

REPRESENTATIONS OF METAPLECTIC GROUPS II: HECKE ALGEBRA CORRESPONDENCES

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ABSTRACT. The metaplectic group is defined by its oscillator or Weil representation. Using the types of the Weil representations we define two Hecke algebras that govern two Bernstein's components containing the even and the odd Weil representation, respectively.

1. INTRODUCTION

Let k be a p -adic field with p odd. Let V^+ and V^- be two quadratic spaces of dimension $2n + 1$, trivial discriminant, and trivial and non-trivial Hasse invariants, respectively. Then $\mathrm{SO}(V^+)$ is a split, adjoint group of type B_n , while $\mathrm{SO}(V^-)$ is its unique non-split inner form. Let \mathcal{S}^\pm denote the category of smooth representations of $\mathrm{SO}(V^\pm)$, and \mathcal{S}_0^\pm the component (in the sense of Bernstein [Be]) of \mathcal{S}^\pm containing the trivial representation of $\mathrm{SO}(V^\pm)$.

Let W be a non-degenerate symplectic space of dimension $2n$ over k . Let $\widetilde{\mathrm{Sp}}(W)$ be a non-trivial 2-fold central extension of $\mathrm{Sp}(W)$. Let \mathcal{G} denote the category of genuine, smooth representations of $\widetilde{\mathrm{Sp}}(W)$. Let ψ be a non-trivial additive character of k . Let ω_ψ be the Weil representation of $\widetilde{\mathrm{Sp}}(W)$, which depends on ψ [We]. The Weil representation decomposes $\omega_\psi = \omega_\psi^e \oplus \omega_\psi^o$ into a sum of two irreducible representations, called even and odd Weil representations. Let \mathcal{G}_ψ^+ and \mathcal{G}_ψ^- be the components of \mathcal{G} containing ω_ψ^e and ω_ψ^o , respectively.

The main result of this paper is an equivalence of categories \mathcal{G}_ψ^+ and \mathcal{S}_0^+ , and an equivalence of categories \mathcal{G}_ψ^- and \mathcal{S}_0^- . Under these equivalences, ω_ψ^e corresponds to the trivial representation of the split group $\mathrm{SO}(V^+)$ while ω_ψ^o corresponds to the trivial representation of $\mathrm{SO}(V^-)$. More precisely, we describe two Hecke algebras, denoted by H_ψ^+ and H_ψ^- , such that the subcategories \mathcal{G}_ψ^+ and \mathcal{G}_ψ^- are equivalent to the categories of H_ψ^+ -modules and H_ψ^- -modules, respectively. Via explicit isomorphisms, the algebras H_ψ^+ and H_ψ^- are isomorphic to the affine Hecke algebras of $\mathrm{SO}(V^+)$ and $\mathrm{SO}(V^-)$, respectively. While much of this approach is standard, a novelty in this paper is that we exploit minimal types of the Weil representation to define generators of H_ψ^+ and H_ψ^- , and quickly prove relations satisfied by them.

The second result of this paper is a comparison of the correspondence of representations obtained in this paper with the theta correspondences. Recall that, by results of Kazhdan and Lusztig, the set of equivalence classes of irreducible

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representations in \mathcal{S}_0^+ and \mathcal{S}_0^- can be partitioned into finite sets parameterized by conjugacy classes of pairs (s, u) ,

$$\text{Irr}(\mathcal{S}_0^+) \cup \text{Irr}(\mathcal{S}_0^-) = \bigcup \text{Irr}(s, u),$$

where s is a semi-simple element in $\text{Sp}_{2n}(\mathbb{C})$ and u is a unipotent element in $\text{Sp}_{2n}(\mathbb{C})$ such that $sus^{-1} = u^q$. Consequently, by the equivalences of categories obtained in this paper, there is a decomposition

$$\text{Irr}(\mathcal{G}_\psi^+) \cup \text{Irr}(\mathcal{G}_\psi^-) = \bigcup \text{Irr}_\psi(s, u).$$

We show that theta correspondences (that depend on ψ) for dual pairs $\text{Sp}(W) \times \text{SO}(V^\pm)$ give a bijection of finite sets $\text{Irr}(s, u)$ and $\text{Irr}_\psi(s, u)$ for every pair (s, u) . In general this bijection is different from that given by the isomorphisms of Hecke algebras. A similar statement for dual pairs involving even orthogonal groups was proved by Aubert [Au].

2. CLASSICAL GROUPS

Let W be a non-degenerate symplectic space over k of dimension $2n$, and let $\langle \cdot, \cdot \rangle$ denote the symplectic form. Pick a basis of W consisting of \mathbf{e}_i and \mathbf{e}_i^* , $1 \leq i \leq n$, such that $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \langle \mathbf{e}_i^*, \mathbf{e}_j^* \rangle = 0$ and $\langle \mathbf{e}_i, \mathbf{e}_j^* \rangle = \delta_{ij}$. We have a decomposition $W = X + X^*$ where X is spanned by \mathbf{e}_i and X^* is spanned by \mathbf{e}_i^* . The group $\text{Sp}(W)$ has a maximal torus $T = (k^*)^n$ where $t = (t_1, \dots, t_n) \in T$ acts on \mathbf{e}_i and \mathbf{e}_i^* by

$$t(\mathbf{e}_i) = t_i \cdot \mathbf{e}_i \quad \text{and} \quad t(\mathbf{e}_i^*) = t_i^{-1} \cdot \mathbf{e}_i^*.$$

Let $\Sigma = \{\pm e_i \pm e_j\} \cup \{\pm 2e_i \mid 1 \leq i \leq n\}$ be the root system corresponding to T where $\alpha = 2e_i$ is given by $\alpha(t) = t_i^2$. For every root $\alpha \in \Sigma$ there is a root subgroup $U_\alpha \cong k$ of $\text{Sp}(W)$. For every $\alpha \in \Sigma$ we fix a map

$$\Phi_\alpha : \text{SL}_2(k) \rightarrow \text{Sp}(W)$$

such that U_α and $U_{-\alpha}$ are the images of the subgroups of unipotent upper and lower triangular matrices, respectively. We assume that Φ_α and $\Phi_{-\alpha}$ are related by

$$\Phi_\alpha \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \Phi_{-\alpha} \left(\begin{pmatrix} d & c \\ b & a \end{pmatrix} \right).$$

If $\alpha = 2e_i$, we fix Φ_α so that $\text{SL}_2(k)$ acts, in the standard way, on the 2-dimensional subspace of W spanned by \mathbf{e}_i and \mathbf{e}_i^* , and fixes other basis vectors in W . If $\alpha = e_i - e_j$, we fix Φ_α so that $\text{SL}_2(k)$ acts, in the standard way, on the 2-dimensional subspace of X spanned by \mathbf{e}_i and \mathbf{e}_j , and by the dual action on the 2-dimensional subspace of X^* spanned by \mathbf{e}_i^* and \mathbf{e}_j^* .

For any l , $0 \leq l \leq n$, let $W_l \subseteq W$ be the span of $\mathbf{e}_{l+1}, \dots, \mathbf{e}_n$ and $\mathbf{e}_{l+1}^*, \dots, \mathbf{e}_n^*$. If $l = n$, we set $W_n = 0$. Let $X_l \subseteq X$ be the span of $\mathbf{e}_1, \dots, \mathbf{e}_l$. If $l = 0$, we set $X_0 = 0$. Similarly, let $X_l^* \subseteq X^*$ be the span of $\mathbf{e}_1^*, \dots, \mathbf{e}_l^*$. Then

$$W = X_l + W_l + X_l^*.$$

The stabilizer of the flag

$$X_1 \subseteq X_2 \subseteq \dots \subseteq X_n$$

in $\text{Sp}(W)$ is a Borel subgroup B containing T . The corresponding set Δ of simple roots consists of $\alpha_1 = e_1 - e_2, \dots, \alpha_{n-1} = e_{n-1} - e_n$ and $\alpha_n = 2e_n$. A parabolic subgroup P is in a standard position if it contains B . All such parabolic subgroups

are stabilizers of subflags of the flag $X_1 \subseteq \dots \subseteq X_n$. For example, the stabilizer of X_l is a maximal parabolic P . We fix a Levi factor M of P so that it contains T . Equivalently, M stabilizes X_l^* . Thus we have canonical isomorphisms

$$M \cong \mathrm{GL}(X_l) \times \mathrm{Sp}(W_l) \cong \mathrm{GL}(X_l^*) \times \mathrm{Sp}(W_l).$$

Let V be an orthogonal space over k of dimension $2n + 1$ and discriminant 1. Up to an isomorphism there are two such spaces: V^+ and V^- , with trivial and non-trivial Hasse invariant, respectively. Let (\cdot, \cdot) denote the corresponding symmetric bilinear form. Let $Y_l \subseteq V$ be a totally isotropic subspace of dimension l . Note that $l \leq n$ if $V \cong V^+$ and $l \leq n - 1$ if $V \cong V^-$. Then there exists another totally isotropic subspace Y_l^* of dimension l and an orthogonal subspace V_l of discriminant 1 such that

$$V \cong Y_l + V_l + Y_l^*.$$

Pick a basis \mathbf{f}_i of Y_l and a basis \mathbf{f}_i^* of Y_l^* , $1 \leq i \leq n$, such that $(\mathbf{f}_i, \mathbf{f}_j^*) = \delta_{ij}$. The group $\mathrm{SO}(V)$ contains a torus $T_l = (k^*)^l$ where $t = (t_1, \dots, t_l) \in T_l$ acts trivially on V_l , and

$$t(\mathbf{f}_i) = t_i \cdot \mathbf{f}_i \quad \text{and} \quad t(\mathbf{f}_i^*) = t_i^{-1} \cdot \mathbf{f}_i^*.$$

Note that T_n is a maximal torus of $\mathrm{SO}(V^+)$, while T_{n-1} is a maximal split torus of $\mathrm{SO}(V^-)$. The stabilizer of Y_l in $\mathrm{SO}(V)$ is a maximal parabolic subgroup Q . We fix a Levi factor L of Q so that it stabilizes Y_l^* . Then

$$L \cong \mathrm{GL}(Y_l) \times \mathrm{SO}(V_l) \cong \mathrm{GL}(Y_l^*) \times \mathrm{SO}(V_l).$$

3. OPEN COMPACT SUBGROUPS OF $\mathrm{Sp}(W)$

Let R be the ring of integers in k and ϖ a uniformizing element. Let q be the order of the residue field $\mathfrak{f} = R/\varpi R$. Let L be a lattice in W . Let L^* be the dual lattice with respect to the symplectic form. A lattice L is called good, if $\varpi L \subseteq L^* \subseteq L$. We fix a sequence of good lattices,

$$L_0 \subseteq L_1 \subseteq \dots \subseteq L_n,$$

where L_i is the R -span of \mathbf{e}_j for $1 \leq j \leq n$, $\frac{\mathbf{e}_j^*}{\varpi}$ for $j \leq i$, and \mathbf{e}_j^* for $j > i$. Let K_i be the stabilizer of the lattice L_i in $\mathrm{Sp}(W)$. It is a maximal open compact subgroup, and contains a pro- p subgroup K_i^+ such that $K_i/K_i^+ \cong \mathrm{Sp}_{2i}(\mathfrak{f}) \times \mathrm{Sp}_{2(n-i)}(\mathfrak{f})$. The intersection of all K_i is an Iwahori subgroup denoted by I . The double cosets with respect to I can be described in terms of the affine Weyl group. If we view a root α as a linear functional on $\mathbb{E} = \mathbb{R}e_1 + \dots + \mathbb{R}e_n$ via the standard dot product then, for any $m \in \mathbb{Z}$, $\beta = \alpha + m$ is an affine functional on \mathbb{E} . Let s_β be the reflection about the hyperplane in \mathbb{E} where β vanishes. The affine Weyl group W_a is the group of affine transformations of \mathbb{E} generated by these reflections. It is a semi-direct product of the Weyl group and a normal subgroup consisting of translations by vectors in \mathbb{Z}^n .

Let $\Delta_a = \Delta \cup \{\alpha_0\}$ be the set of simple affine roots where $\alpha_0 = 1 - 2e_1$. Let $\mathcal{C} \subseteq \mathbb{E}$ consist of all (x_1, \dots, x_n) such that

$$\frac{1}{2} > x_1 > \dots > x_n > 0.$$

Then \mathcal{C} is the chamber corresponding to Δ_a : its elements are all points in \mathbb{E} where all simple roots take positive value.

As an abstract group, W_a is generated by reflections s_0, \dots, s_n corresponding to simple affine roots and satisfying braid relations imposed by the following Coxeter diagram:



Let $l : W_a \rightarrow \mathbb{Z}$ be the usual length function i.e. $l(w)$ is the number of factors in a minimal expression of w in terms of simple reflections. We recall the following useful fact: Let s_i be a simple reflection. Then $l(s_i w) = l(w) + 1$ if and only if the affine simple root α_i is positive when restricted to $w(\mathcal{C})$.

For every affine root $\beta = \alpha + m$, where $\alpha \in \Sigma$ and $m \in \mathbb{Z}$, define a map $\Phi_\beta : \mathrm{SL}_2(k) \rightarrow \mathrm{Sp}(W)$ by

$$\Phi_\beta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Phi_\alpha \begin{pmatrix} a & \varpi^m b \\ c/\varpi^m & d \end{pmatrix}$$

where $\Phi_\alpha : \mathrm{SL}_2(k) \rightarrow \mathrm{Sp}(W)$ is the map defined previously for any root $\alpha \in \Sigma$. The maximal compact group K_i is generated by $\Phi_{\alpha_j}(\mathrm{SL}_2(R))$ for all affine simple roots $\alpha_j \neq \alpha_i$. For any u in k define

$$x_\beta(u) = \Phi_\beta \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}.$$

Then $U_\beta \cong k$ consists of all $x_\beta(u)$, where $u \in k$. Let $U_\beta(R) \cong R$ be the subgroup consisting of all $x_\beta(u)$ where $u \in R$. For every t in k^\times we also define

$$w_\beta(t) = \Phi_\beta \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix} \quad \text{and} \quad h_\beta(t) = \Phi_\beta \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

Now every element $w = s_{i_1} \cdots s_{i_l}$ in W_a can be represented by $w_{\alpha_{i_1}}(1) \cdots w_{\alpha_{i_l}}(1)$ in $\mathrm{Sp}(W)$. Under this identification, I -double cosets in $\mathrm{Sp}(W)$ are represented by elements in W_a . Moreover, if $[IwI : I]$ denotes the number of single I -cosets in IwI , then $[IwI : I] = q^{l(w)}$ (see [IM]).

The length function l can be generalized by giving different “weights” to simple reflections in different conjugacy classes. For example, for every positive integer m , we can define l_m as follows. If $w = s_{i_1} \cdots s_{i_l}$ is a shortest expression then we define $l_m(w) = l_m(s_{i_1}) + \cdots + l_m(s_{i_l})$ where $l_m(s_i) = 1$ for all $i \neq n$ and $l_m(s_n) = m$. The weighted length functions naturally appear throughout the paper.

Let J be an open compact subgroup defined by

$$J = \bigcap_{i=0}^{n-1} K_i = I \cup Is_n I.$$

Let W'_a be the subgroup of W_a consisting of all elements fixing the affine root α_n . This is an affine Weyl group (for $\mathrm{Sp}_{2n-2}(k)$) generated by reflections $s'_0 = s_0, \dots, s'_{n-2} = s_{n-2}$ and s'_{n-1} , where s'_{n-1} corresponds to the root $\alpha'_{n-1} = 2e_{n-1}$. Let l' be the length function on W'_a .

Proposition 1. *Let l be the length function on W_a and l'_3 be the length function on $W'_a \subset W_a$ weighted so that $l'_3(s'_{n-1}) = 3$ and $l'_3(s'_i) = 1$ for $i = 0, \dots, n - 2$. For every $w \in W'_a$:*

- (1) $l(w) = l'_3(w)$.
- (2) $[JwJ : J] = q^{l(w)}$.

Proof. We shall prove the first statement by induction on $l'(w)$. If $l'(w) = 0$ there is nothing to prove. Now assume that $l'(s'_i w) = l'(w) + 1$ for some simple reflection s'_i . We need to show that $l(w) = l'_3(w)$ implies $l(s'_i w) = l'_3(s'_i w)$. Assume that $i = n - 1$. Then $l'_3(s'_{n-1} w) = l'_3(w) + 3$. Let $x = (x_1, \dots, x_n)$ be in the chamber \mathcal{C} . Then $w(x) = (y_1, \dots, y_{n-1}, x_n)$ since w is in W'_a , and $y_{n-1} > 0$ since $l'(s'_{n-1} w) = l'(w) + 1$. Therefore, for some $1 \leq j \leq n - 1$, $y_{n-1} = x_j + m$ for an integer $m \geq 0$ or $y_{n-1} = -x_j + m$ for an integer $m > 0$. In any case, since $\frac{1}{2} > x_j > x_n$, the inequality $y_{n-1} > x_n$ must hold. Thus

$$\alpha_{n-1}(w(x)) = y_{n-1} - x_n, \alpha_n(s_{n-1}w(x)) = y_{n-1}$$

and

$$\alpha_{n-1}(s_n s_{n-1} w(x)) = x_n + y_{n-1}$$

are all positive and, therefore, $s_{n-1}w$, $s_n s_{n-1}w$ and $s_{n-1} s_n s_{n-1}w$ are the shortest expressions in W_a . Since $s'_{n-1} = s_{n-1} s_n s_{n-1}$, it follows that $l(s'_{n-1} w) = l(w) + 3$, as desired. The case $i \neq n - 1$ is easier, and left to the interested reader. This proves the first statement.

To prove the second statement, note that $J = I \cup U_{\alpha_n}(R) s_n I = I \cup I s_n U_{\alpha_n}(R)$. Since w is in W'_a , it commutes with U_{α_n} and s_n . Thus one easily checks that $JwJ = IwI \cup I s_n w I$. Hence

$$[JwJ : I] = [IwI : I] + [I s_n w I : I] = q^{l(w)} + q^{l(s_n w)} = q^{l(w)}(1 + q).$$

Since $[J : I] = 1 + q$ the second statement follows. The proposition is proved. \square

4. STEINBERG GROUP

Let $\pi : \widetilde{\text{Sp}}(W) \rightarrow \text{Sp}(W)$ be a non-trivial 2-fold central extension. In this section we describe some general properties of $\widetilde{\text{Sp}}(W)$ due to Matsumoto [Ma], Moore [Mo] and Steinberg [St]. Most of these properties can be directly derived in the construction of the Weil representation [We]. However, since the purpose of this article is not a detailed exposition, we may as well spell out some general facts first.

For every affine root α the central extension splits over the subgroup $U_\alpha \subseteq \text{Sp}(W)$, that is, there is a homomorphism $i_\alpha : U_\alpha \rightarrow \widetilde{\text{Sp}}(W)$ such that $\pi \circ i_\alpha$ is the identity map on U_α . The map i_α is unique since \widetilde{U}_α is a 2-divisible group. Abusing notation, we shall consider $x_\alpha(u)$ as elements of $\widetilde{\text{Sp}}(W)$. For any t in k^\times define

$$w_\alpha(t) = x_\alpha(t) x_{-\alpha}(-t^{-1}) x_\alpha(t) \quad \text{and} \quad h_\alpha(t) = w_\alpha(t) w_\alpha(1)^{-1}.$$

Then, in contrast with the linear group,

$$h_\alpha(t) h_\alpha(s) = (t, s)^{m_\alpha} h_\alpha(ts)$$

where (t, s) is the Hilbert symbol of order 2, $m_\alpha = 1$ for long roots and $m_\alpha = 2$ for short roots. If the residual characteristic is odd then the Hilbert symbol is tame, i.e., $(s, t) = 1$ for any two elements in R^\times and the central extension splits over $\Phi_\alpha(\text{SL}_2(R))$ for any affine root α . In other words, the map Φ_α can be lifted to

$$\widetilde{\Phi}_\alpha : \text{SL}_2(R) \rightarrow \widetilde{\text{Sp}}(W).$$

Note that the lift $\tilde{\Phi}_\alpha$ is unique. Indeed any two would have to differ by a character of $\mathrm{SL}_2(R)$ of order 2. But $\mathrm{SL}_2(R)$ has no such characters, since it is generated by pro- p groups $U_\alpha(R)$ and $U_{-\alpha}(R)$, and $p \neq 2$. Finally, the map $\tilde{\Phi}_\alpha$ can be extended to $\mathrm{SL}_2(k)$ if and only if α is a short root.

5. METAPLECTIC GROUP

Let ψ be a non-trivial additive character of k . The conductor of ψ is the smallest integer c such that ψ is trivial on $\varpi^c R$. Without loss of generality we may assume that $c = 0$ or $c = 1$. Let $H = W \times k$ be the Heisenberg group. The multiplication in H is defined by

$$(v_1, z_1) \cdot (v_2, z_2) = (v_1 + v_2, \langle v_1, v_2 \rangle + z_1 + z_2).$$

The center of H consists of elements $(0, z)$. In particular, the center is isomorphic to k . There exists a unique, up to isomorphism, irreducible representation ρ_ψ of the Heisenberg group with the central character ψ . If we fix a decomposition $W = X + X^*$ of the symplectic space W as in Section 2, then ρ_ψ can be realized on the space $S(X^*)$ of locally constant, compactly supported functions on X^* , such that

$$\rho_\psi(x, 0)f(y) = \psi(2\langle y, x \rangle)f(y) \quad \text{and} \quad \rho_\psi(x^*, 0)f(y) = f(y + x^*)$$

where $f \in S(X^*)$, $x \in X$ and $x^* \in X^*$.

For every $g \in \mathrm{Sp}(W)$, the map $(v, z) \rightarrow g(v, z) = (gv, z)$ is an automorphism of H . Let ρ_ψ^g be the representation of H defined by $\rho_\psi^g(h) = \rho_\psi(g(h))$. Since the central character of ρ_ψ^g is also ψ , these two representations are isomorphic. Thus, there exists $g' \in \mathrm{GL}(S(X^*))$ such that $g' \circ \rho_\psi(h) = \rho_\psi^g(h) \circ g'$ for all $h \in H$. Since g' is unique up to an element in \mathbb{C}^\times , the set of all g' for all $g \in \mathrm{Sp}(W)$ is a central extension

$$1 \rightarrow \mathbb{C}^\times \rightarrow \mathrm{Sp}'(W) \rightarrow \mathrm{Sp}(W) \rightarrow 1.$$

Weil has shown that the derived group of $\mathrm{Sp}(W)'$ is isomorphic to $\widetilde{\mathrm{Sp}}(W)$, the 2-fold central extension of $\mathrm{Sp}(W)$. We shall now work out the elements $x_\alpha(u)$ in $\widetilde{\mathrm{Sp}}(W)$ for various roots $\alpha \in \Sigma$. Assume first that $\alpha = e_i - e_j$, a short root. Define

$$\tilde{\Phi}_\alpha : \mathrm{SL}_2(k) \rightarrow \mathrm{Sp}'(W)$$

by

$$[\tilde{\Phi}_\alpha(g)(f)](x) = f(g^{-1}x),$$

where $f \in S(X^*)$, and the action of $\mathrm{SL}_2(k)$ on X^* is specified in Section 2. Since $\mathrm{SL}_2(k)$ is perfect, $\tilde{\Phi}_\alpha(\mathrm{SL}_2(k))$ is contained in $\widetilde{\mathrm{Sp}}(W)$. Thus

$$x_\alpha(u) = \tilde{\Phi}_\alpha \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right).$$

Assume now that $\alpha = 2e_i$, a long root. Without loss of generality assume that $i = n$. Recall, from Section 2, that $W_{n-1} \subseteq W$ is the symplectic subspace spanned by \mathbf{e}_n and \mathbf{e}_n^* . Let Z_{n-1}^* be the line spanned by \mathbf{e}_n^* . Then $S(X^*) = S(X_{n-1}^*) \otimes S(Z_{n-1}^*)$ and we have an inclusion $\mathrm{Sp}'(W_{n-1}) \subseteq \mathrm{Sp}'(W)$ where $\mathrm{Sp}'(W_{n-1})$ acts on $S(Z_{n-1}^*)$. Thus, for ease of notation, we may assume that $W_{n-1} = W$, i.e., $\dim(W) = 2$. Then the group $\mathrm{Sp}'(W)$ contains operators defined by

$$\begin{aligned} [x'_\alpha(u)f](x) &= \psi(ux^2)f(x), \\ [h'_\alpha(t)f](x) &= |t|^{\frac{1}{2}}f(tx), \end{aligned}$$

and

$$[w'_\alpha(-1/t)f](x) = [w'_{-\alpha}(t)f](x) = |t|^{\frac{1}{2}} \int_k f(ty)\psi(2xy) dy$$

where dx is a self-dual measure, i.e., $w'_\alpha(1)^2 = h'_\alpha(-1)$. We note some obvious properties of these elements:

$$h'_\alpha(s)h'_\alpha(t) = h'_\alpha(st),$$

$$h'_\alpha(t)x'_\alpha(u)h'_\alpha(t)^{-1} = x'_\alpha(t^2u),$$

and

$$h'_\alpha(t) = w'_\alpha(t)w'_\alpha(1)^{-1}.$$

These elements differ from $x_\alpha(u)$, $h_\alpha(t)$ and $w_\alpha(t)$ by non-zero complex numbers. We claim that $x'_\alpha(u) = x_\alpha(u)$. Pick $t \in k^\times$ such that $t^2 \neq 1$. Since $[h'_\alpha(t), x'_\alpha(u)] = x'_\alpha((t^2 - 1)u)$ for every $u \in k$, $x'_\alpha(u)$ is contained in $\widetilde{\text{Sp}}(W)$. Thus $x_\alpha(u) = x'_\alpha(u)$, since the group U_α has a unique lift to $\widetilde{\text{Sp}}(W)$. Moreover, since $w_\alpha(1)$ is a non-zero multiple of $w'_\alpha(1)$, the element $x_{-\alpha}(t)$ in $\widetilde{\text{Sp}}(W)$ is necessarily given by

$$x_{-\alpha}(t) = w'_\alpha(-1)x_\alpha(-t)w'_\alpha(1).$$

Since $w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$, one can now compute $w_\alpha(t)$ explicitly and compare it with $w'_\alpha(t)$. It turns out that

$$w_\alpha(t) = \gamma(\psi_t \circ x^2) \cdot w'_\alpha(t)$$

where $\gamma(\psi_t \circ x^2)$ is the Weil index [Ku2] attached to the quadratic form x^2 and the character ψ_t . It is an 8-th root of 1. It follows that $h_\alpha(t) = \gamma(t, \psi \circ x^2)h'_\alpha(t)$, where $\gamma(t, \psi \circ x^2) = \gamma(\psi_t \circ x^2)/\gamma(\psi \circ x^2)$, and

$$h_\alpha(s)h_\alpha(t) = (s, t)h_\alpha(st)$$

where (s, t) is the Hilbert symbol, by the properties of the Weil index. In view of appearance of 8-th roots of one, it is natural to work with a group slightly larger than $\widetilde{\text{Sp}}(W)$. Thus, let $\widetilde{\text{Sp}}(W)_8 \subseteq \text{Sp}'(W)$ be the central extension of $\text{Sp}(W)$ by μ_8 , the group of 8-th roots of one:

$$1 \rightarrow \mu_8 \rightarrow \widetilde{\text{Sp}}(W)_8 \rightarrow \text{Sp}(W) \rightarrow 1.$$

An advantage of this group is that it contains the elements $h'_\alpha(t)$ and $w'_\alpha(t)$, defined above, for every long root $\alpha = 2e_i$. These elements, together with $w_\alpha(t)$ for short roots $\alpha = e_i - e_j$ generate a subgroup isomorphic to the normalizer of the torus T in $\text{Sp}(W)$.

It is useful to extend the definition of $w'_\alpha(t)$ to affine roots. If $\beta = \alpha + m$ is an affine root we define $w'_\beta(t)$ to be $w'_\alpha(\varpi^m t)$. We record the following simple result which can be easily verified using the definition of $w'_\alpha(t)$.

Lemma 2. *Let $\alpha = 2e_i$ and $\beta = 1 - 2e_i$. Let $\varphi \in S(X^*)$ be the characteristic function of the lattice spanned by $\mathbf{e}_1^*, \dots, \mathbf{e}_n^*$. Let $\langle \cdot, \cdot \rangle$ denote L^2 -inner product on $S(X^*)$ with respect to the unique Haar measure on X^* such that the Fourier transform, defined by means of ψ and the Haar measure, preserves L^2 -norm.*

- (1) *If the conductor of ψ is 0 then $\langle w'_\beta(1)(\varphi), \varphi \rangle = q^{-\frac{1}{2}} \langle \varphi, \varphi \rangle$.*
- (2) *If the conductor of ψ is 1 then $\langle w'_\alpha(1)(\varphi), \varphi \rangle = q^{-\frac{1}{2}} \langle \varphi, \varphi \rangle$.*

Recall, from Section 2, the decomposition $W = X_l + W_l + X_l^*$ and the maximal parabolic subgroup P defined as the stabilizer of X_l in $\mathrm{Sp}(W)$. Let U_P be the unipotent radical of P . Then we have a Levi decomposition $P = MU_P$ where M is chosen so that it stabilizes X_l^* . Let \widetilde{P} the inverse image of P in $\widetilde{\mathrm{Sp}}(W)_8$. The central extension splits uniquely over U_P , so abusing notation, we shall denote by \overline{U}_P the unipotent radical of \widetilde{P} . Let \widetilde{M} be the inverse image of M in $\widetilde{\mathrm{Sp}}(W)_8$. Let $Z_l^* = X^* \cap W_l$. Then $X^* = X_l^* + Z_l^*$. Using the decomposition $S(X^*) = S(X_l^*) \otimes S(Z_l^*)$, we have

$$\widetilde{M} \cong \mathrm{GL}(X_l) \times \widetilde{\mathrm{Sp}}(W_l)_8 \cong \mathrm{GL}(X_l^*) \times \widetilde{\mathrm{Sp}}(W_l)_8$$

where $\widetilde{\mathrm{Sp}}(W_l)_8$ acts on $S(Z_l^*)$ by the Weil representation denoted by $\omega_{\psi,l}$, and $g \in \mathrm{GL}(X_l^*)$ acts on $f \in S(X_l^*)$ by

$$[g \cdot f](x) = |\det(g)|^{-\frac{1}{2}} f(g^{-1}x).$$

We also define $\widetilde{\mathrm{Sp}}(W_n)_8 = \mu_8$. It acts on $S(Z_n^*) \cong \mathbb{C}$ by the character given by the inclusion $\mu_8 \subseteq \mathbb{C}^\times$. The explicit action of U_P on $S(X^*)$ is not important to us. However, if $S(X^*)_{U_P}$ denotes the space of U_P co-invariants (the Jacquet functor), then $S(X^*)_{U_P} \cong S(Z_l^*)$ and the natural projection from $S(X^*)$ to $S(X^*)_{U_P}$ is given by the restriction of functions from X^* to Z_l^* . In particular, $g \in \mathrm{GL}(X_l^*)$ acts on $S(X^*)_{U_P}$ by multiplication by $|\det(g)|^{1/2}$.

6. MINIMAL TYPES IN WEIL REPRESENTATION

Recall that K_i is the maximal open compact subgroup defined as the stabilizer of the good lattice L_i , where L_i is the R -span of \mathbf{e}_j for $1 \leq j \leq n$, $\frac{\mathbf{e}_j^*}{\varpi}$ for $j \leq i$, and \mathbf{e}_j^* for $j > i$. Then K_i contains a pro- p subgroup such that $K_i/K_i^+ \cong \mathrm{Sp}_{2i}(\mathfrak{f}) \times \mathrm{Sp}_{2(n-i)}(\mathfrak{f})$. Let \widetilde{K}_i be the inverse image of K_i in $\widetilde{\mathrm{Sp}}(W)_8$. We shall see that $\widetilde{K}_i \rightarrow K_i$ splits, but first, we describe the minimal \widetilde{K}_i -types in ω_ψ , the Weil representation of $\widetilde{\mathrm{Sp}}(W)_8$, realized on $S(X^*)$.

Case (i): conductor of ψ is 0. In this case there is a \widetilde{K}_i -invariant submodule τ_i consisting of functions supported on $L_i \cap X^*$ and invariant under translations by the lattice $L_0 \cap X^*$. Thus this space can be written as

$$\tau_i \cong S(L_i/L_0) \otimes \mathbb{C}$$

where $S(L_i/L_0)$ is naturally the Weil representation of $\mathrm{Sp}_{2i}(\mathfrak{f})$ corresponding to the character of $\mathfrak{f} \cong \frac{1}{\varpi}R/R$ obtained by restricting ψ to $\frac{1}{\varpi}R$, and \mathbb{C} is the trivial representation of $\mathrm{Sp}_{2(n-i)}(\mathfrak{f})$. Note that τ_0 is one-dimensional and spanned by φ , the characteristic function of $L_0 \cap X^*$. If $i \neq 0$, then $\tau_i = \tau_i^e \oplus \tau_i^o$ is a sum of even and odd Weil representations of $\mathrm{Sp}_{2i}(\mathfrak{f})$. Note that we have a natural inclusion of spaces $\tau_i \subseteq \tau_{i+1}$ for all $i < n$.

Case (ii): conductor of ψ is 1. In this case there is a \widetilde{K}_i -invariant submodule τ_i consisting of functions supported on $L_0 \cap X^*$ and invariant under translations by the lattice $\varpi L_i \cap X^*$. Thus this space can be written as

$$\tau_i \cong \mathbb{C} \otimes S(L_0/\varpi L_i)$$

where $S(L_0/\varpi L_i)$ is naturally the Weil representation of $\mathrm{Sp}_{2(n-i)}(\mathfrak{f})$ corresponding to the character of $\mathfrak{f} \cong R/\varpi R$ obtained by restricting ψ to R , and \mathbb{C} is the trivial representation of $\mathrm{Sp}_{2i}(\mathfrak{f})$. Note that τ_n is one-dimensional and spanned by φ , the

characteristic function of $L_0 \cap X^*$. If $i \neq n$, then $\tau_i = \tau_i^e \oplus \tau_i^o$ is a sum of even and odd Weil representations of $\mathrm{Sp}_{2(n-i)}(\mathfrak{f})$. Note that we have a natural inclusion of spaces $\tau_{i+1} \subseteq \tau_i$ for all $i < n$.

Now we can prove that the central extension $\tilde{K}_i \rightarrow K_i$ splits for all i . Let d_e and d_o be the determinant characters of \tilde{K}_i acting on τ_i^e and τ_i^o , respectively. Since $\dim \tau_i^e - \dim \tau_i^o = 1$, the character

$$d_e^{\dim \tau_i^e} \cdot d_o^{-\dim \tau_i^e} : \tilde{K}_i \rightarrow \mu_8$$

is the identity on $\mu_8 \subseteq \tilde{K}_i$.

7. PRELIMINARIES ON HECKE ALGEBRAS

Let G be a totally disconnected topological group and K an open compact subgroup. Fix a measure on G such that the volume of K is one. Let (σ, E) be an irreducible, finite dimensional representation of K . Let (σ^*, E^*) be the dual representation. Let $H(\sigma)$ be the space of locally constant, compactly supported functions

$$f : G \rightarrow \mathrm{End}(E^*)$$

such that

$$f(k_1 g k_2) = \sigma^*(k_1) f(g) \sigma^*(k_2)$$

for all $k_1, k_2 \in K$. This is an algebra with respect to the convolution of functions. For any smooth representation (π, V) of G , the space $(V \otimes E^*)^K$ is an $H(\sigma)$ -module, where the action of $f \in H(\sigma)$ is defined by

$$\pi(f)(v \otimes e) = \int_G \pi(g) v \otimes f(g) e \, dg.$$

Let Γ be an open compact subgroup of G containing K . Then K has finite index in Γ . Let $H_\Gamma(\sigma)$ be the subalgebra of $H(\sigma)$ consisting of functions supported on Γ . Then $H_\Gamma(\sigma)$ is finite dimensional and it is isomorphic to the algebra of intertwining operators of the representation

$$\mathrm{Ind}_K^\Gamma(\sigma^*) = \{ \phi : \Gamma \rightarrow E^* : \phi(kg) = \sigma^*(k)\phi(g) \text{ for all } k \in K, g \in \Gamma \}.$$

Indeed, for $f \in H_\Gamma(\sigma)$ and $\phi \in \mathrm{Ind}_K^\Gamma(\sigma^*)$, let

$$(L(f)\phi)(g) = \int_\Gamma f(h)\phi(h^{-1}g) \, dh \text{ for all } g \in \Gamma.$$

The map $f \mapsto L(f)$ gives an isomorphism of $H_\Gamma(\sigma)$ and the endomorphism algebra of $\mathrm{Ind}_K^\Gamma(\sigma^*)$. Assume now that $H_\Gamma(\sigma)$ is commutative. Let V be an irreducible finite dimensional representation of Γ such that $(V \otimes E^*)^K \neq 0$. Then, since $H_\Gamma(\sigma)$ is commutative, $(V \otimes E^*)^K$ is one-dimensional and the action of $H_\Gamma(\sigma)$ on it factors through a maximal ideal $m \subseteq H_\Gamma(\sigma)$. Moreover,

$$\mathrm{Ind}_K^\Gamma(\sigma^*)/L(m) \cdot \mathrm{Ind}_K^\Gamma(\sigma^*) \cong V^*,$$

the dual representation of V . Thus, if $(\pi_1, V_1), \dots, (\pi_l, V_l)$ are all, up to isomorphism, irreducible representations of Γ such that $(V \otimes E^*)^K \neq 0$, then

$$\mathrm{Ind}_K^\Gamma(\sigma^*) \cong V_1^* \oplus \dots \oplus V_l^*.$$

Moreover, for every $f \in H_\Gamma(\sigma)$, $\mathrm{Trace}(L(f)) = \sum_{i=1}^l d_i \lambda_i$ where d_i is the dimension of V_i and $\lambda_i = \pi_i(f)$. Note that, if f is not supported on K , then the trace of $L(f)$ is 0. Indeed, if $\Gamma = \bigcup_{i=1}^r K g_i$, then $\mathrm{Ind}_K^\Gamma(\sigma^*)$ has a basis ϕ_i of functions supported

on Kg_i . If f is supported on a double coset $K\gamma K = \bigcup_{j=1}^s K\gamma_j$, then $L(f)(\phi_i)$ is supported on $\bigcup_{j=1}^s K\gamma_j g_i$ and this does not contain Kg_i if $K\gamma K \neq K$.

Thus we have proven:

Lemma 3. *Assume that the dimension of $H_\Gamma(\sigma)$ is two. Let (π_1, V_1) and (π_2, V_2) be the unique, up to isomorphism, irreducible representations of Γ such that $(V \otimes E^*)^K \neq 0$. Let d_1 and d_2 be their dimensions. Let T be an element in $H_\Gamma(\sigma)$ not supported on K . Let $\lambda_1 = \pi_1(T)$ and $\lambda_2 = \pi_2(T)$. Then*

- (1) $\dim(\text{Ind}_K^\Gamma(\sigma^*)) = d_1 + d_2$.
- (2) $d_1\lambda_1 + d_2\lambda_2 = 0$.
- (3) *The minimal polynomial of T is $(T - \lambda_1)(T - \lambda_2)$.*

8. HECKE ALGEBRA H_ψ^+

Let $I = \bigcap_{i=0}^n K_i$ be the Iwahori subgroup of $\text{Sp}(W)$, and let \tilde{I} be the inverse image of I in $\widetilde{\text{Sp}}(W)_8$. The Hecke algebra H_ψ^+ , we are about to define, depends on ψ . We have two cases to consider. Assume first that the conductor of ψ is 0. Let $\tau_0 \subseteq S(X^*)$ be the one-dimensional minimal \tilde{K}_0 -type. Let $H_\psi^+ = H(\tau_0)$ be the Hecke algebra, defined as in Section 7, for $G = \widetilde{\text{Sp}}(W)_8$, $K = \tilde{I}$ and $\sigma = \tau_0$. Since τ_0 is one-dimensional, some definitions from Section 7 simplify in this case. First, since $\text{End}(\tau_0^*) \cong \mathbb{C}$, the algebra H_ψ^+ consists of complex valued functions. Second, if (π, V) is a smooth representation of $\widetilde{\text{Sp}}(W)_8$, the Hecke module $(V \otimes \tau_0^*)^{\tilde{I}}$ can be identified with

$$V^{\tilde{I}, \tau_0} = \{v \in V \mid \pi(i)v = \tau_0(i)v \text{ for all } i \in \tilde{I}\}.$$

We shall now determine the algebra H_ψ^+ in terms of its generator and relations. If for some x there exists $i_1, i_2 \in \tilde{I}$ such that $i_1x = xi_2$ and $\tau_0^*(i_1) \neq \tau_0^*(i_2)$ then $f(x) = 0$ for any f in the Hecke algebra. Thus, it is not a priori clear that every \tilde{I} -double coset supports an element in the Hecke algebra. We address this delicate issue as follows. For any simple affine root α_i , define an open compact subgroup

$$\tilde{I}_i = \tilde{I} \cup \tilde{I}w_{\alpha_i}(1)\tilde{I}.$$

Let $H_{\psi,i}$ be the subalgebra of H_ψ^+ consisting of functions supported on \tilde{I}_i . This is a one or two-dimensional subalgebra spanned by the identity element of H_ψ^+ and an element T_i supported on $\tilde{I}w_{\alpha_i}(1)\tilde{I}$ if such an element exists. Recall, from Section 7, that the algebra $H_{\psi,i}$ is naturally isomorphic to the endomorphism algebra of the induced representation $\text{Ind}_{\tilde{I}}^{\tilde{I}_i}(\tau_0^*)$, of dimension $q + 1$. Thus, in order to show that T_i exists, it suffices to show that this representation is reducible. Since $\text{Ind}_{\tilde{I}}^{\tilde{I}_i}(\tau_0^*)$ is dual to $\text{Ind}_{\tilde{I}}^{\tilde{I}_i}(\tau_0)$, it suffices to show that $\text{Ind}_{\tilde{I}}^{\tilde{I}_i}(\tau_0)$ is reducible. Assume first that $i \neq 0$. Then $\tilde{I}_i \subseteq \tilde{K}_0$ so $\tau_0 \subseteq S(X^*)$ is naturally a representation of \tilde{I}_i . Thus, by the Frobenius reciprocity, τ_0 is a one-dimensional summand of $\text{Ind}_{\tilde{I}}^{\tilde{I}_i}(\tau_0)$. Assume now $i = 0$. In this case $\tilde{I}_0 \subseteq \tilde{K}_1$. Consider the \tilde{K}_1 -type $\tau_1 \subseteq S(X^*)$. Then $\tau_1 = \tau_1^e \oplus \tau_1^o$ is a sum of an even and odd Weil representation of $\text{SL}_2(\mathfrak{f})$. Since $\tau_0 \subseteq \tau_1^e$, the Frobenius reciprocity implies that τ_1^e is a summand of $\text{Ind}_{\tilde{I}}^{\tilde{I}_0}(\tau_0)$ of dimension $(q + 1)/2$. We have now shown that T_i exists and we normalize it so that $T_i(w_{\alpha_i}(1)) = 1$ if $i \neq 0, n$, $T_0(w'_{\alpha_0}(1)) = q^{-\frac{1}{2}}$, and $T_n(w'_{\alpha_n}(1)) = 1$.

Lemma 4. *Let ω_ψ denote the action of $\widetilde{\text{Sp}}(W)_8$ on $S(X^*)$. Let $\varphi \in \tau_0 \subseteq S(X^*)$ be the characteristic function of the lattice $L_0 \cap X^*$. Then $\omega_\psi(T_i)\varphi = q\varphi$ if $i \neq 0$ and $\omega_\psi(T_0)\varphi = \varphi$.*

Proof. The idea is to compute the inner product $\langle \omega_\psi(T_i)\varphi, \varphi \rangle$. Let I^+ be the pro- p radical of \tilde{I} . Then $\tilde{I}w_{\alpha_i}(1)\tilde{I} = I^+w_{\alpha_i}(1)\tilde{I}$, so we can write

$$\tilde{I}w_{\alpha_i}(1)\tilde{I} = \bigcup_{j=1}^q x_j w_{\alpha_i}(1)\tilde{I}$$

for some elements x_j in I^+ . Assume now that $i \neq 0, n$. Then

$$\langle \omega_\psi(T_i)\varphi, \varphi \rangle = \sum_{j=1}^q \langle x_j w_{\alpha_i}(1)\varphi, \varphi \rangle = \sum_{j=1}^q \langle w_{\alpha_i}(1)\varphi, x_j^{-1}\varphi \rangle = q\langle \varphi, \varphi \rangle$$

where the last equality holds since x_j and $w_{\alpha_i}(1)$ fix φ . A similar argument works for $i = n$, since $w'_{\alpha_n}(1)$ is the Fourier transform in the variable x_n and it fixes φ , since the conductor is 0. For the case $i = 0$, use Lemma 2, part (1). □

Lemma 5. *The operator T_i satisfies the quadratic relations $(T_i - q)(T_i + 1) = 0$ if $i \neq 0$ and $T_0^2 = 1$. In particular, all T_i are invertible.*

Proof. This follows easily from Lemma 3. Assume first that $i \neq 0$. Then τ_0 is a one-dimensional summand of $\text{Ind}_{\tilde{I}}^{\tilde{I}^i}(\tau_0)$. Thus $d_1 = 1$ and $d_2 = q$. Lemma 4 implies that $\lambda_1 = q$. It follows that $\lambda_2 = -1$, by Lemma 3 (2). Assume now $i = 0$. In this case τ_1^e is a summand of $\text{Ind}_{\tilde{I}}^{\tilde{I}^i}(\tau_0)$ of dimension $(q+1)/2$. Thus $d_1 = d_2 = (q+1)/2$. Since $(\tau_1^e)^{\tilde{I}, \tau_0} = \tau_0$, Lemma 4 implies that $\lambda_1 = 1$. It follows that $\lambda_2 = -1$, by Lemma 3 (2). Finally, invertibility of T_i follows from the quadratic relations. For example, if $i \neq 0$, then $qT_i^{-1} = T_i - q + 1$. The lemma is proved. □

Theorem 6. *Assume that the conductor of ψ is 0. The Hecke algebra H_ψ^+ , as an abstract algebra, is generated by elements T_0, T_1, \dots, T_n satisfying quadratic relations $(T_i - q)(T_i + 1) = 0$ for $i \neq 0$, $T_0^2 = 1$, and satisfying the braid relations corresponding to the following Coxeter diagram:*



Proof. We have already shown that the quadratic relations hold. Thus, we need to prove braid relations. Since $s_0s_1s_0s_1 = s_1s_0s_1s_0$ (a braid relation in the affine Weyl group) it follows that $T_0T_1T_0T_1$ and $T_1T_0T_1T_0$ are supported on the same double coset. Thus they must be equal up to a scalar multiple. However, by Lemma 4 both operators act on the type τ_0 by the same positive scalar. Thus they are equal. Other braid relations are, of course, proved in the same manner. The rest of the proof is completely analogous to the case of linear groups. □

If the conductor of ψ is 1, then the role of K_0 is played by K_n . We define $H_\psi^+ = H(\tau_n)$ where τ_n is the one-dimensional \tilde{K}_n -type, viewed as a representation of \tilde{I} . An obvious analogue of Theorem 6 holds.

9. BERNSTEIN’S COMPONENT \mathcal{G}_ψ^+

In this section we use a decomposition of the category of genuine representations of $\widetilde{\mathrm{Sp}}(W)$ given by Bernstein [Be]. As indicated there, the decomposition is also valid for central extensions. In fact, the modifications of arguments for central extensions (2.2 in [Be]) are trivial for $\widetilde{\mathrm{Sp}}(W)$, since \widetilde{T} is commutative.

Let $\widetilde{B} = \widetilde{T}U \subseteq \widetilde{\mathrm{Sp}}(W)_8$ be the inverse image of the Borel subgroup defined in Section 2. For any $\alpha = 2e_i$ and $t \in k^\times$, let us abbreviate $h'_i(t) = h'_\alpha(t)$. Any element $t \in \widetilde{T}$ can be uniquely written as

$$t = h'_1(t_1) \cdots h'_n(t_n) \cdot \zeta$$

where $t_i \in k^\times$ and $\zeta \in \mu_8$. Then any n -tuple of characters $\chi = (\chi_1, \dots, \chi_n)$ defines a genuine character $\chi : \widetilde{T} \rightarrow \mathbb{C}^\times$

$$\chi(t) = \chi_1(t_1) \cdots \chi_n(t_n) \cdot \zeta.$$

Let $I(\chi)$ denote the normalized principal series representation, i.e., we induce the character χ multiplied by the modular character $\rho_U^{\frac{1}{2}}$. If (π, V) is a smooth representation of $\widetilde{\mathrm{Sp}}(W_8)$, let (π_U, V_U) denote the normalized Jacquet module, i.e., the action of \widetilde{T} on V_U is multiplied by the inverse of the modular character. With these conventions the Frobenius reciprocity takes the form

$$\mathrm{Hom}_{\widetilde{\mathrm{Sp}}(W)_8}(V, I(\chi)) \cong \mathrm{Hom}_{\widetilde{T}}(V_U, \chi).$$

Let \mathcal{G}_ψ^+ be the component of the category of smooth representations of $\widetilde{\mathrm{Sp}}(W)_8$ whose irreducible representation are submodules of $I(\chi)$ where χ is unramified. The even Weil representation ω_ψ^e belongs to \mathcal{G}_ψ^+ . Indeed, the functional $\ell : S(X^*)^e \rightarrow \mathbb{C}$ defined by $\ell(f) = f(0)$, via the Frobenius reciprocity, gives an embedding of ω_ψ^e into $I(\chi_e)$, where χ_e is an unramified character.

Let $\widetilde{I}_T = \widetilde{I} \cap \widetilde{T}$, and consider τ_0 as a character of \widetilde{I}_T by restriction. By Proposition 3.5.2 in [Be] the natural projection from V to V_U gives a surjection

$$r : V^{\widetilde{I}, \tau_0} \rightarrow (V_U)^{\widetilde{I}_T, \tau_0}.$$

For any $w \in W_a$, let $T_w = T_{i_1} \cdots T_{i_l}$ where $w = s_{i_1} \cdots s_{i_l}$, is a shortest product of simple reflections. Note that T_w is invertible, since all T_i are. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ be a translation in W_a such that $\lambda_1 \geq \dots \geq \lambda_n \geq 0$, and let

$$\widehat{T}_\lambda = q^{-\frac{l_0(\lambda)}{2}} T_\lambda$$

where l_0 is the weighted length function on W_a such that $l_0(s_0) = 0$ and 1 for other simple reflections. Let $t_\lambda = h'_1(\varpi^{\lambda_1}) \cdots h'_n(\varpi^{\lambda_n})$. Note that, since $T_0(w'_{\alpha_0}(1)) = q^{-\frac{1}{2}}$, the value of \widehat{T}_λ at t_λ is $q^{-\frac{l(\lambda)}{2}}$. Since

$$\widetilde{I} t_\lambda \widetilde{I} = \bigcup_{i=1}^{q^{l(\lambda)}} t_\lambda u_i \widetilde{I}$$

where u_i are representatives of $\widetilde{I} \cap U$ -co sets in $t_\lambda^{-1}(\widetilde{I} \cap U)t_\lambda$, it follows that

$$r \circ \pi(\widehat{T}_\lambda) = \pi_U(t_\lambda) \circ r.$$

Let v be in the kernel of r . Then $\int_{U_c} \pi(u)v \, du = 0$ for some open compact subgroup U_c of U . Pick λ such that $t_\lambda^{-1}(\tilde{I} \cap U)t_\lambda \supseteq U_c$. Then

$$\pi(T_\lambda)(v) = \pi(t_\lambda) \sum_{i=1}^{q^{l(\lambda)}} \pi(u_i)(v) = \int_{t_\lambda^{-1}(\tilde{I} \cap U)t_\lambda} \pi(u)v \, du = 0,$$

hence $v = 0$, since T_λ is invertible. Thus the map r is an isomorphism. In particular, if V is an irreducible representation such that $V^{\tilde{I}, \tau_0} \neq 0$, then V is a submodule of $I(\chi)$ for some character χ . Moreover, χ is unramified since the character τ_0 is trivial on $h'_i(t)$ for $t \in R^\times$ (as these elements fix the vector $\varphi \in \tau_0$). Conversely, if V is a submodule of $I(\chi)$ for some unramified character χ then $(V_U)^{\tilde{I}_T, \tau_0} \neq 0$ and, therefore, $V^{\tilde{I}, \tau_0} \neq 0$. Thus irreducible representations of H_ψ^+ correspond to irreducible constituents of principal series representations $I(\chi)$. This proves (see (3.12) in [BK]):

Corollary 7. *The functor that attaches to every smooth representation V of $\widetilde{\text{Sp}}(W)_8$ the Hecke algebra module $V^{\tilde{I}, \tau_0}$ is an equivalence of Bernstein’s component \mathcal{G}_ψ^+ and the category of H_ψ^+ -modules.*

Operators \hat{T}_λ form a commuting family of operators. Thus any finite dimensional representation F of H_ψ^+ can be decomposed into a sum of generalized eigenspaces. More precisely, for any $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{C}^n$, define F_μ to be the subspace of F consisting of all $v \in F$ such that

$$\pi(\hat{T}_\lambda - q^{-\sum_{i=1}^n \lambda_i \mu_i})^{\dim(F)}(v) = 0.$$

Then $F = \bigoplus F_\mu$ and any μ such that $F_\mu \neq 0$ is called an exponent of F . The representation F is square integrable if

$$\Re(\mu_1) > 0, \Re(\mu_1) + \Re(\mu_2) > 0, \dots, \Re(\mu_1) + \dots + \Re(\mu_n) > 0$$

for all exponents μ . The representation F is tempered if the exponents satisfy the inequalities where all $>$ have been replaced by \geq . These definitions agree with the usual definitions of square integrable and tempered representations of $\widetilde{\text{Sp}}(W)_8$, by Casselman’s criterion.

Hecke algebra functors. Much of what we have discussed in this section can be generalized as follows. Let $\tilde{P} = \tilde{M}U_P$ be a parabolic subgroup of $\widetilde{\text{Sp}}(W)_8$ in a standard position. Let $\Delta_M \subseteq \Delta$ be the set of simple roots of \tilde{M} . Let $\tilde{I}_M = \tilde{I} \cap \tilde{M}$, and consider τ_0 as a character of \tilde{I}_M . We can define a Hecke algebra $H_\psi^{M,+} = H(\tau_0)$ on \tilde{M} . In particular, for every simple root $\alpha_i \in \Delta_M$ we have an element T_i^M satisfying the quadratic relation $(T_i^M - q)(T_i^M + 1) = 0$ and for every $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ such that $\lambda_1 \geq \dots \geq \lambda_n$ we have an element \hat{T}_λ^M whose value at t_λ is $q^{-\frac{l_M(\lambda)}{2}}$. (Here l_M is the length function for the affine Weyl group of M .) It follows from Bernstein’s relations that there is a unique homomorphism of algebras,

$$i_P : H_\psi^{M,+} \rightarrow H_\psi^+,$$

such that $i_P(T_i^M) = T_i$ and $i_P(\hat{T}_\lambda^M) = \hat{T}_\lambda$. Let (π, V) be a smooth representation of $\widetilde{\text{Sp}}(W)_8$. Then the natural projection from V to V_{U_P} (a normalized Jacquet

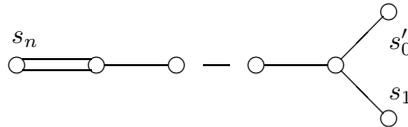
functor) induces an isomorphism

$$r_{U_P} : V^{\tilde{I}, \tau_0} \rightarrow (V_{U_P})^{\tilde{I}_M, \tau_0}.$$

One easily sees, by checking on T_i^M and \hat{T}_λ^M , that $r_{U_P} \circ \pi(i_P(T)) = \pi_{U_P}(T) \circ r_{U_P}$ for all T in $H_\psi^{M,+}$. In other words, the Jacquet functor for the parabolic group \tilde{P} corresponds to the natural (forgetful) functor from H_ψ^+ -modules to $H_\psi^{M,+}$ -modules. On the other hand, tensoring of $H_\psi^{M,+}$ -modules with H_ψ^+ correspond to the induction with respect to the parabolic group opposite to \tilde{P} . For a detailed treatment of this topic the reader can consult [BK] and [Ro].

10. HECKE ALGEBRA OF $SO(V^+)$

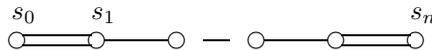
The affine Hecke algebra of $SO(V^+)$ is described in terms of the extended affine Weyl group. To that end, let $\mathbb{E} = \mathbb{R}e_1 + \dots + \mathbb{R}e_n$, and let $\Sigma = \{\pm e_i \pm e_j\} \cup \{\pm e_i\}$ be a root system of type B_n , sitting in \mathbb{E} . Fix a set Δ of simple roots consisting of roots $\alpha_1 = e_1 - e_2, \dots, \alpha_{n-1} = e_{n-1} - e_n$ and $\alpha_n = e_n$. Roots can be viewed as functionals on \mathbb{E} , using the standard dot product on \mathbb{E} . Affine roots are functionals $\alpha + m$ where $\alpha \in \Sigma$ and $m \in \mathbb{Z}$. We have a set of simple affine roots $\Delta_a = \Delta \cup \{\alpha_0\}$ where $\alpha_0 = 1 - e_1 - e_2$. The affine Weyl group (of type B_n) is generated by reflections s'_0 and s_1, \dots, s_n corresponding to the simple affine roots satisfying the braid relations given by the following Coxeter diagram:



The affine Weyl group is a semi-direct product of the Weyl group (generated by s_1, \dots, s_n) and a normal subgroup consisting of translations by $\lambda = (m_1, \dots, m_n) \in \mathbb{Z}^n$ such that $\sum_{i=1}^n m_i$ is even. The extended affine Weyl group, relevant for the adjoint group $SO(V^+)$, contains one additional element s_0 such that $s_0^2 = 1$ and

$$s'_0 = s_0 s_1 s_0.$$

In fact, s_0 is the reflection about the hyperplane $x_1 = \frac{1}{2}$. This shows that the extended affine Weyl group of type B_n coincides with the affine Weyl group of type C_n , denoted by W_a . In particular, it is generated by reflections s_0, \dots, s_n and the braid relations corresponding to the following Coxeter diagram:



Let l_0 be the weighted length function on W_a such that $l_0(s_0) = 0$ and it is 1 on other simple reflections. Let I be the Iwahori subgroup of $SO(V^+)$. Then for every w in the extended affine Weyl group $[IwI : I] = q^{l_0(w)}$.

Let H^+ denote the affine Hecke algebra of $SO(V^+)$. For any $w \in W_a$ let $T_w \in H^+$ be the characteristic function of the double coset parameterized by w . For simplicity, let T_i denote T_{s_i} . Then T_i satisfies $T_0^2 = 1$ and $(T_i - q)(T_i + 1) = 0$, if $i \neq 0$, and the braid relations given by the above Coxeter diagram. In fact, as an

abstract algebra, H^+ is generated by T_0, \dots, T_n modulo these quadratic and the braid relations. Since H and H_ψ^+ are given by “identical” combinatorial data (see Theorem 6), we have obtained the following:

Theorem 8. *The algebra H_ψ^+ is isomorphic, via an explicit isomorphism, to the algebra H^+ .*

Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ be a translation in W_a such that $\lambda_1 \geq \dots \geq \lambda_n \geq 0$. Define $\hat{T}_\lambda = q^{-\frac{i_0(\lambda)}{2}} T_\lambda$. The operators \hat{T}_λ form a commuting family of operators and, as we discussed in the previous section, any finite dimensional representation H^+ can be decomposed in a sum of generalized eigenspaces, parameterized by exponents $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{C}^n$. Again, the corresponding representation of $\text{SO}(V^+)$ is square integrable if and only if

$$\Re(\mu_1) > 0, \Re(\mu_1) + \Re(\mu_2) > 0, \dots, \Re(\mu_1) + \dots + \Re(\mu_n) > 0$$

for all exponents μ , and it is tempered if the exponents satisfy the inequalities where all $>$ have been replaced by \geq .

The trivial representation of $\text{SO}(V^+)$ corresponds to the one-dimensional representation of H^+ where T_w acts by $q^{l_0(w)}$. Thus by Lemma 4, and the isomorphism $H^+ \cong H_\psi^+$, the trivial representation of $\text{SO}(V^+)$ corresponds to the even Weil representation ω_ψ^e .

We summarize what we have proven so far:

Corollary 9. *In the equivalence of categories \mathcal{G}_ψ^+ and \mathcal{S}_0^+ , given by the isomorphism of the respective Hecke algebras, square integrable representations are mapped to square integrable representations, and tempered representations are mapped to tempered representations.*

11. HECKE ALGEBRA H_ψ^-

Assume that the conductor of ψ is 1. Consider the \tilde{K}_{n-1} type τ_{n-1} . Recall that this is the Weil representation of $\text{SL}_2(\mathfrak{f})$, corresponding to the additive character of $\mathfrak{f} = R/\varpi R$ obtained by restricting ψ to R . Let $J = \bigcap_{i=0}^{n-1} K_i$, and let \tilde{J} be the inverse image of J . In particular, we can view the odd Weil representation τ_{n-1}° as a representation of \tilde{J} by restriction. Let $H_\psi^- = H(\tau_{n-1}^\circ)$ be the Hecke algebra, defined as in Section 7, for $G = \widetilde{\text{Sp}}(W)_8$, $K = \tilde{J}$ and $\sigma = \tau_{n-1}^\circ$. We shall now determine this algebra in terms of its generator and relations.

First of all, we will show that this algebra is supported on $\tilde{J}W'_a\tilde{J}$, where W'_a is the subgroup of W_a generated by s_0, \dots, s_{n-2} and s'_{n-1} , the reflection corresponding to the long root $\alpha'_{n-1} = 2e_{n-1}$.

Lemma 10. *The algebra H_ψ^- is supported in $\tilde{J}W'_a\tilde{J}$.*

Proof. Recall that $\alpha_n = 2e_n$. Let f be in H_ψ^- . Let $w \in W_a$. Abusing the notation, we consider w as an element in $\widetilde{\text{Sp}}(W)_8$. We need to show that if $f(w) \neq 0$ then $w(\alpha_n) = \pm\alpha_n$. In particular, either $w \in W'_a$ or $ws_n \in W'_a$. (Note that w and ws_n represent the same \tilde{J} -double coset.) Assume that $w(\alpha_n) = \beta + m$ where m is an integer. Let $u \in R$. Then $w^{-1}x_{\alpha_n}(u)w = x_\beta(t)$ for some $t \in \varpi^{2m}R$. If $m > 0$ or $m = 0$ and β is a positive root different from α_n , then $x_\beta(t)$ is in \tilde{J} and in the

kernel of $\tau_{n-1}^{o,*}$. Thus

$$f(w) = f(wx_\beta(t)) = f(x_{\alpha_n}(u)w) = \tau_{n-1}^{o,*}(x_{\alpha_n}(u))f(w)$$

which implies that $\tau_{n-1}^{o,*}(x_{\alpha_n}(u))$ acts trivially on the image of $f(w)$, for all $u \in R$. Since $\tau_{n-1}^{o,*}$ is cuspidal, this implies that $f(w) = 0$. If $m < 0$ or $m = 0$ and β is a negative root different from $-\alpha_n$ then we repeat the same argument with $x_{-\alpha_n}(u)$ in the place of $x_{\alpha_n}(u)$. The lemma is proved. \square

We now want to prove existence of non-zero elements T_i in H_ψ^- , for $i = 0, \dots, n-2$ supported on $\tilde{J}w_{\alpha_i}(1)\tilde{J}$ and of a non-zero element T_{n-1} supported on $\tilde{J}w_{\alpha'_{n-1}}(1)\tilde{J}$. The case $i \neq n-1$ is easy to treat and similar to what we have done for the algebra H_ψ^+ . We normalize these elements so that $T_i(w_{\alpha_i}(1))$, if $i \neq 0$, and $T_0(w'_{\alpha_0}(1))$ is the identity in $\text{End}(\tau_{n-1}^{o,*})$. The case $i = n-1$ is a real delicacy and we devote our attention to it. Let

$$K = \bigcap_{i=0}^{n-2} K_i.$$

In particular, $J \subseteq K \subseteq K_{n-2}$, and K shares the reductive quotient $\text{Sp}_4(\mathfrak{f})$ with K_{n-2} . Let \tilde{K} be the inverse image of K . Let $H_{\psi, \tilde{K}}$ be the subalgebra of H_ψ^- consisting of functions supported on \tilde{K} . By Lemma 10, this is a one- or two-dimensional subalgebra spanned by the identity element of H_ψ^- and T_{n-1} , if it exists. The algebra $H_{\psi, \tilde{K}}$ is naturally viewed as the endomorphism algebra of $\text{Ind}_{\tilde{J}}^{\tilde{K}}(\tau_{n-1}^{o,*})$. Thus, in order to show that T_{n-1} exists, it suffices to show that this representation is reducible. Since $\text{Ind}_{\tilde{J}}^{\tilde{K}}(\tau_{n-1}^{o,*})$ is dual to $\text{Ind}_{\tilde{J}}^{\tilde{K}}(\tau_{n-1}^o)$ it suffices to show that $\text{Ind}_{\tilde{J}}^{\tilde{K}}(\tau_{n-1}^o)$ is reducible. Since $\tau_{n-1}^o \subseteq \tau_{n-2}^o \subseteq S(X^*)$, it follows from the Frobenius reciprocity that τ_{n-2}^o is a summand of $\text{Ind}_{\tilde{J}}^{\tilde{K}}(\tau_{n-1}^o)$. This is a proper summand since the dimensions of τ_{n-2}^o and $\text{Ind}_{\tilde{J}}^{\tilde{K}}(\tau_{n-1}^o)$ are $(q^2-1)/2$ and $(q^4-1)/2$, respectively. This shows that T_{n-1} exists. Since $w'_{\alpha'_{n-1}}(1)$ commutes with $x_{\pm\alpha_n}(u)$, $T_{n-1}(w'_{\alpha'_{n-1}}(1))$ must be a scalar in $\text{End}(\tau_{n-1}^o)$. We normalize T_{n-1} so that this scalar is $q^{-\frac{1}{2}}$.

Lemma 11. *Let v_1, \dots, v_l be a basis of the \tilde{K}_{n-1} -type $\tau_{n-1}^o \subseteq S(X^*)$, and v_1^*, \dots, v_l^* the dual basis in $\tau_{n-1}^{o,*}$. Then $v = \sum_{i=1}^l v_i \otimes v_i^* \in (\omega_\psi^o \otimes \tau_{n-1}^{o,*})^{\tilde{J}}$ and $\omega_\psi(T_i)v = qv$ if $i \neq 0$ and $\omega_\psi(T_{n-1})v = q^2v$.*

Proof. Let $\langle \cdot, \cdot \rangle$ denote the invariant inner product on $S(X^*)$, as before. Let $\langle \cdot, \cdot \rangle^*$ denote an \tilde{J} -invariant inner product on $\tau_{n-1}^{o,*}$. In order to simplify expressions, assume that v_1, \dots, v_l and v_1^*, \dots, v_l^* are orthonormal bases. Let (\cdot, \cdot) denote the tensor product of the two inner products. As in Lemma 4, the idea is to compute $(\omega_\psi(T_i)v, v)$. Assume first that $i = n-1$. Let J^+ be the pro- p radical of \tilde{J} . Since $\tilde{J}w_{\alpha'_{n-1}}(1)\tilde{J} = J^+w_{\alpha'_{n-1}}(1)\tilde{J}$ and $[\tilde{J}w_{\alpha'_{n-1}}(1)\tilde{J} : \tilde{J}] = q^3$ there exist elements x_j in J^+ such that

$$\tilde{J}w_{\alpha'_{n-1}}(1)\tilde{J} = \bigcup_{j=1}^{q^3} x_j w'_{\alpha'_{n-1}}(1)\tilde{J}.$$

Then, using the definition of $\omega_\psi(T_{n-1})$,

$$\omega_\psi(T_{n-1})(v_i \otimes v_i^*) = \sum_{j=1}^{q^3} x_j w'_{\alpha'_{n-1}}(1) v_i \otimes T_{n-1}((x_j w'_{\alpha'_{n-1}}(1)) v_i^*.$$

Since $x_j \in J^+$ and J^+ is in the kernel of $\tau_{n-1}^{o,*}$,

$$T_{n-1}(x_j w'_{\alpha'_{n-1}}(1)) = \tau_{n-1}^{o,*}(x_j) \circ T_{n-1}(w'_{\alpha'_{n-1}}(1)) = T_{n-1}(w'_{\alpha'_{n-1}}(1)).$$

Since $T_{n-1}(w'_{\alpha'_{n-1}}(1))$ is the scalar $q^{-\frac{1}{2}}$,

$$\omega_\psi(T_{n-1})(v_i \otimes v_i^*) = q^{-\frac{1}{2}} \sum_{j=1}^{q^3} x_j w'_{\alpha'_{n-1}}(1) v_i \otimes v_i^*.$$

Using orthonormality of the basis v_1^*, \dots, v_l^* , it follows that $(\omega_\psi(T_{n-1})v, v)$ is equal to

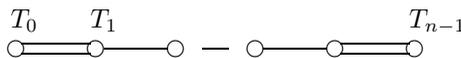
$$\begin{aligned} q^{-\frac{1}{2}} \sum_{i=1}^l \sum_{j=1}^{q^3} \langle x_j w'_{\alpha'_{n-1}}(1) v_i, v_i \rangle &= q^{-\frac{1}{2}} \sum_{i=1}^l \sum_{j=1}^{q^3} \langle w'_{\alpha'_{n-1}}(1) v_i, x_j^{-1} v_i \rangle \\ &= q^{\frac{5}{2}} \sum_{i=1}^l \langle w'_{\alpha'_{n-1}}(1) v_i, v_i \rangle. \end{aligned}$$

We remind the reader that $w'_{\alpha'_{n-1}}(1)$ is the Fourier transform in the variable x_{n-1} . Moreover, since $v_i = \varphi' \otimes v'_i$ where φ' is the characteristic function of the lattice spanned by $\mathbf{e}_1^*, \dots, \mathbf{e}_{n-1}^*$, it follows that $\langle w'_{\alpha'_{n-1}}(1) v_i, v_i \rangle = q^{-\frac{1}{2}} \langle v_i, v_i \rangle$ by Lemma 2, part (2). Thus $(\omega_\psi(T_{n-1})v, v) = q^2(v, v)$, as desired. The case $i \neq n - 1$ is simpler and left to the reader. □

Lemma 12. *The operator T_i satisfies the quadratic relations $(T_i - q)(T_i + 1) = 0$ if $i \neq n - 1$ and $(T_{n-1} - q^2)(T_{n-1} + 1) = 0$. In particular, all T_i are invertible.*

Proof. We use Lemma 3. The case $i \neq n - 1$ is analogous to Lemma 5, so we will omit it. Recall that T_{n-1} is an endomorphism of the induced representation $\text{Ind}_J^{\tilde{K}}(\tau_{n-1}^{o,*})$. The dual representation $\text{Ind}_J^{\tilde{K}}(\tau_{n-1}^o)$ contains two summands. One of them is τ_{n-2}^o . Its dimension is $d_1 = (q^2 - 1)/2$. Thus the other summand has the dimension $d_2 = q^2(q^2 - 1)/2$. By Lemma 11, T_{n-1} acts on $(\tau_{n-2}^o \otimes \tau_{n-1}^{o,*})^{\tilde{J}}$ by $\lambda_1 = q^2$, so it must act on the other summand by $\lambda_2 = -1$, by Lemma 3, (2). Finally, invertibility of T_i follows from the quadratic relation satisfied by T_i . The lemma is proved. □

Theorem 13. *Assume that the conductor of ψ is 1. The Hecke algebra H_ψ^- , as an abstract algebra, is generated by elements T_0, T_1, \dots, T_{n-1} satisfying quadratic relations $(T_i - q)(T_i + 1) = 0$, if $i \neq n - 1$, $(T_{n-1} - q^2)(T_{n-1} + 1) = 0$, and satisfying the braid relations corresponding to the following Coxeter diagram:*



Proof. We need to prove the braid relations. We shall use the weighted length function l_3 on W'_a . Proposition 1 implies that $Jw_1J \cdot Jw_2J = Jw_1w_2J$ if $l_3(w_1) + l_3(w_2) = l_3(w_1w_2)$. Since $s_{n-2}s'_{n-1}s_{n-2}s'_{n-1} = s'_{n-1}s_{n-2}s'_{n-1}s_{n-2}$ (and both are the shortest expressions in W'_a) it follows that

$$T_{n-2}T_{n-1}T_{n-2}T_{n-1} \text{ and } T_{n-1}T_{n-2}T_{n-1}T_{n-2}$$

are supported on one (the same) \tilde{J} -double coset. Since both operators act by the same positive scalar on $(\omega_\psi^o \otimes \tau_{n-1}^{o,*})^{\tilde{J}}$, they have to be equal. This proves one braid relation. Others are proved in the same manner. \square

If the conductor of ψ is 0, then the role of K_{n-1} is played by K_1 . We define $J = \bigcap_{i=1}^n K_i$ and $H_\psi^- = H(\tau_1^o)$ where we view τ_1^o as a representation of \tilde{J} , by restriction. An obvious analogue of Theorem 13 holds.

12. BERNSTEIN'S COMPONENT \mathcal{G}_ψ^-

Let $P_o = M_oU_o$ be the parabolic subgroup of $\mathrm{Sp}(W)$ defined as the stabilizer of the partial flag $X_1 \subseteq X_2 \subseteq \dots \subseteq X_{n-1}$. Recall that W_{n-1} is the symplectic subspace spanned by \mathbf{e}_n and \mathbf{e}_n^* , and Z_{n-1}^* is the line spanned by \mathbf{e}_n^* . Let $\omega_{\psi,n-1}$ denote the Weil representation of $\widetilde{\mathrm{Sp}}(W_{n-1})_8$ acting on $S(Z_{n-1}^*)$. By decomposing $S(Z_{n-1}^*) = S(Z_{n-1}^*)^e \oplus S(Z_{n-1}^*)^o$ into a sum of spaces of even and odd functions, we have a decomposition $\omega_{\psi,n-1} = \omega_{\psi,n-1}^e \oplus \omega_{\psi,n-1}^o$ into two irreducible representations. The odd Weil representation $\omega_{\psi,n-1}^o$ is supercuspidal.

Let $\tilde{P}_o = \tilde{M}_oU_o$ be the inverse image of P_o in $\widetilde{\mathrm{Sp}}(W)_8$. Any element $m \in \tilde{M}_o$ can be uniquely written as

$$m = h'_1(t_1) \cdot \dots \cdot h'_{n-1}(t_{n-1}) \cdot m'$$

where m' is in $\widetilde{\mathrm{Sp}}(W_{n-1})_8$. Thus, any $n - 1$ -tuple of characters $\chi = (\chi_1, \dots, \chi_{n-1})$ defines a genuine representation $\chi \otimes \omega_{\psi,n-1}^o$ of \tilde{M}_o .

Let $I(\chi \otimes \omega_{\psi,n-1}^o)$ denote the normalized principal series representation, i.e., we induce the representation $\chi \otimes \omega_{\psi,n-1}^o$ multiplied by the modular character $\rho_{U_o}^{\frac{1}{2}}$. If (π, V) is a smooth representation of $\widetilde{\mathrm{Sp}}(W)_8$, let (π_{U_o}, V_{U_o}) denote the normalized Jacquet module, i.e., the action of \tilde{M}_o on V_{U_o} is multiplied by the inverse of the modular character. With these conventions the Frobenius reciprocity takes the form

$$\mathrm{Hom}_{\widetilde{\mathrm{Sp}}(W)_8}(V, I(\chi \otimes \omega_{\psi,n-1}^o)) \cong \mathrm{Hom}_{\tilde{M}_o}(V_{U_o}, \chi \otimes \omega_{\psi,n-1}^o).$$

Let \mathcal{G}_ψ^- be the component of the category of smooth representations of $\widetilde{\mathrm{Sp}}(W)_8$ whose irreducible representations are submodules of $I(\chi \otimes \omega_{\psi,n-1}^o)$ where χ is unramified. The odd Weil representation ω_ψ^o belongs to \mathcal{G}_ψ^- . Indeed, the map $\ell : S(X^*)^o \rightarrow S(Z_{n-1}^*)^o$ given by restriction of functions from X^* to Z_{n-1}^* , via the Frobenius reciprocity, gives an embedding of ω_ψ^o into $I(\chi_o \otimes \omega_{\psi,n-1}^o)$, where χ_o is an unramified character.

Let $\tilde{J}_{M_o} = \tilde{J} \cap \tilde{M}_o$, and consider τ_{n-1}^o as a representation of \tilde{J}_{M_o} by restriction. By Proposition 3.5.2 in [Be] the natural projection from V to V_{U_o} gives a surjection

$$r : (V \otimes \tau_{n-1}^{o,*})^{\tilde{J}} \rightarrow (V_{U_o} \otimes \tau_{n-1}^{o,*})^{\tilde{J}_{M_o}}.$$

For any $w \in W'_a$, let $T_w = T_{i_1} \cdots T_{i_l}$ where $w = s_{i_1} \cdots s_{i_l}$, is a shortest product of simple reflections. Note that T_w is invertible since all T_i are. Let

$\lambda = (\lambda_1, \dots, \lambda_{n-1}) \in \mathbb{Z}^{n-1}$ be a translation in W'_a such that $\lambda_1 \geq \dots \geq \lambda_{n-1} \geq 0$, and let

$$\hat{T}_\lambda = q^{-\frac{l_2(\lambda)}{2}} T_\lambda$$

where l_2 is the weighted length function on W'_a such that $l_2(s_{n-1}) = 2$ and 1 for other simple reflections. Let $t_\lambda = h'_1(\varpi^{\lambda_1}) \cdot \dots \cdot h'_{n-1}(\varpi^{\lambda_{n-1}})$. Note that, since $T_0(w'_{\alpha'_{n-1}}(1)) = q^{-\frac{1}{2}}$, the value of \hat{T}_λ at t_λ is $q^{-\frac{l_3(\lambda)}{2}}$. Arguing as in Section 9

$$r \circ \pi(\hat{T}_\lambda) = \pi_{U_o}(t_\lambda) \circ r$$

and, since T_λ is invertible, the map r is an isomorphism. In particular, if V is an irreducible representation such that $(V \otimes \tau_{n-1}^{o,*})^{\tilde{J}} \neq 0$, then τ_{n-1}^o is an \tilde{J}_{M_o} -type of V_{U_o} . Since τ_{n-1}^o is the minimal type of $\omega_{\psi,n-1}^o$, the Frobenius reciprocity implies that V is a submodule of $I(\chi \otimes \omega_{\psi,n-1}^o)$ for some unramified character χ . Conversely, if V is a submodule of $I(\chi \otimes \omega_{\psi,n-1}^o)$ for some unramified character χ then $(V_{U_o} \otimes \tau_{n-1}^{o,*})^{\tilde{J}_{M_o}} \neq 0$ and, therefore, $(V \otimes \tau_{n-1}^{o,*})^{\tilde{J}} \neq 0$. Thus irreducible representations of H_ψ^- correspond to irreducible constituents of principal series representations $I(\chi \otimes \omega_{\psi,n-1}^o)$. This proves (see (3.12) [BK]):

Corollary 14. *The functor that attaches to every smooth representation V of $\widetilde{\text{Sp}}(W)_8$ the Hecke algebra module $(V \otimes \tau_{n-1}^{o,*})^{\tilde{J}}$ is an equivalence of Bernstein's component \mathcal{G}_ψ^- and the category of H_ψ^- -modules.*

Operators \hat{T}_λ form a commuting family of operators. Thus any finite dimensional representation F of H_ψ^- can be decomposed into a sum of generalized eigenspaces. More precisely, for any $\mu = (\mu_1, \dots, \mu_{n-1}) \in \mathbb{C}^{n-1}$, define F_μ to be the subspace of F consisting of all $v \in F$ such that

$$\pi(\hat{T}_\lambda - q^{-\sum_{i=1}^{n-1} \lambda_i \mu_i})^{\dim(F)}(v) = 0.$$

Then $F = \bigoplus F_\mu$ and any μ such that $F_\mu \neq 0$ is called an exponent of F . The representation F is square integrable if

$$\Re(\mu_1) > 0, \Re(\mu_1) + \Re(\mu_2) > 0, \dots, \Re(\mu_1) + \dots + \Re(\mu_{n-1}) > 0$$

for all exponents μ . The representation F is tempered if the exponents satisfy the inequalities where all $>$ have been replaced by \geq . These definitions agree with the usual definitions of square integrable and tempered representations of $\widetilde{\text{Sp}}(W)_8$.

13. HECKE ALGEBRA OF $\text{SO}(V^-)$

The affine Hecke algebra of $\text{SO}(V^-)$ is described in terms of the affine Weyl group W'_a (see [Ti]). Recall that W'_a is generated by reflections s_0, s_1, \dots, s_{n-1} satisfying the braid relations given by the following Coxeter diagram:



Here the vertex marked s_0 is special. The quotient of the corresponding special maximal compact subgroup by its pro- p radical is isomorphic to the quasi-split group $\text{O}_{2n}^{qs}(f)$. Let I be the Iwahori subgroup of $\text{SO}(V^-)$. It is the pointwise stabilizer of a chamber in the Bruhat-Tits building of $\text{SO}(V^-)$. (In particular I is

disconnected.) Then $[IwI : I] = q^{l_2(w)}$ for every w in W'_a , where l_2 is the weighted length function on W_a such that $l_2(s_{n-1}) = 2$ and it is 1 on other simple reflections. Let H^- denote the Iwahori Hecke algebra of $\mathrm{SO}(V^-)$. For any $w \in W_a$ let $T_w \in H^-$ be the characteristic function of the double coset parameterized by w . For simplicity, let T_i denote T_{s_i} . Then T_i satisfy quadratic relations $(T_i - q)(T_i + 1) = 0$, if $i \neq n - 1$, $(T_{n-1} - q^2)(T_{n-1} + 1) = 0$, and the braid relations given by the above Coxeter diagram. In fact, as an abstract algebra, H^- is generated by T_0, \dots, T_{n-1} modulo these quadratic and braid relations.

It is worth mentioning that, in contrast with the case of $\mathrm{SO}(V^+)$, the affine Hecke algebra of $\mathrm{SO}(V^-)$ is isomorphic to the affine Hecke algebra of its simply connected form. There is also an additional difference worth mentioning. Let $\mathbb{N} : \mathrm{SO}(V^-) \rightarrow k^\times / (k^\times)^2$ be the spinor norm. Let χ_{nr} be the unique non-trivial, unramified, quadratic character of k^\times . Then $\chi_{nr} \circ \mathbb{N}$ and the trivial representation of $\mathrm{SO}(V^-)$ belong to two different components of the category of smooth representations of $\mathrm{SO}(V^-)$. Indeed, the restriction of the quadratic character $\chi_{nr} \circ \mathbb{N}$ to the special maximal compact subgroup is the determinant character of the finite group $\mathrm{O}_{2n}^{qs}(\mathfrak{f})$. Thus, the representation $\chi_{nr} \circ \mathbb{N}$ has no non-zero I -fixed vectors. However, each of the two components is equivalent to the category of H^- -modules.

Since H^- and H_{ψ}^- are given by “identical” combinatorial data (see Theorem 13), we have obtained the following:

Theorem 15. *The algebra H_{ψ}^- is isomorphic, via an explicit isomorphism, to the algebra H^- .*

Let $\lambda = (\lambda_1, \dots, \lambda_{n-1}) \in \mathbb{Z}^{n-1}$ be a translation in W'_a such that $\lambda_1 \geq \dots \geq \lambda_{n-1} \geq 0$. Define $\hat{T}_{\lambda} = q^{-\frac{l_2(\lambda)}{2}} T_{\lambda}$. The operators \hat{T}_{λ} form a commuting family of operators and, as we discussed in the previous section, any finite dimensional representation H^- can be decomposed in a sum of generalized eigenspaces, parameterized by exponents $\mu = (\mu_1, \dots, \mu_{n-1}) \in \mathbb{C}^{n-1}$. Again, the corresponding representation of $\mathrm{SO}(V^-)$ is square integrable if and only if

$$\Re(\mu_1) > 0, \Re(\mu_1) + \Re(\mu_2) > 0, \dots, \Re(\mu_1) + \dots + \Re(\mu_{n-1}) > 0$$

for all exponents μ , and it is tempered if the exponents satisfy the inequalities where all $>$ have been replaced by \geq .

The trivial representation of $\mathrm{SO}(V^-)$ corresponds to the one-dimensional representation of H^- where T_w acts by $q^{l_2(w)}$. Thus by Lemma 11, and the isomorphism $H^- \cong H_{\psi}^-$, the trivial representation of $\mathrm{SO}(V^-)$ corresponds to the odd Weil representation ω_{ψ}^o .

We summarize what we have proven so far:

Corollary 16. *In the equivalence of categories \mathcal{G}_{ψ}^- and \mathcal{S}_0^- , given by the isomorphism of the respective Hecke algebras, square integrable representations are mapped to square integrable representations.*

14. STEINBERG REPRESENTATIONS

Let δ_{ψ}^{ϵ} be a representation of $\widetilde{\mathrm{Sp}}(W)_8$ which corresponds to the one-dimensional representation δ of H_{ψ}^{ϵ} such that $\delta(T_i) = -1$ for all i . Let v be a vector in the unique type in δ_{ψ}^{ϵ} isomorphic to τ_0 or τ_{n-1}^o if $\epsilon = +$ or $\epsilon = -$, respectively. Let v^* be a vector in the dual type in the dual representation. Let $c_{v^*v}(g)$ be the corresponding matrix coefficient. Choose v^* so that $c_{v^*v}(1) = 1$.

Proposition 17. *We normalize the Haar measure on $\widetilde{\text{Sp}}(W)_8$ so that the measure of \tilde{I} is 1 if $\epsilon = +$ and so that the measure of \tilde{J} is 1 if $\epsilon = -$. Then*

- (1) *Assume that $\epsilon = +$. Let l_0 be the weighted length function on W_a such that $l_0(s_0) = 0$. Then*

$$\int |c_{v^*v}(g)|^2 dg = \sum_{w \in W_a} \frac{1}{q^{l_0(w)}}.$$

- (2) *Assume that $\epsilon = -$. Let l_2 be the weighted length function on W'_a such that $l_2(s_{n-1}) = 2$. Then*

$$\int |c_{v^*v}(g)|^2 dg = \frac{1}{d(\tau_{n-1}^o)} \sum_{w \in W'_a} \frac{1}{q^{l_2(w)}},$$

where $d(\tau_{n-1}^o) = (q - 1)/2$ is the dimension of the odd Weil representation τ_{n-1}^o of \tilde{J} .

Proof. For (2): Let J^+ be the pro- p radical of \tilde{J} . We shall break up the integral over J^+ -cosets. It is easy to see that the matrix coefficient is supported on $\tilde{J}W'_a\tilde{J}$. Let g_1, \dots, g_m be representatives of all J^+ -cosets in \tilde{J} . Let $w \in W'_a$. Then

$$\tilde{J}w\tilde{J} = \bigcup_{j=1}^{q^{l_3(w)}} \bigcup_{i=1}^m x_j g_i w J^+$$

for some x_j in J^+ . Then

$$\begin{aligned} [\tilde{J} : J^+] \int |c_{v^*v}(g)|^2 dg &= \sum_{w \in W_a} \sum_{j=1}^{q^{l_3(w)}} \sum_{i=1}^m |c_{v^*v}(x_j g_i w)|^2 \\ &= \sum_{w \in W_a} q^{l_3(w)} \sum_{i=1}^m |c_{g_i^{-1}v^*v}(w)|^2. \end{aligned}$$

Claim: $|c_{g_i^{-1}v^*v}(w)|^2 = q^{-l_2(w)-l_3(w)} |c_{g_i^{-1}v^*v}(1)|^2$.

Proof of claim: Let v_1, \dots, v_l be a basis in $\tau_{n-1}^o \subseteq \delta_\psi^-$. Let v_1^*, \dots, v_l^* be the dual basis in $\tau_{n-1}^{o,*}$. Assume, as we can, that $v = v_1$ and $v^* = v_1^*$. Let $w = s_{i_1} \dots s_{i_j}$ be a shortest expression in terms of simple reflections, and let $T_w = T_{s_{i_1}} \dots T_{s_{i_j}}$. Then

$$\delta_\psi^-(T_w) \left(\sum_{i=1}^l v_i \otimes v_i^* \right) = \pm \sum_{i=1}^l v_i \otimes v_i^*.$$

On the other hand, pick a representative of w in $\widetilde{\text{Sp}}(W)_8$ so that $T_w(w)$ is the scalar $q^{\frac{l_2(w)-l_3(w)}{2}}$. Following the calculation in Lemma 11,

$$\delta_\psi^-(T_w) \left(\sum_{i=1}^l v_i \otimes v_i^* \right) = q^{\frac{l_2(w)-l_3(w)}{2}} \sum_{i=1}^l \sum_{j=1}^{q^{l_3(w)}} x_j w v_i \otimes v_i^*.$$

Thus

$$\pm \sum_{i=1}^l v_i \otimes v_i^* = q^{\frac{l_2(w)-l_3(w)}{2}} \sum_{i=1}^l \sum_{j=1}^{q^{l_3(w)}} x_j w v_i \otimes v_i^*.$$

Evaluating $g_i^{-1}v^* \otimes v$ on both sides, using that x_j^{-1} fix $g_i^{-1}v^*$, and squaring both sides proves the claim.

The claim implies that

$$\int |c_{v^*v}(g)|^2 dg = \sum_{w \in W_a} q^{-l_2(w)} \sum_{i=1}^m \frac{|c_{g_i^{-1}v^*v}(1)|^2}{[\tilde{J} : J^+]}$$

The inner sum is equal to $1/d(\tau_{n-1}^o)$ by orthogonality relations of matrix coefficients for the finite group \tilde{J}/J^+ . This proves (2). Part (1) is proved analogously. \square

Corollary 18. *Normalize the Haar measure on $\widetilde{\text{Sp}}(W)_8$ so that the measure of \tilde{I} is 1. Then the formal degrees of δ_ψ^+ and δ_ψ^- coincide and are equal to the formal degree of the Steinberg representation of $\text{SO}(V^+)$ with respect to the Haar measure normalized so that the measure of the Iwahori subgroup is 1. These numbers are equal to the formal degree of the Steinberg representation of $\text{SO}(V^-)$ with respect to the Haar measure normalized so that the measure of the Iwahori subgroup is $2 \cdot \frac{q+1}{q-1}$.*

Proof. Let d be the formal degree of the Steinberg representation of $\text{SO}(V^+)$. Then (see [Bo])

$$\frac{1}{d} = \sum_{w \in W_a} \frac{1}{q^{l_0(w)}}$$

and hence d is equal to the formal degree of δ_ψ^+ , by Proposition 17. On the other hand, using Proposition 17 and formulas of Macdonald ([Mc], page 173), one checks that the formal degrees of δ_ψ^+ and δ_ψ^- are both equal to

$$\prod_{i=0}^{n-1} \frac{(q-1)(q^{n+i}-1)}{(q^{i+1}-1)(q^{i+1}+1)(q^i+1)}$$

Finally, $\sum_{w \in W_a} 1/q^{l_2(w)}$ is the inverse of the formal degree of the Steinberg representation of $\text{SO}(V^-)$ with respect to the Haar measure normalized so that the measure of the Iwahori subgroup is 1. Thus, if we change the Haar measure by the factor $2 \cdot \frac{q+1}{q-1}$ then the formal degree is equal to the formal degree of δ_ψ^- , by Proposition 17. \square

15. PARAMETERS

Let \mathcal{W}_k be the Weil group of the local field k , and let \mathcal{I} be the inertia subgroup of \mathcal{W}_k . Let Fr denote the Frobenius automorphism. Recall that a Langlands parameter for $\text{SO}(V^+)$ (or for $\text{SO}(V^-)$ with some restrictions) is a $\text{Sp}_{2n}(\mathbb{C})$ -conjugacy class of homomorphisms

$$\varphi : \mathcal{W}_k \times \text{SL}_2(\mathbb{C}) \rightarrow \text{Sp}_{2n}(\mathbb{C})$$

such that a finite index subgroup of \mathcal{I} is in the kernel of φ and $\varphi(\mathcal{W}_k)$ consists of semi-simple elements. A parameter φ is unramified if it is trivial on the inertia subgroup \mathcal{I} . It is tempered if $\varphi(\text{Fr})$, when acting on \mathbb{C}^{2n} , has eigenvalues on the unit circle. If φ is unramified define a semi-simple element

$$s = \varphi \left(\text{Fr} \times \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix} \right)$$

and a unipotent element

$$u = \varphi \left(1 \times \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right).$$

These two elements satisfy $sus^{-1} = u^q$, and the conjugacy class of φ is determined by the conjugacy class of the pair (s, u) . The work of Kazhdan and Lusztig [KL] and of Lusztig [Lu] (Corollary 6.5) gives a parameterization of irreducible representations in \mathcal{S}_0^+ and in \mathcal{S}_0^- , respectively, by unramified Langlands parameters.

Proposition 19. *The conjugacy class of a tempered, unramified homomorphism φ is determined by the conjugacy class of the semi simple element s .*

Proof. For every non-negative integer i , let V_i denote the irreducible representation of $SL_2(\mathbb{C})$ of dimension $i + 1$. This representation is symplectic if i is odd and orthogonal if i is even. Let S be a finite set of non-negative integers such that, under the action of $\varphi(SL_2(\mathbb{C}))$,

$$\mathbb{C}^{2n} = \bigoplus_{i \in S} W_i \otimes V_i$$

where $W_i = \text{Hom}_{SL_2(\mathbb{C})}(V_i, \mathbb{C}^{2n})$. The restriction of the symplectic form on \mathbb{C}^{2n} to every isotypic subspace $W_i \otimes V_i$ is nondegenerate and decomposes as a tensor product of a bilinear form on W_i and the unique $SL_2(\mathbb{C})$ -invariant form on V_i . The centralizer of $\varphi(SL_2(\mathbb{C}))$ in $Sp_{2n}(\mathbb{C})$ is the product of classical groups $G(W_i)$. Hence $\varphi(\text{Fr}) = (s_i)$ where s_i is a semi-simple element in $G(W_i)$. We claim that s determines the decomposition of \mathbb{C}^{2n} into irreducible representations given above, and the class of s_i for all i . Indeed, let j be a non-negative integer such that the maximal absolute value of the eigenvalues of s is $q^{j/2}$. Let z_1, \dots, z_m be all eigenvalues of s of absolute value $q^{j/2}$. Then j is the maximal element in S and $z_1 q^{-j/2}, \dots, z_m q^{-j/2}$ are the eigenvalues of s_j on W_j . We now remove the eigenvalues of s on $W_j \otimes V_j$ and continue, in the same fashion using the remaining eigenvalues of s , to determine the next element in S . Since the conjugacy class of $\varphi : SL_2(\mathbb{C}) \rightarrow Sp_{2n}(\mathbb{C})$ is completely determined by the decomposition of \mathbb{C}^{2n} into irreducible representations for $SL_2(\mathbb{C})$, and semi-simple conjugacy classes in the classical groups $G(W_i)$ are determined by their eigenvalues on W_i , the proposition follows. \square

We now describe how to attach the parameter φ to a representation π in these two components in three steps:

- (1) Attach s to π .
- (2) Attach φ to π if π is tempered.
- (3) Attach φ to π if π is a Langlands quotient.

(1) Assume first that π is an irreducible representation in \mathcal{S}_0^+ . Recall, from Section 2, the maximal split torus $T_n = (k^*)^n$ of $SO(V^+)$. Let $\chi = (\chi_1, \dots, \chi_n)$ be an n -tuple of unramified characters. It defines an unramified character χ of T_n by $\chi(t) = \chi_1(t_1) \cdots \chi_n(t_n)$ for every $t = (t_1, \dots, t_n) \in T_n$. If π is a subquotient of the principal series obtained by inducing χ then s , acting on the standard representation of $Sp_{2n}(\mathbb{C})$, has eigenvalues $\chi_1(\varpi)^{\pm 1}, \dots, \chi_n(\varpi)^{\pm 1}$. Now assume that π is an irreducible representation in \mathcal{S}_0^- . Recall, from Section 2, the maximal split torus $T_{n-1} = (k^*)^{n-1}$ of $SO(V^-)$. Its centralizer is $T_{n-1} \times (D^*/k^*)$ where D is a quaternion algebra over k . This is a Levi factor of a minimal parabolic subgroup of $SO(V^-)$. An unramified character of this Levi factor is, by definition, trivial on

the compact factor (D^*/k^*) and is thus given by an $n - 1$ -tuple $\chi = (\chi_1, \dots, \chi_{n-1})$ of unramified characters. In this case s has eigenvalues $\chi_1(\varpi)^{\pm 1}, \dots, \chi_{n-1}(\varpi)^{\pm 1}$ and $q^{\pm \frac{1}{2}}$.

(2) If π is tempered then Proposition 19 shows how φ is determined by s .

(3) A general representation of $\text{SO}(V)$ (V denotes V^+ or V^- here) can be described as a Langlands quotient as follows. As in Section 2, let Q be the maximal parabolic subgroup of $\text{SO}(V)$ defined as the stabilizer of the l -dimensional totally isotropic subspace Y_l , with the Levi factor $L = \text{GL}(Y_l) \times \text{SO}(V_l)$. Let σ be a Langlands quotient for $\text{GL}(Y_l)$ obtained by inducing $\tau_1^{s_1} \otimes \dots \otimes \tau_r^{s_r}$ from a parabolic subgroup stabilizing a partial flag

$$Y_{l_1} \subseteq Y_{l_2} \subseteq \dots \subseteq Y_{l_r} = Y_l$$

where $\tau_i^{s_i}$ is a twist by $|\det_{Y_{l_i}/Y_{l_{i-1}}}|^{s_i}$ of a unitary discrete series representation τ_i of $\text{GL}(Y_{l_i}/Y_{l_{i-1}})$, and s_i are complex numbers satisfying $\Re(s_1) \geq \Re(s_2) \dots \geq \Re(s_r) > 0$. Let τ be an irreducible tempered representation of $\text{SO}(V_l)$. Let $I_Q(\sigma \otimes \tau)$ be the representation of $\text{SO}(V)$ obtained by inducing $\sigma \otimes \tau$ from Q . The representation $I_Q(\sigma \otimes \tau)$ has a unique irreducible quotient denoted by $J_Q(\sigma \otimes \tau)$. Every irreducible representation π is isomorphic to $J_Q(\sigma \otimes \tau)$ for some triple (Q, σ, τ) . If π has a non-zero Iwahori-fixed vector, then τ_i is a Steinberg representation for every i , and τ has a non-zero Iwahori-fixed vector. The Langlands parameter of π is

$$\varphi = \varphi_\sigma + \varphi_\tau + \varphi_\sigma^*$$

where φ_τ is the Langlands parameter of τ and φ_σ is the Langlands parameter of σ .

By our equivalence of categories, we obtain a parameterization of irreducible representations in \mathcal{G}_ψ^+ and \mathcal{G}_ψ^- . Since we shall need it in the next section, we shall now sketch how to attach unramified parameters to representations in these two components. Again, the semi-simple class s is easy to describe. If π is an irreducible representation in \mathcal{G}_ψ^+ , then π is a subquotient of an induced representation $I(\chi)$ where $\chi = (\chi_1, \dots, \chi_n)$ is an n -tuple of unramified characters. Then s has eigenvalues $\chi_1(\varpi)^{\pm 1}, \dots, \chi_n(\varpi)^{\pm 1}$. If π is an irreducible representation in \mathcal{G}_ψ^- , then π is a subquotient of an induced representation $I(\chi \otimes \omega_{\psi, n-1}^o)$ where $\chi = (\chi_1, \dots, \chi_{n-1})$ is an $n - 1$ -tuple of unramified characters. In this case s has eigenvalues $\chi_1(\varpi)^{\pm 1}, \dots, \chi_{n-1}(\varpi)^{\pm 1}$ and $q^{\pm \frac{1}{2}}$. If π is tempered, then s determines u , just as in the case of $\text{SO}(V)$. For all other representations we need a description in terms of Langlands quotients.

As in Section 2, let P be the maximal parabolic subgroup of $\text{Sp}(W)$ defined as the stabilizer of the l -dimensional totally isotropic subspace X_l . Let $\tilde{P} = \tilde{M}U_P$ be the inverse image of P in $\tilde{\text{Sp}}(W)_8$ where $\tilde{M} = \text{GL}(X_l) \times \tilde{\text{Sp}}(W_l)_8$ is the Levi factor. Let σ be a Langlands quotient for $\text{GL}(X_l)$, as above, and let τ be an irreducible tempered representation of $\tilde{\text{Sp}}(W_l)_8$. Let $I_P(\sigma \otimes \tau)$ be the representation of $\tilde{\text{Sp}}(W)_8$ obtained by inducing $\sigma \otimes \tau$ from \tilde{P} . The representation $I_P(\sigma \otimes \tau)$ has a unique irreducible quotient denoted by $J_P(\sigma \otimes \tau)$. Every irreducible representation π is isomorphic to $J_P(\sigma \otimes \tau)$ for some triple (P, σ, τ) . If π is in \mathcal{G}_ψ^+ then τ is in the Bernstein component of $\tilde{\text{Sp}}(W_l)_8$ that contains $\omega_{\psi, l}^e \cong (\omega_\psi^e)_{U_P}$. Similarly, if π is in \mathcal{G}_ψ^- then τ is in the Bernstein component of $\tilde{\text{Sp}}(W_l)_8$ that contains $\omega_{\psi, l}^o \cong (\omega_\psi^o)_{U_P}$. (In particular $l < n$.) In any case, the Langlands parameter of π is $\varphi = \varphi_\sigma + \varphi_\tau + \varphi_\sigma^*$.

16. THETA CORRESPONDENCE

The goal of this section is to show that the theta correspondence preserves unramified Langlands parameters. Let (V, Q) be an orthogonal space of dimension $2n + 1$. Then $(\mathrm{Sp}(W), \mathrm{O}(V))$ is a dual pair in $\mathrm{Sp}(W \otimes V)$. Note that $X^* \otimes V$ is a maximal totally isotropic subspace of $W \otimes V$. Let $S(X^* \otimes V)$ be the space of locally constant, compactly supported functions on $X^* \otimes V$. The Weil representation, defined by means of the additive character ψ , is a representation of $\mathrm{Sp}(W \otimes V)'' \subseteq \mathrm{GL}(S(X^* \otimes V))$, a central extension of $\mathrm{Sp}(W \otimes V)$ by \mathbb{C}^\times . We consider $\mathrm{O}(V)$ as a subgroup of $\mathrm{Sp}(W \otimes V)''$ so that for any $g \in \mathrm{O}(V)$ and $f \in S(X^* \otimes V)$

$$[g \cdot f](x \otimes v) = f(x \otimes g^{-1}v).$$

Let $\mathrm{Sp}(W)'' \subseteq \mathrm{Sp}(W \otimes V)''$ be the inverse image of $\mathrm{Sp}(W)$. Since $\mathrm{Sp}(W)$ is perfect, the inclusion of $\mathrm{Sp}(W)$ into $\mathrm{Sp}(W \otimes V)$ lifts uniquely to an isomorphism i of $\mathrm{Sp}(W)'$ and $\mathrm{Sp}(W)''$ such that $i(\zeta) = \zeta$ for all $\zeta \in \mathbb{C}^\times$. Obviously, the map i is determined by its values on $x_\alpha(u)$ and $x_{-\alpha}(u)$ for all simple roots α . If α is short, then these elements generate a subgroup of $\mathrm{Sp}(W)'$ isomorphic to $\mathrm{SL}_2(k)$. For any g in this group $i(g)$ is necessarily given by

$$[i(g) \cdot f](x \otimes v) = f(g^{-1}x \otimes v).$$

Now assume that α is a long root. For ease of notation assume that $\dim(W) = 2$. Then $\mathrm{Sp}''(W)$ contains operators defined by

$$\begin{aligned} [x''_\alpha(u)f](v) &= \psi(uQ(v))f(v), \\ [h''_\alpha(t)f](v) &= |t|^{\frac{2n+1}{2}}f(tv), \end{aligned}$$

and

$$[w''_\alpha(-1/t)f](v) = [w''_{-\alpha}(t)f](v) = |t|^{\frac{2n+1}{2}} \int_k f(ty)\psi(\langle v, y \rangle) dy$$

where $f \in S(V)$ and $\langle \cdot, \cdot \rangle$ is a symmetric bilinear form such that $2Q(v) = \langle v, v \rangle$. These elements differ from $i(x_\alpha(u))$, $i(h_\alpha(t))$ and $i(w_\alpha(t))$ by non-zero complex numbers. As in Section 5 one argues that $i(x_\alpha(u)) = x''_\alpha(u)$, and there exists an eight root of 1, the Weil index $\gamma(\psi_t \circ Q)$, such that

$$i(w_\alpha(t)) = \gamma(\psi_t \circ Q) \cdot w''_\alpha(t).$$

It follows that $i(h_\alpha(t)) = \gamma(t, \psi \circ Q)h''_\alpha(t)$, where $\gamma(t, \psi \circ Q) = \gamma(\psi_t \circ Q)/\gamma(\psi \circ Q)$.

Now assume that $(-1)^n \det(V)$, the discriminant of V , is 1. Then Lemma 4.2, Chapter I, in Kudla's Hirschberg Castle notes [Ku2] shows that $\gamma(t, \psi \circ Q) = \gamma(t, \psi \circ x^2)$. Thus $i(h'(t)) = h''(t)$ where $h'(t)$ is defined in Section 5.

In particular, if the discriminant of V is 1, then $-1_V \in \mathrm{O}(V)$ and $-1_W = h'_1(-1) \cdots h'_n(-1) \in \mathrm{Sp}(W)'$ coincide as elements in $\mathrm{Sp}(W \otimes V)''$. Note that -1_W can be defined canonically as the unique element in $\mathrm{Sp}(W)'$ acting as 1 on ω_ψ^e and as -1 on ω_ψ^o . In the rest of this section we assume that the discriminant of V is 1.

Let π be an irreducible representation of $\mathrm{SO}(V)$. Since $\mathrm{O}(V) \cong \{\pm 1_V\} \times \mathrm{SO}(V)$, π can be extended to $\mathrm{O}(V)$ in two different ways. Let $\epsilon(V)$ be the Hasse invariant of V , i.e., $\epsilon(V^+) = 1$ and $\epsilon(V^-) = -1$. Let $\epsilon(s, \pi, \psi)$ be the standard epsilon factor attached to π . Its value at $1/2$ does not depend on ψ , and it is equal to 1 or -1 . We extend π to $\mathrm{O}(V)$ so that -1_V acts as $\epsilon(V)\epsilon(1/2, \pi, \psi)$. This and only this extension has a non-zero, irreducible theta lift to $\mathrm{Sp}(W)'$ denoted by $\theta_{W,V}(\pi)$ (see [GS]). It follows that -1_W acts as $\epsilon(V)\epsilon(1/2, \pi, \psi)$ on $\theta_{W,V}(\pi)$. In the notation of Section 2 and Section 15, the following is proved in [GI] and [GS]:

Theorem 20. *Let π be an irreducible representation of $\mathrm{SO}(V)$.*

- (1) *If π is tempered then $\theta_{W,V}(\pi)$ is tempered.*
- (2) *If $\pi \cong J_Q(\sigma \otimes \tau)$ then $\theta_{W,V}(\pi) \cong J_P(\sigma \otimes \theta_{W_i, V_i}(\tau))$.*

We now have:

Corollary 21. *Let π be an irreducible representation of $\mathrm{SO}(V)$ containing non-zero Iwahori-fixed vectors. Then $\theta_{W,V}(\pi)$ is in \mathcal{G}_ψ^+ or in \mathcal{G}_ψ^- and the Langlands parameter of $\theta_{W,V}(\pi)$, as defined in Section 15, coincides with the Langlands parameter of π . In particular, the theta correspondence gives a bijection of finite sets $\mathrm{Irr}(s, u)$ and $\mathrm{Irr}_\psi(s, u)$ for all pairs (s, u) .*

Proof. Kudla’s [Ku1] results on the supercuspidal support for theta correspondences imply that $\theta_{W,V}(\pi)$ must be in one of the two components, and the semi-simple class s is preserved. If π is tempered then $\theta_{W,V}(\pi)$ is also tempered, by Theorem 20, (1). In this case the unipotent class is uniquely determined by s (see Section 15), so the parameters of π and $\theta_{W,V}(\pi)$ are the same. The general case follows from Theorem 20, (2). □

Although the theta correspondence preserves the parameter (s, u) , it does not coincide with the equivalence of categories. This can be seen on the following important example. Assume that the pair (s, u) corresponds to the unique parameter φ trivial on W_k and such that u belongs to the regular unipotent orbit. Let St^\pm be the Steinberg representation of $\mathrm{SO}(V^\pm)$. Then

$$\mathrm{Irr}(s, u) = \{\mathrm{St}^+, \mathrm{St}^-\} \quad \text{and} \quad \mathrm{Irr}_\psi(s, u) = \{\delta_\psi^+, \delta_\psi^-\},$$

where δ_ψ^\pm are the square integrable representations studied in Section 14. Note that -1_W acts as 1 on ω_ψ^e and as -1 on ω_ψ^o . It follows that -1_W acts as 1 on any representation in \mathcal{G}_ψ^+ and as -1 on any representation in \mathcal{G}_ψ^- . In particular, -1_W acts as -1 on δ_ψ^- . On the other hand, since $\epsilon(V^+) = 1$ and $\epsilon(1/2, \mathrm{St}^+, \psi) = -1$, it follows that -1_W acts as -1 on $\theta_{W,V^+}(\mathrm{St}^+)$. This forces

$$\theta_{W,V^+}(\mathrm{St}^+) \cong \delta_\psi^- \quad \text{and} \quad \theta_{W,V^-}(\mathrm{St}^-) \cong \delta_\psi^+.$$

Now let χ be a non-trivial unramified quadratic character of k^* . Using the Hilbert symbol we can write $\chi(x) = (x, a)$ for some non-square unit a . Let St_χ^\pm be the quadratic twist of St^\pm by χ . Let $\delta_{\psi_a}^\pm$ be the square integrable representation of $\mathrm{Sp}(W)'$ which corresponds to St^\pm via the equivalence of components $\mathcal{G}_{\psi_a}^\pm$ and \mathcal{S}_0^\pm . Since a is a unit, the components \mathcal{G}_ψ^+ and $\mathcal{G}_{\psi_a}^+$ coincide, and $\delta_{\psi_a}^+$ corresponds to St_χ^+ by the equivalence of \mathcal{G}_ψ^+ and \mathcal{S}_0^+ . On the other hand, $\delta_{\psi_a}^-$ belongs to the component $\mathcal{G}_{\psi_a}^-$ which is different but equivalent to \mathcal{G}_ψ^- , just as St_χ^- belongs to a component of $\mathrm{SO}(V^-)$ which is different but equivalent to \mathcal{S}_0^- . Since $\epsilon(1/2, \mathrm{St}_\chi^+, \psi) = 1$, it is not difficult to show that

$$\theta_{W,V^+}(\mathrm{St}_\chi^+) \cong \delta_{\psi_a}^+ \quad \text{and} \quad \theta_{W,V^-}(\mathrm{St}_\chi^-) \cong \delta_{\psi_a}^-.$$

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