

DISTINGUISHED TAME SUPERCUSPIDAL REPRESENTATIONS AND ODD ORTHOGONAL PERIODS

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ABSTRACT. We further develop and simplify the general theory of distinguished tame supercuspidal representations of reductive p -adic groups due to Hakim and Murnaghan, as well as the analogous theory for finite reductive groups due to Lusztig. We apply our results to study the representations of $GL_n(F)$, with n odd and F a nonarchimedean local field, that are distinguished with respect to an orthogonal group in n variables. In particular, we determine precisely when a supercuspidal representation is distinguished with respect to an orthogonal group and, if so, that the space of distinguishing linear forms has dimension one.

1. INTRODUCTION

This paper has two objectives: (1) to further develop and simplify the general theory presented in [HMu] of distinguished tame supercuspidal representations, and (2) to apply this general theory to a particularly important class of examples, namely, the representations of $GL_n(F)$, with n odd and F a nonarchimedean local field (under some restrictions), that are distinguished with respect to an orthogonal group in n variables.

1.1. General theory. Generally speaking, we will use the notations and terminology of [HMu]. We also impose the same “tameness” assumptions on the data used to define our representations. For simplicity, we do not recall all of these conventions explicitly in this introduction.

We are interested in the tame supercuspidal representations of a given group $G = \mathbf{G}(F)$, where F is a nonarchimedean local field and \mathbf{G} is a connected reductive F -group. By definition, a supercuspidal representation of G is “tame” if it is one of the representations constructed by Jiu-Kang Yu in [Y]. The basic objects used to parametrize tame supercuspidal representations are called “cuspidal G -data”. (The latter notion was introduced in [Y] though the terminology is from [HMu].) Assume now we have fixed a cuspidal G -datum $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi})$ and let $\pi(\Psi)$ denote the associated representation.

Assume we have also fixed an involution θ of G , that is, an F -automorphism of \mathbf{G} of order two. The central problem considered in [HMu] is the computation of the dimension of $\text{Hom}_{G^\theta}(\pi(\Psi), 1)$, where G^θ is the group of fixed points of θ in G .

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Since this dimension is constant as θ varies over its G -orbit Θ , we write

$$\langle \Theta, \Psi \rangle_G = \dim \text{Hom}_{G^\theta}(\pi(\Psi), 1).$$

(Recall that G acts on involutions by $g \cdot \theta = \text{Int}(g) \circ \theta \circ \text{Int}(g)^{-1}$.)

Let $[\Psi]$ denote the set of refactorizations of Ψ (in the sense of [HMu]) and let $[\theta]$ denote the K^0 -orbit of θ . Note that \vec{G} , y , and the equivalence class of $\pi(\Psi)$ do not vary in the refactorization class $[\Psi]$. On the other hand, the representation ρ of K^0 does vary, however, its twist

$$\rho' = \rho \otimes (\phi|K^0)$$

is an invariant of $[\Psi]$. Recall that ϕ is the quasicharacter of G^0 defined by

$$\phi = \prod_{i=0}^d \phi_i|G^0$$

and note that $\rho'|K_+^0$ is a multiple of the character $\phi|K_+^0$.

Write $[\theta] \sim [\Psi]$ when $\theta(K^0) = K^0$ and $\phi|K_+^{0,\theta} = 1$. Equivalently, $[\theta] \sim [\Psi]$ when there exists $\dot{\Psi} \in [\Psi]$ such that $\dot{\Psi}$ is θ -symmetric in the sense of [HMu]. In the latter case, $\dot{\Psi}$ will be θ' -symmetric for all $\theta' \in [\theta]$.

If $\theta(K^0) = K^0$, we define

$$\langle [\theta], [\Psi] \rangle_{K^0} = \dim \text{Hom}_{K^{0,\theta}}(\rho', \eta'_\theta),$$

where η'_θ is a certain quadratic character defined in [HMu]; otherwise, we take $\langle [\theta], [\Psi] \rangle_{K^0} = 0$. Note that if $\langle [\theta], [\Psi] \rangle_{K^0}$ is nonzero, then $[\theta] \sim [\Psi]$.

We now state some refinements to the main theorem of [HMu, Theorem 5.26] that hold under the same technical assumptions. First of all, we prove in statement 1 of Theorem 3.10 that

$$\langle \Theta, \Psi \rangle_G = \sum_{[\theta]} m_{K^0}([\theta]) \langle [\theta], [\Psi] \rangle_{K^0},$$

where

$$m_{K^0}([\theta]) = [G_\theta : (K^0 \cap G_\theta)G^\theta]$$

and we are summing over the K^0 -orbits $[\theta]$ in Θ such that $[\theta] \sim [\Psi]$. The fact that we are summing over K^0 -orbits of involutions, rather than K -orbits, is a significant improvement over [HMu] since K^0 has a much simpler structure than K . In addition, the explicit formula defining $m_{K^0}([\theta])$ should be easy to evaluate in applications and it corrects a mistake in [HMu].

Next, we state a formula which simplifies Theorem 5.26 (5) of [HMu]. Assume there exists $\theta \in \Theta$ such that $[\theta] \sim [\Psi]$ and fix such a θ . (Such a θ always exists if $\langle \Theta, \Psi \rangle_G$ is nonzero.) Let $g_1, \dots, g_m \in G$ be a maximal (necessarily finite) sequence such that $g_i \theta (g_i)^{-1} \in K^0$ and the K^0 -orbits $[g_i \cdot \theta]$ are distinct. Then we show in statement 2 of Theorem 3.10 that

$$\langle \Theta, \Psi \rangle_G = \sum_{i=1}^m m_{K^0}([g_i \cdot \theta]) \langle [g_i \cdot \theta], [\Psi] \rangle_{K^0}.$$

We also reformulate certain results of Lusztig [Lu] for finite groups of Lie type to make evident that our formulas for $\langle \Theta, \Psi \rangle_G$ have close analogues for representations of finite groups of Lie type.

1.2. A special class of examples. Let $G = \mathbf{G}(F)$, where $\mathbf{G} = \mathrm{GL}_n$ for some odd integer $n \geq 3$ and F is a nonarchimedean local field of characteristic 0 whose residue field has characteristic p with $p \neq 2$. When $\nu \in G$ is symmetric, we define an automorphism of \mathbf{G} by

$$\theta_\nu(g) = \nu^{-1} \cdot {}^t g^{-1} \cdot \nu.$$

Such automorphisms will be called “orthogonal involutions of G ”. In general, we use boldface letters for F -groups and the corresponding nonbold letters for the corresponding subgroups of F -rational points. If θ is an orthogonal involution, let \mathbf{G}^θ be the group of fixed points of θ (and thus G^θ denotes $\mathbf{G}^\theta(F)$).

In this paper, we compute the spaces $\mathrm{Hom}_{G^\theta}(\pi, 1)$ when π is an irreducible tame supercuspidal representation of G and θ is an orthogonal involution of G . We follow the approach of [HMu]. When $\mathrm{Hom}_{G^\theta}(\pi, 1)$ is nonzero, one says that π is G^θ -distinguished. (The property of being G^θ -distinguished is referred to as G^θ -distinction.)

Our main theorem, Theorem 6.8, states that if π has central character ω , then π is G^θ -distinguished precisely when $\omega(-1) = 1$ and θ has the form θ_ν for some symmetric matrix $\nu \in G$ that is similar to the matrix

$$J = \begin{pmatrix} & & & 1 \\ & \cdot & & \\ & & \cdot & \\ 1 & & & \end{pmatrix}.$$

(Note that θ_{ν_1} and θ_{ν_2} are in the same G -orbit if and only if ν_1 is similar to a scalar multiple of ν_2 .) We also show that when π is G^θ -distinguished, the dimension of $\mathrm{Hom}_{G^\theta}(\pi, 1)$ is one. For depth-zero tame supercuspidal representations, the content of Theorem 6.8 constitutes the main result in [HMa].

The case of even n is significantly more complicated and it is treated in [H], which may be regarded as a sequel to this paper. (The main theorem is not nearly as simple to state as our Theorem 6.8.) The reader interested in placing the present paper in a broader context should consult [H], as well as the recent work of Cesar Valverde [V] which characterizes distinction for a class of nonsupercuspidal representations in the same setting we consider.

Our work may be used to characterize the set of tame supercuspidal representations in the image of the local metaplectic correspondence of [FK] on the double cover \tilde{G} of G in terms of distinguished representations. In particular, if π is an irreducible tame supercuspidal representation of G , then π lies in the image of the Flicker-Kazhdan local metaplectic correspondence (from \tilde{G} to G) if and only if π is distinguished by any (hence every) split orthogonal group G^θ .

The local results we obtain are consistent with a global conjecture of Jacquet [Ja] which suggests that, globally, a cuspidal automorphic representation of GL_n should be in the image of the metaplectic correspondence precisely when it is distinguished, in a certain sense, with respect to a split orthogonal similitude group. For a precise global statement, the reader should refer to [Ja] or [Ma].

Let us describe the local analogue of the latter global notion of distinction. Let G^θ be a split orthogonal group and let G_θ be the associated similitude group. When n is odd, the similitude map $\mu_\theta(g) = g\theta(g)^{-1}$ from G_θ to the center Z of G is surjective and we have $G_\theta = G^\theta Z$. Let χ be a quasicharacter of Z . If π is an irreducible tame supercuspidal representation of G , then we say π is (G_θ, χ) -distinguished if

$\text{Hom}_{G^\theta}(\pi, \chi \circ \mu_\theta)$ is nonzero. Clearly, if π is (G_θ, χ) -distinguished, then it is G^θ -distinguished. Conversely, if π is G^θ -distinguished, then π is (G_θ, χ) -distinguished precisely when the central character of π is χ^2 . (This follows immediately from the fact that $G_\theta = G^\theta Z$.)

Jacquet’s conjecture is tied to a potential formulation in terms of relative trace formula of Waldspurger’s works [W1] and [W2] on the nonvanishing at the center of symmetry of the L -functions attached to a quadratic twist of a cuspidal automorphic representation of GL_2 . Work on Jacquet’s conjecture is ongoing with contributions by various authors. (See [Of], for example.)

We would like to also mention the recent work [Mu2] by Fiona Murnaghan that links the existence of distinguished tame supercuspidal representations for a given pair (G, θ) to the existence of certain elements in G that are elliptic regular with respect to θ in a suitable sense. The specific examples we consider are also mentioned in [Mu2].

2. GENERAL NOTATION AND BACKGROUND

Let K be any nonarchimedean local field of characteristic 0. Denote by \mathfrak{O}_K the ring of integers of K and by \mathfrak{P}_K the maximal ideal of \mathfrak{O}_K . Let \mathfrak{k}_K denote the residue field of K . If L/K is a finite extension, we let $N_{L/K}$ denote the norm map from L^\times to K^\times . Let \mathbf{G} be any connected reductive algebraic group defined over K .

For any subgroup \mathbf{H} of \mathbf{G} , let $N_{\mathbf{G}}(\mathbf{H})$ (resp. $Z_{\mathbf{G}}(\mathbf{H})$) denote the normalizer (resp. centralizer) of \mathbf{H} in \mathbf{G} . Similarly, if C is any subgroup of $\mathbf{G}(K)$, we denote by $N_C(\mathbf{H})$ (resp. $Z_K(\mathbf{H})$) the normalizer (resp. centralizer) of \mathbf{H} in C .

Let L be a finite extension of K . Fix an algebraic closure \overline{K} of K containing L . Let Σ denote the set of K -embeddings of L into \overline{K} , and let $\iota : L \rightarrow \overline{K}$ denote the natural inclusion.

Let \mathbf{H} be an algebraic L -group. Let $R_{L/K}\mathbf{H}$ denote the K -group obtained from \mathbf{H} via the restriction of scalars functor from L to K . Then there is a \overline{K} -isomorphism

$$R_{L/K}\mathbf{H} \cong \prod_{\sigma \in \Sigma} \mathbf{H}_\sigma,$$

where $\mathbf{H}_\sigma = \sigma(\mathbf{H})$. The action of $\text{Gal}(\overline{K}/K)$ on $R_{L/K}\mathbf{H}$ corresponds to the following action on $\prod_{\sigma} \mathbf{H}_\sigma$. If $x \in \prod_{\sigma} \mathbf{H}_\sigma$, denote the σ -component of x by x_σ for $\sigma \in \Sigma$. Then, for $\tau \in \text{Gal}(\overline{K}/K)$, define $\tau \cdot x$ to be the element of $\prod_{\sigma} \mathbf{H}_\sigma$ with σ -component

$$(\tau \cdot x)_\sigma = \tau x_{\tau^{-1}\sigma} \quad (\sigma \in \Sigma).$$

(Note that there is a natural action of $\text{Gal}(\overline{K}/K)$ on Σ so the notation $\tau^{-1}\sigma$ is meaningful.) We will identify $R_{L/K}\mathbf{H}$ and $\prod_{\sigma \in \Sigma} \mathbf{H}_\sigma$ (together with the above action of $\text{Gal}(\overline{K}/K)$). We will view each \mathbf{H}_σ as a K -subgroup of G . Note that projection onto the ι -component gives an isomorphism of $(R_{L/K}\mathbf{H})(F)$ with $\mathbf{H}(K)$.

Let θ be a K -involution of \mathbf{G} . Abusing terminology slightly, we will often refer to θ as an involution of $\mathbf{G}(K)$. The group \mathbf{G}^θ of θ -fixed elements in \mathbf{G} is a reductive K -group. For $g \in \mathbf{G}(K)$, let $g \cdot \theta$ be the K -involution $\text{Int}(g) \circ \theta \circ \text{Int}(g)^{-1}$ of \mathbf{G} , where $\text{Int}(g)$ is the automorphism $x \mapsto gxg^{-1}$ of \mathbf{G} . This defines an action of G on the space of K -involutions of \mathbf{G} .

We will make use of much of the above notation in the setting where the fields involved are finite.

For the remainder of the paper, F will denote a fixed nonarchimedean local field of characteristic 0. We will abbreviate $\mathfrak{O}_F, \mathfrak{P}_F,$ and $\mathfrak{f}_F,$ respectively, by $\mathfrak{O}, \mathfrak{P},$ and $\mathfrak{f}.$ Let q be the cardinality of $\mathfrak{f}.$ Let \mathbf{G} be any reductive F -group. We will denote the group $\mathbf{G}(F)$ of F -points of \mathbf{G} by $G.$ In general, we will use boldface letters to denote algebraic groups and corresponding ordinary letters to denote groups of F -rational points (for algebraic groups defined over F). We denote the Lie algebra of \mathbf{G} by $\mathfrak{g},$ and set $\mathfrak{g} = \mathfrak{g}(F).$

Let θ be an involution of $G.$ Let \mathbf{G}_θ be the stabilizer of θ in $\mathbf{G}.$ Then \mathbf{G}_θ is a reductive F -group containing $\mathbf{G}^\theta.$ If \mathbf{Z} denotes the center of $\mathbf{G},$ we have

$$\begin{aligned} G_\theta &= \{g \in G : g \cdot \theta = \theta\} \\ &= \{g \in G : g\theta(g)^{-1} \in Z\}. \end{aligned}$$

Then $g \mapsto g\theta(g)^{-1}$ gives a group homomorphism $\mu : G_\theta \rightarrow Z.$

When $G = \mathrm{GL}_n(F)$ and G^θ is an orthogonal group in n variables, the group G_θ is the associated orthogonal similitude group and μ is the similitude ratio. So it is natural, in general, to view G_θ as a generalized similitude group with similitude ratio $\mu.$

The homomorphism μ yields an isomorphism of abelian groups

$$G_\theta/G^\theta \cong \mu(G_\theta),$$

as well as another such isomorphism

$$G_\theta/ZG^\theta \cong \mu(G_\theta)/\mu(Z).$$

Again, we note that much of this notation will also be used when F is replaced by a fixed finite field (as in §3.2).

Let K be a finite extension of $F.$ Denote by $\mathcal{B}(\mathbf{G}, K)$ the Bruhat-Tits building of \mathbf{G} over $K.$ For any maximal K -split torus \mathbf{T} of $\mathbf{G},$ let $A(\mathbf{G}, \mathbf{T}, K)$ denote the apartment in $\mathcal{B}(\mathbf{G}, K)$ associated to $\mathbf{T}.$ If $y \in \mathcal{B}(\mathbf{G}, K),$ let $[y]$ denote the image of y in the reduced building $\mathcal{B}_{\mathrm{red}}(\mathbf{G}, K).$ For any $y \in \mathcal{B}(\mathbf{G}, K),$ let $\mathbf{G}(K)_{y,0}$ denote the associated parahoric subgroup of $\mathbf{G}(K).$ For a real number $r \geq 0,$ denote by $\mathbf{G}(K)_{y,r}$ the filtration subgroup of $\mathbf{G}(K)_{y,0}$ attached to y and r by Moy and Prasad (see [MP]). (These subgroups are defined with respect to the valuation on K that restricts to the valuation on F mapping F onto $\mathbb{Z}.$) If $K = F,$ we will abbreviate $\mathbf{G}(K)_{y,r}$ by $G_{y,r}.$ Let $\mathbf{G}(K)_{y,r^+} = \bigcup_{s>r} \mathbf{G}(K)_{y,s},$ and let $\mathbf{G}(K)_{y,r:r^+} = \mathbf{G}(K)_{y,r}/\mathbf{G}(K)_{y,r^+}.$ The quotient $\mathbf{G}(K)_{y,0:0^+}$ is the group of \mathfrak{f}_K -rational points of a connected reductive \mathfrak{f}_K -group which we will denote by $\mathbf{G}_y^K,$ i.e., $\mathbf{G}_y^K(\mathfrak{f}_K) = \mathbf{G}(K)_{y,0:0^+}.$ When $K = F,$ we will omit the superscript in this notation, i.e., $\mathbf{G}_y(\mathfrak{f}) = G_{y,0:0^+}.$ The lattices $\mathfrak{g}(K)_{y,r}, \mathfrak{g}(K)_{y,r^+},$ and $\mathfrak{g}(K)_{y,r:r^+}$ (for $r \in \mathbb{R}$) are defined analogously. When $K = F,$ we will abbreviate these lattices, respectively, by $\mathfrak{g}_{y,r}, \mathfrak{g}_{y,r^+},$ and $\mathfrak{g}_{y,r:r^+}.$

The following definition from [HMu] is derived from [Y]:

Definition 2.1. A 5-tuple $(\vec{\mathbf{G}}, y, \vec{r}, \rho, \vec{\phi})$ is called a *cuspidal G -datum* if it satisfies the following conditions:

- D1.** $\vec{\mathbf{G}}$ is a tamely ramified twisted Levi sequence $\vec{\mathbf{G}} = (\mathbf{G}^0, \dots, \mathbf{G}^d)$ in \mathbf{G} and \mathbf{Z}^0/\mathbf{Z} is F -anisotropic, where \mathbf{Z}^0 and \mathbf{Z} are the centers of \mathbf{G}^0 and $\mathbf{G} = \mathbf{G}^d,$ respectively.
- D2.** y is a point in $A(\mathbf{G}, \mathbf{T}, F),$ where \mathbf{T} is a tame maximal F -torus of \mathbf{G}^0 and E' is a Galois tamely ramified extension of F over which \mathbf{T} (hence $\vec{\mathbf{G}}$) splits.

(Here, $A(\mathbf{G}, \mathbf{T}, F) = A(\mathbf{G}, \mathbf{T}, E') \cap \mathcal{B}(\mathbf{G}, F)$, where $A(\mathbf{G}, \mathbf{T}, E')$ denotes the apartment in $\mathcal{B}(\mathbf{G}, E')$ corresponding to \mathbf{T} .)

- D3.** $\vec{r} = (r_0, \dots, r_d)$ is a sequence of real numbers satisfying $0 < r_0 < r_1 < \dots < r_{d-1} \leq r_d$, if $d > 0$, and $0 \leq r_0$ if $d = 0$.
- D4.** ρ is an irreducible representation of the stabilizer $K^0 = G^0_{[y]}$ of $[y]$ in G^0 such that $\rho|_{G^0_{y,0^+}}$ is 1-isotypic and the compactly induced representation $\pi_{-1} = \text{ind}_{K^0}^{G^0} \rho$ is irreducible (hence supercuspidal). Here, $[y]$ denotes the image of y in the reduced building of G .
- D5.** $\vec{\phi} = (\phi_0, \dots, \phi_d)$ is a sequence of quasicharacters, where ϕ_i is a quasicharacter of G^i . We assume that $\phi_d = 1$ if $r_d = r_{d-1}$ (with r_{-1} defined to be 0), and in all other cases if $i \in \{0, \dots, d\}$, then ϕ_i is trivial on G^i_{y,r_i^+} but nontrivial on G^i_{y,r_i} .

As observed in [HMu], the vector \vec{r} is completely determined by $\vec{\psi}$. Consequently, we can and will omit \vec{r} and refer to the resulting 4-tuple $(\vec{\mathbf{G}}, y, \rho, \vec{\phi})$ as a *cuspidal G -datum*.

A cuspidal G -datum $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi})$ determines an open compact-mod-center subgroup $K = K(\Psi)$ of G . As described in §3.1 of [HMu], K can be expressed as a product $K^0 J^1 \cdots J^d$, where K^0 is as defined above, and for $i = 1, \dots, d$, J^i is a certain open compact-mod-center pro- p subgroup of G^i . The datum Ψ also determines a representation $\kappa = \kappa(\Psi)$ of K .

If the cuspidal G -datum $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi})$ satisfies certain genericity conditions (namely, those in Definition 3.11 in [HMu]), then the compactly induced representation $\text{ind}_K^G \kappa$ is irreducible and hence supercuspidal. Such data are called *generic* in [HMu]. For the sake of brevity, in this paper, *all cuspidal G -data will be assumed to be generic*.

Suppose Ψ is a cuspidal G -datum. Let ξ be the K -equivalence class of Ψ , as defined in [HMu]. This consists of cuspidal G -data that are related to Ψ by some combination of K -conjugation, refactorization, and “elementary transformation”, that is, replacing y and ρ by \dot{y} and $\dot{\rho}$, where $[\dot{y}] = [y]$ and $\dot{\rho} \cong \rho$.

Let θ be an involution of G . Following [HMu], we say that Ψ is *θ -symmetric* if:

- $\theta(\vec{\mathbf{G}}) = \vec{\mathbf{G}}$,
- $\vec{\phi} \circ \theta = \vec{\phi}^{-1}$,
- $\theta([y]) = [y]$, where $[y]$ is the point in the reduced building of G corresponding to y .

When the first two conditions are satisfied, but not necessarily the third condition, we say that Ψ is *weakly θ -symmetric*.

If Θ is the G -orbit of θ , define

$$\langle \Theta, \Psi \rangle = \dim \text{Hom}_{G^\theta}(\pi(\Psi), 1).$$

Of course, this dimension is independent of the particular choice of representative θ of Θ . Also, it depends only on the K -equivalence class ξ , and so we also denote it by $\langle \Theta, \xi \rangle$. Let Θ' be a K -orbit of involutions of G . Then $\langle \Theta', \xi \rangle_K$ is defined in [HMu] by

$$\langle \Theta', \xi \rangle_K = \dim \text{Hom}_{K^\theta}(\kappa(\Psi), 1),$$

where θ is an arbitrary element of Θ' . When $\langle \Theta', \xi \rangle_K$ is nonzero, we say that Θ' and ξ are *strongly compatible*. By Propositions 5.7 and 5.20 in [HMu], if Θ' and

ξ are strongly compatible, then, according to Proposition 5.20 of [HMu], they are also *moderately compatible*, which is equivalent to the statement that we can choose a refactorization $\dot{\Psi}$ of Ψ and $\theta \in \Theta'$ such that $\dot{\Psi}$ is θ -symmetric.

3. DISTINGUISHED REPRESENTATIONS: GENERAL THEORY

3.1. A refined multiplicity formula. In this section, we follow the notations of [HMu]. Our first goal is to correct an error in [HMu] (which we thank Shaun Stevens for reporting to us). In particular, the constants $m_K(\Theta)$ do not appear to be well defined and should be replaced by a family of constants $m_K(\Theta')$, as Θ' varies over the set Θ^K of K -orbits in Θ . This error does not affect the theory of [HMu] in a substantial way but it does affect some of the statements of the main results. In particular, the formula

$$\langle \Theta, \xi \rangle_G = m_K(\Theta) \sum_{\Theta' \in \Theta^K} \langle \Theta', \xi \rangle_K$$

which occurs throughout [HMu] should be replaced by

$$(3.1) \quad \langle \Theta, \xi \rangle_G = \sum_{\Theta' \in \Theta^K} m_K(\Theta') \langle \Theta', \xi \rangle_K.$$

A secondary purpose is to obtain formulas for the quantities in (3.1) which involve only K^0 and not the much more complicated group K . This should greatly simplify the computations in examples. In particular, we show that each K -orbit $\Theta' \subset \Theta$ contains a unique K^0 -orbit that contains an involution θ such that $m_K(\Theta') = [G_\theta : (K^0 \cap G_\theta)G^\theta]$. The same is then true for any element of the K^0 -orbit $[\theta]$ of θ , so we denote this index by $m_{K^0}([\theta])$. It is also shown that $m_{K^0}([\theta])$ is a power of two in general.

In addition, if Θ' contributes nontrivially to (3.1), it is shown in [HMu] that

$$\langle \Theta', \xi \rangle_K = \dim \text{Hom}_{K^{0,\theta}}(\rho'(\Psi), \eta'_\theta(\Psi)).$$

This formula also holds with θ replaced by any element of $[\theta]$ and Ψ replaced by any datum in the class $[\Psi]$ of refactorizations of Ψ , and so we denote this dimension by $\langle [\theta], [\Psi] \rangle_{K^0}$. The upshot of these results is that in §3.1.4, we show that it is possible to re-express (3.1) in the form

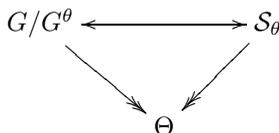
$$\langle \Theta, \Psi \rangle_G = \sum_{[\theta]} m_{K^0}([\theta]) \langle [\theta], [\Psi] \rangle_{K^0},$$

where the summation is over a certain collection of K^0 -orbits $[\theta] \in \Theta$ depending on $[\Psi]$.

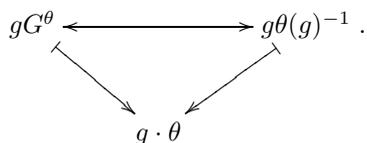
3.1.1. The constants $m_K(\Theta')$. From now on, we assume that we have fixed a G -orbit Θ of involutions of G and an inducing subgroup K , as in [HMu]. Then Θ is a union of K -orbits Θ' and the set of such K -orbits is denoted Θ^K . The rule $g \mapsto g\theta(g)^{-1}$ yields a bijection between G/G^θ and the space \mathcal{S}_θ of elements $g\theta(g)^{-1}$ as g varies over G . Recall that the action of G on the set of involutions of G is given by

$$(g \cdot \alpha)(g') = g\alpha(g^{-1}g')g^{-1}.$$

We have a diagram



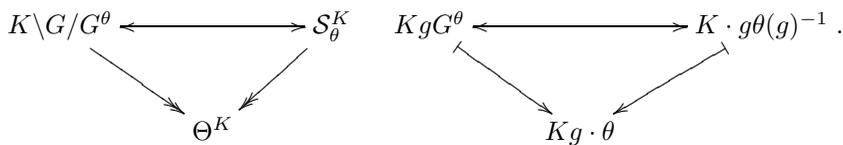
where the maps are given by



Let \mathcal{F}_g be the fiber of $g \cdot \theta \in \Theta$ in G/G^θ . Then $\mathcal{F}_g = g\mathcal{F}_1$ and so there is a canonical bijection between any fiber \mathcal{F}_g and the fiber \mathcal{F}_1 .

The group K acts on G/G^θ by left translations; it acts on \mathcal{S}_θ by $k \cdot x = kx\theta(k)^{-1}$; and it acts on Θ by restricting the action of G on the set of involutions.

The above maps are K -equivariant and we obtain corresponding maps on the sets of K -orbits:



Let \mathcal{F}_g^K be the fiber in $K \backslash G/G^\theta \in \Theta^K$ of $\Theta' = Kg \cdot \theta$. Let $m_K(\Theta')$ be the cardinality of \mathcal{F}_g^K . It is easy to see that $m_K(\Theta')$ is finite. (This will follow from explicit expressions for $m_K(\Theta')$ given below.) We have

$$\mathcal{F}_g^K = g\mathcal{F}_1^{g^{-1}Kg}$$

and thus

$$(3.2) \quad m_K(Kg \cdot \theta) = m_{g^{-1}Kg}(g^{-1}Kg \cdot \theta).$$

Lemma 2.6 of [HMu] asserts that the numbers $m_K(\Theta')$ remain constant as Θ' varies in Θ^K , but the proof appears to be erroneous. Building on the error, the constant $m_K(\Theta)$ is defined to be the common value of the $m_K(\Theta')$'s.

To correct this mistake, we need to use the formula

$$\langle \Theta, \xi \rangle_G = \sum_{\Theta' \in \Theta^K} m_K(\Theta') \langle \Theta', \xi \rangle_K.$$

We remark that in most common applications at most one of the summands $\langle \Theta', \xi \rangle_K$ is nonzero. It is also common that the constants $m_K(\Theta)$ are all 1 since $G_\theta = ZG^\theta$ for some $\theta \in \Theta$. So the error just mentioned is not easily detected by studying examples.

3.1.2. Elementary abelian 2-groups. Let Θ' be the K -orbit of θ . To establish that $m_K(\Theta')$ is a power of two, we will show that it divides the order of the group G_θ/ZG^θ , which turns out to be an elementary finite abelian 2-group.

Let

$$\begin{aligned} Z_{\Theta}^1 &= \{z \in Z : \theta(z) = z^{-1}\}, \\ B_{\Theta}^1 &= \{z\theta(z)^{-1} : z \in Z\} = \mu(Z), \\ H_{\Theta}^1 &= Z_{\Theta}^1/B_{\Theta}^1. \end{aligned}$$

Fix $\theta \in \Theta$ and let $\mathcal{G}_{\theta} = G_{\theta}/ZG^{\theta}$. Let \mathcal{K}_{θ} denote the image of $K \cap G_{\theta}$ in \mathcal{G}_{θ} . The homomorphism μ yields an isomorphism of \mathcal{G}_{θ} with the subgroup $\mu(G_{\theta})/B_{\Theta}^1$ of H_{Θ}^1 . But the group H_{Θ}^1 is an elementary finite abelian 2-group. (See the proof of Lemma 2.8 of [HMu].) It follows that \mathcal{G}_{θ} is an elementary finite abelian 2-group.

Now let Θ' be the K -orbit of θ . We have:

Lemma 3.1. *The constant $m_K(\Theta')$ is identical to the order of the elementary finite abelian 2-group $\mathcal{G}_{\theta}/\mathcal{K}_{\theta}$ and thus it is a power of 2.*

Proof. The constant $m_K(\Theta')$ represents the number of elements of $K \backslash G/G^{\theta}$ that contain a representative g such that $g \cdot \theta = \theta$. But $g \cdot \theta = \theta$ occurs exactly when $g\theta(g)^{-1} \in Z$ or, equivalently, when $g \in G_{\theta}$. Hence we are counting double cosets that have a representative in G_{θ} .

Suppose we have $Kg_1G^{\theta} = Kg_2G^{\theta}$, with $g_1, g_2 \in G_{\theta}$. Then $g_2 = kg_1h$, for some $k \in K$ and $h \in G^{\theta}$. Then k necessarily lies in $K \cap G_{\theta}$. Thus we have $(K \cap G_{\theta})g_1G^{\theta} = (K \cap G_{\theta})g_2G^{\theta}$. It follows that there is a bijection between the set of elements of $K \backslash G/G^{\theta}$ with a representative in G_{θ} and the double coset space $(K \cap G_{\theta}) \backslash G_{\theta}/G^{\theta}$. Since G^{θ} is a normal subgroup of G_{θ} and $Z \subset K \cap G_{\theta}$, our claim follows. \square

We remark that Lemma 2.8 of [HMu] establishes that $m_K(\Theta')$ is finite by showing that

$$m_K(\Theta') \leq |H_{\Theta}^1| < \infty.$$

The proof does not establish that $m_K(\Theta')$ divides $|H_{\Theta}^1|$ or that $m_K(\Theta')$ is a power of two.

We close this section by emphasizing that the expression just given for $m_K(\Theta')$ involves the image of $K \cap G_{\theta}$ in \mathcal{G}_{θ} . In the next section, we show that one can replace K by K^0 .

3.1.3. *A formula for $m_K(\Theta')$.* In this section, we exploit the structure of the inducing group K to obtain a more precise formula for $m_K(\Theta')$.

Having defined G_{θ} , we can now state our desired formula for $m_K(\Theta')$.

Theorem 3.2. *Let $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi})$ be a generic cuspidal G -datum. Let ξ be the K -equivalence class of Ψ , and let Θ' be a K -orbit of F -involutions of \mathbf{G} such that $\langle \Theta', \xi \rangle_K$ is nonzero. Then for any $\theta \in \Theta'$ such that Ψ is θ -symmetric,*

$$m_K(\Theta') = [G_{\theta} : (K^0 \cap G_{\theta})G^{\theta}].$$

We will show later that there exists a unique K^0 -orbit $\Theta'_0 \subset \Theta'$ such that Ψ is symmetric with respect to some (hence every) involution in Θ'_0 . This justifies the notation $m_{K^0}(\Theta'_0)$ for $m_K(\Theta')$ and leads to a reformulation

Fix a generic cuspidal G -datum $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi})$. Assume first that Ψ is θ -symmetric. Let $K = K(\Psi)$ be the inducing group associated to Ψ . Then K has a decomposition

$$K = K^0 J^1 \dots J^d.$$

It is shown in Proposition 3.14 of [HMu] that all of the factors in the latter decomposition are θ -stable. Moreover, we have

$$K^\theta = K^{0,\theta} J^{1,\theta} \dots J^{d,\theta},$$

where S^θ denotes the set of fixed points of θ in S . We need a slight generalization of the latter fact.

Lemma 3.3. *If $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi})$ is a θ -symmetric cuspidal G -datum, then*

$$K \cap G_\theta = (K^0 \cap G_\theta) J^{1,\theta} \dots J^{d,\theta}.$$

The proof of the latter result is identical to that of Proposition 3.14 of [HMu] except that, instead of Lemma 2.9 of [HMu], we substitute the following result whose proof is essentially the same as the proof of Lemma 2.9 of [HMu].

Lemma 3.4. *Suppose α is an automorphism of a group C such that $\alpha^2 = 1$. Assume A, B and Z are α -stable subgroups of C such that $C = AB$ and Z is a subgroup of A that is contained in the center of C . Let*

$$\begin{aligned} A' &= \{a \in A : a\alpha(a)^{-1} \in Z\}, \\ B' &= \{b \in B : \alpha(b) = b\}, \\ C' &= \{c \in C : c\alpha(c)^{-1} \in Z\}. \end{aligned}$$

Then $C' = A'B'$.

Proof of Theorem 3.2. Assume first that Ψ is θ -symmetric. We have shown in Lemma 3.1 that $m_K(\Theta)$ is the index of \mathcal{K}_θ in \mathcal{G}_θ . By definition, \mathcal{K}_θ is the image of $K \cap G_\theta$ in \mathcal{G}_θ . But, according to Lemma 3.3, $K \cap G_\theta$ is a product of $K^0 \cap G_\theta$ with various pro- p -groups $J^{i,\theta}$. Since \mathcal{G}_θ is an elementary finite abelian 2-group, the groups $J^{i,\theta}$ have trivial image in \mathcal{G}_θ . Therefore, \mathcal{K}_θ is identical to the image of $K^0 \cap G_\theta$ in \mathcal{G}_θ . Thus our claim follows when Ψ is θ -symmetric.

If Ψ is not necessarily θ -symmetric but $\langle \Theta', \xi \rangle_K$ is nonzero, then Proposition 5.9 and Lemma 5.19 [HMu] imply that there exists $\theta' \in \Theta'$ and a θ' -symmetric refactorization $\check{\Psi}$ of Ψ . This implies that there exists $k \in K$ such that $k \cdot \check{\Psi}$ is θ -symmetric and our claim follows the argument in the previous paragraph. □

3.1.4. *A simplified formula for $\langle \Theta, \Psi \rangle_G$.* Equation (3.1) can be reformulated in terms of K^0 -orbits of involutions rather than K -orbits. Since K^0 can have a much simpler structure than K , this reformulation should be regarded as a useful simplification in applications. The idea of reducing the theory of distinguished tame supercuspidal representations to objects involving the group \mathbf{G}^0 in the cuspidal G -datum is pursued further in [Mu1].

Suppose $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi})$ is a cuspidal G -datum. Then Ψ determines subgroups $K^0 = K^0(\Psi)$ and $K = K(\Psi)$ as discussed above.

Lemma 3.5. *Let α be an F -automorphism of \mathbf{G} . Then α stabilizes K^0 if and only if it stabilizes both \mathbf{G}^0 and $[y]$.*

Proof. Clearly, if α stabilizes \mathbf{G}^0 and $[y] \subset \mathcal{B}(\mathbf{G}^0, F)$, then α must stabilize $G^0_{[y]} = K^0$.

Conversely, suppose that α stabilizes K^0 . Then $K^0 \subset G^0 \cap \alpha(G^0)$. Since K^0 is an open subgroup of G^0 , it is dense in $\mathbf{G}^0 \cap \alpha(\mathbf{G}^0)$ with respect to the Zariski

topology. (See Lemma 3.2 of [PR].) Thus $\mathbf{G}^0 \cap \alpha(\mathbf{G}^0)$ must have dimension equal to that of \mathbf{G}^0 , which forces $\mathbf{G}^0 = \alpha(\mathbf{G}^0)$. \square

Lemma 3.6. *Let \mathbf{T}^0 be the connected component of the identity in \mathbf{Z}^0 . Then*

$$N_{\mathbf{G}}(\mathbf{T}^0)(F) \cap G_{y,0+} = G_{y,0+}^0.$$

Proof. Since $\mathbf{G}^0 = Z_{\mathbf{G}}(\mathbf{T}^0)$, we have

$$G_{y,0+}^0 \subset Z_{\mathbf{G}}(\mathbf{T}^0)(F) \cap G_{y,0+} \subset N_{\mathbf{G}}(\mathbf{T}^0)(F) \cap G_{y,0+}.$$

It remains to prove the inclusion $N_{\mathbf{G}}(\mathbf{T}^0)(F) \cap G_{y,0+} \subset G_{y,0+}^0$. Since $G_{y,0+}^0 = G^0 \cap G_{y,0+}$, it is enough to show that $N_{\mathbf{G}}(\mathbf{T}^0)(F) \cap G_{y,0+} \subset G^0$. Moreover, it suffices to do this over a splitting field E' of \mathbf{T} , i.e., to show that $N_{\mathbf{G}}(\mathbf{T}^0)(E') \cap \mathbf{G}(E')_{y,0+} \subset \mathbf{G}^0(E')$. We first show that it is, furthermore, enough to prove the analogue of this statement in which \mathbf{T}^0 replaced by the maximal torus \mathbf{T} .

Let $\mathbf{T}' = \text{Int}(g)(\mathbf{T})$. Then \mathbf{T}' is an E' -split maximal torus of \mathbf{G}^0 . Since g fixes y , we have $y \in A(\mathbf{G}^0, \mathbf{T}, E') \cap A(\mathbf{G}^0, \mathbf{T}', E')$. Let \mathbf{T} and \mathbf{T}' be the maximal $f_{E'}$ -tori of \mathbf{G}_y associated respectively to \mathbf{T} and \mathbf{T}' (see the appendix). Since the image of g in $\mathbf{G}_y(f_{E'})$ is trivial, we have $\mathbf{T} = \mathbf{T}'$. It follows that there exists $h \in \mathbf{G}^0(E')_{y,0+}$ such that $\text{Int}(h)(\mathbf{T}') = \mathbf{T}$. Note that $hg \in N_{\mathbf{G}}(\mathbf{T})(E') \cap \mathbf{G}(E')_{y,0+}$. Thus, if we can show that

$$(3.3) \quad N_{\mathbf{G}}(\mathbf{T})(E') \cap \mathbf{G}(E')_{y,0+} \subset \mathbf{G}^0(E'),$$

it will follow that $hg \in \mathbf{G}^0(E')$ so $g \in \mathbf{G}^0(E')$.

It remains to prove (3.3). Suppose $k \in N_{\mathbf{G}}(\mathbf{T})(E') \cap \mathbf{G}(E')_{y,0+}$. Then the image of k in $\mathbf{G}_y(f_{E'})$ is trivial, hence is contained in every Levi subgroup of \mathbf{G}_y containing \mathbf{T} . It follows that k must fix pointwise every facet of $A(\mathbf{G}, \mathbf{T}, E')$ containing y . Thus k acts trivially on $A(\mathbf{G}, \mathbf{T}, E')$ so the image of k in the Weyl group $W(\mathbf{G}, \mathbf{T})$ of \mathbf{T} in \mathbf{G} is trivial. It follows that $k \in \mathbf{T}(E') \subset \mathbf{G}^0(E')$, demonstrating (3.3). \square

Proposition 3.7. *Let θ be an involution of G and suppose $\theta(K^0) = K^0$. Let $k \in K$. Then $(k \cdot \theta)(K^0) = K^0$ if and only if $k \in K^0$.*

Proof. It is clear that if $k \in K^0$, then $k \cdot \theta$ must stabilize K^0 . Conversely, suppose $\theta' = k \cdot \theta$ stabilizes K^0 , where $k \in K$. We have $k = k_0j$, for some $k_0 \in K^0$ and $j \in J^1 \cdots J^d \subset G_{y,0+}$. The condition $\theta'(K^0) = K^0$ is equivalent to the condition that $k\theta(k)^{-1}$ lies in the normalizer $N_K(K^0)$ of K^0 in K .

We claim that $N_K(K^0) = K^0$. Assume, for the moment, that this is the case. Then

$$k_0j\theta(j)^{-1}\theta(k_0)^{-1} \in N_K(K^0) = K^0,$$

and hence $j\theta(j)^{-1} \in K^0$. According to Proposition 2.12 of [HMu], we may choose $g \in K_+^0 = G_{y,0+}^0$ such that $j\theta(j)^{-1} = g\theta(g)^{-1}$. Then $\theta' = \text{Int}(k\theta(k)^{-1}) \circ \theta = k' \cdot \theta$, where $k' = k_0g \in K^0$, and our claim follows.

It therefore suffices to show that $N_K(K^0) = K^0$. Let \mathbf{Z}^0 be the center of \mathbf{G}^0 and let \mathbf{T}^0 be the connected component of the identity in \mathbf{Z}^0 . We first observe that $N_K(K^0) = N_K(\mathbf{G}^0) = N_{\mathbf{G}}(\mathbf{G}^0)(F) \cap K$. This follows from Lemma 3.5 applied to the automorphisms $\text{Int}(k)$ for $k \in K$.

We now show that $N_{\mathbf{G}}(K^0) = N_{\mathbf{G}}(\mathbf{G}^0) = N_{\mathbf{G}}(\mathbf{Z}^0) = N_{\mathbf{G}}(\mathbf{T}^0)$. This is done as follows. Clearly, $N_{\mathbf{G}}(\mathbf{Z}^0) \subseteq N_{\mathbf{G}}(\mathbf{T}^0)$. Now suppose $g \in N_{\mathbf{G}}(\mathbf{T}^0)$. If $t \in \mathbf{T}^0$, then $g^{-1}tg \in \mathbf{T}^0$. But if $h \in \mathbf{G}^0 = Z_{\mathbf{G}}(\mathbf{T}^0)$, we have $hg^{-1}tgh^{-1} = g^{-1}tg$. This implies $ghg^{-1}tgh^{-1}g^{-1} = t$. Thus $ghg^{-1} \in Z_{\mathbf{G}}(\mathbf{T}^0) = \mathbf{G}^0$ and so $g \in N_{\mathbf{G}}(\mathbf{G}^0)$. This

shows that $N_{\mathbf{G}}(\mathbf{T}^0) \subseteq N_{\mathbf{G}}(\mathbf{G}^0)$. Assume next that $g \in N_{\mathbf{G}}(\mathbf{G}^0)$. Then $\text{Int}(g)$ is an automorphism of \mathbf{G}^0 and thus it must preserve the center \mathbf{Z}^0 of \mathbf{G}^0 and its identity component \mathbf{T}^0 . So we have shown $N_{\mathbf{G}}(\mathbf{Z}^0) \subseteq N_{\mathbf{G}}(\mathbf{T}^0) \subseteq N_{\mathbf{G}}(\mathbf{G}^0) \subseteq N_{\mathbf{G}}(\mathbf{Z}^0)$, which implies that the latter inclusions are all equalities.

We now have $N_K(K^0) = N_K(\mathbf{G}^0) \cap K = N_{\mathbf{G}}(\mathbf{T}^0) \cap K$. We are thus reduced to showing that $N_K(\mathbf{T}^0) = K^0$. Clearly, we have $N_K(\mathbf{T}^0) \supset K^0$. So suppose $k \in N_K(\mathbf{T}^0)$. As above, we write $k = k_0j$, with $k_0 \in K^0$ and $j \in J^1 \cdots J^d \subset G_{y,0+}$. To say that k normalizes \mathbf{T}^0 is the same as saying that j normalizes \mathbf{T}^0 . But then, by Lemma 3.6, j must lie in $G_{y,0+}^0$. Thus $k \in K^0$, and so $N_K(\mathbf{T}^0) = K^0$. \square

We now define a refinement of the K -equivalence on cuspidal G -data. Let $[\Psi]$ denote the class of all cuspidal G -data related to Ψ via a combination of refactorization and elementary transformation (as in §5.1 of [HMu]). Observe that the action of an element of K^0 via conjugation on an element of $[\Psi]$ coincides with an elementary transformation. Hence, $[\Psi]$ is preserved by the action of K^0 . We will refer to $[\Psi]$ as the *refactorization class* of Ψ . Note that as Ψ ranges over its refactorization class, $\vec{\mathbf{G}}, K, K^0$, and $[y]$ do not vary, while ρ and $\vec{\phi}$ do vary. Nevertheless, the equivalence class of the representation

$$\rho' = \rho \otimes (\phi|K^0)$$

is an invariant of the refactorization class. Here ϕ is the quasicharacter of G^0 given by

$$\phi = \prod_{i=0}^d \phi_i|G^0.$$

Note that $\rho'|K_+^0$ is a multiple of $\phi|K_+^0$.

For an involution θ of G , let $[\theta]$ denote the K^0 -orbit of θ . Consider the following two conditions on θ and Ψ :

- (1) θ stabilizes K^0 .
- (2) $\phi|K_+^\theta = 1$.

Clearly, if (1) holds for θ , then it must do so for every element of $[\theta]$. Similarly, Lemma 5.5 of [HMu] implies the analogous statement for (2). It follows that both conditions depend only on the K^0 -orbit $[\theta]$ and the refactorization class $[\Psi]$. We write $[\theta] \sim [\Psi]$ when both of the above conditions hold.

Proposition 3.8. *Let θ be an involution of G such that $[\theta] \sim [\Psi]$. Let ξ be the K -equivalence class of Ψ and Θ' the K -orbit of θ .*

- (1) *If $\theta' \in K \cdot \theta$ and $[\theta'] \sim [\Psi]$, then $[\theta'] = [\theta]$.*
- (2) *If $\Psi' \in \xi$ and $[\theta] \sim [\Psi']$, then $[\Psi'] = [\Psi]$.*

Proof. Let $k \in K$ be such that $\theta' = k \cdot \theta$. Since both θ and $k \cdot \theta$ stabilize K^0 , they must stabilize both $[y]$ and \mathbf{G}^0 by Lemma 3.5. It follows that $k\theta(k)^{-1}$ must stabilize $[y]$ as well, hence must lie in K^0 . Again, write $k = k_0j$, with $k_0 \in K^0$ and $j \in J^1 \cdots J^d \in G_{y,0+}$. Then $j\theta(j)^{-1} \in K^0 \cap G_{y,0+} = K_+^0$. As in §2.2 of [HMu], let

$$Z_\theta^1(K_+^0) := \{z \in K_+^0 : \theta(z) = z^{-1}\}.$$

Then $j\theta(j)^{-1} \in Z_\theta^1(K_+^0)$. But since K_+^0 is 2-divisible, Lemma 2.11 of [HMu] implies that $j\theta(j)^{-1} = c\theta(c)^{-1}$ for some $c \in K_+^0$. Thus

$$\theta' = k \cdot \theta = \text{Int}(k\theta(k)^{-1}) \circ \theta = \text{Int}(c\theta(c)^{-1}) \circ \theta = c \cdot \theta.$$

This proves (i).

Now suppose $\Psi' \in \xi$ and $[\theta] \sim [\Psi']$. Then there exist $k \in K$ and a G -datum $\dot{\Psi}$ in the refactorization class of Ψ such that $\Psi' = {}^k\dot{\Psi}$. It follows easily that $[k^{-1}\cdot\theta] \sim [\dot{\Psi}]$. Since $[\theta] \sim [\Psi] = [\dot{\Psi}]$, (ii) implies that $[k^{-1}\cdot\theta] = [\theta]$. Thus both θ and $k^{-1}\cdot\theta$ stabilize K^0 . By Proposition 3.7, it follows that k must lie in K^0 , i.e., that $\Psi' = {}^k\dot{\Psi}$ is in the refactorization class of Ψ , which proves (ii). \square

Proposition 3.9. *Let θ be an involution of G . Then $[\theta] \sim [\Psi]$ if and only if there exists $\dot{\Psi} \in [\Psi]$ such that $\dot{\Psi}$ is θ -symmetric. If this is the case, then $\dot{\Psi}$ is θ' -symmetric for all $\theta' \in [\theta]$.*

Proof. If $[\theta] \sim [\Psi]$, it follows from Proposition 5.9 of [HMu] and Lemma 3.5 that the K -orbit $K \cdot \theta$ and the K -equivalence class ξ of Ψ are moderately compatible. Thus Proposition 5.7 of *loc. cit.* implies that there exists $\dot{\Psi} \in \xi$ such that Ψ is θ -symmetric. Then $[\theta] \sim [\dot{\Psi}]$ so Proposition 3.8(ii) implies that $\dot{\Psi} \in [\Psi]$. In addition, it is clear that $\dot{\Psi}$ must be θ' -symmetric with respect to every $\theta' \in [\theta]$.

Conversely, suppose that there exists $\dot{\Psi} \in [\Psi]$ such that $\dot{\Psi}$ is θ -symmetric. Then $\theta(K^0) = K^0$. Moreover, $K \cdot \theta$ and ξ are moderately compatible by Proposition 5.7 of *loc. cit.* Thus $\phi|_{K^0_\pm} = 1$ by Propositions 5.5 and 5.9 of *loc. cit.* Thus $[\theta] \sim [\Psi]$. \square

Let Ξ be the collection of all (G, K^0) -data, i.e., all G -data Ψ' such that $K^0(\Psi') = K^0$. Let θ be an involution of G , and let $\Theta' \subset \Theta$ be the K - and G -orbits containing θ , respectively. We define (see Theorem 3.2)

$$m_{K^0}([\theta]) := m_K(\Theta') = [G_\theta : (K^0 \cap G_\theta)G^\theta].$$

Of course, this index depends only on Θ' and not on the particular choice of involution θ .

Let $\Psi = (\vec{G}, y, \rho, \vec{\phi}) \in \Xi$. Define

$$\langle [\theta], [\Psi] \rangle_{K^0} := \begin{cases} \dim \text{Hom}_{K^0, \theta}(\rho', \eta'_\theta), & \text{if } [\theta] \sim [\Psi], \\ 0, & \text{otherwise.} \end{cases}$$

Here η'_θ is a certain character of $K^{0, \theta}$ of exponent two defined in §5.6 of [HMu] and described explicitly in §6.3 of the present paper. The latter definition gives a pairing between the set of K^0 -orbits in Θ and the collection of refactorization classes in Ξ . Note that if some $\dot{\Psi} \in [\Psi]$ is θ -symmetric, then $\langle [\theta], [\Psi] \rangle_{K^0} = \langle \Theta', \xi \rangle_K$, where ξ is the K -equivalence class of Ψ . (See §5.6 of [HMu].)

Theorem 3.10. *Let Ψ, ξ_0, K, K_+ , etc. be as above.*

- (1) $\langle \Theta, \Psi \rangle_G = \sum_{[\theta] \sim [\Psi]} m_{K^0}([\theta]) \langle [\theta], [\Psi] \rangle_{K^0}$.
- (2) *Suppose there exists $\theta \in \Theta$ such that $[\theta] \sim [\Psi]$ (as must be the case if $\langle \Theta, \Psi \rangle_G \neq 0$). Let $g_1, \dots, g_m \in G$ be a maximal sequence such that $g_j\theta(g_j)^{-1} \in K^0$ and the $[\theta_j]$ are distinct. Then*

$$\langle \Theta, \Psi \rangle_G = \sum_{j=1}^m m_{K^0}([\theta_j]) \langle [\theta_j], [\Psi] \rangle_{K^0}.$$

Proof. Let Θ' be a K -orbit of involutions of G that is strongly compatible with the K -equivalence class ξ of Ψ , i.e., which gives a nonzero contribution $\langle \Theta', \xi \rangle_K$ to the sum in (3.1). Then Θ' and ξ are moderately compatible by Proposition 5.20

of [HMu]. It follows from Proposition 5.9 of *loc. cit.* that there exists $\theta \in \Theta'$ such that $[\theta] \sim [\Psi]$. By part 1 of Proposition 3.8, $[\theta]$ is the only K^0 -orbit in Θ' with this property. In addition, as discussed above, we have $\langle [\theta], [\Psi] \rangle_{K^0} = \langle \Theta', \xi \rangle_K$, and, by definition, $m_{K^0}([\theta]) = m_K(\Theta')$.

To prove (1), it remains to show that every K^0 -orbit $[\theta] \subset \Theta$ that gives a nonzero contribution to the right-hand side of the formula in 1 arises in the above way, i.e., is contained in some K -orbit Θ' of G -involutions that is strongly compatible with ξ . For such a K^0 -orbit $[\theta]$, we have $\langle [\theta], [\Psi] \rangle_{K^0} \neq 0$, so there must be a refactorization $\dot{\Psi}$ of Ψ which is θ -symmetric by Proposition 3.9. As discussed above this implies that the K -orbit Θ' containing θ satisfies $\langle \Theta', \xi \rangle_K = \langle [\theta], [\Psi] \rangle_{K^0} \neq 0$. This proves (1).

The first part of (2) follows directly from Theorem 5.20 of [HMu] and Proposition 3.9. Since the $[\theta_j]$ are distinct, the second part of (2) will follow from (1) provided that each refactorization class in Θ that contributes nontrivially to the sum in (1) contains one of the θ_j . Thus suppose that $\theta' \in \Theta$ satisfies $\langle [\theta'], [\Psi] \rangle_{K^0} \neq 0$. Then $K \cdot \theta'$ and ξ are moderately compatible by Proposition 5.9 of [HMu] and Lemma 3.5. It follows from Proposition 5.10 (2) of *loc. cit.* that there exists $g \in G$ such that $g\theta(g)^{-1} \in K^0$ and $g \cdot \theta \in K \cdot \theta'$. Then $[g \cdot \theta] \sim [\Psi]$. Since $[\theta'] \sim [\Psi]$, we must have $[g \cdot \theta] = [\theta']$ by Proposition 3.8. In other words, $\theta' = kg \cdot \theta$ for some $k \in K^0$. If we let $h = kg$, then we have $h\theta(h)^{-1} \in K^0$. By the maximality of g_1, \dots, g_m , it follows that $[\theta'] = [h \cdot \theta]$ must contain some θ_j . □

3.2. Finite field theory. In this section only:

- \mathbf{G} will be a connected reductive group defined over a finite field \mathbb{F}_q of odd order q ,
- boldface letters will be used for \mathbb{F}_q -groups and the corresponding nonbold letters for the corresponding groups of \mathbb{F}_q -rational points.

Fix a maximal torus \mathbf{T} of \mathbf{G} that is defined over \mathbb{F}_q and a complex character λ of T . Let $R_{\mathbf{T}}^\lambda = R_{\mathbf{T}}^{\mathbf{G}, \lambda}$ denote the virtual representation of G defined by Deligne-Lusztig [DL] and let $R_{\mathbf{T}, \lambda} = R_{\mathbf{T}, \lambda}^{\mathbf{G}}$ denote its virtual character.

Let θ be an involution of G , that is, an automorphism of \mathbf{G} of order 2 that is defined over \mathbb{F}_q . Fix a closed \mathbb{F}_q -subgroup \mathbf{G}_*^θ of \mathbf{G}^θ that contains the identity component of \mathbf{G}^θ . If $g \in \mathbf{G}$, we define the involution $g \cdot \theta$ in the usual way and we let $\mathbf{G}_*^{g \cdot \theta} = g\mathbf{G}_*^\theta g^{-1}$.

In [Lu], Lusztig gives a formula for the (virtual) dimension of the space of G_*^θ -fixed points of $R_{\mathbf{T}}^\lambda$. We generalize this to a formula for the dimension of the space vectors in the space of $R_{\mathbf{T}}^\lambda$ that transform under G_*^θ by a given (but arbitrary) character χ of G_*^θ .

3.2.1. A generalization of a formula of Lusztig. The results in this section were obtained independently by Fiona Murnaghan and appear in [Mu2]. We generalize results in [Lu] by allowing the inclusion of a certain nontrivial character χ . The need for this extra generality becomes evident in §6.4 and §6.5.

If \mathbf{H} is an \mathbb{F}_q -group, as in [Lu], we define

$$\sigma(\mathbf{H}) = (-1)^{\mathbb{F}_q\text{-rank of } \mathbf{H}}.$$

Suppose \mathbf{S} is a maximal torus in \mathbf{G} that is defined over \mathbb{F}_q . If $s \in \mathbf{S}$, let \mathbf{Z}_s be the identity component of the centralizer of s in \mathbf{G} and let $\varepsilon_{\mathbf{S}} : S \cap G_*^\theta \rightarrow \{\pm 1\}$ by

$$\varepsilon_{\mathbf{S}}(s) = \sigma(Z_{\mathbf{G}}((\mathbf{S} \cap \mathbf{G}_*^\theta)^\circ)) \sigma(Z_{\mathbf{Z}_s}((\mathbf{S} \cap \mathbf{G}_*^\theta)^\circ)).$$

(We warn the reader that our notation \mathbf{Z}_s conflicts with the notations in [Lu].)

Let $\Xi_{\mathbf{T}, \lambda, \chi}$ denote the set of all $g \in G$ such that $(g \cdot \theta)(\mathbf{T}) = \mathbf{T}$ and

$$\lambda(t) = \chi(g^{-1}tg)^{-1} \varepsilon_{g^{-1}\mathbf{T}g}(g^{-1}tg),$$

for all $t \in T \cap G_*^{g \cdot \theta}$. The latter set is a union of double cosets in the space $T \backslash G / G_*^\theta$.

Theorem 3.11. *If χ is a character of G_*^θ , then*

$$\frac{1}{|G_*^\theta|} \sum_{h \in G_*^\theta} R_{\mathbf{T}, \lambda}(h) \chi(h) = \sigma(\mathbf{T}) \sum_{g \in T \backslash \Xi_{\mathbf{T}, \lambda, \chi} / G_*^\theta} \sigma(Z_{\mathbf{G}}((g^{-1}\mathbf{T}g \cap \mathbf{G}_*^\theta)^\circ)).$$

Proof. Our proof is a routine generalization of the proof of Theorem 3.3 of [Lu], but, since the latter proof is rather complicated, we detail the argument.

The first step is to apply the Jordan-Chevalley decomposition to obtain:

$$\sum_{h \in G_*^\theta} R_{\mathbf{T}, \lambda}(h) \chi(h) = \sum_{\substack{s \in G_*^\theta \\ \text{semisimple}}} \sum_{\substack{u \in Z_s \cap G_*^\theta \\ \text{unipotent}}} R_{\mathbf{T}, \lambda}(su) \chi(su).$$

Since u is contained in the commutator subgroup of G_*^θ , we have

$$\sum_{h \in G_*^\theta} R_{\mathbf{T}, \lambda}(h) \chi(h) = \sum_{\substack{s \in G_*^\theta \\ \text{semisimple}}} \chi(s) \sum_{\substack{u \in Z_s \cap G_*^\theta \\ \text{unipotent}}} R_{\mathbf{T}, \lambda}(su).$$

Next, we use the Deligne-Lusztig character formula [DL]

$$R_{\mathbf{T}, \lambda}(su) = \frac{1}{|Z_s|} \sum_{\substack{x \in G \\ x^{-1}sx \in T}} \lambda(x^{-1}sx) R_{x\mathbf{T}x^{-1}, 1}^{Z_s}(u).$$

(Implicit in the latter formula is the fact that $R_{\mathbf{T}, \lambda}$ is supported in the set of elements of G with semisimple part in a conjugate of T .)

First, observe that Theorem 3.4 of [Lu] implies:

$$\sum_{\substack{u \in Z_s \cap G_*^\theta \\ \text{unipotent}}} R_{x\mathbf{T}x^{-1}, 1}^{Z_s}(u) = \frac{\sigma(\mathbf{T})}{|T|} \sum_{\substack{g \in Z_s \\ (x^{-1}g \cdot \theta)(\mathbf{T}) = \mathbf{T}}} \sigma((Z_{Z_s}(g^{-1}x\mathbf{T}x^{-1}g \cap Z_s \cap \mathbf{G}_*^\theta)^\circ)).$$

Note that $x^{-1}sx \in T$ implies that $xTx^{-1} \subset Z_s$ and hence

$$\sigma((Z_{Z_s}(g^{-1}x\mathbf{T}x^{-1}g \cap Z_s \cap \mathbf{G}_*^\theta)^\circ)) = \sigma((Z_{Z_s}(g^{-1}x\mathbf{T}x^{-1}g \cap \mathbf{G}_*^\theta)^\circ)).$$

Let

$$S = \frac{1}{|G_*^\theta|} \sum_{h \in G_*^\theta} R_{\mathbf{T}, \lambda}(h) \chi(h).$$

Putting the above pieces together yields

$$S = \frac{\sigma(\mathbf{T})}{|G_*^\theta||T|} \sum_{(s, x, g)} \frac{\chi(s)}{|Z_s|} \lambda(x^{-1}sx) \sigma((Z_{Z_s}(g^{-1}x\mathbf{T}x^{-1}g \cap \mathbf{G}_*^\theta)^\circ)),$$

where (s, x, g) is summed over the set

$$\{(s, x, g) \in G_*^\theta \times G \times G : x^{-1}sx \in T, g \in Z_s, (x^{-1}g \cdot \theta)(\mathbf{T}) = \mathbf{T}\}.$$

We now change variables by sending (s, x, g) to (t, x', g) , where $t = x^{-1}sx$ and $x' = g^{-1}x$. The latter triples lie in $T \times G \times G$ subject to certain additional conditions that we now describe. First of all, since $s = txt^{-1} = gx'tx'^{-1}g^{-1}$, we have $gx'tx'^{-1}g^{-1} \in G_*^\theta$. The condition $g \in Z_s$ reduces to $g \in Z_{x'tx'^{-1}}$. Thus the condition $gx'tx'^{-1}g^{-1} \in G_*^\theta$ reduces to $x'tx'^{-1} \in G_*^\theta$. We also have $(x'^{-1} \cdot \theta)(\mathbf{T}) = \mathbf{T}$.

Therefore,

$$S = \frac{\sigma(\mathbf{T})}{|G_*^\theta||T|} \sum_{(t, x', g)} \frac{\chi(x'tx'^{-1})}{|Z_{x'tx'^{-1}}|} \lambda(t) \sigma((Z_{Z_{x'tx'^{-1}}}(x'\mathbf{T}x'^{-1} \cap \mathbf{G}_*^\theta)^\circ)),$$

with (t, x', g) summed over

$$\{(t, x', g) \in T \times G \times G : x'tx'^{-1} \in G_*^\theta, g \in Z_{x'tx'^{-1}}, (x'^{-1} \cdot \theta)(\mathbf{T}) = \mathbf{T}\}.$$

This is the same as

$$S = \frac{\sigma(\mathbf{T})}{|G_*^\theta||T|} \sum_{\substack{(t, x') \in T \times G \\ x'tx'^{-1} \in G_*^\theta \\ (x'^{-1} \cdot \theta)(\mathbf{T}) = \mathbf{T}}} \chi(x'tx'^{-1}) \lambda(t) \sigma((Z_{Z_{x'tx'^{-1}}}(x'\mathbf{T}x'^{-1} \cap \mathbf{G}_*^\theta)^\circ)).$$

By the definition of $\varepsilon_{x'\mathbf{T}x'^{-1}}$, we have

$$\varepsilon_{x'\mathbf{T}x'^{-1}}(x'tx'^{-1}) = \sigma(Z_{\mathbf{G}}((x'\mathbf{T}x'^{-1} \cap \mathbf{G}_*^\theta)^\circ)) \sigma(Z_{Z_{x'tx'^{-1}}}((x'\mathbf{T}x'^{-1} \cap \mathbf{G}_*^\theta)^\circ))$$

and thus

$$S = \frac{\sigma(\mathbf{T})}{|G_*^\theta||T|} \sum_{\substack{(t, x') \in T \times G \\ x'tx'^{-1} \in G_*^\theta \\ (x'^{-1} \cdot \theta)(\mathbf{T}) = \mathbf{T}}} \chi(x'tx'^{-1}) \lambda(t) \sigma(Z_{\mathbf{G}}((x'\mathbf{T}x'^{-1} \cap \mathbf{G}_*^\theta)^\circ)) \cdot \varepsilon_{x'\mathbf{T}x'^{-1}}(x'tx'^{-1}).$$

We now change variables by replacing (t, x') by (\bar{t}, g) , where $\bar{t} = x'tx'^{-1}$ and $g = x'^{-1}$. This yields

$$S = \frac{\sigma(\mathbf{T})}{|G_*^\theta||T|} \sum_{\substack{g \in G \\ (g \cdot \theta)(\mathbf{T}) = \mathbf{T}}} \sigma(Z_{\mathbf{G}}((g^{-1}\mathbf{T}g \cap \mathbf{G}_*^\theta)^\circ)) \cdot \sum_{\bar{t} \in g^{-1}Tg \cap G_*^\theta} \chi(\bar{t}) \lambda(g\bar{t}g^{-1}) \varepsilon_{g^{-1}\mathbf{T}g}(\bar{t}).$$

The sum over \bar{t} vanishes unless $g \in \Xi_{\mathbf{T}, \lambda, \chi}$ in which case it equals $|T \cap G_*^{g \cdot \theta}|$. Hence,

$$S = \frac{\sigma(\mathbf{T})}{|G_*^\theta||T|} \sum_{g \in \Xi_{\mathbf{T}, \lambda, \chi}} \sigma(Z_{\mathbf{G}}((g^{-1}\mathbf{T}g \cap \mathbf{G}_*^\theta)^\circ)) \cdot |T \cap G_*^{g \cdot \theta}|.$$

Note that the above summand is constant on double cosets in $T \backslash G / G_*^\theta$. Now let $T \times G^\theta$ act on G by $(t, h) \cdot x = txh^{-1}$. Then TgG_*^θ is the orbit of g . The map $(t, h) \mapsto t$ gives a bijection between the isotropy group of g and $T \cap G_*^{g \cdot \theta}$. Thus

$$|TgG_*^\theta| = \frac{|G_*^\theta||T|}{|T \cap G_*^{g \cdot \theta}|}.$$

Therefore,

$$S = \sigma(\mathbf{T}) \sum_{g \in T \backslash \Xi_{\mathbf{T}, \lambda, \chi} / G_*^\theta} \sigma(Z_{\mathbf{G}}((g^{-1}\mathbf{T}g \cap \mathbf{G}_*^\theta)^\circ))$$

and our claim is proven. □

3.2.2. Reformulation. Let Θ be the G -orbit of some fixed involution θ_0 of G . Above, we have assumed that χ is an arbitrary character of $G_*^{\theta_0}$. In this section, we further require that χ can be extended to a character of G_{θ_0} . Under this assumption, if $g \in G$, then

$$t \mapsto \chi(g^{-1}tg)$$

defines a character of $T \cap G_*^{g \cdot \theta_0}$ that depends only on the involution $g \cdot \theta_0$ and not on g itself. We denote this character by $\chi_{g \cdot \theta_0}$. Similarly,

$$t \mapsto \varepsilon_{g^{-1}\mathbf{T}g}(g^{-1}tg)$$

defines a character $\varepsilon_{\mathbf{T}, g \cdot \theta_0}$ of $T \cap G_*^{g \cdot \theta_0}$ depending only on $g \cdot \theta_0$.

Let

$$\Theta_{\mathbf{T}, \lambda, \chi} = \{\theta \in \Theta : \theta(\mathbf{T}) = \mathbf{T}, \lambda|(T \cap G_*^\theta) = \chi_\theta \cdot \varepsilon_{\mathbf{T}, \theta}\}.$$

Then $gG_{\theta_0} \mapsto g \cdot \theta_0$ gives a bijection between $\Xi_{\mathbf{T}, \lambda, \chi} / G_{\theta_0}$ and $\Theta_{\mathbf{T}, \lambda, \chi}$. (Recall that G_{θ_0} is the stabilizer of θ_0 in G .) It also gives a bijection between $T \backslash \Xi_{\mathbf{T}, \lambda, \chi} / G_{\theta_0}$ and the space of T -orbits in $\Theta_{\mathbf{T}, \lambda, \chi}$.

If $\theta \in \Theta$, then we let $[\theta]$ denote the T -orbit of θ and we take

$$m_T([\theta]) = [G_\theta : G_*^\theta(G_\theta \cap T)].$$

We write $[\theta] \sim \lambda$ when $\theta(\mathbf{T}) = \mathbf{T}$ and $\lambda|((\cap G_*^\theta) = \chi_\theta \cdot \varepsilon_{\mathbf{T}, \theta}$ or, in other words, $[\theta] \subset \Theta_{\mathbf{T}, \lambda, \chi}$. Define

$$\langle [\theta], \lambda \rangle_T^\chi = \begin{cases} \sigma(\mathbf{T}) \sigma(Z_{\mathbf{G}}((\mathbf{T} \cap \mathbf{G}_*^\theta)^\circ)), & \text{if } [\theta] \sim \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$\langle \Theta, \lambda \rangle_G^\chi = \frac{1}{|G_*^{\theta_0}|} \sum_{h \in G_*^{\theta_0}} R_{\mathbf{T}, \lambda}(h) \chi(h).$$

Theorem 3.12.

$$\langle \Theta, \lambda \rangle_G^\chi = \sum_{[\theta] \sim \lambda} m_T([\theta]) \langle [\theta], \lambda \rangle_T^\chi.$$

Proof. Since the set $\Xi_{\mathbf{T}, \lambda, \chi}$ may be expressed as

$$\{g \in G : (g \cdot \theta_0)(\mathbf{T}) = \mathbf{T}, \lambda|(T \cap G_*^{g \cdot \theta_0}) = \chi_{g \cdot \theta_0} \cdot \varepsilon_{\mathbf{T}, g \cdot \theta_0}\},$$

it follows that $\Xi_{\mathbf{T}, \lambda, \chi}$ is a union of double cosets in $T \backslash G / G_{\theta_0}$. If $g \in G^{\theta_0}$, then $gG_*^{\theta_0}g^{-1} = G_*^{g \cdot \theta_0} = G_*^{\theta_0}$ and thus $G_*^{\theta_0}$ is a normal subgroup of G_{θ_0} . Hence, we have an action of $G_{\theta_0} / G_*^{\theta_0}$ on $T \backslash \Xi_{\mathbf{T}, \lambda, \chi} / G_*^{\theta_0}$ by

$$h \cdot (TgG_*^{\theta_0}) = Tgh^{-1}G_*^{\theta_0}.$$

The isotropy group of $TgG_*^{\theta_0}$ is $(G_{\theta_0} \cap g^{-1}Tg) / (G_*^{\theta_0} \cap g^{-1}Tg)$.

We have a projection

$$T \backslash \Xi_{\mathbf{T}, \lambda, \chi} / G_*^{\theta_0} \rightarrow T \backslash \Xi_{\mathbf{T}, \lambda, \chi} / G_{\theta_0}.$$

The G_{θ_0} -orbit of $TgG_*^{\theta_0}$ is the fiber of the double coset TgG_{θ_0} . The cardinality of this fiber is

$$[G_{\theta_0}/G_*^{\theta_0} : (G_{\theta_0} \cap g^{-1}Tg)/(G_*^{\theta_0} \cap g^{-1}Tg)]$$

or, equivalently,

$$[G_{\theta_0} : G_*^{\theta_0}(G_{\theta_0} \cap g^{-1}Tg)].$$

This is also the same as

$$[G_{g \cdot \theta_0} : G_*^{g \cdot \theta_0}(G_{g \cdot \theta_0} \cap T)].$$

We observe

$$\sigma(Z_{\mathbf{G}}((g^{-1}\mathbf{T}g \cap \mathbf{G}_*^{\theta_0})^\circ)) = \sigma(Z_{\mathbf{G}}((\mathbf{T} \cap \mathbf{G}_*^{g \cdot \theta_0})^\circ)),$$

and thus by Theorem 3.11,

$$\begin{aligned} \langle \Theta, \lambda \rangle_G^\chi &= \sigma(\mathbf{T}) \sum_{g \in T \backslash \Xi_{\mathbf{T}, \lambda, \chi} / G_*^{\theta_0}} \sigma(Z_{\mathbf{G}}((\mathbf{T} \cap \mathbf{G}_*^{g \cdot \theta_0})^\circ)) \\ &= \sigma(\mathbf{T}) \sum_{g \in T \backslash \Xi_{\mathbf{T}, \lambda, \chi} / G_{\theta_0}} [G_{g \cdot \theta_0} : G_*^{g \cdot \theta_0}(G_{g \cdot \theta_0} \cap T)] \sigma(Z_{\mathbf{G}}((\mathbf{T} \cap \mathbf{G}_*^{g \cdot \theta_0})^\circ)) \\ &= \sum_{[\theta] \in \Theta_{\mathbf{T}, \lambda, \chi}^T} m_T([\theta]) \langle [\theta], \lambda \rangle_T^\chi. \end{aligned} \quad \square$$

4. PARAMETERS FOR TAME SUPERCUSPIDAL REPRESENTATIONS OF $\mathrm{GL}_n(F)$

From now on, unless otherwise specified, we assume that \mathbf{G} is the group GL_n .

4.1. Howe data. We recall some basic terminology and facts associated with Howe’s construction [Ho] of tame supercuspidal representations of $G = \mathrm{GL}_n(F)$, and then we describe how the latter construction fits within Yu’s framework of constructing tame supercuspidal representations for more general groups [Y]. A more detailed discussion of these matters is contained in [HMu].

For the purposes of this paper, we find it convenient to introduce the notion of a “Howe datum”. This is a GL_n -variant of the notion of a cuspidal G -datum (in the sense of [HMu]).

Definition 4.1. If E is a tamely ramified extension of F of degree n and φ is a quasicharacter of E^\times , then φ is **F -admissible** (or **admissible over F**) if

- there does not exist a proper subfield L of E containing F such that φ factors through the norm map $N_{E/L} : E^\times \rightarrow L^\times$;
- if L is a subfield of E containing F and $\varphi \mid (1 + \mathfrak{P}_E)$ factors through $N_{E/L}$, then E is unramified over L .

If φ and φ' are F -admissible quasicharacters of E^\times and E'^\times , respectively, then φ and φ' are **F -conjugate** if there exists an F -isomorphism of E with E' that takes φ to φ' .

Howe’s construction yields a bijection between the set of equivalence classes of tame supercuspidal representations of G and the set of F -conjugacy classes of F -admissible quasicharacters associated to tamely ramified extensions of F of degree n .

Definition 4.2. If F' is a finite tamely ramified extension of F and φ is a quasicharacter of F'^\times , the **conductor exponent** $f(\varphi)$ of φ is the smallest positive integer such that $\varphi \mid 1 + \mathfrak{P}_{F'}^{f(\varphi)} = 1$.

When F' is a finite tamely ramified extension of F , we let $C_{F'}$ denote the subgroup of F'^{\times} generated by the roots of unity in $\mathfrak{O}_{F'}^{\times}$ of order relatively prime to p and by a prime element $\varpi_{F'}$ in F' such that $\varpi_{F'}^e$ belongs to F , where e is the ramification index of F' over F . If ψ' is a character of F' that is trivial on $\mathfrak{P}_{F'}$ and nontrivial on $\mathfrak{O}_{F'}$ and if $f(\varphi) > 1$, then there exists a unique

$$\gamma_{\varphi} \in C_{F'} \cap (\mathfrak{P}_{F'}^{1-f(\varphi)} - \mathfrak{P}_{F'}^{2-f(\varphi)})$$

such that $\varphi(1+t) = \psi'(\gamma_{\varphi}t)$, $t \in \mathfrak{P}_{F'}^{f(\varphi)-1}$.

Definition 4.3. Let F' be a tamely ramified extension of F and let φ be a quasi-character of F'^{\times} . If $f(\varphi) > 1$, we say that φ is **generic** over F if $F[\gamma_{\varphi}] = F'$. If $f(\varphi) = 1$, then we say that φ is **generic** over F if φ is F -admissible.

We remark that if $f(\varphi) = 1$, then φ is generic over F precisely when F' is unramified over F and φ is not fixed by any nontrivial element of the Galois group $\text{Gal}(F'/F)$. We also observe that, in general, if φ is generic over F , then it is necessarily admissible over F .

Let E be a tamely ramified extension of F of degree n , and let φ be an F -admissible quasicharacter of E^{\times} .

Definition 4.4. A **Howe factorization of φ** consists of

- a tower of fields $F = E_d \subsetneq E_{d-1} \subsetneq \cdots \subsetneq E_0 \subset E$, with $d \geq 0$,
- a collection of quasicharacters φ_i , $i = -1, \dots, d$,

with the following properties:

- For each $i \in \{0, \dots, d\}$, φ_i is a quasicharacter of E_i^{\times} such that the conductor exponent $f_i = f(\varphi_i \circ N_{E/E_i})$ of $\varphi_i \circ N_{E/E_i}$ is greater than 1, and such that φ_i is generic over E_{i+1} if $i \neq d$.
- $f_0 < f_1 < \cdots < f_{d-1}$.
- If φ_d is nontrivial, then $f_d > f_{d-1}$.
- **(The toral case)** If $E_0 = E$, then φ_{-1} is the trivial character of E^{\times} .
- **(The nontoral case)** If $E_0 \subsetneq E$, then φ_{-1} is a quasicharacter of E^{\times} such that $f(\varphi_{-1}) = 1$ and φ_{-1} is generic over E_0 .
- $\varphi = \varphi_{-1} \prod_{i=0}^d \varphi_i \circ N_{E/E_i}$.

Note that E/E_0 is always unramified.

Definition 4.5. A **Howe datum (with respect to G)** consists of:

- a degree n tamely ramified extension E of F ,
- an F -admissible quasicharacter $\varphi : E^{\times} \rightarrow \mathbb{C}^{\times}$,
- a Howe factorization of φ ,
- an F -linear embedding of E in $M(n, F)$.

The latter two ingredients affect the construction but not the equivalence class of the representation that is constructed. If Φ is a Howe datum, then we let $\pi(\Phi)$ denote the associated tame supercuspidal representation.

4.2. Embeddings of E^{\times} in $\text{GL}_n(F)$. The material in this section is quite standard and is well known to experts in the theory of supercuspidal representations and Howe’s construction. It is included for the benefit of nonexperts and to fix some notations.

One can associate an F -embedding $E \hookrightarrow M(n, F)$ to any F -basis e_1, \dots, e_n of E as follows. When $x \in E$ let

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

where $x = x_1e_1 + \dots + x_n e_n$ and $x_1, \dots, x_n \in F$. Thus $x \mapsto \underline{x}$ is an F -linear isomorphism $E \cong F^n$. Multiplication by x is an F -linear transformation of E and hence defines a matrix $\underline{x} \in \mathfrak{g}$. So $x \mapsto \underline{x}$ is the regular representation associated to our basis. Given $x, x' \in E$, we have the relations $\underline{x} \underline{x}' = \underline{xx'}$ and $\underline{x} \underline{x}' = \underline{xx'}$.

Recall that $R_{E/F}\mathrm{GL}_1$ is isomorphic over \overline{F} to a direct product $\prod_{\sigma \in \Sigma} \mathrm{GL}_1$ indexed by the set Σ of F -embeddings of E in \overline{F} . Fix an ordering $\sigma_1, \dots, \sigma_n$ of the n elements of Σ . Then $(R_{E/F}\mathrm{GL}_1)(F) = \{(\sigma_1 x, \dots, \sigma_n x) \mid x \in E^\times\}$ can be identified with E^\times . Under this identification, the embedding $x \mapsto \underline{x}$ for $x \in E^\times$ arises from an embedding of algebraic groups $R_{E/F}\mathrm{GL}_1 \rightarrow \mathrm{GL}_n$.

Let $\iota' : R_{E/F}\mathrm{GL}_1 \rightarrow \mathrm{GL}_n$ be the corresponding embedding

$$(x_1, \dots, x_n) \mapsto \mathrm{diag}(x_1, \dots, x_n).$$

Let $\mu \in \mathrm{GL}_n(E)$ be the matrix

$$(\mu_{ij}) = (\sigma_i(e_j)),$$

and let $\iota = \mathrm{Int}(\mu)^{-1} \circ \iota'$. Then ι is defined over F and

$$\iota(x) = \underline{x} \quad \text{for } x \in (R_{E/F}\mathrm{GL}_1)(F) = E^\times.$$

We will refer to such an embedding $\iota : R_{E/F}\mathrm{GL}_1 \rightarrow \mathrm{GL}_n$ as a *standard F -embedding*.

The next lemma follows from a straightforward argument.

Lemma 4.6. *Given an F -embedding $\iota : E \rightarrow M(n, F)$, there exists an F -basis e_1, \dots, e_n of E such that ι is identical to the embedding associated as above to e_1, \dots, e_n . Thus every F -embedding $\iota : E \rightarrow M(n, F)$ gives rise to a standard F -embedding $R_{E/F}\mathrm{GL}_1 \rightarrow \mathrm{GL}_n$ as above.*

In the following, as in this lemma, we will typically use the same symbol to denote both of these associated embeddings.

We note that in both parts of this lemma, the ordered frame (Fe_1, \dots, Fe_n) is uniquely determined by ι , while the unordered frame $\{Fe_1, \dots, Fe_n\}$ is uniquely determined by the image of ι .

4.3. Attaching a cuspidal G -datum to a Howe datum. Fix a Howe datum $\Phi = (\varphi, E, \{\varphi_i\}, \{E_i\}, \iota : E \hookrightarrow M(n, F))$. The purpose of this section is to associate a cuspidal G -datum $\Psi = (\vec{G}, y, \rho, \vec{\phi})$ to Φ .

By Lemma 4.6, ι determines a unique standard F -embedding (which we also denote by ι) of $R_{E/F}\mathrm{GL}_1$ into GL_n . Let \mathbf{T} be the image of ι in GL_n . Then \mathbf{T} is an elliptic maximal F -torus of GL_n . Identifying $R_{E/F}\mathrm{GL}_1(F)$ with E^\times , we have $T = \iota(E^\times)$.

Given $x \in R_{E/F}\mathrm{GL}_1$ and $\sigma \in \Sigma$, let x_σ denote the σ -component of x . For $i \in \{0, \dots, d\}$, the torus $R_{E_i/F}\mathrm{GL}_1$ embeds naturally in $R_{E/F}\mathrm{GL}_1$ as the subgroup consisting of elements x such that

$$x_\sigma = x_\tau \quad \text{if } \sigma|_{E_i} = \tau|_{E_i}.$$

Let \mathbf{Z}^i be the image of $R_{E_i/F}\mathrm{GL}_1$ under ι . Then \mathbf{Z}^i is an F -subtorus of \mathbf{T} and $Z^i = \iota(E_i^\times)$. Let \mathbf{G}^i be the centralizer of \mathbf{Z}^i in \mathbf{G} . Then our desired tamely ramified twisted Levi sequence is $\vec{\mathbf{G}} = (\mathbf{G}^0, \dots, \mathbf{G}^d)$.

We may assume the embeddings $\sigma_1, \dots, \sigma_n$ are arranged so that for each i , the embeddings that have the same restriction to E_i are grouped consecutively. Then for each i , $\mathbf{M}^i := \mu \mathbf{G}^i \mu^{-1}$ is the standard Levi subgroup $\mathrm{GL}_{n_i} \times \dots \times \mathrm{GL}_{n_i}$ of \mathbf{G} , where $n_i = n[E_i : F]^{-1}$. We may view \mathbf{M}^i as $R_{E_i/F}\mathrm{GL}_{n_i}$ if we alter the usual action of the Galois group as in §2.

The following result is well known.

Lemma 4.7. *For all $i \in \{0, \dots, d\}$, the map $\mathrm{Int}(\mu) : \mathbf{G}^i \rightarrow \mathbf{M}^i$ gives a natural F -isomorphism from \mathbf{G}^i to $R_{E_i/F}\mathrm{GL}_{n_i}$ and hence an isomorphism $G^i \cong \mathrm{GL}_{n_i}(E_i)$.*

Identifying each G^i with $\mathrm{GL}_{n_i}(E_i)$ via $\mathrm{Int}(\mu)$, we obtain a determinant map $\mathrm{det}_i : G^i \rightarrow E_i^\times$. We let

$$\phi_i = \varphi_i \circ \mathrm{det}_i$$

and $\vec{\phi} = (\phi_0, \dots, \phi_d)$.

Let E' be a normal closure of E/F . The space

$$A(\mathbf{G}, \mathbf{T}, F) = A(\mathbf{G}, \mathbf{T}, E')^{\mathrm{Gal}(E'/F)}$$

is 1-dimensional. The point y in our datum Ψ is chosen to be an arbitrary point in $A(\mathbf{G}, \mathbf{T}, F)$. The corresponding point $[y]$ in the reduced building is uniquely determined by \mathbf{T} .

In the toral case, we let ρ be the trivial representation of $G^0 = E^\times$.

Now suppose we are in the nontoral case. Let q_0 be the cardinality of the residue class field of E_0 . Then $G_{y,0}^0$ is conjugate to $\mathrm{GL}_{n_0}(\mathfrak{O}_{E_0})$ and $G_{y,0;0^+}^0 \cong \mathrm{GL}_{n_0}(\mathfrak{f}_{E_0})$. The quasicharacter φ_{-1} is not fixed by any nontrivial element of $\mathrm{Gal}(E/E_0)$, since φ_{-1} is E_0 -admissible and $f(\varphi_{-1}) = 1$. The restriction $\varphi_{-1} | \mathfrak{O}_E^\times$ factors to a character λ of \mathfrak{f}_E^\times that is in general position in the sense that it is not fixed by any nontrivial element of $\mathrm{Gal}(\mathfrak{f}_E/\mathfrak{f}_{E_0})$.

The construction of Deligne and Lusztig yields a bijection between the set of equivalence classes of irreducible cuspidal representations of $\mathrm{GL}_{n_0}(\mathfrak{f}_{E_0})$ and the $\mathrm{Gal}(\mathfrak{f}_E/\mathfrak{f}_{E_0})$ -orbits of characters of \mathfrak{f}_E^\times that are in general position. In particular, the above character λ determines an equivalence class R_λ of irreducible cuspidal representations of $\mathrm{GL}_{n_0}(\mathfrak{f}_{E_0})$. Let ρ° be an irreducible representation of $G_{y,0}^0$ whose restriction to $G_{y,0^+}^0$ is a multiple of the trivial representation and assume that ρ° factors to an irreducible cuspidal representation of $G_{y,0;0^+}^0$ belonging to R_λ . Note that

$$K^0 = G_{[y]}^0 = \underline{E_0}^\times G_{y,0}^0 \cong \langle \underline{\varpi}_{E_0} \rangle \times G_{y,0}^0,$$

for any choice of prime element ϖ_{E_0} in E_0 . Let ρ be the representation of K^0 that restricts to ρ° on $G_{y,0}^0$, and such that $\rho(\underline{\varpi}_{E_0})$ is equal to $\varphi_{-1}(\varpi_{E_0})$ times the identity operator on the space of ρ° .

Let r_i be the depth of ϕ_i , for $i \in \{0, \dots, d-1\}$. Then

$$r_i = \frac{f_i - 1}{e}, \text{ where } f_i = f(\varphi_i \circ N_{E/E_i}),$$

where e is the ramification degree of E over F .

We have now fully constructed our desired cuspidal G -datum Ψ . Note that we had some limited freedom in choosing y and, when $E \neq E_0$, we could vary the choice of ρ so long as $\rho \mid G_{y,0}^0$ factors to an element of R_λ .

5. ORTHOGONAL INVOLUTIONS

For a symmetric matrix $\nu \in G$, let θ_ν be the F -involution of GL_n given by $x \mapsto \nu^{-1} \cdot {}^t x^{-1} \cdot \nu$. Here $X \mapsto {}^t X$ denotes the usual transpose on $n \times n$ matrices.

We will refer to involutions of G of the form θ_ν as *orthogonal involutions*.

5.1. Restrictions of orthogonal involutions. In this section, we prove that if θ is an orthogonal involution of G and if $\vec{G} = (G^0, \dots, G^d)$ is a tamely ramified twisted Levi sequence associated to G such that $\theta(\vec{G}) = \vec{G}$, then θ restricts to an orthogonal involution of each group G^i . Implicit in this statement is that each G^i is isomorphic to a general linear group, however, the choice of isomorphism $G^i \cong \text{GL}(n_i, E_i)$ is irrelevant for our result.

It is important to stress that a given element g of some G^i has a transpose with respect to $G = \text{GL}(n, F)$ and another transpose with respect to $G^i \cong \text{GL}(n_i, E_i)$ (once a specific isomorphism is chosen). Therefore, if $\nu \in G$ is symmetric (as an element of G) and if ν lies in G^i , then it is not necessarily the case that θ_ν restricts to an orthogonal involution of G^i .

Our assertion about restrictions of orthogonal involutions amounts to showing that an orthogonal involution of $G = G^d$ restricts to an orthogonal involution of $G^{d-1} \cong \text{GL}(n_{d-1}, E_{d-1})$, since once this is established one can apply the same result to G^{d-1} and G^{d-2} , and so forth, until one deduces that θ restricts to an orthogonal involution of G^0 . For notational simplicity, we write G' instead of G^{d-1} in this section.

Proposition 5.1. *If θ is an orthogonal involution of G such that $\theta(G') = G'$, then θ restricts to an orthogonal involution of G' .*

Fix an orthogonal involution θ of G such that $\theta(G) = G$. Fix a symmetric matrix ν in G such that $\theta(g) = \nu^{-1} \cdot {}^t g^{-1} \cdot \nu$, for all $g \in G$.

We can (and do) fix an isomorphism $G' \cong \text{GL}(n', E')$, where E' is an intermediate field of E/F and $n' = [E : E']$. We observe that our proof of Proposition 5.1 uses the fact that $n' = n_{d-1} = [E : E_{d-1}]$ is odd, but otherwise it does not use our assumption that n is odd.

Let $X \mapsto {}^\tau X$ be the transpose on $M(n', E')$. The proposition we are considering asserts that there exists $\xi \in G'$ such that ${}^\tau \xi = \xi$ and $\theta(g) = \xi^{-1} \cdot {}^\tau g^{-1} \cdot \xi$, for all $g \in G'$.

Lemma 5.2. *Under the assumptions of Proposition 5.1, $X \mapsto \nu^{-1} \cdot {}^t X \cdot \nu$ preserves $M(n', E')$.*

Proof. Denote the anti-automorphism $X \mapsto \nu^{-1} \cdot {}^t X \cdot \nu$ of $M(n, F)$ by α . Then $\alpha(G') = G'$. It is easy to choose an E' -basis of $M(n', E')$ consisting of elements of G' . But α maps such a basis to another such basis. Therefore, α preserves $M(n', E')$. □

Proof of Proposition 5.1. Define an E' -algebra automorphism of $M(n', E')$ by

$$\beta(X) = \nu^{-1} ({}^t ({}^\tau X)) \nu.$$

By the Skölem-Noether Theorem, there exists $\xi \in G'$ such that $\beta(X) = \xi^{-1}X\xi$, for all $X \in M(n', E')$.

Taking $X = {}^\tau g^{-1}$, we obtain $\theta(g) = \nu^{-1}({}^t g^{-1})\nu = \xi^{-1}({}^\tau g^{-1})\xi$. Therefore, $g = \theta(\theta(g)) = \xi^{-1}({}^\tau \xi)g({}^\tau \xi^{-1})\xi$. This says that the element $z = \xi^{-1}({}^\tau \xi)$ lies in the center Z' of G' . It now suffices to show that $z = 1$.

We note that $\xi z = {}^\tau \xi$ and thus $z = ({}^\tau \xi)\xi^{-1}$. Therefore, $z^{-1} = \tau z^{-1} = \xi^{-1} \cdot {}^\tau \xi = z$. Thus, $z = \pm 1$. But we have (identifying Z' with $(E')^\times$) $1 = \det_{G'}(({}^\tau \xi)\xi^{-1}) = \det_{G'}(z) = z^{n'} = z$. □

5.2. θ -split embeddings of E^\times . In this section, we work in the following generality: E/F is a finite separable extension of degree n of arbitrary fields, where n is an integer (possibly even) greater than 1. The separability assumption is required because we need to know that the trace $\text{tr}_{E/F}$ is not identically zero and, in addition, we need E/F to have primitive elements.

5.2.1. *Parametrization of θ -split embeddings.* Fix an F -basis e_1, \dots, e_n of E . We refer the reader to Section §4.2 for the notation \underline{x} and \underline{x} (for $x \in E^\times$) defined with respect to this basis.

Given $x, x' \in E$ and $a \in E^\times$, then

$$\langle x, x' \rangle_a = \text{tr}_{E/F}(axx')$$

defines a symmetric F -bilinear form on E . The matrix of this inner product is the symmetric matrix $\nu^a = (\nu_{ij}^a)$ in G defined by

$$\nu_{ij}^a = \text{tr}_{E/F}(ae_i e_j).$$

Thus

$$\langle x, x' \rangle_a = \text{tr}_{E/F}(axx') = {}^t \underline{x} \cdot \nu^a \cdot \underline{x'},$$

for all $x, x' \in E$.

The following standard result (whose proof we omit) depends upon the separability of E/F .

Lemma 5.3. *The inner product $\langle \cdot, \cdot \rangle_a$ is nondegenerate or, equivalently, ν^a is invertible.*

We now consider the mapping

$$E^\times \rightarrow \{\text{nondegenerate symmetric } F\text{-bilinear forms on } E\}$$

given by $a \mapsto \langle \cdot, \cdot \rangle_a$, as well as variations on this mapping. If an F -basis of E has been fixed, then $\langle \cdot, \cdot \rangle_a$ determines a symmetric matrix ν^a in G . So we have a map

$$E^\times \rightarrow \mathcal{S} := \{\text{symmetric matrices in } G\}.$$

There is a natural action of G on \mathcal{S} : for $g \in G$ and $\nu \in \mathcal{S}$, define $g \cdot \nu = g\nu{}^t g$. We will say that two elements of \mathcal{S} in the same G -orbit are *similar*.

Changing the basis chosen above has the effect of replacing ν^a by another matrix that is similar to ν^a , so we obtain a

$$E^\times \rightarrow \mathcal{O}_G(\mathcal{S}) := \{G\text{-orbits in } \mathcal{S}\}.$$

If $a, b \in E^\times$, then

$$\nu^{ab} = {}^t \underline{b} \nu^a = \nu^a \underline{b},$$

from which it follows that

$$\nu^{ab^2} = {}^t \underline{b} \nu^a \underline{b}$$

and thus ν^{ab^2} is similar to ν^a . Therefore, our map $E^\times \rightarrow \mathcal{O}_G(\mathcal{S})$ gives rise to a canonical map

$$X_E \rightarrow \mathcal{O}_G(\mathcal{S}),$$

where

$$X_E := E^\times / (E^\times)^2.$$

Each $\nu \in \mathcal{S}$ determines an involution θ_ν of G by

$$\theta_\nu(g) = \nu^{-1} \cdot {}^t g^{-1} \cdot \nu.$$

For simplicity, we write θ^a instead of θ_{ν^a} .

Let \mathbf{T} be a torus in \mathbf{G} . For simplicity, we will often refer to the group $T = \mathbf{T}(F)$ as a *torus* in G . If θ is an involution of G , such a torus T is said to be θ -split if all of its elements g satisfy $\theta(g) = g^{-1}$. Since T is dense in \mathbf{T} with respect to the Zariski topology, this is equivalent to the condition $\theta(g) = g^{-1}$ for all $g \in \mathbf{T}$, and we will also say that \mathbf{T} is θ -split in this case.

Now fix $T = \underline{E}^\times$. Then T is θ^a -split, according to the calculation:

$$\theta^a(\underline{x}^{-1}) = (\nu^a)^{-1} \cdot {}^t \underline{x} \cdot \nu^a = (\nu^a)^{-1} \cdot \nu^a \cdot \underline{x} = \underline{x}.$$

Lemma 5.4. *The map $a \mapsto \theta^a$ gives a bijection between E^\times / F^\times and the set of orthogonal involutions θ of G for which $T = \underline{E}^\times$ is θ -split.*

Proof. We first consider injectivity. Suppose $a_1, a_2 \in E^\times$. Then the condition $\theta^{a_1} = \theta^{a_2}$ is equivalent to the condition that ν^{a_1} and ν^{a_2} (or the associated inner products) are scalar multiples of each other. It is easy to see that this is equivalent to the existence of $z \in F^\times$ such that $\text{tr}_{E/F}((a_1 - za_2)x) = 0$ for all $x \in E$. Separability of E/F then says that this is equivalent to $a_1 = za_2$, which proves injectivity.

We now consider surjectivity. Suppose θ is an orthogonal involution such that T is θ -split. Choose a symmetric matrix $\nu \in G$ such that $\theta(g) = \nu^{-1} \cdot {}^t g^{-1} \cdot \nu$, for all $g \in G$. (Up to scalar multiples, ν is uniquely determined by θ .) We need to show that there exists $a \in E^\times$ such that $\nu = \nu^a$.

For $x, y \in E$, define $\langle x, y \rangle_\nu = {}^t \underline{x} \cdot \nu \cdot \underline{y}$ and let $\phi_\nu \in \text{Hom}_F(E, F)$ be defined by $\phi_\nu(z) = \langle 1, z \rangle_\nu$. We now observe that every nonzero element of $\text{Hom}_F(E, F)$ is associated to an element of E^\times in the following manner. Define an F -linear map $E \rightarrow \text{Hom}_F(E, F)$ by mapping $a \in E$ to $\text{tr}_{E/F} \circ \mu_a$, where $\mu_a : E \rightarrow E$ is given by $\mu_a(x) = ax$. This map is clearly injective. Therefore, it defines an F -linear isomorphism $E \cong \text{Hom}_F(E, F)$ since E and $\text{Hom}_F(E, F)$ both have F -dimension $[E : F]$. This implies that there exists $a \in E^\times$ such that $\phi_\nu(x) = \text{tr}_{E/F}(ax)$, for all $x \in E$.

Now suppose $x, y \in E$. Note that $\theta(\underline{x}^{-1}) = \underline{x}$, from which it follows that $\nu \cdot \underline{x} = {}^t \underline{x} \cdot \nu$. Therefore, we have $\langle x, y \rangle_\nu = {}^t \underline{x} \cdot \nu \cdot \underline{y} = {}^t \underline{1} \cdot {}^t \underline{x} \cdot \nu \cdot \underline{y} = {}^t \underline{1} \cdot \nu \cdot \underline{xy} = \langle 1, xy \rangle_\nu = \phi_\nu(xy) = \text{tr}_{E/F}(axy) = \langle x, y \rangle_a$. We now deduce that $\nu = \nu^a$ which completes the proof. \square

We now observe that

$$\theta^{ab^{-2}} = \underline{b} \cdot \theta^a$$

and interpret this as an equivariance property of the map $a \mapsto \theta^a$. More precisely, let E^\times act on E^\times / F^\times by $b \cdot (aF^\times) = ab^{-2}F^\times$, and let T act on the set of involutions of G by restricting the usual action of G on involutions. Then the mapping $a \mapsto \theta^a$ becomes equivariant with respect to $T = E^\times$, where we identify E^\times with T via $x \mapsto \underline{x}$. This yields:

Corollary 5.5. *Let $T = \underline{E^\times}$. The map $a \mapsto \theta^a$ gives a bijection from $E^\times / (E^\times)^2 F^\times$ to the set of T -orbits of orthogonal involutions θ of G for which T is θ -split.*

5.2.2. $Y_{E/F}$. We have just shown that $a \mapsto \theta^a$ gives a bijection

$$\mu_{E/F} : Y_{E/F} \rightarrow \mathcal{O}^T,$$

where $Y_{E/F} = E^\times / (E^\times)^2 F^\times$ and \mathcal{O}^T is the set of T -orbits of orthogonal involutions θ such that T is θ -split. Now let $y_{E/F}$ denote the cardinality of $Y_{E/F}$.

Lemma 5.6. *$y_{E/F} - 1$ is the number of quadratic extensions of F contained in E . In particular, $y_{E/F} = 1$ if n is odd and $y_{E/F} = 2$ if $n = 2$.*

Proof. We start by noting that the nontrivial elements of $X_E = E^\times / (E^\times)^2$ represent quadratic extensions of E . So the elements of $Y_{E/F}$ may be viewed as quadratic extensions of E modulo those of the form EF' , where F' is a quadratic extension of F .

Now assume E/F is a degree n extension of p -adic fields of characteristic zero with $p \neq 2$. Here is another interpretation of $Y_{E/F}$ and $y_{E/F}$. We rewrite $Y_{E/F}$ as

$$\frac{E^\times / (F^\times \cap (E^\times)^2)}{((E^\times)^2 / (F^\times \cap (E^\times)^2)) \times (F^\times / (F^\times \cap (E^\times)^2))}$$

and then see that

$$y_{E/F} = \frac{|E^\times / (F^\times \cap (E^\times)^2)|}{|(E^\times)^2 / (F^\times \cap (E^\times)^2)| \cdot |F^\times / (F^\times \cap (E^\times)^2)|}.$$

This implies

$$y_{E/F} = \frac{4}{|F^\times / (F^\times \cap (E^\times)^2)|}.$$

Now the nontrivial elements of $(F^\times \cap (E^\times)^2) / (F^\times)^2$ are in bijective correspondence with the quadratic extensions of F that are contained in E and we have

$$\begin{aligned} |F^\times / (F^\times \cap (E^\times)^2)| &= \frac{|F^\times / (F^\times)^2|}{|(F^\times \cap (E^\times)^2) / (F^\times)^2|} \\ &= \frac{4}{|(F^\times \cap (E^\times)^2) / (F^\times)^2|}. \end{aligned}$$

Our claim follows. □

5.2.3. *Split orthogonal involutions.* Let

$$J = J_n = \begin{pmatrix} & & & 1 \\ & \ddots & & \\ & & \ddots & \\ 1 & & & \end{pmatrix}.$$

In this section, we consider the orthogonal involution θ_J and its G -orbit Θ_J . We assume throughout that E/F is a degree n tamely ramified extension of characteristic-zero p -adic fields. Note that if $\theta \in \Theta_J$, then G^θ is a split orthogonal group. We will prove:

Proposition 5.7. *Given an embedding of E^\times in G with image T , then there exists $\theta \in \Theta_J$ such that T is θ -split. Consequently, Θ_J must contain a T -orbit that lies in \mathcal{O}^T . Given $\theta \in \Theta_J$ there exists an embedding of E^\times in G whose image T is θ -split.*

Our approach to the proof of Proposition 5.7 involves the characterization of G -orbits in \mathcal{S} using discriminants and Hasse invariants. Since there is some inconsistency in the literature regarding the use of the terms “discriminant” and “Hasse invariant”, we start by defining these terms.

If $s \in \mathcal{S}$, then the *discriminant of s* , which we denote by $\text{disc } s$, is the class of $\det s$ in $X_F = F^\times / (F^\times)^2$. Another notion of discriminant (that arises in the theory of the Witt group of F) is the notion of the *signed discriminant of s* . This is the class of $(-1)^{n(n-1)/2} \det s$ in X_F . Note that

$$\det J_n = (-1)^{n(n-1)/2}.$$

If A is a symmetric matrix in $\text{GL}(m, F)$, $m \in \mathbb{N}$, then we define the *Hasse invariant of A* by

$$\text{Hasse}(A) = \prod_{i \leq j} (a_i, a_j),$$

where $\text{diag}(a_1, \dots, a_m)$ is a diagonal matrix in the G -orbit of A and $(,)$ is the Hilbert symbol

$$(a, b) = \begin{cases} 1, & \text{if } z^2 = ax^2 + by^2 \text{ has a solution } (x, y, z) \in F^3 - \{0\}, \\ -1, & \text{otherwise.} \end{cases}$$

The following classical result is Theorem 63.20 of [O]:

Lemma 5.8. *The G -orbits in \mathcal{S} are characterized by the discriminant and Hasse invariant. There are eight possibilities for the pair $(\text{disc}(\nu), \text{Hasse}(\nu))$. When $n > 2$ each of these possibilities corresponds to a different G -orbit in \mathcal{S} and these eight orbits give all the G -orbits in \mathcal{S} . When $n = 2$, there are only seven orbits since it is impossible to have both $\text{disc}(\nu) = -1$ and $\text{Hasse}(\nu) = -1$.*

Note that the Hasse invariant is often defined as a product over $i < j$, instead of $i \leq j$. We will let $\text{Hasse}_0(A)$ denote the latter version of the Hasse invariant. Though these two definitions are not equivalent, either may be used to classify quadratic forms. The discrepancy between these two definitions is the product

$$\begin{aligned} \prod_{i=1}^m (a_i, a_i) &= \prod_{i=1}^m (a_i, -1) = (\text{disc}(A), -1) \\ &= \begin{cases} 1, & \text{if } \prod a_i \text{ is a sum of two squares,} \\ -1, & \text{otherwise.} \end{cases} \\ &= \begin{cases} -1, & \text{if } -1 \notin (F^\times)^2 \text{ and } \text{disc}(A) \text{ has odd valuation,} \\ 1, & \text{otherwise.} \end{cases} \end{aligned}$$

Lemma 5.9. *If $\nu \in \text{GL}_n(\mathfrak{O}_F) \cap \mathcal{S}$, then $\text{Hasse}(\nu) = \text{Hasse}_0(\nu) = 1$.*

Proof. Fix $\nu \in \text{GL}_n(\mathfrak{O}) \cap \mathcal{S}$. Using the fact that $(a, b) = 1$ whenever $a, b \in \mathfrak{O}^\times$, we see that it suffices to show that ν is similar to a diagonal matrix in $\text{GL}_n(\mathfrak{O})$.

We pass to the residue field \mathfrak{f} of F . If γ is a matrix (or vector) with entries in \mathfrak{O} , let $\bar{\gamma}$ denote the corresponding matrix (or vector) with entries in \mathfrak{f} . Then $\gamma \mapsto \bar{\gamma}$ defines a homomorphism from $\text{GL}_n(\mathfrak{O})$ onto $\text{GL}_n(\mathfrak{f})$. Let K_1 denote the kernel. Diagonalizing $\bar{\nu}$ in $\text{GL}_n(\mathfrak{f})$, we see that there exists $\alpha, \delta \in \text{GL}_n(\mathfrak{O})$ with δ diagonal such that $\bar{\delta} = {}^t \bar{\alpha} \bar{\nu} \bar{\alpha} = {}^t \bar{\alpha} \bar{\nu} \bar{\alpha}$. Our assertion now follows from Lemma 4 of [HMa]. □

We have observed that we have an identity

$$\nu^a = {}^t \underline{a} \nu^1 = \nu^1 \underline{a}.$$

Taking determinants yields the identity

$$\det(\nu^a) = N_{E/F}(a) \cdot \det(\nu^1).$$

It is perhaps of some interest to note that the latter identity can also be deduced from the following standard result.

Lemma 5.10. *Let $\sigma_1, \dots, \sigma_n$ be the distinct F -embeddings of E into a fixed algebraic closure \overline{F} of F . Let A be the diagonal matrix whose i -th diagonal entry is $\sigma_i(a)$ and let B be the matrix whose ij -th entry is $\sigma_j(e_i)$. Then $\nu^a = B \cdot A \cdot {}^t B$.*

The next lemma is also quite well known (cf., Proposition 12.1.4 of [IR]).

Lemma 5.11. *Let β be a primitive element for E/F and take $e_1 = 1, e_2 = \beta, e_3 = \beta^2, \dots, e_n = \beta^{n-1}$. Let f be the minimal polynomial for β over F . Then $\det(\nu^1) = (-1)^{n(n-1)/2} N_{E/F}(f'(\beta))$.*

Let us now examine the discriminant of the G -orbit \mathcal{S}^a in \mathcal{S} of ν^a . First we note that, by definition, the discriminant of \mathcal{S}^1 is just the discriminant $\text{disc}(E/F)$ of the extension E/F . Therefore,

$$\text{disc}(\mathcal{S}^a) = N_{E/F}(a) \cdot \text{disc}(E/F).$$

By Lemma 5.10, $\text{disc}(E/F)$ lies in the image of $(-1)^{n(n-1)/2} N_{E/F}(E^\times)$ in X_F . The same must therefore be true of $\text{disc}(\mathcal{S}^a)$. In other words, the signed discriminant of \mathcal{S}^a lies in the subset $N_{E/F}(E^\times)/(F^\times)^2$ of X_F .

If n is odd, then $N_{E/F}$ defines a surjective map from E^\times to X_F since, in fact, the restriction of this map to F^\times is identical to the natural projection $F^\times \rightarrow X_F$. The following result is now immediate.

Lemma 5.12. *The matrices in $\nu^1 \underline{E^\times} = \{\nu^a : a \in E^\times\}$ are symmetric. The discriminant classes represented by these elements comprise the image of*

$$(-1)^{n(n-1)/2} N_{E/F}(E^\times)$$

in X_F . If n is odd, then disc maps $\nu^1 \underline{E^\times}$ onto X_F .

To show that there exists $a \in E^\times$ such that ν^a is similar to J , it suffices to show that for some a the matrices ν^a and J have the same discriminant and the same Hasse invariant. We start with the case in which E/F is unramified, then we settle the totally and tamely ramified case, and finally we combine the latter cases to obtain the desired result for general tamely ramified extensions.

Lemma 5.13. *If E/F is an unramified extension of degree n , then there exists an element $a \in E^\times$ and an F -basis of E such that the associated matrix ν^a is identical to J .*

Proof. When $a \in E^\times$ and β is a primitive element for E/F , we have

$$\det(\nu^a) = N_{E/F}(af'(\beta))(-1)^{n(n-1)/2},$$

where f is the minimal polynomial of β and ν^a is defined with respect to the basis $e_1 = 1, e_2 = \beta, e_3 = \beta^2, \dots, e_n = \beta^{n-1}$. We may choose such a β which, in addition, lies in \mathfrak{D}_E^\times . Take $a = f'(\beta)^{-1}$. Since f is irreducible modulo \mathfrak{P} , the image of $f'(\beta)$ in $\mathfrak{D}_E/\mathfrak{P}_E$ is nonzero, and hence $f'(\beta)$ and a are units.

Since $\text{tr}_{E/F}$ takes \mathfrak{D}_E to \mathfrak{D} , ν^a has entries in \mathfrak{D} . Moreover, since $\det \nu^a = (-1)^{n(n-1)/2} \in \mathfrak{D}^\times$, it follows that $\nu^a \in \text{GL}_n(\mathfrak{D})$. Thus, according to Lemma 5.9, we have $\text{Hasse}(\nu^a) = 1 = \text{Hasse}(J)$. We also have

$$\det(\nu^a) = (-1)^{n(n-1)/2} = \det J.$$

Therefore, by Lemma 5.8, ν^a must be similar to J . Now choose $g \in G$ such that $g \cdot \nu^a \cdot {}^t g = J$. Define an F -basis e'_1, \dots, e'_n of E by $e'_i = \sum_j g_{ij} e_j$. Then the matrix ν^a with respect to e'_1, \dots, e'_n is precisely J . □

Lemma 5.14. *If E/F is a totally and tamely ramified extension of degree n , then there exists an element $a \in E^\times$ and an F -basis of E such that the associated matrix ν^a is identical to J .*

Proof. As in the proof of Lemma 5.13, we choose a certain $a \in E^\times$ and a certain primitive element β for E/F , and we use the F -basis $e_1 = 1, e_2 = \beta, e_3 = \beta^2, \dots, e_n = \beta^{n-1}$ of E . In the present case, we take β to be an element of E that is an n -th root of a prime element ϖ_F in F . (The fact that this is possible follows from Proposition 12 [La].)

We observe that

$$\underline{\underline{\beta}} = \begin{pmatrix} 0 & \cdots & 0 & \varpi_F \\ 1 & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{pmatrix}.$$

It is easy to evaluate $\text{tr}_{E/F}(\beta^k) = \text{tr}(\underline{\underline{\beta}}^k)$ for any k and to verify that the trace is zero unless k is a multiple of n . Taking $a = \beta^{1-n}/n$, we obtain $\nu^a = J$. □

Proposition 5.15. *If E/F is a tamely ramified extension of degree n , then there exists an element $a \in E^\times$ and an F -basis of E such that the associated matrix ν^a is identical to J .*

Proof. Let K/F be the maximal unramified subextension of E/F . Let $f = [K : F]$ and $e = n/f = [K : E]$. We define a tensor product map

$$M(e, F) \times M(f, F) \rightarrow M(n, F)$$

by taking $A \otimes B$ to be the $e \times e$ block matrix whose ij -th block is $A_{ij}B \in M(f, F)$.

Now suppose $a \in E^\times$ and let α be a K -basis $\alpha_1, \dots, \alpha_e$ of E . Define a symmetric matrix $\nu_\alpha^a \in \text{GL}_e(K)$ by $(\nu_\alpha^a)_{ij} = \text{tr}_{E/K}(a\alpha_i\alpha_j)$. Similarly, suppose $b \in K^\times$ and let β be a F -basis β_1, \dots, β_f of K . Define a symmetric matrix $\nu_\beta^b \in \text{GL}_f(F)$ by $(\nu_\beta^b)_{kl} = \text{tr}_{K/F}(b\beta_k\beta_l)$. Let $\alpha \otimes \beta$ be the F -basis of E given by

$$\alpha_1\beta_1, \dots, \alpha_1\beta_f, \alpha_2\beta_1, \dots, \alpha_2\beta_f, \dots, \alpha_e\beta_1, \dots, \alpha_e\beta_f.$$

Suppose that ν_α^a has entries in F . Then the tensor product $\nu_\alpha^a \otimes \nu_\beta^b$ is defined and, according to the following calculation, it is identical to $\nu_{\alpha \otimes \beta}^{ab}$:

$$\begin{aligned}
 (\nu_\alpha^a \otimes \nu_\beta^b)_{ijkl} &= (\nu_\alpha^a)_{ij} (\nu_\beta^b)_{kl} \\
 &= \operatorname{tr}_{E/K}(a\alpha_i\alpha_j) \operatorname{tr}_{K/F}(b\beta_k\beta_l) \\
 &= \operatorname{tr}_{K/F}(b\beta_k\beta_l \operatorname{tr}_{E/K}(a\alpha_i\alpha_j)) \\
 &= \operatorname{tr}_{K/F}(\operatorname{tr}_{E/K}(b\beta_k\beta_l a\alpha_i\alpha_j)) \\
 &= \operatorname{tr}_{E/F}(ab\alpha_i\alpha_j\beta_k\beta_l) \\
 &= (\nu_{\alpha \otimes \beta}^{ab})_{ijkl}.
 \end{aligned}$$

By Lemma 5.13, we may choose a and α such that $\nu_\alpha^a = J_e$. By Lemma 5.14, we may choose b and β such that $\nu_\beta^b = J_f$. Then $J_n = J_e \otimes J_f = \nu_\alpha^a \otimes \nu_\beta^b = \nu_{\alpha \otimes \beta}^{ab}$ which proves our claim. \square

Proof of Proposition 5.7. Proposition 5.15 and Lemma 5.4 imply that there exists an embedding $x \mapsto \underline{x}$ of E^\times in G whose image T is θ_J -split. If $g \in G$ and $\theta = g \cdot \theta_J$, then $g \mapsto g\underline{x}g^{-1}$ defines an embedding of E^\times in G whose image is θ -split.

Now suppose we are given an embedding of E^\times in G and let T denote its image. Lemma 4.6 implies that the embedding must come from an F -basis e_1, \dots, e_n of E . Proposition 5.15 says that there must exist another basis e'_1, \dots, e'_n and $a \in E^\times$ such that the associated matrix ν^a is J . The change-of-basis matrix in G between these bases sends θ_J to a matrix $\theta \in \Theta_J$ such that T is θ -split. Consequently, Θ_J must contain a T -orbit that lies in \mathcal{O}^T which completes the proof. \square

5.2.4. *Refined results when n is odd.* In this section, we assume n is odd.

Lemma 5.16. *Suppose L is a subring of $M(n, F)$ that is a field extension of F of odd degree. Assume θ is an orthogonal involution of $G = \operatorname{GL}_n(F)$ such that $\theta(L^\times) = L^\times$. Then $\theta(t) = t^{-1}$ for all $t \in L^\times$.*

Proof. Choose a symmetric matrix ν such that $\theta = \theta_\nu$. Then $\sigma(x) = \nu^{-1} \cdot {}^t x \cdot \nu$ defines an F -automorphism of L whose square is the identity map. Since $\operatorname{Gal}(L/F)$ has odd order, σ must be identity map on L . This is equivalent to our assertion. \square

The latter result shows that every θ -stable torus in $\mathbf{G} = \operatorname{GL}_n$ must in fact be θ -split.

In the next result, we continue to assume that we have fixed an embedding of E in $M(n, F)$ and we let T denote the image of E^\times .

Proposition 5.17. *Assume n is odd. The G -orbit Θ_J is the unique G -orbit of orthogonal involutions of G that contains an involution θ for which T is θ -stable. For every such involution θ , the torus T must in fact be θ -split. The set of all $\theta \in \Theta_J$ such that T is θ -split comprises a single T -orbit in Θ_J . The orthogonal group associated to any element of Θ_J is a split orthogonal group.*

Proof. According to Corollary 5.5 and Lemma 5.6, the map $\mu_{E/F} : Y_{E/F} \rightarrow \mathcal{O}^T$ of §5.2.2 reduces to a bijection between two singleton sets when n is odd. This says that there is a unique T -orbit of orthogonal involutions θ such that T is θ -split and every involution of the form θ^a , for $a \in E^\times$, lies in this orbit. The fact that θ -stable tori must be θ -split follows from Lemma 5.16.

Proposition 5.7 implies that the latter orbit lies in Θ_J . It is well known and easily verified that the orthogonal group associated to θ_J is split. The orthogonal groups associated to other elements of Θ_J are G -conjugate to the latter group and hence they must also be split. □

Corollary 5.18. *If n is odd and θ is an orthogonal involution, then the following are equivalent:*

- $\theta(\vec{\mathbf{G}}) = \vec{\mathbf{G}}$,
- Z^0 is a θ -split torus,
- Ψ is weakly θ -symmetric.

Proof. Assume Z^0 is a θ -split torus. Then Z^i is a θ -split torus for all i , since it is a torus and it is contained in Z^0 . Since Z^i is θ -split, it is θ -stable and hence so is its stabilizer in G . So $\vec{\mathbf{G}}$ is θ -stable. Conversely, if $\vec{\mathbf{G}}$ is θ -stable, then Z_0 must be θ -stable and hence θ -split by the Lemma 5.16. This establishes the equivalence of the first two conditions.

Now consider the quasicharacter ϕ_i in $\vec{\phi} = (\phi_0, \dots, \phi_d)$ and assume $\theta(G^i) = G^i$. Then θ restricts to an orthogonal involution of G^i with respect to any isomorphism $G^i \cong \mathrm{GL}_{n_i}(E_i)$, according to Proposition 5.1. Thus if $g \in G^i = \theta(G^i)$, then $\det_i(g\theta(g)) = 1$ so $g\theta(g)$ lies in the commutator subgroup of G^i . This implies that our first and third conditions are equivalent. □

6. ORTHOGONAL PERIODS

Suppose n is odd from now on. Fix a G -orbit Θ of orthogonal involutions of G and fix a Howe datum $\Phi = (\varphi, E, \{\varphi_i\}, \{E_i\}, \iota : E \hookrightarrow M(n, F))$ in the sense of Definition 4.5. Let $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi})$ be a cuspidal G -datum that is associated to Φ as in §4.3. Recall from Theorem 3.10 the formula

$$\langle \Theta, \Psi \rangle_G = \sum_{[\theta] \sim [\Psi]} m_{K^0}([\theta]) \langle [\theta], [\Psi] \rangle_{K^0}.$$

Our objective in this section is to compute all of the terms on the right-hand side of the latter formula.

Let us briefly sketch our strategy. From Proposition 5.17, it follows that if $\langle \Theta, \xi \rangle_G$ is nonzero, then $\Theta = \Theta_J$. So let us assume $\Theta = \Theta_J$. In §6.1, we show that $m_{K^0}([\theta]) = 1$ for all orbits $[\theta]$ in our examples. (This is not true for even n .)

By definition, if a summand $\langle [\theta], [\Psi] \rangle_{K^0}$ is nonzero, then it is equal to the dimension of the space $\mathrm{Hom}_{K^0, \theta}(\rho', \eta'_\theta)$. (See §3.1.4.) Using our generalized version of Lusztig’s results (Theorem 3.11), we then show that we can assume that our torus \mathbf{T} in Definition 2.1 is θ -split.

Next, we use Proposition 5.17 to identify a particular summand as the only summand that can be nonzero. In §6.3, we give an explicit formula for η'_θ and show that it is always trivial for our purposes. Finally, to compute the only relevant summand, we appeal to Lusztig’s formula. (In this case, we do not need to use the generalized form of the formula from §3.2.1.)

6.1. Triviality of $m_{K^0}([\theta])$.

Lemma 6.1. *Let θ be an orthogonal involution of G . Then $\mu(G_\theta) = Z^2$ and $G_\theta = ZG^\theta$. Consequently, $m_{K^0}([\theta]) = 1$.*

Proof. Choose a symmetric matrix $\nu \in G$ such that $\theta = \theta_\nu$. The similitude ratio defines a homomorphism $\mu : G_\theta \rightarrow Z$. We have

$$\mu(g) = g\theta(g)^{-1} = g \cdot \nu^{-1} \cdot {}^t g \cdot \nu.$$

If $z \in Z$, then $\mu(z) = z^2$ and thus $\mu(G_\theta) \supset \mu(Z) = Z^2$. Let us identify Z with F^\times in the obvious way. If $g \in G_\theta$, then $\det \mu(g) = (\det g)^2 \in Z^2$. On the other hand, if $z \in Z$, then $\det z = z^n \equiv z \pmod{Z^2}$. So, $Z^2 \supset \mu(G_\theta) \supset Z^2$ and hence $\mu(G_\theta) = Z^2$.

The similitude ratio μ defines an exact sequence

$$1 \rightarrow G^\theta \rightarrow G_\theta \rightarrow \mu(G_\theta) \rightarrow 1.$$

This yields an exact sequence

$$1 \rightarrow G^\theta / \{\pm 1\} \rightarrow G_\theta / Z \rightarrow \mu(G_\theta) / Z^2 \rightarrow 1,$$

since $Z \subset G_\theta$, $G^\theta \cap Z = \{\pm 1\}$ and $\mu(Z) = Z^2$. Hence, we have an isomorphism

$$G_\theta / ZG^\theta \cong \mu(G_\theta) / Z^2.$$

Since $\mu(G_\theta) = Z^2$, we deduce that $G_\theta = ZG^\theta$.

By Theorem 3.2, we have

$$m_{K^0}([\theta]) = [G_\theta : (K^0 \cap G_\theta)G^\theta].$$

But now $Z \subseteq K^0 \cap G_\theta$ implies

$$[G_\theta : (K^0 \cap G_\theta)G^\theta] \leq [G_\theta : ZG^\theta] = 1.$$

Therefore, $m_K(\Theta') = 1$. □

6.2. Relevant Involutions. Suppose θ is an involution of G such that

$$\langle [\theta], [\Psi] \rangle_{K^0} = \dim \text{Hom}_{K^{0,\theta}}(\rho', \eta'_\theta) \neq 0,$$

where K^0 , ρ' and η'_θ are associated to Ψ as in [HMu]. (Only orbits $[\theta]$ with this property can contribute to the formula for $\langle \Theta, \Psi \rangle_G$ in Theorem 3.10.) Then we must have $[\theta] \sim [\Psi]$, that is, $\theta(K^0) = K^0$, and the character ϕ of K^0 given by $\phi(g) = \prod_{i=0}^d \phi_i(g)$ restricts trivially to $K^{0,\theta}$. In particular, by Lemma 3.5, θ must stabilize \mathbf{G}^0 and $[y]$.

Let \mathbf{T} be the F -torus in \mathbf{G} such that $T = \iota(E^\times)$. Then \mathbf{T} can be taken to be the torus appearing in Definition 2.1. We want to show that there always exists a θ -split maximal F -torus \mathbf{T}' of \mathbf{G} with the properties in Definition 2.1. By Lemma 5.16, it suffices to show that there is a θ -stable torus \mathbf{T}' with these properties.

If Ψ is toral, then θ stabilizes $T = K^0$, and we are done. So suppose Ψ is nontoral. Then there exists an \mathfrak{f} -group \mathbf{G}_y^0 such that $\mathbf{G}_y^0(\mathfrak{f}) = G_{y,0:0+}^0 \cong \text{GL}_{n_0}(\mathfrak{f}_{E_0})$. Let \mathbf{T} be the \mathfrak{f} -torus in \mathbf{G}_y^0 determined by T . (See the Appendix.) Thus $\mathbf{T}(\mathfrak{f}) = T_{0:0+}$. The character $\varphi_{-1} | \mathfrak{D}_E^\times$ projects to a character λ of $\mathbf{T}(\mathfrak{f})$.

Recall that the Deligne-Lusztig virtual representation $R_\mathbf{T}^\lambda$ of $\mathbf{G}_y^0(\mathfrak{f})$ associated to (\mathbf{T}, λ) is an irreducible cuspidal representation that corresponds to the representation ρ° of $G_{y,0}^0$. In addition, ρ is the representation of K^0 that restricts to ρ° on $G_{y,0}^0$ and acts according to $\rho(\underline{\varpi}_{E_0}) = \varphi_{-1}(\varpi_{E_0})$ for any prime element ϖ_{E_0} in E_0 .

We note that $K^{0,\theta} = G_{y,0}^{0,\theta}$. The involution θ determines an involution of \mathbf{G}_y^0 that we also denote by θ . It follows from Proposition 2.12 of [HMu] that the group of fixed points of θ in $\mathbf{G}_y^0(\mathfrak{f})$ is the same as the image of $G_{y,0}^{0,\theta}$ in $\mathbf{G}_y^0(\mathfrak{f})$. Moreover, if we identify $\mathbf{G}_y^0(\mathfrak{f})$ with $\text{GL}_{n_0}(\mathfrak{f}_{E_0})$, then there exists an \mathfrak{f}_{E_0} -involution θ_0 of GL_{n_0}

such that θ_0 coincides with θ on $\mathrm{GL}_{n_0}(\mathfrak{f}_{E_0})$ under this identification. Observe that θ , and hence θ_0 , are nontrivial on the center of $\mathrm{GL}_{n_0}(\mathfrak{f}_{E_0})$. It follows that θ_0 is an outer involution of GL_{n_0} and thus that $\mathbf{G}_y^{0,\theta} \cong \mathbf{O}_{n_0}$.

Recall that $\rho' = \rho \otimes \phi$ and $\eta'_\theta = \eta \otimes \phi$. Note that $\eta_\theta(g) = \phi(g)\eta'_\theta(g)$ defines a character of exponent two of $\mathbf{G}_y^0(\mathfrak{f})$. We therefore have

$$\langle [\theta], [\Psi] \rangle_{K^0} = \dim \mathrm{Hom}_{K^{0,\theta}}(\rho, \eta_\theta) = \dim \mathrm{Hom}_{\mathbf{G}_y^0(\mathfrak{f})^\theta}(R_{\mathbf{T}(\mathfrak{f})}^\lambda, \eta_\theta).$$

Thus Theorem 3.11 now implies

$$\langle [\theta], [\Psi] \rangle_{K^0} = \sigma(\mathbf{T}) \sum_{\gamma \in \mathbf{T}(\mathfrak{f}) \backslash \Xi_{\mathbf{T},\lambda,\eta_\theta} / \mathbf{G}_y^0(\mathfrak{f})^\theta} \sigma \left(Z_{\mathbf{G}_y^0} \left((\gamma^{-1} \mathbf{T} \gamma \cap \mathbf{G}_y^{0,\theta})^\circ \right) \right).$$

Since $\langle [\theta], [\Psi] \rangle_{K^0}$ is nonzero, by the definition of $\Xi_{\mathbf{T},\lambda,\eta_\theta}$, we see that there exists $\gamma \in \mathbf{G}_y^0(\mathfrak{f})$ such that $(\gamma \cdot \theta)(\mathbf{T}) = \mathbf{T}$ and the summand above associated to γ is nonzero (as are all the summands).

Suppose that $g \in \mathbf{G}_{y,0}^0$ projects to γ . Then $[g \cdot \theta] = [\theta]$. Therefore, there is no essential loss in generality in replacing $g \cdot \theta$ by θ . In other words, we may assume $g = 1$ and therefore $\theta(\mathbf{T}) = \mathbf{T}$.

Lemma 6.2. *Assuming $\theta(\mathbf{T}) = \mathbf{T}$, there exists a θ -stable elliptic maximal F -torus \mathbf{T}' of \mathbf{G}^0 such that*

- (1) $y \in A(\mathbf{G}^0, \mathbf{T}', F)$.
- (2) *The image of $\mathbf{T}' \cap \mathbf{G}_{y,0}^0$ in $\mathbf{G}_y^0(\mathfrak{f})$ is $\mathbf{T}(\mathfrak{f})$.*
- (3) \mathbf{T} and \mathbf{T}' are conjugate in $\mathbf{G}_{y,0^+}^0$.

Proof. Let $\mathbf{H} = \mathrm{GL}_{n_0}$. The group \mathbf{G}^0 is isomorphic to the group $R_{E_0/F} \mathbf{H}$ obtained from \mathbf{H} via restriction of scalars from E_0 to F . As discussed in §2, over an algebraic closure \overline{F} of F ,

$$R_{E_0/F} \mathbf{H} \cong \prod_{\sigma \in \Sigma} \mathbf{H}_\sigma,$$

where Σ is the set of F -embeddings of E_0 in \overline{F} . Moreover, the identification of the F -group \mathbf{G}^0 and $\prod_{\sigma \in \Sigma} \mathbf{H}_\sigma$ (together with the above action of $\mathrm{Gal}(\overline{F}/F)$) determines an identification of $\mathcal{B}(\mathbf{G}^0, F)$ with $\mathcal{B}(\mathbf{H}, E_0)$.

It is easily checked that since $[E_0 : F]$ is odd and θ is defined over F , θ must stabilize each factor in the above decomposition of \mathbf{G}^0 . Thus, for each $\sigma \in \Sigma$, θ determines an E_0 -automorphism θ_σ of \mathbf{H} . In fact, $\theta_\sigma = {}^\sigma \theta_e$, where $e \in \Sigma$ is the identity embedding, and ${}^\sigma \theta_e$ is the map $x \mapsto \sigma(\theta_e(\sigma^{-1}(x)))$.

Recall that $\mathbf{T} \cong R_{E/F} \mathrm{GL}_1 = R_{E_0/F}(R_{E/E_0} \mathrm{GL}_1)$. In fact, this isomorphism is compatible with the identification of \mathbf{G}^0 with $R_{E_0/F} \mathbf{H}$ in the sense that \mathbf{T} can be identified with $R_{E_0/F} \mathbf{S}$, where $\mathbf{S} \cong R_{E/E_0} \mathrm{GL}_1$ is a unramified elliptic maximal E_0 torus of \mathbf{H} . The existence of a torus \mathbf{T}' with the above-stated properties now follows immediately from Proposition A.3. \square

We have thus demonstrated the following result.

Proposition 6.3. *Suppose θ is an involution of G and Ψ is a cuspidal G -datum such that $\langle [\theta], [\Psi] \rangle_{K^0} \neq 0$. Then there is a θ -split maximal F -torus \mathbf{T} of \mathbf{G} with the properties given in Definition 2.1.*

6.3. Triviality of η'_θ . In this section, we establish that the character η'_θ in the application of the theory of [HMu] to $(\mathrm{GL}_n, \mathrm{O}_n)$ is trivial, when n is odd.

Assume θ is an orthogonal involution of $G = \mathrm{GL}_n(F)$, where n is odd. Let $\Psi = (\mathbf{G}, y, \rho, \vec{\phi})$ be a cuspidal G -datum. Let $\Phi = (\varphi, E, \{\varphi_i\}, \{E_i\}, \iota : E \hookrightarrow M(n, F))$ be an associated Howe datum. Let \mathbf{T} be the elliptic maximal F -torus of \mathbf{G} such that $\mathbf{T}(F) = \iota(E^\times)$. We may assume that \mathbf{T} is θ -split by Proposition 6.3.

Proposition 6.4. *The character η'_θ is trivial.*

In the toral case, this follows immediately from the fact that $K^{0,\theta} = \{\pm 1\}$ lies in the center of G .

In general, the character η'_θ of $K^{0,\theta}$ has an expression

$$\eta'_\theta(k) = \prod_{i=0}^{d-1} \chi^{\mathcal{M}_i}(f'_i(k))$$

in the notation of [HMu]. Here the i th factor is given explicitly as

$$\det(\mathrm{Int}(k)|W_i^+)^{(p-1)/2},$$

where

$$W_i^+ = J^{i+1,\theta} / J_+^{i+1,\theta},$$

and J_+^{i+1} is a certain subgroup of finite index in J^{i+1} . (See §3.1 in [HMu].) In the above determinant, we are viewing W_i^+ as an \mathfrak{f}^* -vector space, where \mathfrak{f}^* is the field of prime order contained in \mathfrak{f} . We will show that each of the factors in the definition of η'_θ is trivial.

It is more convenient to work on the Lie algebra \mathfrak{g} . The groups J^{i+1} and J_+^{i+1} have obvious analogues \mathfrak{J}^{i+1} and \mathfrak{J}_+^{i+1} in the Lie algebra \mathfrak{g} , and it is easily seen that

$$\det(\mathrm{Int}(k)|W_i^+)^{(p-1)/2} = \det(\mathrm{Ad}(k)|\mathfrak{W}_i^+)^{(p-1)/2},$$

where

$$\mathfrak{W}_i^+ = \mathfrak{J}^{i+1,\theta} / \mathfrak{J}_+^{i+1,\theta}.$$

As above, we view \mathfrak{W}_i^+ as an \mathfrak{f}^* -vector space. In fact, the \mathfrak{f}^* -linear structure on \mathfrak{W}_i^+ extends naturally to an \mathfrak{f} -linear structure. Moreover, $\mathrm{Ad}(k)$ is \mathfrak{f} -linear. According to a classical “transitivity of norms” formula (see §7.4 in [J]), we have

$$\det_{\mathfrak{f}^*}(\mathrm{Ad}(k)|\mathfrak{W}_i^+) = N_{\mathfrak{f}/\mathfrak{f}^*}(\det_{\mathfrak{f}}(\mathrm{Ad}(k)|\mathfrak{W}_i^+)).$$

To establish that η'_θ is trivial, we will show that for all i and for all $k \in K^{0,\theta}$, the determinant $\det_{\mathfrak{f}}(\mathrm{Ad}(k)|\mathfrak{W}_i^+)$ is trivial.

6.3.1. Some notations. There is no loss of generality in assuming that $\mathbf{G} = \mathbf{G}^{i+1}$ and doing so will allow us to simplify our notations. In particular, we take $\mathbf{G}' = \mathbf{G}^i$ and routinely drop subscripts and superscripts involving i by using notations such as $\Phi = \Phi(\mathbf{G}, \mathbf{T}) \cup \{0\}$ and $\Phi' = \Phi(\mathbf{G}', \mathbf{T}) \cup \{0\}$.

Note that the fact that \mathbf{T} is θ -split implies that $\theta a = -a$, for all $a \in \Phi$. Let $(\Phi - \Phi')^+$ be any set of representatives for the various pairs $\{a, -a\}$ as a ranges over $\Phi - \Phi'$. For each $a \in \Phi - \Phi'$, we have the 1-dimensional space

$$\mathfrak{g}_a^\theta = (\mathfrak{g}_a + \mathfrak{g}_{-a})^\theta.$$

For any extension K of F contained in \overline{F} , let

$$\begin{aligned} \mathfrak{W}(K) &= \bigoplus_{a \in \Phi - \Phi'} \mathfrak{g}_a(K)_{y,s:s^+}, \\ \mathfrak{W}^+(K) &= \bigoplus_{a \in (\Phi - \Phi')^+} \mathfrak{g}_a^\theta(K)_{y,s:s^+}. \end{aligned}$$

Let \dot{E}/F denote the Galois closure of E/F in \overline{F} . Then \mathfrak{W} and \mathfrak{W}^+ are the spaces of $\text{Gal}(\dot{E}/F)$ -fixed points in $\mathfrak{W}(\dot{E})$ and $\mathfrak{W}^+(\dot{E})$, respectively.

6.3.2. *The structure of the proof.* Let $\delta : K^{0,\theta} \rightarrow \mathfrak{f}^\times$ be the map

$$k \mapsto \det_{\mathfrak{f}}(\text{Ad}(k)|\mathfrak{W}^+).$$

To show δ is trivial, first observe that it is trivial on $K_+^{0,\theta}$. Abbreviate \mathfrak{f}_{E_0} by \mathfrak{f}_0 . In §6.2, we observed that

$$K^{0,\theta}/K_+^{0,\theta} \cong \text{O}_{n_0}(\mathfrak{f}_0).$$

Note that δ must be trivial on the negative of the identity matrix. Thus it suffices to show that δ is trivial as a homomorphism $\text{SO}_{n_0}(\mathfrak{f}_0) \rightarrow \mathfrak{f}^\times$.

Our basic strategy can now be described as follows. Let F' be a (unique up to isomorphism) unramified quadratic extension of F . Let $\mathfrak{f}' = \mathfrak{f}_{F'}$ and let $\mathfrak{f}'_0 = \mathfrak{f}_{F'E_0}$. Taking F' -rational points, we show that δ has a natural extension to a homomorphism

$$\delta' : \text{SO}_{n_0}(\mathfrak{f}'_0) \rightarrow (\mathfrak{f}')^\times.$$

Then the triviality of δ follows from the fact (shown below) that $\text{SO}_{n_0}(\mathfrak{f}_0)$ is contained in the commutator subgroup of $\text{SO}_{n_0}(\mathfrak{f}'_0)$.

6.3.3. *The spinor norm.* Let p be an odd prime and let \mathbb{F}_p denote the field of order p . As in §3.2.1, for any power q of p , let \mathbb{F}_q denote the finite field of order q (inside a fixed algebraic closure of \mathbb{F}_p). Let $\nu : \text{O}(n_0, \mathbb{F}_q) \rightarrow \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$ be the spinor norm. Recall that an element of $\text{O}(n_0, \mathbb{F}_q)$ lies in the kernel of ν precisely if it can be expressed as a product of reflections $r_{v_1} \cdots r_{v_m}$ through anisotropic vectors $v_1, \dots, v_m \in \mathbb{F}_q^{n_0}$ such that

$$Q(v_1) \cdots Q(v_m) \in (\mathbb{F}_q^\times)^2,$$

where Q is the quadratic form on $\mathbb{F}_q^{n_0}$ that is used to define SO_{n_0} . It is well known that the commutator subgroup of $\text{SO}_{n_0}(\mathbb{F}_q)$ is the group $B_k(q)$ consisting of the elements in the kernel of ν that also lie in $\text{SO}_{n_0}(\mathbb{F}_q)$, where $k = (n_0 - 1)/2$. The group $B_k(q)$ is also the commutator subgroup of $\text{O}_{n_0}(\mathbb{F}_q)$ and it has index two in $\text{SO}_{n_0}(\mathbb{F}_q)$.

Lemma 6.5. *For any power q of p , $\text{SO}_{n_0}(\mathbb{F}_q)$ is contained in the commutator subgroup $B_k(q^2)$ of $\text{SO}_{n_0}(\mathbb{F}_{q^2})$.*

Proof. Given $g \in \text{SO}_{n_0}(\mathbb{F}_q)$, we can write $g = r_{v_1} \cdots r_{v_m}$, where each v_i in $\mathbb{F}_q^{n_0}$ is anisotropic with respect to the quadratic form defining SO_{n_0} , and r_{v_i} is the associated reflection.

Let \tilde{Q} be the obvious extension of the above quadratic form to $\mathbb{F}_{q^2}^{n_0}$. Then $\tilde{Q}(v_i) \in \mathbb{F}_q^\times \subset (\mathbb{F}_{q^2}^\times)^2$. Therefore, if ν' is the spinor norm on $\text{O}_{n_0}(\mathbb{F}_{q^2})$ then $\nu'(g) = 1$. \square

A general reference for the material in this section is [Lm].

6.3.4. *Extension of scalars.* Let ε be a unit in F whose image in the residue field \mathfrak{f} generates \mathfrak{f}^\times . Let $F' = F[\sqrt{\varepsilon}]$ and $E' = \dot{E}[\sqrt{\varepsilon}]$. Then E'/\dot{E} and F'/F are unramified quadratic extensions and $E' = \dot{E}F'$. Note that restriction from E' to \dot{E} defines an isomorphism

$$\text{Gal}(E'/F') \cong \text{Gal}(\dot{E}/F)$$

whose inverse is

$$\alpha \mapsto (x + y\sqrt{\varepsilon} \mapsto \alpha(x) + \alpha(y)\sqrt{\varepsilon}).$$

Note that $\mathfrak{W}(F')$ and $\mathfrak{W}^+(F')$ are the spaces of $\text{Gal}(E'/F')$ -fixed points in $\mathfrak{W}(E')$ and $\mathfrak{W}^+(E')$, respectively. All of these spaces may be regarded as \mathfrak{f}' -vector spaces and we have

$$\begin{aligned} \mathfrak{W}(F') &= \mathfrak{W} \otimes_{\mathfrak{f}} \mathfrak{f}', \\ \mathfrak{W}^+(F') &= \mathfrak{W}^+ \otimes_{\mathfrak{f}} \mathfrak{f}'. \end{aligned}$$

Let

$$K^0(F') = \mathbf{G}^0(F')_{[y]}, \quad K^0(F')_+ = \mathbf{G}^0(F')_{y,0+}.$$

Then

$$K^0(F')^\theta = \mathbf{G}^0(F')_{y,0}^\theta, \quad K^0(F')_+^\theta = \mathbf{G}^0(F')_{y,0+}^\theta.$$

By the discussion in §6.2, we have

$$K^0(F')^\theta / K^0(F')_+^\theta = \mathbf{G}^0(F')_{y,0:0+}^\theta \cong \text{O}_{n_0}(\mathfrak{f}'_0).$$

For $k \in K^0(F')^\theta$, define a homomorphism $\delta' : K^0(F')^\theta \rightarrow (\mathfrak{f}')^\times$ by

$$\delta'(k) = \det_{\mathfrak{f}'}(\text{Ad}(k)|\mathfrak{W}^+(F')).$$

We regard δ' also as a homomorphism

$$\delta' : \text{SO}_{n_0}(\mathfrak{f}'_0) \rightarrow (\mathfrak{f}')^\times.$$

Observe that since $\mathfrak{W}^+(F') = \mathfrak{W}^+ \otimes_{\mathfrak{f}} \mathfrak{f}'$, the restriction of δ' to $\text{SO}_{n_0}(\mathfrak{f}_0)$ is δ . Since $\text{SO}_{n_0}(\mathfrak{f}_0)$ is contained in the commutator subgroup of $\text{SO}_{n_0}(\mathfrak{f}'_0)$ by Lemma 6.5, δ is trivial. It follows that η'_θ must be trivial.

6.4. Lusztig’s theory for our examples. To simplify notations, we assume in this section that $n = n_0$. Later, we use the results of this section with n replaced by n_0 . Our objective is to apply the results of §3.2.1 to the finite groups that arise from the tame supercuspidal representations considered in this paper. What we do turns out to be a routine generalization of §2 in [HMa] analogous to our generalization of the theory in [Lu].

We resume the notations of §3.2.1 with $\mathbb{F}_q = \mathfrak{f}_0$ and $\mathbf{G} = \text{GL}_n$ (where n is an odd integer greater than 1). Let θ be the involution $\theta(g) = {}^t g^{-1}$ of G . Then $\mathbf{G}^\theta = \text{O}_n$ and $(\mathbf{G}^\theta)^\circ = \text{SO}_n$. Let \mathcal{J} be the set of all θ -split maximal tori in \mathbf{G} . The group $(\mathbf{G}^\theta)^\circ$ acts transitively on \mathcal{J} by conjugation. (See §1.5 [Lu].)

Let \mathbf{T} be a θ -stable elliptic maximal torus in \mathbf{G} . Let λ and χ be complex characters of T and G^θ , respectively. Assume that λ is nonsingular (in the sense of [Lu]). Let $\Xi_{\mathbf{T}}$ denote the set of all $g \in G$ such that $(g \cdot \theta)(\mathbf{T}) = \mathbf{T}$. Like $\Xi_{\mathbf{T},\lambda,\chi}$, this set is a union of double cosets in $T \backslash G / G^\theta$.

Lemma 6.6. *The set $\Xi_{\mathbf{T}}$ consists of a single double coset in $T \backslash G / G^\theta$. The set $\Xi_{\mathbf{T},\lambda,\chi}$ is empty unless $\lambda(-1) = \chi(-1)$ in which case it equals $\Xi_{\mathbf{T}}$.*

Proof. The first assertion is Lemma 2 of [HMa]. As stated, this lemma only applies to a certain specific elliptic maximal \mathbb{F}_q -torus of \mathbf{G} . However, the lemma holds for any such torus since all such tori are conjugate in G . We now prove the second assertion (which generalizes Lemma 1 of [HMa]).

Suppose $g \in \Xi_{\mathbf{T}}$. Then $g^{-1}\mathbf{T}g$ is θ -stable and hence θ -split by Lemma 5.16 (which applies equally well when the local field F is replaced by the finite field \mathbb{F}_q). Hence $g^{-1}\mathbf{T}g \in \mathcal{J}$. Let \mathbf{A} be the θ -stable (hence θ -split) maximal \mathbb{F} -torus of \mathbf{G} consisting of the diagonal matrices. We may choose $h \in (\mathbf{G}^\theta)^\circ$ such that $g^{-1}\mathbf{T}g = h\mathbf{A}h^{-1}$. It follows that $G^\theta \cap g^{-1}Tg \subset h(G^\theta \cap \mathbf{A})h^{-1}$. But the elements of $G^\theta \cap \mathbf{A}$ are diagonal matrices whose diagonal entries are ± 1 . Thus the squares of all elements of $G^\theta \cap g^{-1}Tg$ are trivial. Since $g^{-1}Tg \cong \mathbb{F}_{q^n}^\times$, we deduce that $G^\theta \cap g^{-1}Tg = \{\pm 1\}$. Since $\varepsilon_{g^{-1}\mathbf{T}g}(\pm 1) = 1$, we see that $g \in \Xi_{\mathbf{T},\lambda,\chi}$ if and only if $\lambda(-1) = \chi(-1)$. But the latter condition does not depend on g . Therefore, if it is satisfied we have $\Xi_{\mathbf{T},\lambda,\chi} = \Xi_{\mathbf{T}}$ and if it is not satisfied $\Xi_{\mathbf{T},\lambda,\chi}$ is empty. \square

Proposition 6.7. *Suppose θ is an orthogonal involution of G and \mathbf{T} is an elliptic maximal \mathbb{F}_q -torus in \mathbf{G} . If λ is a character of T and χ is a character of G^θ , then*

$$\frac{1}{|G^\theta|} \sum_{h \in G^\theta} R_{\mathbf{T},\lambda}(h) \chi(h) = \begin{cases} 1, & \text{if } \lambda(-1) = \chi(-1), \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We first assume that θ is chosen as above. Theorem 3.11 says

$$\frac{1}{|G^\theta|} \sum_{h \in G^\theta} R_{\mathbf{T},\lambda}(h) \chi(h) = \sigma(\mathbf{T}) \sum_{g \in T \setminus \Xi_{\mathbf{T},\lambda,\chi} / G^\theta} \sigma(Z_{\mathbf{G}}((g^{-1}\mathbf{T}g \cap \mathbf{G}^\theta)^\circ)).$$

If $\lambda(-1) \neq \chi(-1)$, then our claim follows from Lemma 6.6 since the latter sum over $T \setminus \Xi_{\mathbf{T},\lambda,\chi} / G^\theta$ is an empty sum. Now assume $\lambda(-1) = \chi(-1)$. Then Lemma 6.6 implies that $\Xi_{\mathbf{T},\lambda,\chi} = Tg_0G^\theta$. We have $\sigma(\mathbf{T}) = -1$ and

$$\sigma(Z_{\mathbf{G}}((g^{-1}\mathbf{T}g \cap \mathbf{G}^\theta)^\circ)) = \sigma(\mathbf{G}) = -1.$$

This establishes our claim for the given θ . The case of general orthogonal involutions follows upon applying an inner automorphism to the formula in the special case already proven. \square

6.5. Main results. We now prove the main theorem:

Theorem 6.8. *Let π be an irreducible tame supercuspidal representation of G with central character ω and let θ be an orthogonal involution of G . Then π is G^θ -distinguished precisely when θ lies in Θ_J and $\omega(-1) = 1$. When π is G^θ -distinguished, the dimension of $\text{Hom}_{G^\theta}(\pi, 1)$ is one. If π is associated to an F -admissible quasicharacter φ , then the condition $\omega(-1) = 1$ can also be stated as $\varphi(-1) = 1$. Similarly, if π is associated to a cuspidal G -datum $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi})$ and if ω' is the central character of $\rho' = \rho \otimes \phi$, then $\omega(-1) = 1$ can also be stated as $\omega'(-1) = 1$.*

Proof. Let Θ be a G -orbit of orthogonal involutions of G and let Ψ be a cuspidal G -datum to which π is associated. From §3.1, we have

$$\langle \Theta, \Psi \rangle_G = \sum_{[\theta] \sim [\Psi]} m_{K^0}([\theta]) \langle [\theta], [\Psi] \rangle_{K^0},$$

which simplifies to

$$\langle \Theta, \Psi \rangle_G = \sum_{[\theta] \sim [\Psi]} \langle [\theta], [\Psi] \rangle_{K^0},$$

according to Lemma 6.1.

Suppose we have a nonzero summand $\langle [\theta], [\Psi] \rangle_{K^0}$. According to Proposition 6.3, there exists a θ -split maximal F -torus \mathbf{T} with the properties described in Definition 2.1. Proposition 5.17 implies that the G -orbit Θ_J is the unique G -orbit of orthogonal involutions of G that contains an involution θ' for which \mathbf{T} is θ' -stable (and hence θ' -split). Therefore, the existence of a nonzero summand implies that $\Theta = \Theta_J$. In other words, since $\langle \Theta, \Psi \rangle_G$ is nonzero, then $\Theta = \Theta_J$.

Now fix a maximal torus \mathbf{T} as above. Proposition 5.17 states that the set of all $\theta' \in \Theta_J$ such that T is θ' -split comprises a single T -orbit in Θ_J . Since $T \subseteq K$, we see that there can be at most one nonzero summand. Let θ be an element of the unique orbit parametrizing this summand. Then

$$\langle \Theta, \Psi \rangle_G = \langle [\theta], [\Psi] \rangle_{K^0}.$$

By Proposition 6.4, η'_θ is trivial and thus

$$\langle [\theta], [\Psi] \rangle_{K^0} = \dim \text{Hom}_{K^{0,\theta}}(\rho', \eta'_\theta) = \dim \text{Hom}_{K^{0,\theta}}(\rho', 1).$$

Assume that the datum Ψ is toral. Then $\rho = 1$ and $\rho' = \rho \otimes \phi = \phi$. If Ψ comes from a Howe datum Φ and we identify T with E^\times , then the quasicharacter ϕ of T corresponds to the F -admissible quasicharacter φ of E^\times .

But $K^{0,\theta} = \{\pm 1\}$, $\rho' = \phi$ so

$$\text{Hom}_{K^{0,\theta}}(\rho', 1) = \text{Hom}_{\{\pm 1\}}(\phi, 1),$$

and, in the toral case, we have

$$\langle \Theta, \Psi \rangle_G = \begin{cases} 1, & \text{if } \phi(-1) = 1, \\ 0, & \text{if } \phi(-1) = -1, \end{cases}$$

or, in other words,

$$\langle \Theta, \Psi \rangle_G = \begin{cases} 1, & \text{if } \omega'(-1) = 1, \\ 0, & \text{if } \omega'(-1) = -1, \end{cases}$$

where ω' is the central character of ρ' .

Now assume Ψ is not toral. Referring back to the discussion in §6.2, we obtain the formula

$$\langle \Theta, \Psi \rangle_G = \langle [\theta], [\Psi] \rangle_{K^0} = \dim \text{Hom}_{\mathbf{G}_y^0(\mathfrak{f})^\theta}(R_{\Gamma(\mathfrak{f})}^\lambda, \eta_\theta).$$

We now apply Proposition 6.7 to the \mathfrak{f}_0 -group \mathbf{G}_y^0 using the fact that $\eta_\theta = \phi|_{G_{y,0}^{0,\theta}}$ and again we obtain

$$\langle \Theta, \Psi \rangle_G = \begin{cases} 1, & \text{if } \omega'(-1) = 1, \\ 0, & \text{if } \omega'(-1) = -1, \end{cases}$$

where ω' is the central character of ρ' .

The assertions of the theorem now follow directly for representations π that are G^θ -distinguished. It remains to show that our necessary conditions for G^θ -distinction are also sufficient conditions. Now suppose $\theta \in \Theta_J$ and $\Psi = (\vec{\mathbf{G}}, y, \rho, \vec{\phi})$ is a cuspidal G -datum with $\omega'(-1) = 1$. Let E/F be a tame extension of degree n appearing in a Howe datum associated to Ψ (as in §4.3). The maximal torus \mathbf{T} appearing in Definition 2.1 must be isomorphic to $R_{E/F}\text{GL}_1$.

Since $\theta \in \Theta_J$, Proposition 5.17 implies that there exists a θ -stable embedding of E in $M(n, F)$. The results of §4.2 show that after conjugating by an appropriate element of G , we may assume that \mathbf{T} , and hence $[y]$, are θ -stable. Moreover, \mathbf{T} must be θ -split by 5.17. The same must be true for $\mathbf{Z}^0 \subset \mathbf{T}$. By Corollary 5.18, we must have

- (1) θ stabilizes $\vec{\mathbf{G}}$,
- (2) $\phi \circ \theta = \phi^{-1}$.

In particular, (1) implies that θ must stabilize \mathbf{G}^0 , hence K^0 , by Lemma 3.5. Moreover, (2) implies that for $g \in K_+^{0,\theta}$, we have $\phi(g) = \phi(\theta(g)) = \phi(g)^{-1}$. Thus $\phi(g) = \pm 1$, and since $K_+^{0,\theta}$ is a pro- p group, we must have $\phi|_{K_+^{0,\theta}} = 1$. We have therefore shown that $[\theta] \sim [\Psi]$. As in the above discussion, we see that the assumption that $\omega'(-1) = 1$ then implies that $\langle [\theta], [\Psi] \rangle_{K^0} = 1$ and thus π is G^θ -distinguished. □

APPENDIX A. LIFTING TORI FROM \mathbf{G}_y TO \mathbf{G}

Let \mathfrak{F} denote the algebraic closure of \mathfrak{f} and let F^{un} denote the maximal unramified extension of F . The following result is a direct consequence of [D, §2].

Lemma A.1. *Let \mathbf{G} be a reductive group over F . Suppose that \mathbf{T} is an elliptic maximal unramified F -torus of \mathbf{G} and y is a vertex in $A(\mathbf{G}, \mathbf{T}, F)$.*

- (1) *The torus \mathbf{T} determines a minisotropic maximal \mathfrak{f} -torus \mathbb{T} of \mathbf{G}_y such that the image of $\mathbf{T}(F^{un}) \cap \mathbf{G}(F^{un})_{y,0}$ in $\mathbf{G}_y(\mathfrak{F})$ is $\mathbb{T}(\mathfrak{F})$.*
- (2) *If \mathbf{T}' is another elliptic maximal unramified F -torus of \mathbf{G} such that $y \in A(\mathbf{G}, \mathbf{T}', F)$, and \mathbb{T}' is the associated torus of \mathbf{G}_y , then $\mathbb{T}' = \mathbb{T}$ if and only if \mathbf{T}' is conjugate to \mathbf{T} by an element of $G_{y,0+}$.*
- (3) *Every minisotropic maximal \mathfrak{f} -torus of \mathbf{G}_y is associated to some elliptic maximal unramified F -torus of \mathbf{G} in this way.*

Suppose now that θ is an involution of G . The following result is an analogue of Lemma A.1 (3) for θ -stable tori.

Lemma A.2. *Let \mathbf{G} be a reductive algebraic group defined over F , and let θ be an F -involution of \mathbf{G} . Let y be a vertex in $\mathcal{B}(\mathbf{G}, F)$ and suppose that $\theta([y]) = [y]$. Let \mathbb{T} be a θ -stable minisotropic maximal \mathfrak{f} -torus of \mathbf{G}_y . Then there exists a θ -stable elliptic maximal unramified F -torus \mathbf{T} of \mathbf{G} such that $y \in A(\mathbf{G}, \mathbf{T}, F)$ and the image of $\mathbf{T}(F^{un}) \cap \mathbf{G}(F^{un})_{y,0}$ in $\mathbf{G}_y(\mathfrak{F})$ is $\mathbb{T}(\mathfrak{F})$.*

Proof. Let \mathcal{S}_0 be the set of elliptic maximal F -unramified tori \mathbf{S} of \mathbf{G} such that

- $A(\mathbf{G}, \mathbf{S}, F)$ contains y ,
- The maximal torus of \mathbf{G}_y determined by \mathbf{S} is \mathbb{T} .

This set is nonempty by Lemma A.1 (3). Note that \mathcal{S}_0 is θ -stable. By Lemma A.1 (2), $G_{y,0+}$ acts transitively by conjugation on \mathcal{S}_0 . We may therefore topologize \mathcal{S}_0 by giving it the quotient topology inherited from $G_{y,0+}$. With this topology, it is clear that $G_{y,0+}$ acts continuously on \mathcal{S}_0 . Moreover, \mathcal{S}_0 is compact and metrizable.

Let $X = \{r \in \mathbb{R} : r \geq 0, G_{y,r} \neq G_{y,r+}\}$. The elements of X can be written as a sequence r_0, r_1, r_2, \dots , where $r_0 = 0$. We now inductively define a nested sequence of compact subsets of \mathcal{S}_0 . Suppose we have already defined a sequence $\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_i$ of compact subsets of \mathcal{S}_0 such that each \mathcal{S}_j is a θ -stable orbit of G_{y,r_j^+} in \mathcal{S}_0 . (To

begin with, note that this is true for \mathcal{S}_0 .) We claim that there is some θ -stable orbit \mathcal{S}_{i+1} of G_{y,r_{i+1}^+} in \mathcal{S}_i .

Consider the collection \mathcal{T} of G_{y,r_{i+1}^+} -orbits in \mathcal{S}_i . The group G_{y,r_i^+} acts transitively on \mathcal{T} , and therefore \mathcal{T} has size dividing $[G_{y,r_i^+} : G_{y,r_{i+1}^+}]$, which is a power of q , hence odd. Since θ acts as a permutation of \mathcal{T} of order dividing 2, some G_{y,r_{i+1}^+} -orbit $\mathcal{S}_{i+1} \in \mathcal{T}$ must be fixed by θ , proving the claim.

Note that the \mathcal{S}_i form a nested sequence of nonempty compact subspaces. Hence, $\mathcal{S} = \bigcap_i \mathcal{S}_i$ is nonempty. Moreover, \mathcal{S} is θ -stable and is contained in a single G_{y,r_i^+} -orbit for each i , hence must be a singleton. In other words, \mathcal{S} consists of a single θ -stable torus. □

Now consider the following situation. Let E be a finite extension of F . Let \mathbf{H} be an unramified reductive group defined over E , and let \mathbf{G} be the group $R_{E/F}\mathbf{H}$ obtained from \mathbf{H} via restriction of scalars. Let θ_0 be an E -involution of \mathbf{H} . Then θ_0 naturally determines an F -involution of \mathbf{G} . Let \mathbf{S} be an elliptic unramified maximal E -torus in \mathbf{H} and let \mathbf{T} be the torus $R_{E/F}\mathbf{S}$ in \mathbf{G} . Then \mathbf{T} is an elliptic maximal torus of \mathbf{G} which contains a maximal unramified torus of \mathbf{G} . Let y be a vertex in $A(\mathbf{G}, \mathbf{T}, F)$ such that $\theta([y]) = [y]$. Since $A(\mathbf{G}, \mathbf{T}, F) = A(\mathbf{H}, \mathbf{S}, E)$, we can also view y as a θ_0 -fixed point of $A(\mathbf{H}, \mathbf{S}, E)$. Note that θ descends to an \mathfrak{f} -involution of \mathbf{G}_y (which we will also denote by θ). Similarly, θ_0 descends to an \mathfrak{f}_E -involution of the \mathfrak{f}_E -group \mathbf{H}_y^E .

Proposition A.3. *In the above situation, if the \mathfrak{f} -torus \mathbf{T} in \mathbf{G}_y determined by \mathbf{T} is θ -stable, then there is an element $g \in G_{y,0^+}$ such that $g\mathbf{T}g^{-1}$ is θ -stable.*

Proof. Let K/F be the maximal unramified subextension of E/F . Let $\tilde{\mathbf{H}} = R_{E/K}\mathbf{H}$ and $\tilde{\mathbf{S}} = R_{E/K}\mathbf{S}$. By the transitivity of restriction of scalars, $\mathbf{G} = R_{K/F}\tilde{\mathbf{H}}$ and $\mathbf{T} = R_{K/F}\tilde{\mathbf{S}}$. Note that θ_0 determines a K -involution $\tilde{\theta}_0$ of $\tilde{\mathbf{H}}$, which descends to an \mathfrak{f}_E -involution of the \mathfrak{f}_E -group $\tilde{\mathbf{H}}_y^K$.

Let $\tilde{\mathbf{S}}$ be the k_E -torus in $\tilde{\mathbf{H}}_y^K$ determined by $\tilde{\mathbf{S}}$. Since K/F is unramified, it follows that $\mathbf{G}_y = R_{\mathfrak{f}_E/\mathfrak{f}}\tilde{\mathbf{H}}_y^K$ and $\mathbf{T} = R_{\mathfrak{f}_E/\mathfrak{f}}\tilde{\mathbf{S}}$. Moreover, the involution of \mathbf{G}_y determined by the involution θ_0 of $\tilde{\mathbf{H}}_y^K$ is precisely θ . Since \mathbf{T} is θ -stable, it follows that $\tilde{\mathbf{S}}$ must be $\tilde{\theta}_0$ -stable.

Since E/K is totally ramified, it follows from Lemma 2.1.1 of [AD] that $\tilde{\mathbf{H}}(K)_y = \mathbf{H}(E)_y$ and hence that $\tilde{\mathbf{H}}_y^K = \mathbf{H}_y^E$. Similarly, $\tilde{\mathbf{S}} = \mathbf{S}$. It is easily seen that the actions of $\tilde{\theta}_0$ on $\tilde{\mathbf{H}}_y^K$ and θ_0 on \mathbf{H}_y^E coincide under the above identification. Thus since $\tilde{\mathbf{S}}$ is $\tilde{\theta}_0$ -stable, it follows that \mathbf{S} is θ_0 -stable.

By Lemmas A.1 and A.2, there exists $g \in \mathbf{H}(E)_{y,0^+}$ such that $\mathbf{S}' = g\mathbf{S}g^{-1}$ is θ_0 -stable. Let $\mathbf{T}' = R_{E/F}\mathbf{S}' \subset \mathbf{G}$. Then $\mathbf{T}' = g\mathbf{T}g^{-1}$, where g here is viewed as an element of $G_{y,0^+} = \mathbf{H}(E)_{y,0^+}$. Moreover, since \mathbf{S}' is θ_0 -stable, \mathbf{T}' must be θ -stable. □

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