$. We define the set \mathcal{B} as in Theorem 4.12 and we set*

$$\mathcal{J}_\tau = \langle \mathcal{B} \rangle_A.$$

The following diagram commutes and all unmarked arrows are identities or natural inclusions:

$$\begin{array}{ccccc} \mathcal{S}_q(n, r) & \longrightarrow & \mathcal{J}_\tau & \longrightarrow & KS_q(n, r) \\ \parallel & & \downarrow \cong & & \downarrow \cong \\ \mathcal{S}_q(n, r) & \xrightarrow{\Phi} & \mathcal{J}(n, r)_A & \longrightarrow & \mathcal{J}(n, r)_K \end{array}$$

Thus, the asymptotic algebra $\mathcal{J}(n, r)_A$ is nothing but the A -span of our Wedderburn basis and the Du-Lusztig homomorphism Φ can simply be interpreted as the inclusion of $\mathcal{S}_q(n, r)$ into $\langle \mathcal{B} \rangle_A$. Furthermore, our results directly and explicitly show that $\langle \mathcal{B} \rangle_A$ is isomorphic as an A -algebra to a direct sum of full matrix rings over A .

6. A HYPOTHESIS IMPLYING JAMES' CONJECTURE

In this section we show how our results provide an equivalent formulation of a conjecture about the representation theory of specialisations of the q -Schur algebra. We first recall the conjecture.

The construction of the Iwahori-Hecke algebra of type A and of the q -Schur algebra as in Section 2 together with their Kazhdan-Lusztig bases can be carried out over an arbitrary integral domain R with quotient field k and with an arbitrary invertible parameter $q \in R$ having a square root in that domain. We denote the resulting algebra by $\mathcal{S}_q(n, r)_R$ and its extension of scalars to k by $\mathcal{S}_q(n, r)_k$.

The case of the Laurent polynomial ring $A = \mathbb{Z}[v, v^{-1}]$ and $q = v^2$ is called the “generic” case, since for every other choice (R, q) there is a ring homomorphism $\varphi : \mathbb{Z}[v, v^{-1}] \rightarrow R$ mapping v^2 to $q \in R$, which induces a ring homomorphism $\mathcal{S}_{v^2}(n, r)_A \rightarrow \mathcal{S}_q(n, r)_R \subseteq \mathcal{S}_q(n, r)_k$. This is called a “specialisation”.

It is known, that $\mathcal{S}_q(n, r)_k$ is semisimple unless q is an e -th root of unity. If q is a root of unity, then there is a decomposition matrix, which records the multiplicities of the simple modules in the so-called “standard modules”. For the case that k has characteristic zero, recent work by Lascoux, Leclerc and Thibon, and Varagnolo and Vasserot yields a complete determination of these decomposition matrices (see

[18], [10] and the references there). However, the case of positive characteristic is still open.

James' conjecture is a statement about this modular case (see [13, §4] and [10, §3]). Roughly speaking, it asserts that if k is a field of characteristic ℓ and e is the multiplicative order of the parameter $q \in k$ and ℓ is greater than r , then the decomposition matrix of $\mathcal{S}_q(n, r)_k$ does not depend on the particular value of ℓ but only on e . Note that James originally used the condition $\ell e > r$ which is a bit weaker, but the important point is that the condition holds for all but finitely many values of ℓ .

We now want to make this statement more precise. Both the simple modules and the standard modules have a labelling by the set $\Lambda^+(n, r)$ or partitions of r with at most n parts. Let $V_{k, q}^\lambda$ denote the standard module and $M_{k, q}^\mu$ the simple module of $\mathcal{S}_q(n, r)_k$ corresponding to λ and μ , respectively. Then the decomposition matrix for $\mathcal{S}_q(n, r)_k$ consists of the numbers

$$d_{\lambda, \mu}^{k, q} := \text{multiplicity of } M_{k, q}^\mu \text{ in } V_{k, q}^\lambda.$$

Conjecture 6.1 (James, see [13, §4] and [10, §3]). *If $\ell > r$ and e is the multiplicative order of $q \in k$, then $d_{\lambda, \mu}^{k, q} = d_{\lambda, \mu}^{\mathbb{Q}(\zeta_e), \zeta_e}$ for all $\lambda, \mu \in \Lambda(n, r)$, where ζ_e is a complex primitive e -th root of unity.*

Meinolf Geck has shown in [11, Theorem 1.2] that this statement is equivalent to the fact that, for $\ell > r$, the rank of the Du-Lusztig homomorphism $\Phi : \mathcal{S}_q(n, r)_k \rightarrow \mathcal{J}(n, r)_k$ with respect to the two bases $(\theta_{\underline{a}})_{\underline{a} \in M(n, r)}$ and $(t_{\underline{a}})_{\underline{a} \in M(n, r)}$ is equal to the rank of the corresponding Du-Lusztig homomorphism $\mathcal{S}_{\zeta_{2e}^2}(n, r)_{\mathbb{Q}(\zeta_{2e})} \rightarrow \mathcal{J}(n, r)_{\mathbb{Q}(\zeta_{2e})}$ with respect to the corresponding bases, where e is the multiplicative order of $q \in k$ and ζ_{2e} is a primitive $2e$ -th root of unity in \mathbb{C} . In particular, the rank does not depend on the characteristic ℓ of k .

In view of our Theorem 5.5 this immediately implies:

Theorem 6.2 (An equivalent formulation of James' conjecture). *Let $\{\theta_{\underline{a}} \mid \underline{a} \in M(n, r)\}$ be the Du-Kazhdan-Lusztig basis of $\mathcal{S}_q(n, r)$, let τ be a symmetrising trace form for $K\mathcal{S}_q(n, r)$ in the sense of Section 4 and $\{c_{\underline{d}}\}_{\underline{d} \in \mathcal{D}(n, r)}$ the corresponding Schur elements as above. Let $\{\theta_{\underline{a}}^\vee \mid \underline{a} \in M(n, r)\}$ be the dual basis of $\{\theta_{\underline{a}} \mid \underline{a} \in M(n, r)\}$ with respect to τ . Let \mathcal{B} be the basis defined in Theorem 4.12. Let $s := |M(n, r)|$ and $M = (m_{\underline{a}, \underline{b}})_{\underline{a}, \underline{b} \in M(n, r)} \in A^{s \times s}$ be the matrix, for which*

$$\theta_{\underline{a}} = \sum_{\underline{c} \in M(n, r)} m_{\underline{a}, \underline{c}} \cdot c_{\underline{d}}^{-1} \theta_{\underline{c}} \theta_{\underline{d}}^\vee$$

with $c_{\underline{d}}^{-1} \theta_{\underline{c}} \theta_{\underline{d}}^\vee \in \mathcal{B}$ holds for all $\underline{a} \in M(n, r)$.

Let ℓ be a prime and $\varphi_\ell : \mathbb{Z}[v, v^{-1}] \rightarrow \mathbb{F}_\ell$ a ring homomorphism, such that the multiplicative order of $\varphi_\ell(v)$ is equal to $2e$. Denote by $\varphi_\ell(M)$ the matrix in $\mathbb{F}_\ell^{s \times s}$ that one gets by applying the ring homomorphism φ_ℓ to every entry of M .

Let ζ_{2e} be a primitive $2e$ -th root of unity in \mathbb{C} and $\varphi_e : \mathbb{Z}[v, v^{-1}] \rightarrow \mathbb{Q}(\zeta_{2e})$ be the ring homomorphism mapping v to ζ_{2e} . Then there is a ring homomorphism $\varphi_e^e : \mathbb{Z}[\zeta_{2e}] \rightarrow \mathbb{F}_\ell$ with $\varphi_e = \varphi_e^e \circ \varphi_e$. Denote by $\varphi_e(M)$ the matrix in $\mathbb{Q}(\zeta_{2e})^{s \times s}$ that one gets by applying the ring homomorphism φ_e to every entry of M .

Then James' conjecture is equivalent to the fact that for $\ell > r$ the ranks of $\varphi(M)$ (over \mathbb{F}_ℓ) and of $\varphi_e(M)$ (over $\mathbb{Q}(\zeta_{2e})$) are equal.

Let τ be a symmetrising trace form on $KS_q(n, r)$. We denote by $\{\theta_{\underline{a}} \mid \underline{a} \in M(n, r)\}$ the Du-Kazhdan-Lusztig basis of $\mathcal{S}_q(n, r)$ and by $\{\theta_{\underline{a}}^\vee \mid \underline{a} \in M(n, r)\}$ its dual basis relative to τ . As above, we denote by \mathcal{B} the Wedderburn basis obtained in Theorem 4.12. Moreover, we denote by $M = (m_{\underline{a}, \underline{b}})_{\underline{a}, \underline{b} \in M(n, r)}$ the change of basis matrix from $\{\theta_{\underline{a}} \mid \underline{a} \in M(n, r)\}$ to \mathcal{B} as above and by $P_\tau = (p_{\underline{a}, \underline{b}})_{\underline{a}, \underline{b} \in M(n, r)}$ the change of basis matrix from $\{\theta_{\underline{a}} \mid \underline{a} \in M(n, r)\}$ to $\{\theta_{\underline{a}}^\vee \mid \underline{a} \in M(n, r)\}$, that is,

$$\theta_{\underline{a}} = \sum_{\underline{b} \in M(n, r)} p_{\underline{a}, \underline{b}} \cdot \theta_{\underline{b}}^\vee$$

for all $\underline{a} \in M(n, r)$. Formula (4.2) implies that

$$P_\tau = (\tau(\theta_{\underline{a}} \theta_{\underline{b}}))_{\underline{a}, \underline{b} \in M(n, r)} \quad \text{and} \quad P_\tau^{-1} = (\tau(\theta_{\underline{a}}^\vee \theta_{\underline{b}}^\vee))_{\underline{a}, \underline{b} \in M(n, r)}.$$

Lemma 6.3. *With the above notation, the matrix*

$$D = M^T P_\tau^{-1} M$$

is monomial and its entries are the Schur elements $c_{\underline{d}}$ associated to $\underline{d} \in \mathcal{D}(n, r)$ as in Definition 4.11.

Proof. The matrix M^T is the change of basis matrix from \mathcal{B}^\vee to $\{\theta_{\underline{a}}^\vee \mid \underline{a} \in M(n, r)\}$ and thus the matrix D is the change of basis matrix from \mathcal{B}^\vee to \mathcal{B} , that is,

$$\theta_{\underline{c}} \theta_{\underline{d}}^\vee = \sum_{\underline{c}' \in M(n, r)} d_{\underline{c}, \underline{c}'} c_{\underline{d}'}^{-1} \theta_{\underline{c}'} \theta_{\underline{d}'}^\vee$$

for all $\theta_{\underline{c}} \theta_{\underline{d}}^\vee \in \mathcal{B}^\vee$. Using Proposition 4.16, the result follows. \square

Proposition 6.4 (A hypothesis implying James' conjecture). *Let τ be a symmetrising trace form on $KS_q(n, r)$ in the sense of Section 4 and $\{c_{\underline{d}}\}_{\underline{d} \in \mathcal{D}(n, r)}$ the corresponding Schur elements as above. Let $\varphi_e : A \rightarrow \mathbb{Z}[\zeta_{2e}], v \mapsto \zeta_{2e}$ be a specialisation to characteristic 0 where v^2 is mapped to a primitive e -th root of unity in a cyclotomic field and $\varphi_\ell : A \rightarrow \mathbb{F}_\ell$ is a second specialisation to characteristic ℓ such that there is a ring homomorphism $\varphi_\ell^e : \mathbb{Z}[\zeta_{2e}] \rightarrow \mathbb{F}_\ell$ with $\varphi_\ell = \varphi_\ell^e \circ \varphi_e$. We suppose that $\ell > r$ and the following hypotheses on τ :*

- *The Schur elements $c_{\underline{d}}$ for $\underline{d} \in \mathcal{D}(n, r)$ lie in A .*
- *The coefficients of the matrix P_τ^{-1} lie in A .*
- *Let a be the number of Schur elements $c_{\underline{d}}$ for $\underline{d} \in \mathcal{D}(n, r)$ that do not vanish under φ_e and b the number of Schur elements that do not vanish under φ_ℓ . The numbers a and b are both equal to the rank over $\mathbb{Q}(\zeta_{2e})$ of the matrix $\varphi_e(M)$ for M from above.*

Note that we denote with the notation $\varphi_e(M)$ the matrix one gets from M by applying the ring homomorphism φ_e on every entry.

If τ can be found by fulfilling all these hypotheses, then James' conjecture holds for all $\ell > r$ for which φ_ℓ as above exist.

Proof. We denote by M the change of basis matrix from $\{\theta_{\underline{a}} \mid \underline{a} \in M(n, r)\}$ to \mathcal{B} as above. Then Lemma 6.3 asserts that

$$D = M^T P_\tau^{-1} M.$$

Thanks to Theorem 4.12, the coefficients of the matrix M lie in A . By hypothesis, the matrix P_τ^{-1} has coefficients in A . By Lemma 6.3 and the first hypothesis the entries of D are also in A .

Since the matrices D , M , M^T , and P_τ^{-1} have coefficients in A , the matrices $\varphi_e(D)$, $\varphi_e(M)$, $\varphi_\ell(D)$, $\varphi_\ell(M)$, $\varphi_\ell(M^T)$ and $\varphi_\ell(P_\tau^{-1})$ are well defined. We then have the equality

$$\varphi_\ell(D) = \varphi_\ell(M^T) \cdot \varphi_\ell(P_\tau^{-1}) \cdot \varphi_\ell(M),$$

implying that $\text{rk}_{\mathbb{F}_\ell}(\varphi_\ell(D)) \leq \text{rk}_{\mathbb{F}_\ell}(\varphi_\ell(M))$. Moreover, we have $\varphi_\ell(M) = \varphi_\ell^e(\varphi_e(M))$. Since φ_ℓ^e is a ring homomorphism, we deduce that

$$\text{rk}_{\mathbb{F}_\ell}(\varphi_\ell(M)) \leq \text{rk}_{\mathbb{Q}(\zeta_{2e})}(\varphi_e(M)).$$

Since D is a monomial matrix containing only the Schur elements as non-zero entries, the numbers a and b from the hypotheses are the ranks of $\varphi_e(D)$ and $\varphi_\ell(D)$, respectively. However, if as in the last hypothesis the ranks of $\varphi_e(M)$ and $\varphi_\ell(D)$ are equal, then it follows that $\text{rk}_{\mathbb{F}_\ell}(\varphi_\ell(M)) \leq \text{rk}_{\mathbb{F}_\ell}(\varphi_\ell(D))$. We then deduce that

$$\text{rk}_{\mathbb{F}_\ell}(\varphi_\ell(M)) = \text{rk}_{\mathbb{F}_\ell}(\varphi_\ell(D)),$$

and the result now follows from Theorem 6.2. \square

Remark 6.5. To prove James' conjecture it is enough to find a symmetrising trace form τ on $K\mathcal{S}_q(n, r)$ such that the hypotheses of Proposition 6.4 are satisfied. We notice that the assumption on P_τ in the statement of Proposition 6.4 is “generic” in the sense that this property only depends on the “generic” q -Schur algebra, but not on specialisations over finite fields.

Remark 6.6. We can replace the second assumption of Proposition 6.4 by the fact that the matrix $P_\tau^{-1}M$ (or $M^T P_\tau^{-1}$) has its coefficients in A .

Remark 6.7. For the usual trace form τ_0 on Iwahori-Hecke algebras of type A , we note that the assumptions of Proposition 6.4 hold. Then using [16], we can prove in a way similar to the one of the proof of Proposition 6.4 that the rank of the Lusztig homomorphism (specialised in a finite field \mathbb{F}_ℓ by $\varphi_\ell : A \rightarrow \mathbb{F}_\ell$ mapping v^2 to an element $q \in \mathbb{F}_\ell$ with multiplicative order e as above) does not depend on ℓ . However, as noted by Geck in [11] an analogue result such as Theorem 6.2 in Iwahori-Hecke algebras does not imply the Iwahori-Hecke algebras of James' conjecture.

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