CARTAN SUBALGEBRAS FOR QUANTUM SYMMETRIC PAIR COIDEALS

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ABSTRACT. There is renewed interest in the coideal subalgebras used to form quantum symmetric pairs because of recent discoveries showing that they play a fundamental role in the representation theory of quantized enveloping algebras. However, there is still no general theory of finite-dimensional modules for these coideals. In this paper, we establish an important step in this direction: we show that every quantum symmetric pair coideal subalgebra admits a quantum Cartan subalgebra which is a polynomial ring that specializes to its classical counterpart. The construction builds on Kostant and Sugiura's classification of Cartan subalgebras for real semisimple Lie algebras via strongly orthogonal systems of positive roots. We show that these quantum Cartan subalgebras act semisimply on finite-dimensional unitary modules and identify particularly nice generators of the quantum Cartan subalgebra for a family of examples.

CONTENTS

1. INTRODUCTION

In the mid-1980s, Drinfel′d and Jimbo ([Dr], [Ji]) introduced a family of algebras, referred to as quantized enveloping algebras, which are Hopf algebra deformations of enveloping algebras of semisimple Lie algebras. Shortly afterwards, Noumi, Sugitani, Dijkhuizen, and the author developed a theory of quantum symmetric spaces based on the construction of quantum analogs of symmetric pairs ([D], [NS], [N], [L2], [L3], [L4], [L5]). Classically, a symmetric pair consists of a semisimple Lie

algebra and a Lie subalgebra fixed by an involution. Quantum symmetric pairs are formed using one-sided coideal subalgebras because quantized enveloping algebras contain very few Hopf subalgebras. Although quantum symmetric pair coideal subalgebras can be viewed as quantum analogs of enveloping algebras of the underlying fixed Lie subalgebras, which are either semisimple or reductive, there is still no general classification of finite-dimensional modules for these coideal subalgebras. In this paper, we establish a significant step in this direction: we show that every quantum symmetric pair coideal subalgebra admits a quantum Cartan subalgebra which is a polynomial ring that specializes to its classical counterpart and acts semisimply on finite-dimensional unitary modules. It should be noted that Kolb has extended the theory of quantum symmetric pair coideal subalgebras to quantized Kac-Moody algebras ([Ko]). Although we do not consider this more general situation here, we expect many of the ideas and constructions of this paper to carry over to the Kac-Moody setting.

One of the original goals in developing the theory of quantum symmetric spaces, realizing Macdonald-Koornwinder polynomials as quantum zonal spherical functions, has in large part been completed (see [N], [NS], [DN], [NDS], [DS], [L3], [L4]). Recently, one-sided coideal subalgebras used to form quantum symmetric pairs have gained renewed attention because of the discovery of their fundamental roles in the representation theory of quantized enveloping algebras. Ehrig and Stroppel ([ES]) show that a family of these coideal subalgebras are just the right algebras needed in the categorification of certain quantized enveloping algebra representations arising in Schur duality and skew Howe duality. In [BW], Bao and Wang obtain canonical bases for representations of this family of coideal subalgebras and use it to settle conjectures concerning irreducible characters of Lie superalgebras. Expanding greatly on these ideas, Bao and Wang present a general theory of canonical bases that works for all types of quantum symmetric pair coideal subalgebras in [BW2]. Following a program initiated in [BW], Balagovic and Kolb ([BK]) use quantum symmetric pair coideal subalgebras to construct universal K-matrices which provide solutions of the reflection equation arising in inverse scattering theory. These K-matrices (are expected to) play a similar role as the universal R-matrices for quantized enveloping algebras.

The body of work described in the previous paragraph relies on a variety of representations with respect to the quantum symmetric pair coideal subalgebras. However, these representations are mainly modules or submodules of the larger quantized enveloping algebras. For example, the analysis of zonal spherical functions on quantum symmetric spaces uses spherical modules, a special family of finite-dimensional simple representations for the quantized enveloping algebras (see for example [L1], [L2], and [L5]). Ehrig and Stroppel establish dualities involving quantum symmetric pair coideal subalgebras using exterior products of the natural representations for the larger quantized enveloping algebras ([ES]). Bao and Wang obtain canonical basis with respect to a quantum symmetric pair coideal subalgebra for tensor products of finite-dimensional simple modules of the quantized enveloping algebra ([BW2]). One of the beautiful properties of Lusztig's canonical bases (Chapter 25, [Lu]) is that they yield compatible bases for each finite-dimensional simple module of a quantized enveloping algebra. Without a highest weight theory and corresponding classification of finite-dimensional simple modules for the quantum symmetric pair coideal subalgebras, a comparable result remains elusive.

The Cartan subalgebra of a quantized enveloping algebra is the Laurent polynomial ring generated by the group-like elements. The group-like property enables one to easily break down many modules into direct sums of weight spaces in the context of a highest weight module theory for the quantized enveloping algebra. This Cartan is unique in the sense that it is the only Laurent polynomial ring contained inside the quantized enveloping algebra of the correct size. Unfortunately, in most cases, a quantum symmetric pair coideal subalgebra does not inherit a Cartan subalgebra in the form of a Laurent polynomial ring of group-like elements from the larger quantized enveloping algebra. This is because the Cartan subalgebra of the larger quantized enveloping algebra corresponds to a maximally split one with respect to the involution defining the quantum symmetric pair coideal subalgebra. Hence the Laurent polynomial ring generated by the group-like elements inside a quantum symmetric pair coideal subalgebra is typically too small to play the role of a Cartan subalgebra. This lack of an obvious Cartan subalgebra consisting of group-like elements has been the primary reason so little progress has been made on a general representation theory for quantum symmetric pair coideal subalgebras. In this paper, we show that, despite this fact, there are good Cartan subalgebra candidates for quantum symmetric pair coideal subalgebras (as stated in Theorem B below).

In order to find good choices for quantum symmetric pair Cartan subalgebras, we first revisit the classical case. It is well known that there is a one-to-one correspondence between isomorphism classes of real forms for a complex semisimple Lie algebra $\mathfrak{g}$ and conjugacy classes of involutions on $\mathfrak{g}$. Thus, understanding the Cartan subalgebras of $\mathfrak{g}^\theta$ is closely connected to understanding Cartan subalgebras of real semisimple Lie algebras. Unlike the complex setting, the set of Cartan subalgebras of a real semisimple Lie algebra do not form a single conjugacy class with respect to automorphisms of the Lie algebra. A theorem of Kostant and Sugiura ([OV, Section 4.7], [Su], and [K]) classifies these conjugacy classes using subsets of the positive roots that satisfy strongly orthogonal properties with respect to the Cartan inner product. Using Cayley transforms, which are a special family of Lie algebra automorphisms, the Cartan subalgebra $\mathfrak{h}$ can be converted into $\mathfrak{t}$, a Cartan subalgebra of $\mathfrak{g}$ whose intersection with $\mathfrak{g}^\theta$ is a Cartan subalgebra of this smaller Lie algebra. In other words, for any symmetric pair, one can use Cayley transforms to express a Cartan subalgebra of $\mathfrak{g}^\theta$ in terms of a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$.

The Cartan subalgebra $\mathfrak{t} \cap \mathfrak{g}^\theta$ is spanned by elements in the union of $\{e_\beta + f_{-\beta}, \beta \in \Gamma_\theta\}$ and $\mathfrak{g}^\theta \cap \mathfrak{h}$ where Γ_θ is a system of strongly orthogonal positive roots associated to θ (see Section 2). Note that elements in $\mathfrak{g}^\theta \cap \mathfrak{h}$ correspond in the quantum setting to the group-like elements of $\mathcal{B}_\theta$. Lifting elements of the form $e_\beta + f_{-\beta}$ is more complicated. In order to accomplish these lifts, we use particularly nice systems of strongly orthogonal positive roots that satisfy extra orthogonality conditions with respect to the Cartan inner product.

Theorem A (Theorem 2.7). *For each maximally split symmetric pair $\mathfrak{g}, \mathfrak{g}^\theta$, there exists a maximum-sized strongly orthogonal subset $\Gamma_\theta = \{\beta_1, \ldots, \beta_m\}$ of positive roots such that for all j we have $\theta(\beta_j) = -\beta_j$, and for all $i > j$ and all simple roots α in the support of β_i, α is strongly orthogonal to β_j. Moreover, each β_j is strongly orthogonal to all but at most two distinguished simple roots contained in its support.*

For $\mathfrak{g}, \mathfrak{g}^\theta$ maximally split and Γ_θ chosen as in Theorem A, the elements $e_\beta + f_{-\beta}$, $\beta \in \Gamma_\theta$ are lifted to the quantum case in a two-step process. The first step lifts $f_{-\beta}$ to the lower triangular part of the quantized enveloping algebra so that the lift commutes with all generators for $U_q(\mathfrak{g})$ defined by simple roots strongly orthogonal to β. In many cases, this lift is a member of one of Lusztig's PBW basis ([Ja, Chapter 8]). For other cases, the construction is more subtle. It relies on showing that elements of special submodules of $U_q(\mathfrak{g})$ with respect to the adjoint action satisfy nice commutativity properties. The desired Cartan element associated to β is formed by considering elements of $\mathcal{B}_\theta$ with a lowest weight term equal to the lift of $f_{-\beta}$. The process uses fine information on the possible weights and biweights in the expansion of elements in $\mathcal{B}_\theta$ in terms of weight and biweight vectors and a special projection map. (See Section 3.1 for the precise notion of biweight.) We further insist that the quantum analogs of the $e_\beta + f_{-\beta}$ are fixed by a quantum version of the Chevalley antiautomorphism. This property ensures that elements of the quantum Cartan subalgebra of a quantum symmetric pair coideal subalgebra acts semisimply on a large family of finite-dimensional modules. These lifts are the substance behind our main result, Theorem 7.2, which can be summarized as follows.

Theorem B. *Every quantum symmetric pair coideal subalgebra $\mathcal{B}_\theta$ contains a commutative subalgebra $\mathcal{H}$ that is a polynomial ring over the algebra of group-like elements inside $\mathcal{B}_\theta$. Moreover, $\mathcal{H}$ specializes to the enveloping algebra of a Cartan subalgebra of the underlying fixed Lie subalgebra, $\mathcal{H}$ acts semisimply on finite-dimensional simple unitary $\mathcal{B}_\theta$-modules, and $\mathcal{H}$ satisfies certain uniqueness conditions related to its expression as a sum of weight vectors.*

There is one family of quantum symmetric pair coideal subalgebras that has its own separate history, namely, the one consisting of the nonstandard q-deformed enveloping algebras of orthogonal Lie algebras introduced in a paper by Gavrilik and Klimyk [GK]. The fact that the nonstandard q-deformed enveloping algebra $U_q'(\mathfrak{so}_n)$ agrees with one of the coideal subalgebras was realized from the beginning of the theory of quantum symmetric pairs (see for example [N, Section 2.4] and [L4, end of Section 2]). In a series of papers, Gavrilik, Klimyk, and Iorgov classify the finite-dimensional simple modules for $U_q'(\mathfrak{so}_n)$ using quantum versions of Gel'fand-Tsetlin basis (see [GK], [IK], and the references therein). Rowell and Wenzl study q-deformed algebras $U_q'(\mathfrak{so}_n)$ in the context of fusion categories ([W], [RW]). Taking advantage of the finite-dimensional representation theory for $U_q'(\mathfrak{so}_n)$, they prove parts of a conjecture on braided fusion categories ([RW]). An interesting aspect of their work is that they redo the classification of finite-dimensional modules for $U_q'(\mathfrak{so}_n)$ by developing a Verma module theory based on actions of a quantum Cartan subalgebra that agrees with the one studied here (see Remark 7.4). Preliminary investigations suggest a similar Verma module theory can be realized for other symmetric pair coideal subalgebras by analyzing the action of the quantum Cartan subalgebras presented in this paper. We intend to explore this further in future work.

There is also recent work on the representation theory of $\mathcal{B}_\theta$ for another infinite family of quantum symmetric pairs that form a subset of those of type AIII. In [Wa], H. Watanabe constructs a triangular decomposition for these coideal subalgebras, uses this decomposition to develop a Verma module theory, and then classifies irreducible modules living inside a large category of representations. The Cartan

part of this triangular decompositon plays a standard role: Cartan elements act as scalars on highest weight generating vectors for Verma modules and these scalars are the weights used in the classification. Building on this foundation, Watanabe studies analogs of Kashiwara's crystal bases and shows that they have particularly nice combinatorial properties. We see in Section 8 that the Cartan part of this triangular decomposition in [Wa] agrees with the one presented here. We expect that the Cartan subalgebras of this paper can be used to classify families of irreducible modules for all quantum symmetric pair coideal subalgebras of type AIII using very similar arguments to those in [Wa]. This would be a first step towards generalizing Watanabe's crystal basis results to other symmetric pairs, especially others of type AIII.

In analogy to Lusztig's well-known braid group action on quantized enveloping algebras, Kolb and Pellegrini ([KoP]) introduce a braid group action on certain families of quantum symmetric pair coideal subalgebras. One might expect that it is possible to construct generators for quantum Cartan subalgebras by starting with one element and taking images of it under braid group operators. However, as explained in Remark 8.4, the situation is more subtle. Indeed, this method does not necessarily result in elements that commute with each other. Nevertheless, the braid group action could be quite useful in computing the action of the Cartan subalgebra on various representations. In particular, Watanabe uses the braid group action in order to compute the action of Cartan elements on highest weight vectors and this might generalize to other symmetric pair families (see [Wa, Section 4.3]).

There are many connections with the quantum Cartan subalgebra of this paper and the center of $\mathcal{B}_\theta$ studied in [KoL]. Both constructions take advantage of various projection maps, weight and biweight expansions, the structure of $U_q(\mathfrak{g})$ with respect to the adjoint action, and certain commutativity conditions. In the last section of this paper, we comment on a particularly close connection between the center and the quantum Cartan subalgebra of $\mathcal{B}_\theta$ for a symmetric pair of type AIII/AIV (see Remark 8.3). In [Ko2], Kolb shows that central elements yield solutions to reflection equations. It would be interesting to see whether the quantum Cartan subalgebra elements of this paper, which share many properties of central elements, yield solutions to a related family of equations.

This paper is organized as follows. Section 2 is devoted to the classical case. After introducing basic notation in Section 2.1, we review in Section 2.2 some of the classical theory of Cartan subalgebras in the real setting, explain the notion of strongly orthogonal systems, and present part of the Kostant-Sugiura classification of real Cartan subalgebras in terms of strongly orthogonal systems of positive roots. In Section 2.3, we show how to use this classification and Cayley transforms to identify the Cartan subalgebra of a fixed Lie subalgebra $\mathfrak{g}^\theta$ with respect to a given triangular decomposition of $\mathfrak{g}$. Maximally split involutions are the focus of Section 2.4 where we present Theorem A (Theorem 2.7) with case-by-case explicit descriptions of the desired maximum-sized strongly orthogonal systems of positive roots.

Section 3 reviews basic properties of quantized enveloping algebras. We set notation in Section 3.1, discuss adjoint module structures and dual Vermas in Section 3.2, and describe Lusztig's automorphisms in Section 3.3. In Section 3.4, we review the quantum Chevalley antiautomorphism and the fact that subalgebras invariant

with respect to this automorphism act semisimply on finite-dimensional unitary modules.

In Section 4, we turn our attention to quantum symmetric pair coideal subalgebras. After setting notation in Section 4.1, we establish in Section 4.2 a result on the biweight space expansions of elements of a coideal subalgebra $\mathcal{B}_\theta$ associated to a maximally split involution θ. In Section 4.3, we introduce a projection map on the quantized enveloping algebra. We then show how to construct elements of $\mathcal{B}_\theta$ equal to a given lower triangular element plus terms that vanish with respect to this projection map.

The main result of Section 5 is a version of the first assertion of Theorem B for the lower triangular part of $U_q(\mathfrak{g})$. The proof is based on lifting root vectors associated to weights in strongly orthogonal root systems (as specified by Theorem A) to elements in the lower triangular part that satisfy strong commutativity properties. These lifts rely on an analysis of certain finite-dimensional simple modules in the classical setting (Section 5.1) and lifts of special families of root vectors in Sections 5.2 and 5.3. Details of the lift are given in Section 5.4.

Section 6 sets up the basic tools needed to create Cartan elements in $\mathcal{B}_\theta$. Section 6.1 introduces the notion of generalized normalizers in order to analyze when elements of $\mathcal{B}_\theta$ commute with elements of carefully chosen sub-quantized enveloping algebras of $U_q(\mathfrak{g})$. Sections 6.2 and 6.3 establish properties of lowest and highest weight summands, respectively, of elements inside special generalized normalizers associated to $\mathcal{B}_\theta$. Building on the prior sections, Section 6.4 presents methods for constructing Cartan elements with desired commutativity properties in $\mathcal{B}_\theta$ from the lift of the root vectors $f_{-\beta}$.

Theorem B, the main result of the paper, is the heart of Section 7. First, we establish in Section 7.1 a specialization criteria that is later used to show that the quantum Cartan subalgebras specialize to their classical counterparts. We show how to construct the desired Cartan elements, and thus prove the detailed version of Theorem B (Theorem 7.2 and Corollary 7.3), in Section 7.2.

In the final section (Section 8) of the paper, we analyze in detail the quantum Cartan subalgebras for a family of symmetric pairs of type AIII/AIV. After a brief overview of the generators and relations for this family in Section 8.1, we present the $n = 2$ and $n = 3$ cases in Sections 8.2 and 8.3, respectively. For general n (Section 8.4), we give explicit formulas for a set of generators for the quantum Cartan subalgebra. We also establish a close connection between the quantum Cartan subalgebra and central elements of subalgebras of $\mathcal{B}_\theta$.

We conclude the paper with an appendix listing commonly used notation by order of appearance.

2. Cartan subalgebras of real semisimple Lie algebras

Let $\mathbb{C}$ denote the complex numbers, let $\mathbb{R}$ denote the real numbers, and let $\mathbb{N}$ denote the nonnegative integers. We recall facts about normal real forms of a complex semisimple Lie algebra $\mathfrak{g}$ and present part of Kostant and Sugiura's results on the classification of Cartan subalgebras in the real setting in terms of strongly orthogonal systems of positive roots. Much of the material in this section is based on the presentation in [OV, Section 4.7] and [Kn, Chapter VI, Sections 6 and 7] (see also [Su], [AK], and [DFG]). However, we take a slightly different perspective, focusing on Cartan subalgebras of a subalgebra $\mathfrak{g}^\theta$ of $\mathfrak{g}$ fixed by an involution θ

rather than Cartan subalgebras of related real forms. We conclude this section with Theorem 2.7, the detailed version of Theorem A, which identifies strongly orthogonal systems of positive roots with special properties.

2.1. Basic notation: The classical case.

Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be a fixed triangular decomposition of the complex semisimple Lie algebra $\mathfrak{g}$. Let Δ denote the set of roots of $\mathfrak{g}$ and recall that Δ is a subset of the dual space $\mathfrak{h}^*$ to $\mathfrak{h}$. Write Δ^+ for the subset of positive roots. Let $h_i, e_\alpha, f_{-\alpha}, i = 1, \ldots, n, \alpha \in \Delta^+$ be a Chevalley basis for $\mathfrak{g}$. Set $\pi = \{\alpha_1, \ldots, \alpha_n\}$ equal to a fixed choice of simple roots for the root system Δ. As is standard, we view roots as elements of $\mathfrak{h}^*$ (see for example [H, Chapter 14]). Let $\mathcal{W}$ denote the Weyl group associated to the root system Δ and let w_0 denote the longest element of $\mathcal{W}$. Given a simple root α_i, we often write e_i for e_{α_i} and f_i for $f_{-\alpha_i}$. Given a positive root $\alpha \in \mathfrak{h}^*$, we write h_α for the coroot in $\mathfrak{h}$. Let $(\ ,\ )$ denote the Cartan inner product on $\mathfrak{h}^*$.

We define notation associated to a subset π' of π as follows. Let $\mathfrak{g}_{\pi'}$ be the semisimple Lie subalgebra of $\mathfrak{g}$ generated by e_i, f_i, h_i for all $\alpha_i \in \pi'$. Write $Q(\pi')$ (resp., $Q^+(\pi')$) for the set of linear combinations of elements in π' with integer (resp., nonnegative integer) coefficients. In particular, $Q(\pi)$ is just the root lattice for $\mathfrak{g}$ with respect to the choice of simple roots π. Let $\Delta(\pi')$ be the subset of Δ equal to the root system generated by π' with respect to the Cartan inner product $(\ ,\ )$. Write $\mathcal{W}_{\pi'}$ for the subgroup of $\mathcal{W}$ generated by the reflections defined by the simple roots in π'. In other words, $\mathcal{W}_{\pi'}$ can be viewed as the Weyl group for the root system $\Delta(\pi')$. Write $w(\pi')_0$ for the longest element of $\mathcal{W}_{\pi'}$. If π' is the empty set, we set $w(\pi')_0 = 1$.

Let $P(\pi)$ denote the weight lattice, and let $P^+(\pi)$ denote the subset of $P(\pi)$ consisting of dominant integral weights with respect to the choice of simple roots π of the root system Δ. Recall that $P(\pi)$ has a standard partial order defined by $\beta \geq \gamma$ if and only if $\beta - \gamma \in Q^+(\pi)$. Similarly, let $P(\pi')$ denote the weight lattice and let $P^+(\pi')$ denote the set of dominant integral weights associated to the root system $\Delta(\pi')$ with a set of simple roots π'. Since $\pi \subset \mathfrak{h}^*$, $P(\pi)$ and $Q(\pi)$ are subsets of $\mathfrak{h}^*$. Thus, given say $\lambda \in P(\pi)$, it makes sense to also consider $\lambda/2 \in \mathfrak{h}^*$ as we do at the beginning of Section 3.2.

Given a weight $\beta = \sum_i m_i \alpha_i \in \mathfrak{h}^*$ and $\alpha_i \in \pi$, we set $\mathrm{mult}_{\alpha_i}(\beta) = m_i$. Write $\mathrm{Supp}(\beta)$ for the subset of simple roots α_i in π such that $\mathrm{mult}_{\alpha_i}(\beta) = m_i \neq 0$. Define the height of β by $\mathrm{ht}(\beta) = \sum_i m_i$. Since the set π of simple roots form a basis for $\mathfrak{h}^*$, the notion of height works for all elements of $\mathfrak{h}^*$ and hence for all elements in the weight lattice $P(\pi)$ and the root lattice $Q(\pi)$.

2.2. Cartan subalgebras and the normal real form.

The normal real form of the semisimple Lie algebra $\mathfrak{g}$ is the real Lie subalgebra

$$\mathfrak{g}_{\mathbb{R}} = \sum_{i=1}^n \mathbb{R} h_i + \sum_{\alpha \in \Delta^+} \mathbb{R} e_\alpha + \sum_{\alpha \in \Delta^+} \mathbb{R} f_{-\alpha}.$$

Set $\mathfrak{h}_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{h} = \sum_{i=1}^n \mathbb{R} h_i$.

Consider a Lie algebra involution θ of $\mathfrak{g}$. Conjugating with a Lie algebra automorphism of $\mathfrak{g}$ if necessary, we may assume that both $\mathfrak{h}$ and $\mathfrak{h}_{\mathbb{R}}$ are stable under the action of θ, and, moreover, θ restricts to an involution of $\mathfrak{g}_{\mathbb{R}}$. The Cartan

decomposition associated to θ is the decomposition of $\mathfrak{g}_\mathbb{R}$ into subspaces

$$\mathfrak{g}_\mathbb{R} = \mathfrak{t}_\mathbb{R} \oplus \mathfrak{p}_\mathbb{R}$$

where $\mathfrak{t}_\mathbb{R} = \{t \in \mathfrak{g}_\mathbb{R} | \ \theta(t) = t\}$ and $\mathfrak{p}_\mathbb{R} = \{t \in \mathfrak{g}_\mathbb{R} | \ \theta(t) = -t\}$.

A Cartan subalgebra $\mathfrak{s}_\mathbb{R}$ of the real semisimple Lie algebra $\mathfrak{g}_\mathbb{R}$ is called *standard* if

$$\mathfrak{s}_\mathbb{R} = \mathfrak{s}_\mathbb{R}^+ \oplus \mathfrak{s}_\mathbb{R}^-$$

where $\mathfrak{s}_\mathbb{R}^+ \subseteq \mathfrak{t}_\mathbb{R}$ and $\mathfrak{s}_\mathbb{R}^- \subseteq \mathfrak{p}_\mathbb{R}$. The space $\mathfrak{s}_\mathbb{R}^-$ is called the *vector part* of the Cartan subalgebra $\mathfrak{s}_\mathbb{R}$. The dimension of $\mathfrak{s}_\mathbb{R}^+$ is called the *compact dimension* of $\mathfrak{s}_\mathbb{R}$ and the dimension of $\mathfrak{s}_\mathbb{R}^-$ is called the *noncompact dimension* of $\mathfrak{s}_\mathbb{R}$. A standard Cartan subalgebra is called *maximally compact* if its compact dimension is the maximum possible value and is called *maximally noncompact* if its noncompact dimension is the largest possible value. Maximally noncompact Cartan subalgebras are also called maximally split Cartan subalgebras.

Let $\mathfrak{a}_\mathbb{R}$ be a maximal commutative real Lie algebra that is a subset of $\mathfrak{p}_\mathbb{R}$. Note that $\mathfrak{s}_\mathbb{R}$ is maximally split if and only if

$$\mathfrak{s}_\mathbb{R}^- = \mathfrak{s}_\mathbb{R} \cap \mathfrak{p}_\mathbb{R} = \mathfrak{s}_\mathbb{R} \cap \mathfrak{a}_\mathbb{R} = \mathfrak{a}_\mathbb{R}.$$

Recall that θ has been adjusted so that the Cartan subalgebra $\mathfrak{h}$ is stable with respect to θ. It follows that θ sends root vectors to root vectors. Hence θ induces an involution, which we also call θ, on the root system of $\mathfrak{g}$. In particular, $\theta(\alpha)$ equals the weight of $\theta(e_\alpha)$ for all $\alpha \in \Delta$. This involution extends to the lattices $Q(\pi)$ and $P(\pi)$ by insisting that $\theta(\sum_i \eta_i \alpha_i) = \sum_i \eta_i \theta(\alpha_i)$ for all $\eta = \sum_i \eta_i \alpha_i$. We write $Q(\pi)^\theta$ (resp., $P(\pi)^\theta$) for the set of elements in $Q(\pi)$ (resp., $P(\pi)$) fixed by θ. Let Δ_θ denote the subset of the positive roots Δ^+ such that $\theta(\alpha) = -\alpha$.

We say that two elements α and β in $\mathfrak{h}^*$ are orthogonal if they are orthogonal with respect to the Cartan inner product (i.e., $(\alpha, \beta) = 0$). The following definition introduces a stronger form of orthogonality and specifies important subsets of positive roots used in the classification of the Cartan subalgebras of real semisimple Lie algebras.

Definition 2.1. Let Γ be a subset of the positive roots Δ^+.

(i) Two positive roots α and β are strongly orthogonal if $(\alpha, \beta) = 0$ and $\alpha + \beta$ is not a root.

(ii) Γ is a strongly orthogonal system of positive roots if every pair α, β in Γ is strongly orthogonal.

(iii) Γ is a strongly orthogonal system of positive roots associated to the involution θ if $\Gamma \subseteq \Delta_\theta$ and Γ is a strongly orthogonal system of positive roots.

Given $\beta \in \mathfrak{h}^*$, write $\mathrm{Orth}(\beta)$ for the set consisting of all $\alpha \in \pi$ such that α, β are orthogonal. For $\beta \in \Delta^+$, we write $\mathrm{StrOrth}(\beta)$ for the set of all $\alpha \in \pi$ such that α, β are strongly orthogonal.

Now consider a strongly orthogonal system of positive roots Γ. To simplify notation, we refer to Γ as a *strongly orthogonal system* with the property that it consists of positive roots understood. Similarly, we call Γ a *strongly orthogonal θ-system* if it is a strongly orthogonal system of positive roots associated to the involution θ. A classification of strongly orthogonal θ-systems is given in [AK] and [DFG, Section 4.2].

The next theorem gives necessary and sufficient conditions for two Cartan sub-algebras of a normal real form $\mathfrak{g}_{\mathbb{R}}$ to be conjugate to each other.

Theorem 2.2 (Kostant-Sugiura's theorem [K], [Su]; see also [OV, Section 4.7]). *A subspace $\mathfrak{b}_{\mathbb{R}}$ of $\mathfrak{a}_{\mathbb{R}}$ is a vector part of a standard Cartan subalgebra $\mathfrak{s}_{\mathbb{R}}$ if and only if*

$$\mathfrak{b}_{\mathbb{R}} = \{x \in \mathfrak{a}_{\mathbb{R}} |\ \alpha(x) = 0 \text{ for all } \alpha \in \Gamma\}$$

where Γ is a strongly orthogonal θ-system. Moreover, if $\mathfrak{b}'_{\mathbb{R}}$ is the vector part of another Cartan subalgebras $\mathfrak{s}'_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}}$ defined by the strongly orthogonal θ-system Ω, then $\mathfrak{s}_{\mathbb{R}}$ and $\mathfrak{s}'_{\mathbb{R}}$ are conjugate if and only if there is an element w in the Weyl group $\mathcal{W}$ that commutes with θ and $w\Omega = \Gamma$.

Since θ restricts to an involution of $\mathfrak{g}_{\mathbb{R}}$, we can lift the Cartan decomposition to the complex setting, yielding a decomposition

$$\mathfrak{g} = \mathfrak{g}^\theta \oplus \mathfrak{p}$$

where $\mathfrak{g}^\theta = \{\theta(t) = t |\ t \in \mathfrak{g}\} = \mathfrak{t}_{\mathbb{R}} \oplus i\mathfrak{t}_{\mathbb{R}}$ and $\mathfrak{p} = \{\theta(t) = -t |\ t \in \mathfrak{g}\} = \mathfrak{p}_{\mathbb{R}} \oplus i\mathfrak{p}_{\mathbb{R}}$. We say that a Cartan subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ is *standard* if it is the complexification of a standard Cartan subalgebra $\mathfrak{s}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}}$. In particular, a Cartan subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ is standard if and only if $\mathfrak{s} = (\mathfrak{s} \cap \mathfrak{g}^\theta) \oplus (\mathfrak{s} \cap \mathfrak{p})$.

Since θ restricts to an involution on $\mathfrak{h}_{\mathbb{R}}$, it follows that $\mathfrak{h}_{\mathbb{R}}$ is a standard Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$ and so $\mathfrak{h}$ is a standard Cartan subalgebra of $\mathfrak{g}$. Write $\mathfrak{h}_\theta$ for $\mathfrak{h} \cap \mathfrak{g}^\theta$ and $\mathfrak{h}^-$ for $\mathfrak{h} \cap \mathfrak{p}$. We have

$$\mathfrak{h} = \mathfrak{h}_\theta \oplus \mathfrak{h}^-$$

and we refer to $\mathfrak{h}^-$ as the vector part of $\mathfrak{h}$ since it is the complexification of the vector part of $\mathfrak{h}_{\mathbb{R}}$. Set $\mathfrak{a} = \mathfrak{a}_{\mathbb{R}} + i\mathfrak{a}_{\mathbb{R}}$ and note that $\mathfrak{a}$ is a maximal commutative complex Lie algebra subset of $\mathfrak{p}$.

2.3. **Cayley transforms.** We review here the process for converting a Cartan subalgebra into a maximally compact one via Cayley transforms as presented in [Kn, Chapter VI, Section 7]. Our setup is as before. We start with an involution θ of $\mathfrak{g}$ that restricts to an involution of $\mathfrak{g}_{\mathbb{R}}$ and to an involution of $\mathfrak{h}_{\mathbb{R}}$.

Let $\alpha \in \Delta_\theta$ and so $\theta(\alpha) = -\alpha$. Note that $e_\alpha, f_{-\alpha}, h_\alpha$ are all in $\mathfrak{p}$ and, moreover, $h_\alpha \in \mathfrak{h}$ and so $h_\alpha \in \mathfrak{a}$. Applying an automorphsim to $\mathfrak{g}$ that fixes elements of $\mathfrak{h}$ and sends root vectors to scalar multiples of themselves if necessary, we may assume that

$$\theta(e_\alpha) = f_{-\alpha}$$

and thus $e_\alpha + f_{-\alpha} \in \mathfrak{g}^\theta$ for all $\alpha \in \Delta_\theta$. Given a Lie subalgebra $\mathfrak{s}$ of $\mathfrak{h}$, let $\ker_\alpha(\mathfrak{s})$ denote the kernel of α restricted to $\mathfrak{s}$. Since $\theta(\alpha) = -\alpha$, we have $\mathfrak{h}_\theta \subset \ker_\alpha(\mathfrak{h})$. Hence $\ker_\alpha(\mathfrak{h}) = \mathfrak{h}_\theta \oplus \ker_\alpha(\mathfrak{a})$ and so

$$\mathfrak{h} = \mathbb{C}h_\alpha \oplus \mathfrak{h}_\theta \oplus \ker_\alpha(\mathfrak{a}).$$

Definition 2.3. Given a positive root γ, the Cayley transform $\mathbf{d}_\gamma$ (as in [Kn, Chapter VI, Section 7]) is the Lie algebra isomorphism of $\mathfrak{g}$ defined by $\mathbf{d}_\gamma = \mathrm{Ad}(\exp\frac{\pi}{4}(f_{-\gamma} - e_\gamma))$.

It follows from the above definition that the Cayley transform $\mathbf{d}_\gamma$ sends h_γ to $e_\gamma + f_{-\gamma}$. Also $\mathbf{d}_\gamma(h) = h$ for all $h \in \ker_\gamma(\mathfrak{h})$. Therefore, for $\alpha \in \Delta_\theta$ we have

$$\mathbf{d}_\alpha(\mathfrak{h}) = \mathbb{C}(e_\alpha + f_{-\alpha}) \oplus \mathfrak{h}_\theta \oplus \ker_\alpha(\mathfrak{a}).$$

Moreover, this direct sum decomposition restricts to the analogous one for $\mathfrak{h}_\mathbb{R}$. We have

$$\mathbf{d}_\alpha(\mathfrak{h}) \cap \mathfrak{g}^\theta = \mathbb{C}(e_\alpha + f_{-\alpha}) \oplus \mathfrak{h}_\theta.$$

Hence $\mathbf{d}_\alpha$ transforms $\mathfrak{h}$ into a Cartan subalgebra $\mathbf{d}_\alpha(\mathfrak{h})$ of compact dimension one higher than that of $\mathfrak{h}$.

We can repeat this process using the root space decomposition of $\mathfrak{g}$ with respect to this new Cartan subalgebra $\mathfrak{s} = \mathbf{d}_\alpha(\mathfrak{h})$. In particular, one picks a positive root β (if possible) with respect to $\mathfrak{s}$ that vanishes on $\mathfrak{s} \cap \mathfrak{g}^\theta$. Applying the Cayley transform defined by β yields another Cartan subalgebra where once again the compact dimension increases by one. Thus we have the following result as explained in the presentation of Cayley transforms in [Kn, Chapter VI, Section 7] (see also [DFG]).

Lemma 2.4. *A standard Cartan subalgebra $\mathfrak{h}$ with respect to the involution θ can be transformed into a maximally compact Cartan subalgebra by repeated applications of Cayley transforms defined by roots that vanish on the nonvector part of the Cartan subalgebra.*

Let Γ be a strongly orthogonal θ-system. We can express $\mathfrak{h}$ as

$$\mathfrak{h} = \mathfrak{h}_\theta \oplus \sum_{\alpha \in \Gamma} \mathbb{C}h_\alpha \oplus \mathfrak{b},$$

where $\mathfrak{b} \subseteq \mathfrak{a}$ and $[h, e_\alpha] = 0$ (and so $\alpha(h) = 0$) for all $h \in \mathfrak{b}$ and $\alpha \in \Gamma$. Note also that the Cayley transforms $\mathbf{d}_\alpha, \alpha \in \Gamma$ are a commutative set of automorphisms on $\mathfrak{g}$. Applying all of them to $\mathfrak{g}$ yields a Cartan subalgebra of the form

$$\mathfrak{h}_\theta \oplus \left(\bigoplus_{\alpha \in \Gamma} \mathbb{C}(e_\alpha + f_{-\alpha}) \right) \oplus \mathfrak{b}.$$

It is straightforward to see that this Cartan subalgebra is the complexification of a real Cartan subalgebra of $\mathfrak{g}_\mathbb{R}$, and, moreover, $\mathfrak{b}$ is the complexification of the vector part.

We say that a strongly orthogonal θ-system Γ is a maximum strongly orthogonal θ-system if $|\Gamma| \geq |\Omega|$ for all strongly orthogonal θ-systems Ω.

Theorem 2.5. *Let θ be an involution of $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ compatible with the normal real structure such that $\mathfrak{h}$ is a standard Cartan subalgebra with respect to θ and $\theta(e_\alpha) = f_{-\alpha}$ whenever $\theta(\alpha) = -\alpha$. Then Γ is a maximum strongly orthogonal θ-system if and only if*

$$\mathfrak{h}_\theta \oplus \left(\bigoplus_{\alpha \in \Gamma} \mathbb{C}(e_\alpha + f_{-\alpha}) \right)$$

is a Cartan subalgebra of $\mathfrak{g}^\theta$. Moreover, if Ω is another maximum strongly orthogonal θ-system, then $\Omega = w\Gamma$ for some element w in $\mathcal{W}$ that commutes with θ.

Proof. Consider a strongly orthogonal θ-system Γ. Since $\Gamma \subseteq \Delta_\theta$, we have $\theta(\alpha) = -\alpha$ and so $h_\alpha \in \mathfrak{p}$ for all $\alpha \in \Gamma$. It follows that $\mathfrak{h}_\Gamma = \sum_{\alpha \in \Gamma} \mathbb{C}h_\alpha$ is a subspace

of $\mathfrak{a}$. The fact that Γ is a strongly orthogonal system ensures that $\{h_\alpha | \ \alpha \in \Gamma\}$ is a linearly independent set and so $\dim(\mathfrak{h}_\Gamma) = |\Gamma|$. Moreover, the subspace

$$\bigoplus_{\alpha \in \Gamma} \mathbb{C}(e_\alpha + f_{-\alpha})$$

of $\mathfrak{g}^\theta$ is a commutative Lie algebra and has the same dimension $|\Gamma|$. Since

$$(2.1) \qquad \mathfrak{h}_\theta \oplus \left(\bigoplus_{\alpha \in \Gamma} \mathbb{C}(e_\alpha + f_{-\alpha}) \right)$$

is a commutative Lie subalgebra of $\mathfrak{g}^\theta$, and hence a commutative Lie subalgebra of a Cartan subalgebra of $\mathfrak{g}^\theta$, we must have

$$(2.2) \qquad \mathrm{rank}(\mathfrak{g}^\theta) - \dim(\mathfrak{h}_\theta) \geq |\Gamma|,$$

where $\mathrm{rank}(\mathfrak{g}^\theta)$ is the rank of $\mathfrak{g}^\theta$ which is just the dimension of any Cartan subalgebra of $\mathfrak{g}^\theta$. It follows that equality holds in (2.2) if and only if (2.1) is a Cartan subalgebra of $\mathfrak{g}^\theta$. The first part of the theorem then follows once we show that there exists a strongly orthogonal θ-system of maximum possible size $\mathrm{rank}(\mathfrak{g}^\theta) - \dim(\mathfrak{h}_\theta)$.

Let $\mathfrak{s}$ be a maximally compact Cartan subalgebra of $\mathfrak{g}$ with respect to θ. It follows that $\mathfrak{s} \cap \mathfrak{g}^\theta$ is a Cartan subalgebra of $\mathfrak{g}^\theta$ and so $\dim(\mathfrak{s} \cap \mathfrak{g}^\theta) = \mathrm{rank}(\mathfrak{g}^\theta)$. Let $\mathfrak{b}$ be the complexification of the vector part of $\mathfrak{s}$ and let $\mathfrak{a}$ be a maximal commutative Lie algebra containing $\mathfrak{b}$ and contained in $\mathfrak{p}$. We can write $\mathfrak{h} = \mathfrak{h}_\theta \oplus \mathfrak{a}$ and $\mathfrak{s} = (\mathfrak{s} \cap \mathfrak{g}^\theta) \oplus \mathfrak{b}$.

By Theorem 2.2, there exists a strongly orthogonal θ-system Γ such that

$$(2.3) \qquad \mathfrak{b} = \{x \in \mathfrak{a} | \alpha(x) = 0 \text{ for all } \alpha \in \Gamma\}.$$

The decomposition (2.3) ensures that $\mathfrak{h}_\Gamma \cap \mathfrak{b} = 0$. Hence

$$(2.4) \qquad \mathfrak{h}_\Gamma + \mathfrak{b} = \mathfrak{h}_\Gamma \oplus \mathfrak{b} \subseteq \mathfrak{a}.$$

Let $\mathfrak{b}'$ be a subspace of $\mathfrak{a}$ such that

$$\mathfrak{h}_\Gamma \oplus \mathfrak{b}' \oplus \mathfrak{b} = \mathfrak{a}.$$

Recall that $\alpha(h_\alpha) > 0$ for all $\alpha \in \Gamma$. Since all pairs of roots in Γ are strongly orthogonal, it follows that $\alpha(h_\beta) = 0$ for all $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$. Thus the map $(\alpha, x) \mapsto \alpha(x)$ defines a nondegenerate pairing on $\mathbb{C}\Gamma \times \mathfrak{h}_\Gamma$. This nondegenerate pairing allows us to adjust the space $\mathfrak{b}'$ if necessary so that $\alpha(x) = 0$ for all $x \in \mathfrak{b}'$ and $\alpha \in \Gamma$. But then we see by (2.3), $\mathfrak{b}'$ must equal zero and the inclusion of (2.4) is actually an equality. Thus we have

$$\mathfrak{h} = \mathfrak{h}_\theta \oplus \mathfrak{h}_\Gamma \oplus \mathfrak{b}.$$

It follows that

$$\mathrm{rank}(\mathfrak{g}^\theta) = \dim(\mathfrak{s} \cap \mathfrak{g}^\theta) = \dim(\mathfrak{h}) - \dim(\mathfrak{b}) = \dim(\mathfrak{h}_\theta \oplus \mathfrak{h}_\Gamma) = \dim(\mathfrak{h}_\theta) \oplus |\Gamma|.$$

Hence if $\mathfrak{s}$ is a maximally compact Cartan subalgebra with respect to θ and Γ is a strongly orthogonal system satisfying (2.3), then

$$|\Gamma| = \mathrm{rank}(\mathfrak{g}^\theta) - \dim(\mathfrak{h}_\theta).$$

This completes the proof of the first assertion.

The second assertion follows from the fact that all maximally compact Cartan subalgebras with respect to θ are conjugate ([Kn, Proposition 6.61] and Theorem 2.2). $\qquad \square$

2.4. **Maximally split involutions.** In this section, we define the notion of maximally split involutions as presented in [L2, Section 7]. After reviewing basic properties from Section 7 of [L2], we identify a maximum strongly orthogonal θ-system Γ_θ for each choice of simple Lie algebra $\mathfrak{g}$ and maximally split involution θ. The choice of Γ_θ also satisfies certain conditions that will allow us to lift the Cartan subalgebra of Theorem 2.5 to the quantum setting compatible with quantum analogs of $U(\mathfrak{g}^\theta)$.

Definition 2.6 ([L2, Section 7])**.** The involution θ is called maximally split with respect to a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ if it satisfies the following three conditions:

   (i) $\theta(\mathfrak{h}) = \mathfrak{h}$;
   (ii) if $\theta(h_i) = h_i$, then $\theta(e_i) = e_i$ and $\theta(f_i) = f_i$;
   (iii) if $\theta(h_i) \neq h_i$, then $\theta(e_i)$ (resp., $\theta(f_i)$) is a root vector in $\mathfrak{n}^-$ (resp., $\mathfrak{n}^+$).

We call a pair $\mathfrak{g}, \mathfrak{g}^\theta$ a maximally split symmetric pair provided that $\mathfrak{g}$ is a complex semisimple Lie algebra and θ is a maximally split involution with respect to a known triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. (The triangular decomposition is not included in the notation of such symmetric pairs, but rather understood from context.) By Section 7 of [L2], every involution θ of the semisimple Lie algebra $\mathfrak{g}$ is conjugate to a maximally split involution.

The pair $\mathfrak{g}, \mathfrak{g}^\theta$ is irreducible if $\mathfrak{g}$ cannot be written as the direct sum of two complex semisimple Lie subalgebras which both admit θ as an involution. By [A], the pair $\mathfrak{g}, \mathfrak{g}^\theta$ is irreducible if and only if $\mathfrak{g}$ is simple or $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic as Lie algebras. Moreover, in this latter case, θ is the involution that sends e_i to f_i^*, f_i to e_i^*, and h_i to $-h_i^*$ where $e_i, h_i, f_i, i = 1, \ldots, n$ (resp., $e_i^*, h_i^*, f_i^*, i = 1, \ldots, n$) are the generators for $\mathfrak{g}_1$ (resp., $\mathfrak{g}_2$).

As in previous sections, we modify θ via conjugation with an automorphism of $\mathfrak{g}$ preserving root spaces so that $\theta(e_\alpha) = f_{-\alpha}$ and thus $e_\alpha + f_{-\alpha} \in \mathfrak{g}^\theta$ for all $\alpha \in \Delta_\theta$. The vector space $\mathfrak{h} \cap \mathfrak{p}$ is a maximal abelian Lie subalgebra of $\mathfrak{p}$ which we can take for $\mathfrak{a}$. In particular, $\mathfrak{h} = \mathfrak{h}_\theta \oplus \mathfrak{a}$ and $\mathfrak{h}$ is a maximally split Cartan subalgebra with respect to θ.

Let π_θ denote the subset of π fixed by θ. There exists a permutation p on $\{1, \ldots, n\}$ which induces a diagram automorphism (also called p) on π such that

$$\theta(-\alpha_i) - \alpha_{p(i)} \in Q^+(\pi_\theta)$$

for all $\alpha_i \in \pi \setminus \pi_\theta$. Note that p restricts to a permutation on $\pi \setminus \pi_\theta$. We sometimes write $p(\alpha_i)$ for $\alpha_{p(i)}$. We extend p to a map on $\mathfrak{h}^*$ by setting $p(\sum_i m_i \alpha_i) = \sum_i m_i p(\alpha_i)$.

Theorem 2.7. *For each maximally split symmetric pair* $\mathfrak{g}, \mathfrak{g}^\theta$, *there exists a maximum strongly orthogonal* θ-*system* $\Gamma_\theta = \{\beta_1, \ldots, \beta_m\}$ *and a set of simple roots* $\{\alpha_\beta, \alpha'_\beta | \; \beta \in \Gamma_\theta\}$ *such that for all* $\beta \in \Gamma_\theta$, *we have*

(2.5) $$\beta = \alpha_\beta, \quad \beta = \alpha'_\beta + w_\beta \alpha_\beta, \quad or \quad \beta = \alpha'_\beta + \alpha_\beta + w_\beta \alpha_\beta,$$

where $w_\beta = w(\mathrm{Supp}(\beta) \setminus \{\alpha'_\beta, \alpha_\beta\})_0$ *and for all* $j = 1, \ldots, m$, *the following conditions hold:*

   (i) *for all* $i > j$, $\mathrm{Supp}(\beta_i) \subset \mathrm{StrOrth}(\beta_j)$;
   (ii) $\mathrm{Supp}(\beta_j) \setminus \{\alpha_{\beta_j}, \alpha'_{\beta_j}\} \subseteq \mathrm{StrOrth}(\beta_j)$;
   (iii) θ *restricts to an involution on* $\mathrm{Supp}(\beta_j)$;
   (iv) $-w_{\beta_j}$ *restricts to a permutation on* $\pi_\theta \cap \mathrm{Supp}(\beta_j)$.

Moreover, we can further break down the expression for each β in Γ_θ given in (2.5) into the following five cases:

(1) $\beta = \alpha_\beta = \alpha'_\beta$.

(2) $\beta = \alpha_\beta + w_\beta \alpha_\beta$ *and* $\alpha_\beta = \alpha'_\beta$.

(3) $\beta = p(\alpha_\beta) + w_\beta \alpha_\beta = w\alpha_\beta$ *where* $w = w(\mathrm{Supp}(\beta) \setminus \{\alpha_\beta\})_0$. *In this case,* $\Delta(\mathrm{Supp}(\beta))$ *is a root system of type A_r where $r = |\mathrm{Supp}(\beta)|$ and $\alpha_\beta \neq \alpha'_\beta = p(\alpha_\beta)$.*

(4) $\beta = \alpha'_\beta + w_\beta \alpha_\beta = w\alpha'_\beta$ *where* $w = w(\mathrm{Supp}(\beta) \setminus \{\alpha'_\beta\})_0$. *In this case,* $\Delta(\mathrm{Supp}(\beta))$ *is a root system of type B_r where $r = |\mathrm{Supp}(\beta)|$, $\{\alpha_\beta\} = \mathrm{Supp}(\beta) \setminus \pi_\theta$ and α'_β is the unique short root in $\mathrm{Supp}(\beta)$.*

(5) $\beta = \alpha'_\beta + \alpha_\beta + w_\beta \alpha_\beta$ *and* $\alpha'_\beta \neq \alpha_\beta$. *In this case,* $\alpha'_\beta \in \pi_\theta$, α'_β *is strongly orthogonal to all $\alpha \in \mathrm{Supp}(\beta) \setminus \{\alpha'_\beta, \alpha_\beta\}$, and α'_β is not strongly orthogonal to β but the two roots are orthogonal (i.e., $(\alpha'_\beta, \beta) = 0$).*

Proof. Note that it is sufficient to prove the theorem for the case where $\mathfrak{g}, \mathfrak{g}^\theta$ is irreducible since, in the general case, we can simply take a disjoint union of strongly orthogonal systems corresponding to each irreducible component that satisfies the desired properties of the theorem. Consider the irreducible symmetric pair $\mathfrak{g}, \mathfrak{g}^\theta$ where $\mathfrak{g}$ is the direct sum of two isomorphic copies of the same Lie algebra and θ is the involution descibed above. Then θ sends roots of $\mathfrak{g}_1$ to those of $\mathfrak{g}_2$ and so Δ_θ is the empty set. It follows that the only choice for Γ_θ in this case is the empty set which clearly satisfies the conditions of the theorem. Thus, for the remainder of the proof, we assume that $\mathfrak{g}$ is simple.

The proof for $\mathfrak{g}$ simple is by cases. We use the notation in [H, Section 12.1] for the set of simple roots and orthonormal unit vectors associated to root systems of various types of simple Lie algebras. For each case, we define θ, define p if p does not equal the identity, state $\Delta_\theta, \mathfrak{h}_\theta$, and provide a choice for Γ_θ that satisfies the above conditions. If β is a simple root, then $\alpha_\beta = \beta = \alpha'_\beta$. For β not equal to a simple root in π, we specify α_β and α'_β. For the latter root, we also state whether $\alpha'_\beta \in \pi_\theta$ or whether it equals α_β or $p(\alpha_\beta)$. The relationship between β, α_β, and α'_β can then be read from context. For example, $\beta = \alpha'_\beta + w_\beta \alpha_\beta$ if $\alpha'_\beta = p(\alpha_\beta)$ while $\beta = \alpha_\beta + w_\beta \alpha_\beta$ if $\mathrm{mult}_{\alpha_\beta}(\beta) = 2$ and $\alpha'_\beta = \alpha_\beta$.

Recall that the automorphism group of a root system associated to a semisimple Lie algebra is the semidirect product of the Weyl group and the group of diagram automorphisms ([H, Section 12.2]). Moreover, -1 is not in the Weyl group associated to a simple Lie algebra if and only if this simple Lie algebra is one of the following types: $A_n(n > 1)$, D_n (for $n \geq 5$ and n odd), and E_6 ([H2, Section 4]). Hence, $-w_\beta$ restricts to the identity for all other cases, including D_n, n even. For type A_n and type E_6, $-w_\beta$ restricts to the unique permutation on the simple roots corresponding to the unique nonidentity diagram automorphism. Using this information, it is straightforward to check that (iv) holds in all cases.

Type AI: $\mathfrak{g}$ is of type A_n, $\theta(\alpha_i) = -\alpha_i$ for $1 \leq i \leq n$, $\Delta_\theta = \Delta^+$, $\mathfrak{h}_\theta = 0$, and we may choose $\Gamma_\theta = \{\beta_1, \ldots, \beta_s\}$ where $\beta_j = \alpha_{2j-1}$ for $j = 1, \ldots, s$, $s = \lfloor (n+1)/2 \rfloor$.

Type AII: $\mathfrak{g}$ is of type A_n where $n = 2m + 1$ is odd and $n \geq 3$, $\theta(\alpha_i) = \alpha_i$ for $i = 2j + 1$, $0 \leq j \leq m$, $\theta(\alpha_i) = -\alpha_{i-1} - \alpha_i - \alpha_{i+1}$ for $i = 2j$, $1 \leq j \leq m$. In this case, $\mathfrak{h}_\theta = \mathrm{span}_{\mathbb{C}}\{h_1, h_3, \ldots, h_{2m+1}\}$ and Δ_θ is empty, so Γ_θ is also the empty set.

Type AIII/AIV: $\mathfrak{g}$ is of type A_n, r is an integer with $1 \leq r \leq (n+1)/2$, $\theta(\alpha_j) = \alpha_j$ for all $r + 1 \leq j \leq n - r$, $\theta(\alpha_i) = -\alpha_{n-i+1}$ for $1 \leq i \leq r - 1$ and

$n - r + 2 \leq i \leq n$, $\theta(\alpha_r) = -\alpha_{r+1} - \alpha_{r+2} - \cdots - \alpha_{n-r} - \alpha_{n-r+1}$ and $\theta(\alpha_{n-r+1}) = -\alpha_{n-r} - \alpha_{n-r-1} - \cdots - \alpha_{r+1} - \alpha_r$. In this case, $p(i) = n - i + 1$ for all $i = 1, \ldots, n$, $\mathfrak{h}_\theta = \mathrm{span}_{\mathbb{C}}\{h_j, h_i - h_{n-i+1} | \, r + 1 \leq j \leq n - r, 1 \leq i \leq r\}$, $\Delta_\theta = \{\alpha_k + \alpha_{k+1} + \cdots + \alpha_{n-k+1} | \, 1 \leq k \leq r\}$, and the only choice for Γ_θ is $\Gamma_\theta = \Delta_\theta$. Hence we have $\Gamma_\theta = \{\beta_1, \cdots, \beta_r\}$, where $\beta_j = \alpha_j + \alpha_{j+1} + \cdots + \alpha_{n-j+1}$ for $j = 1, \cdots, r$. For each j, we may set $\alpha_{\beta_j} = \alpha_j$ and $\alpha'_{\beta_j} = p(\alpha_{\beta_j}) = \alpha_{n-j+1}$.

Type BI/BII: $\mathfrak{g}$ is of type B_n, r is an integer such that $1 \leq r \leq n$, $\theta(\alpha_i) = \alpha_i$ for all $r + 1 \leq i \leq n$, $\theta(\alpha_i) = -\alpha_i$ for all $1 \leq i \leq r - 1$ if $r < n$ and for all $1 \leq i \leq n$ if $r = n$, and $\theta(\alpha_r) = -\alpha_r - 2\alpha_{r+1} - \cdots - 2\alpha_{n-1} - 2\alpha_n$ if $r < n$. In this case, $\mathfrak{h}_\theta = \mathrm{span}_{\mathbb{C}}\{h_{r+1}, \ldots, h_n\}$, Δ_θ is the subset of the positive roots Δ^+ generated by the set $\{\alpha_1, \alpha_2, \ldots, \alpha_{r-1}, \alpha_r + \cdots + \alpha_n\}$ which is a root system of type B_r, and a choice for Γ_θ is $\Gamma_\theta = \{\beta_1, \cdots, \beta_r\}$ where $\beta_{2j-1} = \alpha_{2j-1} + 2\alpha_{2j} + \cdots + 2\alpha_n$ with $\alpha_{\beta_{2j-1}} = \alpha_{2j} = \alpha'_{\beta_{2j-1}}$ and $\beta_{2j} = \alpha_{2j-1}$ for $j = 1, \ldots, \lfloor r/2 \rfloor$, and if r is odd, then $\beta_r = \alpha_r + \cdots + \alpha_n$ with $\alpha_{\beta_r} = \alpha_r$ and $\alpha'_{\beta_r} = \alpha_n$.

Type CI: $\mathfrak{g}$ is of type C_n, $\theta(\alpha_i) = -\alpha_i$ for $1 \leq i \leq n$. We have $\Delta_\theta = \Delta^+$ and $\mathfrak{h}_\theta = 0$. There is one choice for Γ_θ, namely $\{2\epsilon_1, 2\epsilon_2, \ldots, 2\epsilon_n\}$. Rewriting this in terms of the simple roots $\{\alpha_1, \ldots, \alpha_n\}$, we have $\Gamma_\theta = \{\beta_1, \ldots, \beta_n\}$ where $\beta_j = 2\alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_{n-1} + \alpha_n$ for $1 \leq j \leq n - 1$ and $\beta_n = \alpha_n$. We have $\alpha_{\beta_j} = \alpha_j = \alpha'_{\beta_j}$ for $j = 1, \ldots, n$.

Type CII, Case 1: $\mathfrak{g}$ is of type C_n, r is even and satisfies $1 \leq r \leq n - 1$, $\theta(\alpha_j) = \alpha_j$ for $j = 1, 3, \cdots, r - 1$ and $j = r + 1, r + 2, \ldots, n$, $\theta(\alpha_i) = -\alpha_{i-1} - \alpha_i - \alpha_{i+1}$ for $i = 2, 4, \ldots, r - 2$, and $\theta(\alpha_r) = -\alpha_{r-1} - \alpha_r - 2\alpha_{r+1} - \cdots - 2\alpha_{n-1} - \alpha_n$. We have $\Delta_\theta = \{\alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_{n-1} + \alpha_n | \, j = 1, 3, \ldots, r - 1\}$ and $\mathfrak{h}_\theta = \mathrm{span}_{\mathbb{C}}\{h_1, h_3, \ldots, h_{r-1}, h_{r+1}, h_{r+2}, \ldots, h_n\}$. The only choice for Γ_θ is Δ_θ and so $\Gamma_\theta = \{\beta_1, \beta_2, \ldots, \beta_{r/2}\}$ where $\beta_j = \alpha_{2j-1} + 2\alpha_{2j} + \cdots + 2\alpha_{n-1} + \alpha_n$ with $\alpha'_{\beta_j} = \alpha_{2j-1}$ and $\alpha_{\beta_j} = \alpha_{2j}$ for $j = 1, 2, \ldots, r/2$.

Type CII, Case 2: $\mathfrak{g}$ is of type C_n, n is even, $\theta(\alpha_j) = \alpha_j$ for $j = 1, 3, \cdots, n - 1$, $\theta(\alpha_i) = -\alpha_{i-1} - \alpha_i - \alpha_{i+1}$ for $i = 2, 4, \ldots, n - 2$, and $\theta(\alpha_n) = -2\alpha_{n-1} - \alpha_n$. One checks that $\Delta_\theta = \{\alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_{n-1} + \alpha_n | \, j = 1, 3, \ldots, n - 3\} \cup \{\alpha_{n-1} + \alpha_n\}$ and $\mathfrak{h}_\theta = \mathrm{span}_{\mathbb{C}}\{h_1, h_3, \ldots, h_{n-1}\}$. Moreover, the only choice for Γ_θ is Δ_θ. Hence we have $\Gamma_\theta = \{\beta_1, \beta_2, \ldots, \beta_t\}$ where $t = n/2$, $\beta_j = \alpha_{2j-1} + 2\alpha_{2j} + \cdots + 2\alpha_{n-1} + \alpha_n$ for $j = 1, \ldots, t - 1$ and $\beta_t = \alpha_{n-1} + \alpha_n$. We have $\alpha'_{\beta_j} = \alpha_{2j-1}$ and $\alpha_{\beta_j} = \alpha_{2j}$ for $j = 1, \ldots, t$.

Type DI/DII, Case 1: $\mathfrak{g}$ is of type D_n, r is an integer such that $1 \leq r \leq n - 2$, $\theta(\alpha_i) = \alpha_i$ for $i = r + 1, r + 2, \ldots, n$, $\theta(\alpha_i) = -\alpha_i$ for $i = 1, \ldots, r - 1$, and $\theta(\alpha_r) = -\alpha_r - 2\alpha_{r+1} \cdots - 2\alpha_{n-2} - \alpha_{n-1} - \alpha_n$. In this case, $\mathfrak{h}_\theta = \mathrm{span}_{\mathbb{C}}\{h_i | \, r + 1 \leq i \leq n\}$ and Δ_θ is the set of all positive roots contained in the span of the union of the following two sets: $\{\alpha_j | \, 1 \leq j \leq r - 1\}$ and $\{\alpha_i + 2\alpha_{i+1} + 2\alpha_{i+2} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n | \, 1 \leq i \leq r - 1\}$. A choice for Γ_θ is $\Gamma_\theta = \{\beta_1, \ldots, \beta_{2t}\}$ where $\beta_{2j} = \alpha_{2j-e}$ and $\beta_{2j-1} = \alpha_{2j-e} + 2\alpha_{2j-e+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ with $\alpha_{\beta_{2j-1}} = \alpha_{2j-e+1} = \alpha'_{\beta_{2j-1}}$ for $j = 1, \ldots, t$ where $t = \lfloor r/2 \rfloor$, $e = 0$ if r is odd and $e = 1$ if r is even.

Type DI, Case 2: $\mathfrak{g}$ is of type D_n with $n \geq 4$ (case $n = 3$ is the same as type AI), $\theta(\alpha_i) = -\alpha_i$ for $1 \leq i \leq n - 2$, $\theta(\alpha_n) = -\alpha_{n-1}$ and $\theta(\alpha_{n-1}) = -\alpha_n$. In this case, $p(i) = i$ for $i = 1, \ldots, n - 2$, $p(n - 1) = n$, $p(n) = n - 1$, $\mathfrak{h}_\theta = \mathbb{C}(h_{n-1} - h_n)$ and Δ_θ is the set consisting of those positive roots in the span of the union of the following two sets: $\{\alpha_i | \, i = 1, \ldots, n - 2\}$ and $\{\alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n | \, 1 \leq i \leq n - 2\}$. A choice for Γ_θ is $\{\beta_1, \ldots, \beta_{2t}\}$ where $\beta_{2j} = \alpha_{2j-e}$ and $\beta_{2j-1} = \alpha_{2j-e} + 2\alpha_{2j-e+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ for $j = 1, \ldots, t$ where $t = \lfloor (n-1)/2 \rfloor$,

$e = 1$ if n is odd and $e = 0$ if n is even. Note that $\beta_{n-3+e} = \alpha_{n-2} + \alpha_{n-1} + \alpha_n$. We have $\alpha_{\beta_{2j-1}} = \alpha_{2j-e+1} = \alpha'_{\beta_{2j-1}}$ for $j = 1, \ldots, t-1$, $\alpha_{\beta_{n-3+e}} = \alpha_{n-1}$ and $\alpha'_{\beta_{n-3+e}} = p(\alpha_{n-1}) = \alpha_n$.

Type DI, Case 3: $\mathfrak{g}$ is of type D_n, $\theta(\alpha_i) = -\alpha_i$ all $i = 1, \ldots, n$. Here, we have $\mathfrak{h}_\theta = 0$ and $\Gamma_\theta = \{\beta_1, \ldots, \beta_{2t}\}$ where $t = \lfloor n/2 \rfloor$, $\beta_{2j} = \alpha_{2j-e}$ and $\beta_{2j-1} = \alpha_{2j-e} + 2\alpha_{2j-e+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ for $j = 1, \ldots, t-1$, $\beta_{2t} = \alpha_{n-1}$, and $\beta_{2t-1} = \alpha_n$ where $e = 0$ if n is odd and $e = 1$ if n is even. We have $\alpha_{\beta_{2j-1}} = \alpha_{2j-e+1} = \alpha'_{\beta_{2j-1}}$ for $j = 1, \ldots, t-1$.

Type DIII, Case 1: $\mathfrak{g}$ is of type D_n, n is even, $\theta(\alpha_i) = \alpha_i$ for $i = 1, 3, \ldots, n-1$, $\theta(\alpha_i) = -\alpha_{i-1} - \alpha_i - \alpha_{i+1}$ for $i = 2, 4, \ldots, n-2$, and $\theta(\alpha_n) = -\alpha_n$. In this case, $\Delta_\theta = \{\alpha_k + 2\alpha_{k+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n | \ k = 1, 3, 5, \cdots, n-3\} \cup \{\alpha_n\}$. and $\mathfrak{h}_\theta = \text{span}_{\mathbb{C}}\{h_1, h_3, \ldots, h_{n-1}\}$. It follows that $\Gamma_\theta = \Delta_\theta = \{\beta_1, \ldots, \beta_{n/2}\}$ where $\beta_j = \alpha_{2j-1} + 2\alpha_{2j} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ for $j = 1, \ldots, n/2 - 1$ and $\beta_{n/2} = \alpha_n$. We have $\alpha_{\beta_j} = \alpha_{2j} = \alpha'_{\beta_j}$ for $j = 1, \ldots, n/2 - 1$.

Type DIII, Case 2: $\mathfrak{g}$ is of type D_n, n is odd, $\theta(\alpha_i) = \alpha_i$ for $i \in \{1, 3, 5, \ldots, n-2\}$, $\theta(\alpha_i) = -\alpha_{i-1} - \alpha_i - \alpha_{i+1}$ for $i = 2, 4, 6, \ldots, n-3$, $\theta(\alpha_{n-1}) = -\alpha_{n-2} - \alpha_n$, and $\theta(\alpha_n) = -\alpha_{n-2} - \alpha_{n-1}$. In this case, $p(i) = i$ for $i = 1, \ldots, n-2$, $p(n-1) = n$, $p(n) = n-1$. We have $\Delta_\theta = \{\alpha_k + 2\alpha_{k+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n | \ k = 1, 3, 5, \cdots, n-2\}$ and $\mathfrak{h}_\theta = \text{span}_{\mathbb{C}}(\{h_i | \ i = 1, 3, \ldots, n-2\} \cup \{h_{n-1} - h_n\})$. It follows that $\Gamma_\theta = \Delta_\theta$ and so $\Gamma_\theta = \{\beta_1, \ldots, \beta_t\}$ where $t = (n-1)/2$, $\beta_j = \alpha_{2j-1} + 2\alpha_{2j} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ with $\alpha_{\beta_j} = \alpha_{2j} = \alpha'_{\beta_j}$ for $j = 1, \ldots, t-1$ and $\beta_t = \alpha_{n-2} + \alpha_{n-1} + \alpha_n$ with $\alpha_{\beta_t} = \alpha_{n-1}, \alpha'_{\beta_t} = p(\alpha_{\beta_{n-1}}) = \alpha_n$.

Type EI, EV, EVIII: $\mathfrak{g}$ is of type E6, E7, E8, respectively, and $\theta(\alpha_i) = -\alpha_i$ for all i. Hence $\Delta_\theta = \Delta^+$ and $\mathfrak{h}_\theta = 0$. Set

$$\gamma_1 = \epsilon_8 + \epsilon_7 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8, \quad \alpha_{\gamma_1} = \alpha_8,$$

$$\gamma_2 = \epsilon_8 - \epsilon_7 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \quad \alpha_{\gamma_2} = \alpha_1,$$

$$\gamma_3 = \epsilon_6 + \epsilon_5 = \alpha_7 + 2\alpha_6 + 2\alpha_5 + 2\alpha_4 + \alpha_3 + \alpha_2, \quad \alpha_{\gamma_3} = \alpha_6,$$

$$\gamma_4 = \epsilon_6 - \epsilon_5 = \alpha_7,$$

$$\gamma_5 = \epsilon_4 + \epsilon_3 = \alpha_5 + 2\alpha_4 + \alpha_3 + \alpha_2, \quad \alpha_{\gamma_5} = \alpha_4,$$

$$\gamma_6 = \epsilon_4 - \epsilon_3 = \alpha_5,$$

$$\gamma_7 = \epsilon_2 - \epsilon_1 = \alpha_3,$$

$$\gamma_8 = \epsilon_2 + \epsilon_1 = \alpha_2.$$

Case 1. $\mathfrak{g}$ is of type E8. A choice for Γ_θ is the set $\{\beta_1, \ldots, \beta_8\}$ where $\beta_i = \gamma_i$ for $i = 1, 2, \ldots, 8$ and $\alpha_{\beta_j} = \alpha_{\gamma_j} = \alpha'_{\beta_j}$ for $j = 1, 2, 3, 5$.

Case 2. $\mathfrak{g}$ is of type E7. A choice for Γ_θ is the set $\{\beta_1, \ldots, \beta_7\}$ where $\beta_i = \gamma_{i+1}$ for $i = 1, 2, \ldots, 7$ and $\alpha_{\beta_j} = \alpha_{\gamma_{j+1}} = \alpha'_{\beta_j}$ for $j = 1, 2, 4$.

Case 3. $\mathfrak{g}$ is of type E6. A choice for Γ_θ is $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ where $\beta_i = \gamma_{i+4}$ for $i = 1, 2, 3, 4$ and $\alpha_{\beta_j} = \alpha_{\gamma_{j+4}} = \alpha'_{\beta_j}$ for $j = 1$.

Type EII: $\mathfrak{g}$ is of type E6, $\theta(\alpha_1) = -\alpha_6, \theta(\alpha_2) = -\alpha_2, \theta(\alpha_3) = -\alpha_5, \theta(\alpha_4) = -\alpha_4, \theta(\alpha_5) = -\alpha_3, \theta(\alpha_6) = -\alpha_1$. In this case, $p(1) = 6, p(2) = 2, p(3) = 5, p(4) = 4, p(5) = 3, p(6) = 1$, $\mathfrak{h}_\theta = \text{span}_{\mathbb{C}}\{h_1 - h_6, h_3 - h_5\}$ and $\Delta_\theta = \Delta^+ \cap \text{span}\{\alpha_1 + \alpha_6, \alpha_3 + \alpha_5, \alpha_4\}$. Here, we may choose $\Gamma_\theta = \{\beta_1, \beta_2, \beta_3\}$ where $\beta_1 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$ with $\alpha_{\beta_1} = \alpha_1$ and $\alpha'_{\beta_1} = p(\alpha_{\beta_1}) = \alpha_6$, $\beta_2 = \alpha_3 + \alpha_4 + \alpha_5$ with $\alpha_{\beta_2} = \alpha_3$ and $\alpha'_{\beta_2} = p(\alpha_{\beta_2}) = \alpha_5$, and $\beta_3 = \alpha_4$.

Type EIII: $\mathfrak{g}$ is of type E6, $\theta(\alpha_j) = \alpha_j$ for $j = 3, 4, 5$, $\theta(\alpha_1) = -\alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$, $\theta(\alpha_6) = \alpha_5 - \alpha_4 - \alpha_3 - \alpha_1$, and $\theta(\alpha_2) = -2\alpha_4 - \alpha_3 - \alpha_5 - \alpha_2$. In this case $p(1) = 6, p(2) = 2, p(3) = 5, p(4) = 4, p(5) = 3, p(6) = 1$, $\mathfrak{h}_\theta = \text{span}_{\mathbb{C}}\{h_3, h_4, h_5, h_1 - h_6\}$ and $\Delta_\theta = \{\beta_1, \beta_2\} = \Gamma_\theta$ where $\beta_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$, $\beta_2 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$, $\alpha_{\beta_1} = \alpha_2 = \alpha'_{\beta_1}$, $\alpha_{\beta_2} = \alpha_1$, and $\alpha'_{\beta_2} = p(\alpha_{\beta_2}) = \alpha_6$.

Type EIV: $\mathfrak{g}$ is of type E6, $\theta(\alpha_1) = -\alpha_1 - 2\alpha_3 - 2\alpha_4 - \alpha_2 - \alpha_5$, $\theta(\alpha_6) = -\alpha_6 - 2\alpha_5 - 2\alpha_4 - \alpha_2 - \alpha_3$, $\theta(\alpha_j) = \alpha_j$ for $j = 2, 3, 4, 5$. We have $\mathfrak{h}_\theta = \text{span}_{\mathbb{C}}\{h_2, h_3, h_4, h_5\}$. One checks that Δ_θ must be in the span of the two vectors $2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$ and $2\alpha_6 + \alpha_2 + 2\alpha_4 + 2\alpha_5 + \alpha_3$. Since this span does not contain any positive roots, we get Δ_θ is the empty set and so Γ_θ is also the empty set.

Type EVI: $\mathfrak{g}$ is of type E7, $\theta(\alpha_j) = \alpha_j$ for $j = 2, 5, 7$, $\theta(\alpha_6) = -\alpha_6 - \alpha_5 - \alpha_7$, $\theta(\alpha_4) = -\alpha_2 - \alpha_5 - \alpha_4$, $\theta(\alpha_i) = -\alpha_i$ for $i = 1, 3$. In this case, $\Delta_\theta = \text{span}\{\alpha_1, \gamma_2, \gamma_3, \gamma_5, \gamma_7\} \cap \Delta^+$ and $\mathfrak{h}_\theta = \text{span}_{\mathbb{C}}\{h_2, h_5, h_7\}$. A choice for Γ_θ is $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ where $\beta_1 = \gamma_2$ with $\alpha_{\beta_1} = \alpha_{\gamma_2}$ and $\beta_j = \gamma_{2j-1}$ with $\alpha_{\beta_j} = \alpha_{\gamma_{2j-1}}$ for $j = 2, 3, 4$.

Type EVII: $\mathfrak{g}$ is of type E7, $\theta(\alpha_i) = \alpha_i$ for $i = 2, 3, 4, 5$, $\theta(\alpha_1) = -\alpha_1 - 2\alpha_3 - 2\alpha_4 - \alpha_2 - \alpha_5$, $\theta(\alpha_6) = -\alpha_6 - 2\alpha_5 - 2\alpha_4 - \alpha_2 - \alpha_3$, $\theta(\alpha_7) = -\alpha_7$. We have $\mathfrak{h}_\theta = \text{span}_{\mathbb{C}}\{h_2, h_3, h_4, h_5\}$ and $\Delta_\theta = \Gamma_\theta = \{\beta_1, \beta_2, \beta_3\}$ where following the notation for types EI, EV, EVIII, we have $\beta_1 = \gamma_2$ with $\alpha_{\beta_1} = \alpha_1 = \alpha'_{\beta_1}$, $\beta_2 = \gamma_3$ with $\alpha_{\beta_2} = \alpha_6 = \alpha'_{\beta_2}$, and $\beta_3 = \gamma_4 = \alpha_7$.

Type EIX: $\mathfrak{g}$ is of type E8, $\theta(\alpha_i) = \alpha_i$ for $i = 2, 3, 4, 5$, $\theta(\alpha_i) = -\alpha_i$ for $i = 7, 8$, $\theta(\alpha_1) = -\alpha_1 - 2\alpha_3 - 2\alpha_4 - \alpha_2 - \alpha_5$, $\theta(\alpha_6) = -\alpha_6 - 2\alpha_5 - 2\alpha_4 - \alpha_2 - \alpha_3$. The vector space $\mathfrak{h}_\theta = \text{span}_{\mathbb{C}}\{h_2, h_3, h_4, h_5\}$ and $\Delta_\theta = \text{span}\{\beta_1, \beta_2, \beta_3, \beta_4\} = \Gamma_\theta$ where following the notation for types EI, EV, EVIII, we have $\beta_1 = \gamma_1$ with $\alpha_{\beta_1} = \alpha_8 = \alpha'_{\beta_1}$, $\beta_2 = \gamma_2$ with $\alpha_{\beta_1} = \alpha_1 = \alpha'_{\beta_1}$, $\beta_3 = \gamma_3$ with $\alpha_{\beta_2} = \alpha_6 = \alpha'_{\beta_2}$, and $\beta_4 = \gamma_4 = \alpha_7$.

Type FI: $\mathfrak{g}$ is of type F4, $\theta(\alpha_i) = -\alpha_i$, $i = 1, 2, 3, 4$. In this case, $\mathfrak{h}_\theta = 0$ and $\Delta_\theta = \Delta^+$. A choice for $\Gamma_\theta = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ where $\beta_1 = \epsilon_1 + \epsilon_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$ with $\alpha_{\beta_1} = \alpha_1 = \alpha'_{\beta_1}$, $\beta_2 = \epsilon_1 - \epsilon_2 = \alpha_2 + 2\alpha_3 + 2\alpha_4$ with $\alpha_{\beta_2} = \alpha_4 = \alpha'_{\beta_2}$, $\beta_3 = \epsilon_3 + \epsilon_4 = \alpha_2 + 2\alpha_3$ with $\alpha_{\beta_3} = \alpha_3 = \alpha'_{\beta_3}$, and $\beta_4 = \epsilon_3 - \epsilon_4 = \alpha_2$.

Type FII: $\mathfrak{g}$ is of type F4, $\theta(\alpha_i) = \alpha_i$ for $i = 1, 2, 3$, $\theta(\alpha_4) = -\alpha_4 - 3\alpha_3 - 2\alpha_2 - \alpha_1$. In this case, $\mathfrak{h}_\theta = \text{span}\{h_1, h_2, h_3\}$ and $\Delta_\theta = \{\beta_1\} = \Gamma_\theta$ where $\beta_1 = \epsilon_1 = 2\alpha_4 + 2\alpha_2 + \alpha_1 + 3\alpha_3$ and $\alpha_{\beta_1} = \alpha_4 = \alpha'_{\beta_1}$.

Type G: $\mathfrak{g}$ is of type G2, $\theta(\alpha_i) = -\alpha_i$ for $i = 1, 2$. In this case, $\mathfrak{h}_\theta = 0$ and $\Delta_\theta = \Delta^+$. There are three possibilities for a maximum strongly orthogonal θ-system inside Δ_θ, namely

(1) $\{-2\epsilon_1 + \epsilon_2 + \epsilon_3, -\epsilon_2 + \epsilon_3\} = \{\alpha_2, \alpha_2 + 2\alpha_1\}$;
(2) $\{-2\epsilon_2 + \epsilon_1 + \epsilon_3, -\epsilon_1 + \epsilon_3\} = \{\alpha_2 + 3\alpha_1, \alpha_2 + \alpha_1\}$;
(3) $\{2\epsilon_3 - \epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_2\} = \{2\alpha_2 + 3\alpha_1, \alpha_1\}$.

The first subset is a choice for Γ_θ. So we have $\Gamma_\theta = \{\beta_1, \beta_2\}$ where $\beta_1 = \alpha_2 + 2\alpha_1$, $\alpha_{\beta_1} = \alpha_1 = \alpha'_{\beta_1}$ and $\beta_2 = \alpha_2$. $\qquad\square$

The next remark expands on properties of the roots described in the above theorem in light of the specific examples of Γ_θ given for each symmetric pair.

Remark 2.8. Let $\mathfrak{g}, \mathfrak{g}^\theta$ be an irreducible maximally split symmetric pair and let Γ_θ be a maximum strongly orthogonal θ-system satisfying the conditions of Theorem 2.7. Let $\beta \in \Gamma_\theta$ and set $r = |\text{Supp}(\beta)|$. Then in the explicit versions of Γ_θ given in

the proof of the theorem, we see that

- if β satisfies (3), then $\Delta(\text{Supp}(\beta))$ is a root system of type A_r where $r = |\text{Supp}(\beta)|$ and θ restricts to an involution on $\mathfrak{g}_{\text{Supp}(\beta)}$ of type AIII/AIV;
- if β satisfies (4), then β is the final root in Γ_θ where either $\mathfrak{g}, \mathfrak{g}^\theta$ is of type BI/BII and r is odd or $\mathfrak{g}, \mathfrak{g}^\theta$ is of type CII Case 2;
- if β satisfies (5), then $\Delta(\text{Supp}(\beta))$ is a root system of type C_r, θ restricts to an involution of type CII on $\mathfrak{g}_{\text{Supp}(\beta)}$, and $\mathfrak{g}, \mathfrak{g}^\theta$ is of type CII.

3. Quantized enveloping algebras

In this section, we turn our attention to the quantized enveloping algebra of the complex semisimple Lie algebra $\mathfrak{g}$ with a chosen triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. After setting basic notation, we describe the structure of the locally finite part and a realization of dual Verma modules as subspaces of the quantized enveloping algebra. We also discuss Lusztig's automorphisms and Chevalley antiautomorphisms as well as the concept of specialization.

3.1. **Basic notation: The quantum case.** Let q be an indeterminate. The quantized enveloping algebra, $U_q(\mathfrak{g})$, is the Hopf algebra over $\mathbb{C}(q)$ generated as an algebra by $E_i, F_i, K_i^{\pm 1}, i = 1, \ldots, n$ with relations and Hopf algebra structure as defined in [L2, Section 1] (with x_i, y_i, t_i replaced by E_i, F_i, K_i) or [Ko, Section 3.1]. Sometimes we write E_α (resp., $F_{-\alpha}$) for E_j (resp., F_j) where α is the simple root α_j. Note that a counit ϵ, which is an algebra homomorphism from $U_q(\mathfrak{g})$ to the scalars, is part of the Hopf structure. In our setting, the counit is defined by $\epsilon(E_i) = \epsilon(F_i) = 0$ and $\epsilon(K_i^{\pm 1}) = 1$.

Let U^+ be the $\mathbb{C}(q)$ subalgebra of $U_q(\mathfrak{g})$ generated by $E_i, i = 1, \ldots, n$. Similarly, let U^- be the $\mathbb{C}(q)$ subalgebra of $U_q(\mathfrak{g})$ generated by $F_i, i = 1, \ldots, n$. Let $\mathcal{T}$ denote the group generated by the $K_1^{\pm 1}, \ldots, K_n^{\pm 1}$ and write U^0 for the group algebra over $\mathbb{C}(q)$ generated by $\mathcal{T}$. The algebra $U_q(\mathfrak{g})$ admits a triangular decomposition which is an isomorphism of vector spaces

$$(3.1) \qquad\qquad U_q(\mathfrak{g}) \cong U^- \otimes U^0 \otimes U^+$$

via the multiplication map. Let G^+ (resp., G^-) be the subalgebra of $U_q(\mathfrak{g})$ generated by $E_i K_i^{-1}, i = 1, \ldots, n$ (resp., $F_i K_i, i = 1, \ldots, n$). One gets similar triangular decompositions upon replacing U^+ by G^+ or U^- by G^-.

The (left) adjoint action of $U_q(\mathfrak{g})$, which makes $U_q(\mathfrak{g})$ into a left module over itself, comes from the Hopf algebra structure and is defined on generators by

- $(\text{ad } E_i)a = E_i a - K_i a K_i^{-1} E_i$,
- $(\text{ad } K_i)a = K_i a K_i^{-1}$ and $(\text{ad } K_i^{-1})a = K_i^{-1} a K_i$,
- $(\text{ad } F_i)a = F_i a K_i - a F_i K_i = F_i K_i K_i^{-1} a K_i - a F_i K_i$

for all $a \in U_q(\mathfrak{g})$ and each $i = 1, \ldots, n$. Temporarily fix i. It is straightforward to check that $(\text{ad } E_i)a = \epsilon(E_i)a = 0$, $(\text{ad } F_i)a = \epsilon(F_i)a = 0$ and $(\text{ad } K_i^{\pm 1})a = \epsilon(K_i^{\pm 1})a = a$ if and only if $[E_i, a] = [F_i, a] = [K_i^{\pm 1}, a] = 0$.

Set $A = \mathbb{C}[q, q^{-1}]_{(q-1)}$ and let $\hat{U}$ be the A subalgebra generated by

$$E_i, F_i, K_i^{\pm 1}, \frac{(K_i - 1)}{(q - 1)}$$

for $i = 1, \ldots, n$. The specialization of $U_q(\mathfrak{g})$ at $q = 1$ is the $\mathbb{C}$ algebra $\hat{U} \otimes_A \mathbb{C}$. It is well known that $\hat{U} \otimes_A \mathbb{C}$ is isomorphic to $U(\mathfrak{g})$ (see for example [L2]). Given

a subalgebra S of $U_q(\mathfrak{g})$, we say that S specializes to the subalgebra $\bar{S}$ of $U(\mathfrak{g})$ provided the image of $S \cap \hat{U}$ in $\hat{U} \otimes_A \mathbb{C}$ is $\bar{S}$.

Given a $U_q(\mathfrak{g})$-module V, the λ weight space of V, denoted V_λ, is the subspace consisting of vectors v such that $K_i v = q^{(\alpha_i, \lambda)} v$ for each i. Elements of V_λ are called weight vectors with respect to this module structure. We say that $u \in U_q(\mathfrak{g})$ is a weight vector of weight λ if it is a weight vector of weight λ with respect to the adjoint action. In other words, $K_i u K_i^{-1} = q^{(\alpha_i, \lambda)} u$ for all $i = 1, \ldots, n$. Given any (ad $U_q(\mathfrak{g})$)-module M and a weight $\lambda \in Q(\pi)$, let M_λ denote the λ weight space of M with respect to the adjoint action.

We can write $U_q(\mathfrak{g})$ as a direct sum of weight spaces with respect to the adjoint action where all weights are in $Q(\pi)$. Suppoe that $a = \sum_\lambda a_\lambda \in U_q(\mathfrak{g})$ where each $a_\lambda \in (U_q(\mathfrak{g}))_\lambda$. We say that v is a weight summand of a with respect to the expression of a as a sum of weight vectors (or, simply, v is a weight summand of a) if $v = a_\lambda$ for some λ. Note that U^-, U^+, G^-, G^+ can all also be written as a sum of weight spaces with respect to the adjoint action. Using (3.1), we have the following direct sum:

$$(3.2) \qquad U_q(\mathfrak{g}) = \bigoplus_{\lambda, \mu \in Q^+(\pi)} G_{-\lambda}^- U^0 U_\mu^+.$$

Note that

$$G_{-\lambda}^- U^0 = U_{-\lambda}^- U^0$$

for each $\lambda \in Q^+(\pi)$ and so we can replace $G_{-\lambda}^-$ with $U_{-\lambda}^-$ in the above direct sum decomposition. Similarly, we could replace U_μ^+ with G_μ^+ for $\mu \in Q^+(\pi)$. We refer to $G_{-\lambda}^- U^0 U_\mu^+$ as a biweight subspace of $U_q(\mathfrak{g})$. An element $v \in G_{-\lambda}^- U^0 U_\mu^+$ is said to have biweight $(-\lambda, \mu)$. Note that a vector of biweight $(-\lambda, \mu)$ has weight $\mu - \lambda$. We call $a = \sum_{\lambda, \mu \in Q^+(\pi)} a_{\lambda, \mu}$ where each $a_{\lambda, \mu} \in G_{-\lambda}^- U^0 U_\mu^+$ the expansion of a as a sum of biweight vectors and call v a biweight summand of a if $v = a_{\lambda, \mu}$ for some $\lambda, \mu \in Q(\pi)$.

We have the following coarse version of (3.2):

$$(3.3) \qquad U_q(\mathfrak{g}) = \bigoplus_{\lambda \in Q^+(\pi)} G_{-\lambda}^- U^0 U^+.$$

Suppose that $\sum_{\lambda, \mu \in Q^+(\pi)} a_{\lambda, \mu}$ is the expansion of a as a sum of biweight vectors. Then

$$a = \sum_{\lambda \in Q^+(\pi)} a_\lambda \text{ where } a_\lambda = \sum_{\mu \in Q^+(\pi)} a_{\lambda, \mu}$$

is the expression of a as a sum using the direct sum decomposition (3.3). Given an element u in $G_{-\zeta}^- U^0 U^+$, set l-weight$(u) = -\zeta$. We say that u is a minimal l-weight summand of a if $u \in G_{-\zeta}^- U^0 U^+$ and $a - u \in \sum_{\lambda \not\geq \zeta} G_{-\lambda}^- U^0 U^+$.

Set $K_\xi = K_1^{\xi_1} \cdots K_n^{\xi_n}$ for each $\xi = \sum_i \xi_i \alpha_i$ in the root lattice $Q(\pi)$. Note that the map $\xi \mapsto K_\xi$ defines an isomorphism from $Q(\pi)$ to $\mathcal{T}$. We can enlarge the group $\mathcal{T}$ to a group $\check{\mathcal{T}}$ so that this isomorphism extends to an isomorphism from the weight lattice $P(\pi)$ to $\check{\mathcal{T}}$. In particular, let r be a positive integer such that $r\eta \in Q(\pi)$ for each $\eta \in P(\pi)$ and form the free abelian group of rank n generated by $K_1^{1/r}, \ldots, K_n^{1/r}$ where $(K_j^{1/r})^r = K_j$ for each j. Note that $\mathcal{T}$ embeds in this group in the obvious way. Given $\eta \in P(\pi)$, we set $K_\eta = K_1^{\eta_1/r} \cdots K_n^{\eta_n/r}$ where

$\eta = (\eta_1/r)\alpha_1 + \cdots + (\eta_n/r)\alpha_n$ and each η_j is an integer. The group $\check{\mathcal{T}}$ is the subgroup of $\langle K_1^{1/r}, \ldots, K_n^{1/r} \rangle$ consisting of all elements of the form K_ξ where $\xi \in P(\pi)$ and thus the map $\xi \mapsto K_\xi$ defines an isomorphism from $P(\pi)$ to $\check{\mathcal{T}}$.

Let N be a positive integer so that $(\mu, \alpha_i) \in \frac{1}{N}\mathbb{Z}$ for all $\mu \in P(\pi)$ and $\alpha_i \in \pi$. We will often work with elements in the simply connected quantized enveloping algebra $\check{U}$, a Hopf algebra containing $U_q(\mathfrak{g})$. As an algebra, $\check{U}$ is generated over $\mathbb{C}(q^{1/N})$ by $U_q(\mathfrak{g})$ and $\check{\mathcal{T}}$ such that $(\mathrm{ad}\, K_\mu)v = K_\mu v K_\mu^{-1} = q^{(\mu,\lambda)}v$ for all $v \in \check{U}_\lambda$. Note that the counit ϵ satisfies $\epsilon(K_\mu) = 1$ for all $K_\mu \in \check{\mathcal{T}}$. (For more details, see [J, Section 3.2.10]). Thus elements of $\check{U}$ are linear combinations of terms of the form uK_μ where $u \in U_q(\mathfrak{g})$ and $\mu \in P(\pi)$. Note that (3.2) and thus the notion of biweight extends to $\check{U}$.

Given $\lambda \in P(\pi)$, write $M(\lambda)$ for the universal highest weight $U_q(\mathfrak{g})$-module (i.e., the Verma module) generated by a highest weight vector of weight λ and let $L(\lambda)$ be the simple highest weight $U_q(\mathfrak{g})$-module of highest weight λ. When λ is in $P^+(\pi)$, the simple module $L(\lambda)$ is finite-dimensional.

Suppose that π' is a subset of π. Set $U_{\pi'}$ equal to the subalgebra of $U_q(\mathfrak{g})$ generated by $E_i, K_i^{\pm 1}, F_i$ for all $\alpha_i \in \pi'$. Note that we can identify $U_{\pi'}$ with the quantized enveloping algebra $U_q(\mathfrak{g}_{\pi'})$. Given a subalgebra M of $U_q(\mathfrak{g})$, set $M_{\pi'} = U_{\pi'} \cap M$. For example, the subalgebra $U_{\pi'}^+$ of U^+ is the algebra generated by E_i with $\alpha_i \in \pi'$. Given $\lambda \in P^+(\pi')$, write $L_{\pi'}(\lambda)$ for the simple highest weight $(U_{\pi'}U^0)$-module of highest weight λ. Given a subset S of $U_q(\mathfrak{g})$, we write $S^{U_{\pi'}}$ for the set

$$\{s \in S|\ [a, s] = 0 \text{ for all } a \in U_{\pi'}\}.$$

By the definition of the adjoint action and the counit ϵ, we see that $S^{U_{\pi'}}$ also equals the set

$$\{s \in S|\ (\mathrm{ad}\ a)s = \epsilon(a)s \text{ for all } a \in U_{\pi'}\}.$$

We refer to $S^{U_{\pi'}}$ as a trivial $(\mathrm{ad}\ U_{\pi'})$-module. The set $S^{U_{\pi'}}$ is also called the centralizer of $U_{\pi'}$ inside of S.

We call a lowest weight vector f with respect to the action of $U_q(\mathfrak{g})$ (i.e., $F_i f = 0$ for all $\alpha_i \in \pi$) inside a module for this algebra a $U_q(\mathfrak{g})$ *lowest weight vector*. In the special case where the action is the adjoint action, we call f an $(\mathrm{ad}\ U_q(\mathfrak{g}))$ *lowest weight vector*. We use the same language for subsets π' of π with $U_q(\mathfrak{g})$ replaced by $U_{\pi'}$.

3.2. Dual vermas and the locally finite part.
We recall here the construction of a number of $(\mathrm{ad}\, U_q(\mathfrak{g}))$-modules. (A good reference with more details is [J, Section 7.1].) Given $\lambda \in P(\pi)$ we can realize G^- as an $(\mathrm{ad}\, U_q(\mathfrak{g}))$-module, which we refer to as $G^-(\lambda)$, via the action

$$(\mathrm{ad}_\lambda E_i)1 = 0, \quad (\mathrm{ad}_\lambda K_i)1 = q^{\frac{1}{2}(\lambda,\alpha_i)}1, \quad (\mathrm{ad}_\lambda F_i)1 = (1 - q^{(\lambda,\alpha_i)})F_i K_i$$

for $i = 1, \ldots, n$. (Here we are using a slightly different Hopf structure and so a slightly different adjoint action which is why we have G^- instead of U^- as in [J, Section 7.1].) With respect to this action, $G^-(\lambda)$ is isomorphic to the dual module $\delta M(\lambda/2)$ of $M(\lambda/2)$. Thus $G^-(\lambda)$ contains a unique simple submodule isomorphic to $L(\lambda/2)$. Note that we are following the construction in [J, Section 7.1] in choosing $\lambda \in P(\pi)$ as opposed to $\lambda \in 2P(\pi)$. This means that there is a

possibility that $\lambda/2$ is not in $P(\pi)$. However, $\lambda/2$ is still a well-defined element of $\mathfrak{h}^*$ (as explained in Section 2.1) and the construction works fine when this happens.

If γ is dominant integral, then $L(\gamma)$ is finite-dimensional and hence can also be viewed as a module generated by a lowest weight vector. In this case, a lowest weight generating vector for $L(\gamma)$ has weight $w_0\gamma$. Note that 1 has weight γ viewed as an element in the module $G^-(2\gamma)$. Moreover, the $U_q(\mathfrak{g})$ lowest weight vector of the submodule of $G^-(2\gamma)$ isomorphic to $L(\gamma)$ is of the form $(\mathrm{ad}_{2\gamma}y)1$ where $y \in U^-_{-\gamma'}$ and $\gamma' = \gamma - w_0\gamma$. Now consider an arbitrary weight γ and a weight $\beta \in Q^+(\pi)$. The dimension of the subspace of $\delta M(\gamma)$ spanned by vectors of weight greater than or equal to $\gamma - \beta$ is finite-dimensional. Therefore any $U_q(\mathfrak{g})$ lowest weight vector of $\delta M(\gamma)$ generates a finite-dimensional submodule of $\delta M(\gamma)$. Thus $\delta M(\gamma)$, and hence $G^-(2\gamma)$, contains a $U_q(\mathfrak{g})$ lowest weight vector of weight $\gamma - \beta$ if and only if γ is a dominant integral and $\beta = \gamma - w_0\gamma$.

The degree function defined by

- $\deg_{\mathcal{F}}(F_iK_i) = \deg_{\mathcal{F}}(E_i) = 0$ for all i,
- $\deg_{\mathcal{F}} K_i = -1$ and $\deg_{\mathcal{F}} K_i^{-1} = 1$ for all i,

yields an $(\mathrm{ad}\, U_q(\mathfrak{g}))$-invariant filtration $\mathcal{F}$ on $\check{U}$ such that $G^-K_{-\lambda}$ is a homogenous subspace of degree $\mathrm{ht}(\lambda)$. This degree function, and hence this filtration, extends to $\check{U}$ by setting $\deg_{\mathcal{F}} K_\xi = -\mathrm{ht}(\xi)$ and $\deg_{\mathcal{F}} K_\xi^{-1} = \mathrm{ht}(\xi)$ for all $\xi \in P(\pi)$. (This is just the ad-invariant filtration on $U_q(\mathfrak{g})$ as in [J, Section 7.1], though with a slightly different form of the adjoint action.)

Now

$$(\mathrm{ad}\, F_i)K_{-\lambda} = F_iK_{-\lambda}K_i - K_{-\lambda}F_iK_i = (1 - q^{(\lambda,\alpha_i)})F_iK_iK_{-\lambda}.$$

In particular, $G^-K_{-\lambda}$ and $G^-(\lambda)$ are isomorphic as (U^-)-modules where the action on the first module is via $(\mathrm{ad}\, U^-)$ and the action on the second module is via $(\mathrm{ad}_\lambda U^-)$. This is not true if we replace U^- with U^+. Instead, the twisted adjoint action of U^+ corresponds to a version of the graded adjoint action with respect to the $(\mathrm{ad}\, U_q(\mathfrak{g}))$-invariant filtration $\mathcal{F}$. More precisely, given $f \in G^-$ and $i \in \{1, \ldots, n\}$, there exists f' and f'' in G^- and a scalar c such that we can write

$$(\mathrm{ad}\, E_i)fK_{-\lambda} = f'K_{-\lambda} + f''K_{-\lambda}K_i^2 + cfK_{-\lambda}E_i,$$

where here we are considering $fK_{-\lambda}$ as an element of $U_q(\mathfrak{g})$ and using the ordinary adjoint action of E_i. We obtain a graded action of $(\mathrm{ad}\, E_i)$ on $G^-K_{-\lambda}$ by dropping the lower degree term with respect to $\mathcal{F}$ (i.e., $f''K_{-\lambda}K_i^2$) as well as the contribution from $\sum_i U_q(\mathfrak{g})E_i$ (i.e., $cfK_{-\lambda}E_i$). Thus we may equip $G^-K_{-\lambda}$ with a graded adjoint action of U^+. Using this graded adjoint action of U^+ combined with the ordinary action of $(\mathrm{ad}\, U^-)$ and setting $(\mathrm{ad}\, K_\beta) \cdot K_{-\lambda} = q^{\frac{1}{2}(\lambda,\beta)}K_{-\lambda}$ for all $\beta \in Q^+(\pi)$ makes $G^-K_{-\lambda}$ into an $(\mathrm{ad}\, U_q(\mathfrak{g}))$-module. (We will often refer to this action as the graded adjoint action, but still use the same notation for the adjoint action, where the fact that $G^-K_{-\lambda}$ is being viewed as a module rather than a subset of the larger module $U_q(\mathfrak{g})$ is understood from context.) Note that in the above description of $(\mathrm{ad}\, E_i)fK_{-\lambda}$, the term f' is the top degree term of $(\mathrm{ad}\, E_i)f$ (more precisely, $(\mathrm{ad}\, E_i)f = f' +$ terms of degree strictly less than 0) and is independent of λ. Hence, we may view the graded adjoint action of U^+ on $G^-K_{-\lambda}$ as independent of λ. Moreover, $G^-K_{-\lambda}$ and $G^-(\lambda)$ are isomorphic $U_q(\mathfrak{g})$-modules where the action on the first module is the graded adjoint and on the second module is the twisted

adjoint action. (The fact that we can make $G^- K_{-\lambda}$ into an (ad $U_q(\mathfrak{g})$)-module in this way is implicit in [J, Lemma 7.1.1].)

Now consider any weight λ. By the above discussion, $G^- K_{-\lambda}$ and $G^-(\lambda)$ are isomorphic as (ad U^-)-modules and the latter is isomorphic to $\delta M(\lambda/2)$. It follows that $G^- K_{-\lambda}$ contains a nonzero (ad $U_q(\mathfrak{g})$) lowest weight vector $g K_{-\lambda}$ where g is of weight $-\beta$ if and only if $\lambda = 2\gamma$ and $\beta = \gamma - w_0 \gamma$ for some $\gamma \in P^+(\pi)$. Moreover, this lowest weight vector is unique (up to scalar multiple) and is contained in $(\operatorname{ad} U^-) K_{-\lambda}$.

Let π' be a subset of π. For each $\lambda \in P(\pi)$, write $\tilde{\lambda}$ for the restriction of the weight λ to an element of $P(\pi')$, the weight lattice for π'. In particular, $\tilde{\lambda}$ is the element in $P(\pi')$ that satisfies $(\lambda, \alpha_i) = (\tilde{\lambda}, \alpha_i)$ for all $\alpha_i \in \pi'$. Here, the subset π' is understood from context. Note that $G^-_{\pi'} K_{-\lambda}$ and $G^-_{\pi'} K_{-\tilde{\lambda}}$ are isomorphic as (ad $U_{\pi'}$)-modules. Hence $G^-_{\pi'} K_{-\lambda}$ contains a nonzero (ad $U_{\pi'}$) lowest weight vector, say $g K_{-\lambda}$, if and only if $\tilde{\lambda} = 2\tilde{\gamma}$ for some choice of $\tilde{\gamma} \in P^+(\pi')$. Moreover, the weight of $g K_{-\lambda}$ (viewed as an element in the (ad $U_q(\mathfrak{g})$)-module $G^- K_{-\lambda}$) is $w_{\pi'} \tilde{\gamma}$ where $w_{\pi'} = w(\pi')_0$ and so the weight of g (considered as an element of the (ad $U_q(\mathfrak{g})$)-module $U_q(\mathfrak{g})$) is $-\tilde{\gamma} + w_{\pi'} \tilde{\gamma}$. Since $(\gamma - \tilde{\gamma}, \alpha) = 0$ for all $\alpha \in \pi'$, we have $w_{\pi'}(\gamma - \tilde{\gamma}) = \gamma - \tilde{\gamma}$. Hence

$$-\gamma + w_{\pi'}\gamma = -(\gamma - \tilde{\gamma}) - \tilde{\gamma} + (\gamma - \tilde{\gamma}) + w_{\pi'}\tilde{\gamma} = -\tilde{\gamma} + w_{\pi'}\tilde{\gamma}.$$

Thus the weight of g is $-\gamma + w_{\pi'}\gamma$. We may similarly analyze (ad $U_{\pi'}$) lowest weight vectors in $G^- K_{-\lambda}$ where we view this space as an (ad $U_{\pi'}$)-module. In particular, if $g \in G^-$ and $g K_{-\lambda}$ is an (ad $U_{\pi'}$) lowest weight vector, then g generates a finite-dimensional (ad$_\lambda U_{\pi'}$)-submodule of $G^-(\lambda)$ and the restriction $-\tilde{\mu}$ of the weight $-\mu$ of g (considered as an element of $G^-(\lambda)$) to π' must satisfy $\tilde{\mu} \in P^+(\pi')$. Note that $g K_{-\lambda}$ generates a finite-dimensional $U_q(\mathfrak{g})$-submodule of $G^- K_{-\lambda}$ when we use the graded adjoint action. Thus if $b \in U_q(\mathfrak{g})$ and $(\operatorname{ad}_\lambda b)g = 0$, we have

(3.4) $$\deg_{\mathcal{F}}((\operatorname{ad} b)g K_{-\lambda}) \leq \deg_{\mathcal{F}}(K_{-\lambda}) = \operatorname{ht}(\lambda).$$

Let $F(\check{U})$ denote the locally finite part of $\check{U}$ with respect to the (left) adjoint action. By [J, Section 7.1], $F(\check{U})$ admits a direct sum decomposition

$$F(\check{U}) = \bigoplus_{\gamma \in P^+(\pi)} (\operatorname{ad} U_q(\mathfrak{g})) K_{-2\gamma}$$

as (ad $U_q(\mathfrak{g})$)-modules. The subspace of $G^- K_{-2\gamma}$ corresponding to the finite-dimensional (ad$_{2\gamma} U_q(\mathfrak{g})$)-submodule of $G^-(2\gamma)$ is the subspace (ad U^-)$K_{-2\gamma}$ of $G^- K_{-2\gamma}$.

Let π' be a subset of π as above. Set $F_{\pi'}(\check{U})$ equal to the locally finite part of $\check{U}$ with respect to the action of (ad $U_{\pi'}$). By the defining relations for $U_q(\mathfrak{g})$ (see for example Lemma 4.5 of [JL]) and the definition of the adjoint action, we see that $E_j, F_j K_j \in F_{\pi'}(\check{U})$ for all $\alpha_j \notin \pi'$ and $F(\check{U}) \subseteq F_{\pi'}(\check{U})$. Also, $K_{-2\gamma} \in F_{\pi'}(\check{U})$ for all γ that restricts to a dominant integral weight in $P^+(\pi')$.

Since $F_{\pi'}(\check{U})$ contains all the finite-dimensional (ad $U_{\pi'}$)-submodules of $\check{U}$, it follows that the subspace of elements in $F_{\pi'}(\check{U})$ that admit a trivial (ad $U_{\pi'}$) action is the same as the subspace of elements in $\check{U}$ admitting such a trivial action. In

particular, we have

$$F_{\pi'}(\check{U})^{U_{\pi'}} = \{a \in F_{\pi'}(\check{U}) | \ (\mathrm{ad}\ u)a = \epsilon(u)a \ \text{for all} \ u \in U_{\pi'}\}$$
$$= \{a \in \check{U} | \ (\mathrm{ad}\ u)a = \epsilon(u)a \ \text{for all} \ u \in U_{\pi'}\} = \check{U}^{U_{\pi'}}.$$

It follows from the discussion at the end of Section 3.1, that this subspace agrees with the centralizer of $U_{\pi'}$ inside $\check{U}$. In particular,

$$\check{U}^{U_{\pi'}} = \{a \in \check{U} | ua - au = 0 \ \text{for all} \ u \in U_{\pi'}\}$$
$$= \{a \in \check{U} | \ (\mathrm{ad}\ u)a = \epsilon(u)a \ \text{for all} \ u \in U_{\pi'}\}.$$

3.3. **Lusztig's automorphisms.** For each $j \in \{1, \ldots, n\}$, let T_j denote Lusztig's automorphism associated to the simple root α_j ([Lu, Section 37.1]; see also Section 3.4 of [Ko]). For each w in $\mathcal{W}$, the automorphism T_w is defined to be the composition

$$T_w = T_{r_1} \cdots T_{r_m},$$

where $s_{r_1} \cdots s_{r_m}$ is a reduced expression for w and each s_{r_j} is the reflection defined by the simple root α_{r_j}. By [Lu] (or [Ja, Section 8.18]) T_w is independent of reduced expression for w. One of the properties satisfied by the T_j, $j = 1, \ldots, n$ is $T_j^{-1} = \sigma T_j \sigma$ where σ is the $\mathbb{C}(q)$ algebra antiautomorphism defined by

$$(3.5) \qquad \sigma(E_j) = E_j, \quad \sigma(F_j) = F_j, \quad \sigma(K_j) = K_j^{-1}, \quad \sigma(K_j^{-1}) = K_j.$$

The following result of [Ko] relates Lusztig's automorphisms and submodules of $U_q(\mathfrak{g})$ with respect to the action of $(\mathrm{ad}\ U_{\pi'})$ where $\pi' \subset \pi$.

Lemma 3.1 ([Ko, Lemma 3.5]). *Let π' be a subset of the simple roots π and let α_i be a simple root not in π'. Set $w_{\pi'} = w(\pi')_0$.*

  (i) *The subspace $(\mathrm{ad}\ U_{\pi'})(E_i)$ of U^+ is a simple finite-dimensional $(\mathrm{ad}\ U_{\pi'})$-submodule of $U_q(\mathfrak{g})$ with highest weight vector $T_{w_{\pi'}}(E_i)$ and lowest weight vector E_i.*
  (ii) *The subspace $(\mathrm{ad}\ U_{\pi'})(F_i K_i)$ of G^- is a simple finite-dimensional $(\mathrm{ad}\ U_{\pi'})$-submodule of $U_q(\mathfrak{g})$ with highest weight vector $F_i K_i$ and lowest weight vector $T_{w_{\pi'}}^{-1}(F_i K_i)$.*

Assume π', α_i, and $w_{\pi'}$ are chosen as in the above lemma. Note that $\widetilde{w_{\pi'}\alpha_i}$, the restriction of the weight $w_{\pi'}\alpha_i$ with respect to π', is an element in $P^+(\pi')$. In particular, as $(U_{\pi'})$-modules $(\mathrm{ad}\ U_{\pi'})(E_i)$ is isomorphic to $L_{\pi'}(w_{\pi'}\alpha_i)$, the finite-dimensional simple $(U_{\pi'}U^0)$-module of highest weight $w_{\pi'}\alpha_i$.

3.4. **The quantum Chevalley antiautomorphism.** Let $U_{\mathbb{R}(q)}$ be the $\mathbb{R}(q)$ subalgebra of $U_q(\mathfrak{g})$ generated by $E_i, F_i, K_i,$ and K_i^{-1}. Let κ denote the algebra antiautomorphism of $U_{\mathbb{R}(q)}$ defined by

$$\kappa(E_i) = F_i K_i \quad \kappa(F_i) = K_i^{-1} E_i, \quad \kappa(K) = K$$

for all $i = 1, \ldots, n$ and $K \in \mathcal{T}$. Note that $\kappa^2 = \mathrm{Id}$ and so κ is an involution. We define the notion of complex conjugate for $\mathbb{C}(q)$ by setting $\bar{s} = s$ for all $s \in \mathbb{C}$ and $\bar{q} = q$ and insisting that conjugation is an algebra automorphism. Extend κ to a conjugate linear antiautomorphism of $U_q(\mathfrak{g})$ by setting $\kappa(au) = \bar{a}\kappa(u)$ for all $a \in \mathbb{C}(q)$ and $u \in U_{\mathbb{R}(q)}$. We refer to κ as the quantum Chevalley antiautomorphism as in [L2]. Note that

$$\kappa((\mathrm{ad}\ E_j)b) = \kappa(E_j b - K_j b K_j^{-1} E_j) = \kappa(b)F_j K_j - F_j \kappa(b)K_j = -(\mathrm{ad}\ F_j)\kappa(b).$$

It follows that

(3.6) $\qquad \kappa((\text{ad } E_{i_1} \cdots E_{i_m})E_k K) = (-1)^m (\text{ad } F_{i_1} \cdots F_{i_m})(KF_k K_k)$

for all $K \in \mathcal{T}$.

We recall information about unitary modules for quantum groups as presented in [L2]. Let C be a subalgebra of $U_q(\mathfrak{g})$ such that $\kappa(C) = C$. Given a C-module M, a mapping S_M from $M \times M$ to $\mathbb{R}(q)$ is called sesquilinear if S_M is linear in the first variable and conjugate linear in the second. We say that the C-module M is *unitary* provided M admits a sesquilinear form S_M that satisfies

- $S_M(av, w) = S_M(v, \kappa(a)w)$ for all $a \in C$ and $v, w \in M$;
- $S_M(v, v)$ is a positive element of $\mathbb{R}(q)$ for each nonzero vector $v \in M$;
- $S_M(v, w) = \overline{S_M(w, v)}$ for all v and w in M.

Theorem 3.2. *Let C be a subalgebra of $U_q(\mathfrak{g})$ such that $\kappa(C) = C$. Every finite-dimensional unitary C-module can be written as a direct sum of finite-dimensional simple C-modules.*

Proof. This is just Theorem 2.4 and Corollary 2.5 of [L2] except that here we only assume that C is a κ invariant subalgebra of $U_q(\mathfrak{g})$ where as in [L2], C is additionally assumed to be a (left) coideal subalgebra. Nevertheless, the proofs in [L2] do not use the coideal assumption and thus apply to this more general setting. $\qquad \square$

The next result is an immediate consequence of Theorem 3.2 and the fact that a commutative polynomial ring H acts semisimply on an H-module V if and only if V can be written as a direct sum of eigenspaces with respect to the action of H.

Corollary 3.3. *Let C be a subalgebra of $U_q(\mathfrak{g})$ such that $\kappa(C) = C$. Let H be a commutative subalgebra of C such that*

- $\kappa(H) = H$;
- $H \cap U^0$ *is the Laurent polynomial ring generated by $H \cap \mathcal{T}$;*
- H *is isomorphic to a polynomial ring over $H \cap U^0$.*

Then any finite-dimensional unitary C-module can be written as a direct sum of eigenspaces with respect to the action of H.

Recall that every finite-dimensional simple $U_q(\mathfrak{g})$-module $L(\lambda)$ admits a sesquilinear form S defined by $S(av, bv) = S(v, \mathcal{P}_{HC}(\kappa(a)b)v)$ where v is a highest weight generating vector for $L(\lambda)$ and $\mathcal{P}_{HC}$ is the Harish-Chandra projection as defined in (1.14) of [L2]. Thus, the following result is an immediate consequence of Theorem 3.2.

Corollary 3.4. *Let C be a subalgebra of $U_q(\mathfrak{g})$ such that $\kappa(C) = C$. The finite-dimensional simple $U_q(\mathfrak{g})$-module $L(\lambda)$ admits a semisimple C action for all $\lambda \in P^+(\pi)$.*

The next corollary is a special case of Corollary 3.3.

Corollary 3.5. *Let H be a commutative subalgebra of $U_q(\mathfrak{g})$ such that $\kappa(H) = H$. Assume further that $H \cap U^0$ is the Laurent polynomial ring generated by $H \cap \mathcal{T}$ and that H is isomorphic to a polynomial ring over $H \cap U^0$. Then any finite-dimensional $U_q(\mathfrak{g})$-module can be written as a direct sum of eigenspaces with respect to the action of H.*

4. Quantum symmetric pairs

We review the construction of quantum symmetric pairs following [Ko, (Sections 4 and 5)], which is closely based on [L2] but uses right coideal subalgebras instead of left ones. Afterwards, we analyze elements of these coideal subalgebras through biweight space expansions and the introduction of a special projection map.

4.1. **Definitions.** Let θ be a maximally split involution for the semisimple Lie algebra $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ as defined in Section 2.4. Let $\mathcal{M}$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $E_i, F_i, K_i^{\pm 1}$ for all $\alpha_i \in \pi_\theta$. Note that $\mathcal{M} = U_q(\mathfrak{m})$ where $\mathfrak{m}$ is the Lie subalgebra of $\mathfrak{g}$ generated by e_i, f_i, h_i for all $\alpha_i \in \pi_\theta$. By Section 4 of [Ko], we can lift θ to an algebra automorphism θ_q of $U_q(\mathfrak{g})$ and use this automorphism to define quantum analogs of $U(\mathfrak{g}^\theta)$. In particular, the quantum symmetric pair (right) coideal subalgebra $\mathcal{B}_{\theta,\mathbf{c},\mathbf{s}}$ is generated by

- the group $\mathcal{T}_\theta = \{ K_\beta | \ \beta \in Q(\pi)^\theta \}$,
- $\mathcal{M} = U_q(\mathfrak{m})$,
- the elements $B_i = F_i + c_i \theta_q(F_i K_i) K_i^{-1} + s_i K_i^{-1}$ for all i with $\alpha_i \notin \pi_\theta$,

where $\mathbf{c} = (c_1, \ldots, c_n)$ and $\mathbf{s} = (s_1, \ldots, s_n)$ are n-tuples of scalars with $\mathbf{c} \in \mathbf{C}$ and $\mathbf{s} \in \mathbf{S}$ where $\mathbf{C}$ and $\mathbf{S}$ are defined as in [Ko, (5.9) and (5.11)], respectively, (see also [L2, Section 7, Variations 1 and 2]). We write $\mathcal{B}_\theta$ for $\mathcal{B}_{\theta,\mathbf{c},\mathbf{s}}$ where $\mathbf{c}$ and $\mathbf{s}$ are understood from context. We further assume that each entry in $\mathbf{s}$ and each entry in $\mathbf{c}$ is in $\mathbb{R}[q, q^{-1}]_{(q-1)}$. This guarantees that each B_i is contained in $U_{\mathbb{R}(q)}$ and in $\hat{U}$. Thus $\mathcal{B}_\theta$ admits a conjugate linear antiautomorphism (see the discussion below) and specializes to $U(\mathfrak{g}^\theta)$ as q goes to 1 ([L2, Section 7]).

Note that there are extra conditions on both $\mathbf{C}$ and $\mathbf{S}$. Only the latter affect the arguments of this paper. For the entries of $\mathbf{s}$ it is important to note that $s_i \neq 0$ implies α_i is an element of a distinguished subset $\mathcal{S}$ of $\pi \setminus \pi_\theta$ (see [L2, Variation 2] for the definition of $\mathcal{S}$). By Section 7 of [L3], we have $\mathcal{S}$ is empty except in the following cases:

- **Type AIII:** $n = 2m + 1$, π_θ is empty and $\mathcal{S} = \{\alpha_{m+1}\}$.
- **Type CI:** $\mathcal{S} = \{\alpha_n\}$.
- **Type DIII:** Case 1, $\mathcal{S} = \{\alpha_n\}$.
- **Type EVII:** $\mathcal{S} = \{\alpha_7\}$.

Let Γ_θ be the choice of the strongly orthogonal θ-system of Theorem 2.7. By the above description for $\mathcal{S}$ with respect to irreducible symmetric pairs and the proof of Theorem 2.7, we see that if $\alpha_i \in \mathcal{S}$ and $\alpha_i = \alpha_\beta$ for some $\beta \in \Gamma_\theta$, then $\beta = \alpha_\beta = \alpha_i$. In other words, $\alpha_\beta \in \mathcal{S}$ implies that β is a simple root.

Write $\mathbb{N}\mathcal{S}$ for the set of linear combinations of elements in $\mathcal{S}$ with nonnegative integer coefficients. Note that $\mathcal{S}$ is a subset of Δ_θ and in particular, $\theta(\alpha) = -\alpha$ for all $\alpha \in \mathcal{S}$. This follows immediately by inspection of the list above, but was also part of the original definition of $\mathcal{S}$ in [L3, Section 7]. We will find this property useful in some of the arguments below which include generic elements of $\mathbb{N}\mathcal{S}$.

By [L2], there exists a Hopf algebra automorphism ψ of $U_q(\mathfrak{g})$ that restricts to a real Hopf algebra automorphism of $U_{\mathbb{R}(q)}$ and so that $\psi \kappa \psi^{-1}$ restricts to a conjugate linear antiautomorphism of $\mathcal{B}_\theta$. Set $\kappa_\theta = \psi \kappa \psi^{-1}$ and note that κ_θ is a conjugate linear antiautomorphism of $U_q(\mathfrak{g})$. Thus Theorem 3.2, Corollary 3.3, and Corollary 3.4 hold for κ replaced by κ_θ and apply to $C = \mathcal{B}_\theta$. Therefore, $\mathcal{B}_\theta$ acts semisimply on finite-dimensional unitary $\mathcal{B}_\theta$-modules and the set of finite-dimensional unitary

$\mathcal{B}_\theta$-modules includes the set of finite-dimensional $U_q(\mathfrak{g})$-modules. Note that $\kappa_\theta(X)$ is a nonzero scalar multiple of $\kappa(X)$ for any nonzero weight vector X in U^+, G^+, U^- or G^-. Thus, if we show that Y is a nonzero scalar multiple of $\kappa(X)$ for weight vectors $X \in U^+$ (resp., $X \in G^+$) and $Y \in G^-$ (resp., $Y \in U^-$), then the same assertion holds for κ replaced by κ_θ.

By Theorem 4.4 of [Ko], θ_q restricts to the identity on $\mathcal{M}$ and $\theta_q(K_\beta) = K_{\theta(\beta)}$ for all $\beta \in Q(\pi)$. Also, in the case where $\theta(\alpha_i) = -\alpha_{p(i)}$, we get $\theta_q(F_iK_i) = cE_{p(i)}$ for some nonzero scalar c. More generally, by Theorem 4.4 of [Ko], for $\theta(\alpha_i) \neq \alpha_i$ there exists a nonnegative integer r and elements

$$Z_i^- = F_{i_1} \cdots F_{i_r} \quad \text{and} \quad Z_i^+ = E_{j_1} \cdots E_{j_r}$$

with α_{i_k} and α_{j_k} in π_θ for $1 \leq k \leq r$ and nonzero scalars u_i and v_i such that

$$\theta_q(E_i) = u_i\sigma((\text{ad}\,Z_{p(i)}^-)(F_{p(i)}K_{p(i)}))$$

and

$$(4.1) \qquad\qquad \theta_q(F_iK_i) = v_i(\text{ad}\,Z_{p(i)}^+)(E_{p(i)})$$

where σ is the $\mathbb{C}(q)$ algebra antiautomorphism defined in (3.5). Moreover, $\theta_q(F_iK_i)$ is the highest weight vector of the (ad $\mathcal{M}$)-module generated by $E_{p(i)}$ as described in Lemma 3.1(ii). An analogous assertion holds for $\theta_q(E_i)$.

Set $\check{\mathcal{T}}_\theta = \{K_\mu|\ \mu \in P(\pi)^\theta\}$ and note that $\check{\mathcal{T}}_\theta$ is a subgroup of $\check{\mathcal{T}}$ which is contained in the simply connected quantized enveloping algebra (see Section 3.1). It is sometimes convenient to extend $\mathcal{B}_\theta$ to a "simply connected version" by including the elements $\check{\mathcal{T}}_\theta$. We denote this extended algebra by $\check{\mathcal{B}}_\theta$. By definition, $\check{\mathcal{B}}_\theta = \mathcal{B}_\theta\check{\mathcal{T}}_\theta$.

4.2. Biweight space expansions.
Set $\mathcal{M}^+ = \mathcal{M} \cap U^+$, the subalgebra of $\mathcal{M}$ generated by E_i for all $\alpha_i \in \pi_\theta$. Note that we have only defined the generators B_i for $\alpha_i \notin \pi_\theta$. It is useful to set $B_i = F_i$ for $\alpha_i \in \pi_\theta$. Given an m-tuple $J = (j_1, \ldots, j_m)$ with entries in $j_k \in \{0,1\}$, we set $F_J = F_{j_1} \cdots F_{j_m}$ and $B_J = B_{j_1} \cdots B_{j_m}$.

Let $\mathcal{J}$ be the subset of tuples with entries in $\{0,1\}$ so that $\{F_J, J \in \mathcal{J}\}$ forms a basis for U^-. We have the following direct sum decomposition for $\mathcal{B}_\theta$ ([L2], (7.17)]):

$$(4.2) \qquad\qquad \mathcal{B}_\theta = \bigoplus_{J \in \mathcal{J}} B_J \mathcal{M}^+ \mathcal{T}_\theta.$$

Given an m-tuple J, write $\text{wt}(J) = \alpha_{j_1} + \alpha_{j_2} + \cdots + \alpha_{j_m}$. The next result provides us with fine information concerning the biweight expansion of elements B_J in $\mathcal{B}_\theta$.

Lemma 4.1. *Let $J = (j_1, \ldots, j_m)$ be an m-tuple of nonnegative integers such that* $\text{wt}(J) = \beta$. *Then* $\deg_{\mathcal{F}}(B_J) = m = \text{ht}(\beta)$ *and*

$$(4.3) \qquad B_J \in F_J + \sum_{(\lambda,\eta,\gamma,\gamma') \in \mathcal{N}} G_{-\beta+\lambda+\eta+\gamma}^- U_{\theta(-\lambda)-\gamma}^+ K_{-\beta+2\gamma'},$$

where $\mathcal{N}$ is the set of four-tuples $(\lambda, \eta, \gamma, \gamma')$ in $Q^+(\pi \setminus \pi_\theta) \times \mathbb{N}\mathcal{S} \times Q^+(\pi) \times Q^+(\pi)$ satisfying

- $0 < \lambda \leq \beta$,
- $\eta \leq \lambda$,
- $\gamma \leq \beta - \lambda$ *and* $\gamma \leq \theta(-\lambda) - \eta$,
- $\gamma' \leq \gamma$.

Proof. The claim about degree follows immediately from (4.3) and the fact that elements in G^- and U^+ have degree 0. By assumption, $K_{j_1}^{-1} \cdots K_{j_m}^{-1} = K_{-\beta}$. Recall that for $\alpha_i \notin \pi_\theta$, B_i is a linear combination of F_i, $\theta_q(F_i K_i) K_i^{-1}$ and K_i^{-1} with the final term only appearing if $\alpha_i \in \mathcal{S}$. Also, by the definition of $\theta_q(F_{j_r} K_{j_r})$ (see (4.1) and the related discussion), we have

$$\theta_q(F_{j_r} K_{j_r}) \in [(\mathrm{ad}\, \mathcal{M}^+) E_{p(j_r)}] \subseteq U^+_{\theta(-\alpha_{j_r})}.$$

To prove (4.3), note that B_J can be written as a linear combination

$$(4.4) \qquad B_J \in F_J + \sum_{\mathcal{I}_2 \cup \mathcal{I}_3 \neq \emptyset} \mathbb{C}(q) A(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3) K_{-\beta},$$

where $\{\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\}$ forms a partition of $\{1, \ldots, m\}$ such that $\{\alpha_i | i \in \mathcal{I}_2\} \cap \pi_\theta = \emptyset$ and $\{\alpha_i | i \in \mathcal{I}_3\} \subseteq \mathcal{S}$, and

$$A(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3) = A_{j_1} A_{j_2} \cdots A_{j_m},$$

where

- $A_{j_i} = F_{j_i} K_{j_i}$ if $i \in \mathcal{I}_1$,
- $A_{j_i} = \theta_q(F_{j_i} K_{j_i})$ if $i \in \mathcal{I}_2$,
- $A_{j_i} = s_{j_i}$ if $i \in \mathcal{I}_3$.

Assume that $\mathcal{I}_2$ is not empty. Set $\gamma_k = \sum_{i \in \mathcal{I}_k} \alpha_{j_i}$ for $k = 1, 2$, and 3 and note that $\gamma_1 + \gamma_2 + \gamma_3 = \beta$. The assumption on $\mathcal{I}_2$ further forces $\gamma_2 > 0$. Consider first the case where $\mathcal{I}_3$ is empty and so $\gamma_1 = \beta - \gamma_2$. If we simply reorder the A_{j_i} so that elements with $j_i \in \mathcal{I}_1$ are on the left and elements with $j_i \in \mathcal{I}_2$ are on the right we get a term in

$$G^-_{-\beta+\gamma_2} U^+_{\theta(-\gamma_2)} K_{-\beta}.$$

In order to understand what other spaces of the form $G^-_\xi U^+_{\xi'} K_{-\xi''}$ show up in the biweight expansion of $A(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3) K_{-\beta}$, we need to understand what happens when we commute an element A_{j_r} in U^+ (i.e., $j_r \in \mathcal{I}_2$) pass an element A_{j_k} in G^- (i.e., $j_k \in \mathcal{I}_1$).

The following identity is easily derived from one of the standard defining relations for $U_q(\mathfrak{g})$:

$$(4.5) \qquad E_i F_j K_j - q^{-2} F_j K_j E_i = -\delta_{ij}(q-q)^{-1}(1 - K_i^2).$$

Suppose that $A_{j_r} = \theta_q(F_{j_i} K_{j_i}) \in U^+_{\theta(-\alpha_{j_r})}$ and $A_{j_k} = F_i K_i$. Relation (4.5) implies that there exists a scalar c such that

$$A_{j_r} A_{j_k} \in c A_{j_k} A_{j_r} + U^+_{\theta(-\alpha_{j_r})-\alpha_i} + U^+_{\theta(-\alpha_{j_r})-\alpha_i} K_i^2.$$

Thus, moving the A_{j_r} terms in U^+ to the right of the A_{j_k} terms in G^- yields

$$(4.6) \qquad A(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3) K_{-\beta} \in G^-_{-\beta+\gamma_2} U^+_{\theta(-\gamma_2)} K_{-\beta}$$

$$+ \sum_{\substack{\gamma \leq \beta - \gamma_2 \\ \gamma \leq \theta(-\gamma_2)}} \sum_{\gamma' \leq \gamma} G^-_{-\beta+\gamma_2+\gamma} U^+_{\theta(-\gamma_2)+\gamma} K_{-\beta+2\gamma'}$$

when $\mathcal{I}_2 \neq \emptyset$ and $\mathcal{I}_3 = \emptyset$. Moreover, since $\mathcal{I}_2 \neq \emptyset$, we have $\gamma_2 > 0$.

Now suppose that $\mathcal{I}_3$ is nonempty. We can write

$$A(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3) = s A_{j'_1} \cdots A_{j'_r},$$

where $s = \prod_{j_i \in \mathcal{I}_3} s_{j_i}$ and $j'_1, \ldots, j'_r$ is the sequence obtained from $j_1, \ldots, j_m$ by removing all j_k with $k \in \mathcal{I}_3$. In other words, we have consolidated the terms A_{j_i} with $j_i \in \mathcal{I}_3$ into a single scalar s, and then renumbered the remaining A_{j_i} accordingly. Note that $\mathrm{wt}(j'_1, \ldots, j'_r) = \beta - \gamma_3$. Arguing as above we get

$$(4.7) \qquad A(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)K_{-\beta} = s A_{j'_1} \cdots A_{j'_r} K_{-\beta}$$

$$\in G^-_{-\beta+\gamma_2+\gamma_3} U^+_{\theta(-\gamma_2)} K_{-\beta}$$

$$+ \sum_{\substack{\gamma \le \beta-\gamma_2-\gamma_3 \\ \gamma \le \theta(-\gamma_2)}} \sum_{\gamma' \le \gamma} G^-_{-\beta+\gamma_2+\gamma_3+\gamma} U^+_{\theta(-\gamma_2)+\gamma} K_{-\beta+2\gamma'}$$

when $\mathcal{I}_3 \ne \emptyset$. Note that in this case, $\gamma_3 > 0$.

The lemma now follows by expanding out the second term of the right hand side of (4.4) using (4.6), (4.7) with $\gamma_2 = \lambda$ and $\gamma_3 = \eta$. $\square$

Remark 4.2. It is sometimes useful to use coarser versions of Lemma 4.1. It particular, it follows from (4.3) that

$$B_J \in F_J + \sum_{0 < \lambda \le \beta} \sum_{0 \le \gamma \le \beta} G^-_{-\beta+\lambda} U^+ K_{-\beta+\gamma}$$

and

$$B_J \in F_J + \sum_{0 < \lambda \le \beta} G^-_{-\beta+\lambda} U^0 U^+$$

for each m-tuple J with $\mathrm{wt}(J) = \beta$. It follow from the direct sum decomposition (4.2) and the definition of l-weight (Section 3.1) that if g is a minimal l-weight summand of an element $b \in \mathcal{B}_\theta$ of l-weight $-\lambda$, then $g \in U^-_{-\lambda} \mathcal{M}^+ \mathcal{T}_\theta$.

4.3. **Decompositions and projections.** Set N^+_θ equal to the subalgebra of U^+ generated by $(\mathrm{ad}\,\mathcal{M}^+) U^+_{\pi \setminus \pi_\theta}$. By [L2, Section 6], N^+_θ is an (ad $\mathcal{M}$)-module. Multiplication induces an isomorphism (this is just [L2, (6.2)] with G^- replaced by U^+)

$$N^+_\theta \otimes \mathcal{M}^+ \cong U^+.$$

Let $\mathcal{T}'$ be the subgroup of $\mathcal{T}$ generated by $\{K_i \mid \alpha_i \notin \pi_\theta$ and $i \le p(i)\}$. Note that $\mathcal{T} = \mathcal{T}_\theta \times \mathcal{T}'$ and hence the multiplication map defines an isomorphism

$$\mathbb{C}(q)[\mathcal{T}_\theta] \otimes \mathbb{C}(q)[\mathcal{T}'] \cong U^0.$$

It follows from the triangular decomposition (3.1) for $U_q(\mathfrak{g})$ and the above tensor product decompositions, that multiplication also induces an isomorphism

$$U_q(\mathfrak{g}) \cong U^- \otimes \mathbb{C}(q)[\mathcal{T}_\theta] \otimes \mathbb{C}(q)[\mathcal{T}'] \otimes N^+_\theta \otimes \mathcal{M}^+.$$

Using the fact that elements of $\mathcal{M}^+$ are eigenvectors with respect to the adjoint action of $\mathcal{T}$, we may reorder this tensor product decomposition as

$$(4.8) \qquad U_q(\mathfrak{g}) \cong U^- \otimes \mathcal{M}^+ \otimes \mathbb{C}(q)[\mathcal{T}_\theta] \otimes \mathbb{C}(q)[\mathcal{T}'] \otimes N^+_\theta.$$

Let $(N^+_\theta)_+$ denote the span of the weight vectors in N^+_θ of positive weight in $Q^+(\pi)$. Set

$$(\mathcal{T}')_+ = \sum_{\substack{K \in \mathcal{T}' \\ K \ne Id}} \mathbb{C}(q)K$$

and let $(\mathcal{T}'N_\theta^+)_+$ denote the subspace of $\mathcal{T}'N_\theta$ defined by

$$(\mathcal{T}'N_\theta^+)_+ = (\mathcal{T}')_+N_\theta^+ + \mathcal{T}'(N_\theta^+)_+.$$

The tensor product decomposition (4.8) yields a direct sum decomposition

$$(4.9) \qquad U_q(\mathfrak{g}) = \left(U^-\mathcal{M}^+\mathcal{T}_\theta\right) \oplus \left(U^-\mathcal{M}^+\mathcal{T}_\theta(\mathcal{T}'N_\theta^+)_+\right).$$

Let $\mathcal{P}$ denote the projection of $U_q(\mathfrak{g})$ onto $U^-\mathcal{M}^+\mathcal{T}_\theta$ with respect to this decomposition.

Proposition 4.3. *Given a positive weight β and an element $b \in \mathcal{B}_\theta$ such that*

$$b = \sum_{\mathrm{wt}(I)=\beta} B_I a_I$$

where each $a_I \in \mathcal{M}^+\mathcal{T}_\theta$, there exists an element $\tilde{b} \in \mathcal{B}_\theta$ such that
  (i) $\tilde{b} = b + \sum_{\mathrm{wt}(J)<\beta} B_J c_J$ *for $c_J \in \mathcal{M}^+\mathcal{T}_\theta$.*
  (ii) $\mathcal{P}(\tilde{b}) = \sum_{\mathrm{wt}(I)=\beta} F_I a_I.$

Proof. If $\mathrm{wt}(I) = \alpha_i$ for some i, then $b = B_i a$ for some $a \in \mathcal{M}^+\mathcal{T}_\theta$ and the lemma follows immediately from the definitions of the generator B_i (both when $\alpha_i \in \pi_\theta$ and when $\alpha_i \notin \pi_\theta$) and the projection map $\mathcal{P}$. We prove the lemma using induction on $\mathrm{ht}(\beta)$. In particular, we assume that (i) and (ii) hold for all β' with $\beta' < \beta$.

By Remark 4.2 and the definition of G^-, we can write

$$b = \sum_{\mathrm{wt}(I)=\beta} F_I a_I + Y$$

with

$$Y \in \sum_{0<\lambda\leq\beta} G^-_{-\beta+\lambda}U^+U^0 = \sum_{0<\lambda\leq\beta} U^-_{-\beta+\lambda}U^+U^0.$$

Using the direct sum decompositon (4.9), we see that

$$\mathcal{P}(Y) \in \sum_{0<\lambda\leq\beta} U^-_{-\beta+\lambda}\mathcal{M}^+\mathcal{T}_\theta.$$

Thus, there exist $c_J \in \mathcal{M}^+\mathcal{T}_\theta$ such that

$$\mathcal{P}(Y) = \sum_{\beta'<\beta}\sum_{\mathrm{wt}(J)=\beta'} F_J c_J.$$

It follows that

$$(4.10) \qquad \mathcal{P}(b) = \sum_{\mathrm{wt}(I)=\beta} F_I a_I + \mathcal{P}(Y) = \sum_{\mathrm{wt}(I)=\beta} F_I a_I + \sum_{\beta'<\beta}\sum_{\mathrm{wt}(J)=\beta'} F_J c_J.$$

Set

$$(4.11) \qquad b_{\beta'} = \sum_{\beta'<\beta}\sum_{\mathrm{wt}(J)=\beta'} B_J c_J$$

and note that $b_{\beta'} \in \mathcal{B}_\theta$. For $\beta' < \beta$, we may apply the inductive hypothesis, yielding elements $\tilde{b}_{\beta'}$ satisfying (i) and (ii) with respect to $b_{\beta'}$. In particular, we have

$$(4.12) \qquad \mathcal{P}(\tilde{b}_{\beta'}) = \sum_{\mathrm{wt}(J)=\beta'} F_J c_J.$$

Set

$$\tilde{b} = b - \sum_{\beta' < \beta} \tilde{b}_{\beta'}$$

and note that $\tilde{b} \in \mathcal{B}_\theta$. The definition (4.11) of $b_{\beta'}$ and the fact that each $\tilde{b}_{\beta'}$ satisfies (i) with β replaced by β' guarantees that $\tilde{b}$ also satisfies (i). Condition (ii) for $\tilde{b}$ follows from (4.10) and (4.12).   □

It follows from Proposition 4.3 that the map $b \mapsto \mathcal{P}(b)$ defines a vector space isomorphism from $\mathcal{B}_\theta$ onto $U^- \mathcal{M}^+ \mathcal{T}_\theta$. This is the essence of the next result.

Corollary 4.4. *The projection map $\mathcal{P}$ restricts to a vector space isomorphism from $\mathcal{B}_\theta$ onto $U^- \mathcal{M}^+ \mathcal{T}_\theta$. Thus, if $b \in \mathcal{B}_\theta$, then $\mathcal{P}(b) = 0$ if and only if $b = 0$.*

Let β_1 be a maximal weight such that $c_J \neq 0$ for some J such that $\mathrm{wt}(J) = \beta_1$ in the right hand side of (4.10). The right hand side of (4.10) can be written as

$$\sum_{\mathrm{wt}(I)=\beta} F_I a_I + \sum_{\mathrm{wt}(J)=\beta_1} F_J c_J + \sum_{\substack{\beta' < \beta \\ \beta' \not\geq \beta_1}} \sum_{\mathrm{wt}(J)=\beta'} F_J c_J.$$

Set $d_1 = b - \sum_{\mathrm{wt}(J)=\beta_1} B_J c_J$ and apply induction on the set of $\beta' \not\geq \beta_1$. The end result is an element d_2 in $\mathcal{B}_\theta$ with the same image under $\mathcal{P}$ as $\tilde{b}$. Hence, by Corollary 4.4, we have $d_2 = \tilde{b}$.

Now assume that each of the a_I terms in the above expression are scalars. We can expand c_J as a sum of weight vectors in $\mathcal{M}^+ \mathcal{T}_\theta$. For each of these weight vectors, say c'_J, we can find $\lambda_0, \gamma_0, \gamma'_0, \eta_0$ satisfying the assumptions of Lemma 4.1 with respect to β such that

$$\beta_1 = \beta - \lambda_0 - \eta_0 - \gamma_0$$

and $c'_J \in U^+_{\theta(-\lambda_0)-\gamma_0} K_{-\lambda_0-\eta_0-\gamma_0+2\gamma'_0}$. Now suppose that $\lambda_1, \gamma_1, \gamma'_1, \eta_1$ also satisfy the assumptions of this lemma with β replaced by β_1. Set $\lambda_2 = \lambda_0 + \lambda_1$, $\gamma_2 = \gamma_0 + \gamma_1, \gamma'_2 = \gamma'_0 + \gamma'_1$, and $\eta_2 = \eta_0 + \eta_1$. It is straightforward to check that $\lambda_2, \gamma_2, \gamma'_2, \eta'_2$ satisfy the conditions of Lemma 4.1 with respect to β. Moreover,

$$(G^-_{-\beta_1+\lambda_1+\eta_1+\gamma_1} U^+_{\theta(-\lambda_1)-\gamma_1} K_{-\beta_1+2\gamma'_1}) c'_J \subseteq G^-_{-\beta+\lambda_2+\eta_2+\gamma_2} U^+_{\theta(-\lambda_2)-\gamma_2} K_{-\beta+2\gamma'_2}.$$

It follows that, just as for b of Proposition 4.3, the term d_1 is contained in a sum of spaces of the form

$$G^-_{-\beta+\lambda+\eta+\gamma} U^+_{\theta(-\lambda)-\gamma} K_{-\beta+2\gamma'},$$

where $\beta, \lambda, \eta, \gamma, \gamma'$ satisfy the conditions of Lemma 4.1. By induction, the same is true for $d_2 = \tilde{b}$.

5. Lifting to the lower triangular part

We begin the process of lifting Cartan elements of the form $e_\beta + f_{-\beta}$ to the quantum setting. The main result of this section identifies a lift with nice properties of $f_{-\beta}$ for nonsimple roots β in the special strongly orthogonal θ-systems of Theorem 2.7.

5.1. **Properties of special root vectors.** Consider a subset π' of π. Given a weight $\lambda \in P^+(\pi)$, we write $\bar{L}(\lambda)$ for the finite-dimensional simple $U(\mathfrak{g})$-module with highest weight λ and let $\bar{L}_{\pi'}(\lambda)$ denote the finite-dimensional simple $U(\mathfrak{g}_{\pi'} + \mathfrak{h})$-module generated by a vector of highest weight λ. Recall that $U(\mathfrak{g}_{\pi'} + \mathfrak{h})$ acts semisimply on $U(\mathfrak{g})$-modules. For all weights γ that are in $P(\pi)$ and restrict to an integral dominant weight in $P^+(\pi')$, we write $[\bar{L}(\lambda) : \bar{L}_{\pi'}(\gamma)]$ for the multiplicity of $\bar{L}_{\pi'}(\gamma)$ in $\bar{L}(\lambda)$ where both are viewed as $U(\mathfrak{g}_{\pi'} + \mathfrak{h})$-modules. Suppose that $\beta \in P^+(\pi)$ and $(\beta, \alpha) = 0$ for all $\alpha \in \pi'$. It follows that the finite-dimensional simple $U(\mathfrak{g}_{\pi'} + \mathfrak{h})$-module $\bar{L}_{\pi'}(\beta)$ of highest weight β is a trivial one-dimensional $U(\mathfrak{g}_{\pi'} + \mathfrak{h})$-module.

Set $\mathfrak{n}_{\pi'}^- = \mathfrak{g}_{\pi'} \cap \mathfrak{n}^-$. Let M be a finite-dimensional $U(\mathfrak{g}_{\pi'})$-module. We can decompose M into a direct sum of finite-dimensional simple $U(\mathfrak{g}_{\pi'})$-modules, each generated by a highest weight vector. Note that any nonzero highest weight vector with respect to the action of $U(\mathfrak{g}_{\pi'})$ inside M is not an element of $\mathfrak{n}_{\pi'}^- M$. In particular,

$$(5.1) \qquad\qquad M = M_0 \oplus \mathfrak{n}_{\pi'}^- M$$

where M_0 is the space spanned by the $U(\mathfrak{g}_{\pi'})$ highest weight vectors in M.

For each simple root α, let ν_α denote the corresponding fundamental weight.

Theorem 5.1. *Let $\beta \in \Delta^+$ and $\alpha_\beta \in \mathrm{Supp}(\beta)$ such that*

    (i) $\beta = \alpha_\beta + w_\beta \alpha_\beta$,
    (ii) $\pi' \subseteq \mathrm{StrOrth}(\beta)$,

where $\pi' = \mathrm{Supp}(\beta) \setminus \{\alpha_\beta\}$ and $w_\beta = w(\pi')_0$. Then for all $m \geq 1$, $\bar{L}(m\nu_{\alpha_\beta})$ has a unique nonzero vector (up to scalar multiple) of weight $m\nu_{\alpha_\beta} - \beta$ that generates a trivial $U(\mathfrak{g}_{\pi'})$-module.

Proof. Fix $m \geq 1$. Set $\nu = \nu_{\alpha_\beta}$. Note that w_β is a product of reflections defined by roots in π'. Since $\alpha_\beta \notin \pi'$, we must have $w_\beta \alpha_\beta - \alpha_\beta \in Q(\pi')$. Thus, the coefficient of α_β in $\beta = \alpha_\beta + w_\beta \alpha_\beta$ written as a linear combination of simple roots is 2. By assumption (ii) and the fact that ν is the fundamental weight associated to α_β, we have $(m\nu - \beta, \alpha) = 0$ for all $\alpha \in \pi'$. Hence any weight vector in $\bar{L}(m\nu)$ of weight $m\nu - \beta$ that is a highest weight vector with respect to the action of $U(\mathfrak{g}_{\pi'})$ generates a trivial $U(\mathfrak{g}_{\pi'})$-module. Thus we only need to prove

$$(5.2) \qquad\qquad [\bar{L}(m\nu) : \bar{L}_{\pi'}(m\nu - \beta)] = 1.$$

Let v denote a highest weight generating vector of $\bar{L}(m\nu)$. Since $(\nu, \alpha) = 0$ for all $\alpha \in \pi'$, $U(\mathfrak{g}_{\pi'})$ acts trivially on v. Moreover, $\alpha_\beta \notin \pi'$ ensures that $f_{-\alpha_\beta} v$ is a highest weight vector with respect to the action of $U(\mathfrak{g}_{\pi'})$. It follows that $U(\mathfrak{g}_{\pi'}) f_{-\alpha_\beta} v$ and $\bar{L}_{\pi'}(m\nu - \alpha_\beta)$ are isomorphic as $U(\mathfrak{g}_{\pi'} + \mathfrak{h})$-modules. Thus $U(\mathfrak{g}_{\pi'}) f_{-\alpha_\beta} v$ contains a unique nonzero lowest weight vector (up to scalar multiple) of weight $w_\beta(m\nu - \alpha_\beta) = m\nu - w_\beta \alpha_\beta = m\nu - \beta + \alpha_\beta$ with respect to the action of $U(\mathfrak{g}_{\pi'} + \mathfrak{h})$.

By the previous paragraph, $y f_{-\alpha_\beta} v$ is a (possibly zero) scalar multiple of the lowest weight vector of $U(\mathfrak{g}_{\pi'}) f_{-\alpha_\beta} v$ for any choice of $y \in U(\mathfrak{n}_{\pi'}^-)$ of weight $-\beta + 2\alpha_\beta$. Thus, there is at most one nonzero vector (up to scalar multiple) in $\bar{L}(m\nu)$ of weight $m\nu - \beta$ of the form

$$f_{-\alpha_\beta} y f_{-\alpha_\beta} v,$$

where $y \in U(\mathfrak{n}_{\pi'}^-)$. Hence the $m\nu - \beta$ weight space of $\bar{L}(m\nu)$ satisfies

(5.3) $$\bar{L}(m\nu)_{m\nu-\beta} \subseteq \mathbb{C}f_{-\alpha_\beta}yf_{-\alpha_\beta}v + \mathfrak{n}_{\pi'}^-U(\mathfrak{n}^-)v.$$

Since there are no highest weight $U(\mathfrak{g}_{\pi'})$ vectors inside $\mathfrak{n}_{\pi'}^-U(\mathfrak{n}^-)v$, (5.3) ensures that $\bar{L}(m\nu)$ contains at most one highest weight vector of weight $m\nu - \beta$. In particular,

(5.4) $$[\bar{L}(m\nu) : \bar{L}_{\pi'}(m\nu - \beta)] \le 1.$$

By assumption (ii), $(\alpha, \beta) = 0$ and $\alpha + \beta$ is not a root for all $\alpha \in \pi'$. It follows that $\beta - \alpha$ is also not a root for all $\alpha \in \pi'$. In particular, we have

$$[e_\alpha, f_{-\beta}] = 0 = [f_{-\alpha}, f_{-\beta}]$$

for all $\alpha \in \pi'$. Thus $e_\alpha f_{-\beta}v = [e_\alpha, f_{-\beta}]v = 0$ for all $\alpha \in \pi'$. Similarly, $f_{-\alpha}f_{-\beta}v = 0$ for all $\alpha \in \pi'$. Therefore, $\mathbb{C}f_{-\beta}v$ is a trivial $U(\mathfrak{g}_{\pi'} + \mathfrak{h})$-submodule of $\bar{L}(m\nu)$ of the desired weight. This shows that the inequality in (5.4) is actually an equality which proves (5.2). $\qquad\square$

Let $\lambda \in P^+(\pi)$ with $m_i = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i)$ for each i and let v denote the highest weight generating vector for $\bar{L}(\lambda)$. By Section 21.4 of [H], we have

$$fv = 0 \text{ if and only if } f \in \sum_i U(\mathfrak{n}^-)f_i^{m_i+1}$$

for all $f \in U(\mathfrak{n}^-)$. Consider $\beta \in Q^+(\pi)$. Suppose that π' is a proper subset of $\mathrm{Supp}(\beta)$, $m_i = 0$ for all $\alpha_i \in \pi'$ and $m_j \ge \mathrm{mult}_{\alpha_j}\beta$ for all $\alpha_j \in \mathrm{Supp}(\beta) \setminus \pi'$. Then

(5.5) $$fv = 0 \text{ if and only if } f \in \sum_{\alpha_i \in \pi'} U(\mathfrak{n}^-)f_i$$

for all elements f of weight $-\beta$ in $U(\mathfrak{n}^-)$.

Corollary 5.2. *Let $\beta \in \Delta^+$ and $\alpha_\beta \in \mathrm{Supp}(\beta)$ such that*

(i) *$\beta = \alpha_\beta + w_\beta\alpha_\beta$,*
(ii) *$\pi' \subseteq \mathrm{StrOrth}(\beta)$,*

where $\pi' = \mathrm{Supp}(\beta) \setminus \{\alpha_\beta\}$ and $w_\beta = w(\pi')_0$. Then the root vector $f_{-\beta}$ is the unique nonzero element (up to scalar multiple) of weight $-\beta$ in $U(\mathfrak{n}^-)$ such that $[a, f_{-\beta}] = 0$ for all $a \in \mathfrak{g}_{\pi'}$.

Proof. Suppose that f is a nonzero element of weight $-\beta$ in $U(\mathfrak{n}^-)$ and assume that $[e_i, f] = [f_i, f] = [h_i, f] = 0$ for all $\alpha_i \in \pi'$. Choose $m \ge 2$. By (i) we have $m \ge \mathrm{mult}_{\alpha_\beta}\beta$. Set $\nu = \nu_{\alpha_\beta}$ and let v be the highest weight generating vector for $\bar{L}(m\nu)$. Since $[e_i, f] = 0$ for all $\alpha_i \in \pi'$, it follows that fv is a highest weight vector of weight $m\nu - \beta$. By assumptions on β, we also have $[e_i, f_{-\beta}] = 0$ for all $\alpha_i \in \pi'$. By Theorem 5.1 and its proof, any highest weight vector in $\bar{L}(m\nu)$ of highest weight $m\nu - \beta$ with respect to the action of $U(\mathfrak{g}_{\pi'})$ is a scalar multiple of $f_{-\beta}v$. Hence

$$(f - cf_{-\beta})v = 0$$

for some scalar c. Set $f' = f - cf_{-\beta}$. By (5.5) and the related discussion, it follows that

(5.6) $$f' \in \sum_{\alpha_i \in \pi'} U(\mathfrak{n}^-)f_i.$$

Let ι be the Lie algebra antiautomorphism of $\mathfrak{g}_{\pi'}$ sending e_i to e_i, f_i to f_i, and h_i to $-h_i$ for all $i = 1, \ldots, n$. Since $[a, f'] = 0$ for all $a \in \mathfrak{g}_{\pi'}$, the same is true for $\iota(f')$. By (5.6), we have

$$\iota(f') \in \sum_{\alpha_i \in \pi'} f_i U(\mathfrak{n}^-). \tag{5.7}$$

Choose $\lambda \in P^+(\pi)$ so that $2(\lambda, \alpha_i)/(\alpha_i, \alpha_i) \geq \mathrm{mult}_{\alpha_i} \beta$ for all $\alpha_i \in \pi'$. Let w be a nonzero highest weight generating vector for $\bar{L}(\lambda)$. Since $[a, \iota(f')] = 0$ for all $a \in \mathfrak{g}_{\pi'}$, it follows that $\iota(f')w$ is a $U(\mathfrak{g}_{\pi'})$ highest weight vector with respect to the action of $U(\mathfrak{g}_{\pi'})$. By (5.7), we see that $\iota(f')w \in \mathfrak{n}_{\pi'}^- U(\mathfrak{n}^-)w$. It follows from (5.1) and the related discussion that $\iota(f') = 0$. Hence $f' = 0$ and f is a scalar multiple of $f_{-\beta}$. $\qquad\square$

5.2. **Quantum analogs of special root vectors.** In analogy to the classical setting, we write $[L(\lambda) : L_{\pi'}(\gamma)]$ for the multiplicity of $L_{\pi'}(\gamma)$ in $L(\lambda)$ where $\lambda \in P^+(\pi)$, γ is in $P(\pi)$ and restricts to a dominant integral weight in $P^+(\pi')$, and both modules are viewed as $U_{\pi'}U^0$-modules. Note that the character formula for the finite-dimensional simple $U_q(\mathfrak{g})$-module $L(\lambda)$ of highest weight λ is the same as its classical counterpart ([J, Chapter 4] or [JL, Section 5.10]). Hence

$$[L(\lambda) : L_{\pi'}(\gamma)] = [\bar{L}(\lambda) : \bar{L}_{\pi'}(\gamma)]$$

for all subsets π' of π, all dominant integral weights $\lambda \in P^+(\pi)$, and all weights $\gamma \in P(\pi)$ so that $\tilde{\gamma} \in P^+(\pi')$. Thus, we have the following quantum analog of Theorem 5.1.

Theorem 5.3. *Let* $\beta \in \Delta^+$ *and* $\alpha_\beta \in \mathrm{Supp}(\beta)$ *such that*

(i) $\beta = \alpha_\beta + w_\beta \alpha_\beta$,
(ii) $\pi' \subseteq \mathrm{StrOrth}(\beta)$,

where $\pi' = \mathrm{Supp}(\beta) \setminus \{\alpha_\beta\}$ *and* $w_\beta = w(\pi')_0$. *Then for all* $m \geq 1$, $L(m\nu_{\alpha_\beta})$ *has a unique nonzero vector (up to scalar multiple) of weight* $m\nu_{\alpha_\beta} - \beta$ *that generates a trivial* $U_{\pi'}$*-module.*

Consider a subset π' of π and a weight $\lambda \in P^+(\pi)$ such that $(\lambda, \alpha_i) = 0$ for all $\alpha_i \in \pi'$. It follows from the description of the (ordinary) adjoint action given in Section 3.1 that $(\mathrm{ad}\, F_i)K_{-2\lambda} = (\mathrm{ad}\, E_i)K_{-2\lambda} = 0$ and $(\mathrm{ad}\, K_i)K_{-2\lambda} = K_{-2\lambda}$ for all $\alpha_i \in \pi'$. Hence, for all $\alpha_i \in \pi'$, the ordinary adjoint action of E_i, namely $(\mathrm{ad}\, E_i)$, on $(\mathrm{ad}\, U^-)K_{-2\lambda}$ agrees with the action of $(\mathrm{ad}_\lambda E_i)$ on $(\mathrm{ad}_\lambda U^-)1$, which is the $(\mathrm{ad}\, U_q(\mathfrak{g}))$-submodule of $G^-(\lambda)$ (from Section 3.2) generated by 1. The analogous assertion holds with E_i replaced by K_i. In particular, as $U_{\pi'}$-modules, $(\mathrm{ad}\, U^-)K_{-\lambda}$ is isomorphic to $(\mathrm{ad}_\lambda U^-)1$ where the action on the first module is using the ordinary adjoint action and the action on the latter is using the twisted adjoint action. (Note that this means that the ordinary adjoint action of $U_{\pi'}$ on $G^-K_{-\lambda}$ is the same as the graded action of $U_{\pi'}$ on $G^-K_{-\lambda}$ as presented in Section 3.2.)

Corollary 5.4. *Let* $\beta \in \Delta^+$ *and* $\alpha_\beta \in \mathrm{Supp}(\beta)$ *such that*

(i) $\beta = \alpha_\beta + w_\beta \alpha_\beta$,
(ii) $\pi' \subseteq \mathrm{StrOrth}(\beta)$,

where $\pi' = \mathrm{Supp}(\beta) \setminus \{\alpha_\beta\}$ *and* $w_\beta = w(\pi')_0$. *Then there exists a unique nonzero element (up to scalar multiple)* Y *in* $(U_{-\beta}^-)^{U_{\pi'}}$. *Moreover, a scalar multiple of* Y *specializes to* $f_{-\beta}$ *as* q *goes to* 1 *and* $Y K_\beta K_{-2\nu_{\alpha_\beta}} \in (\mathrm{ad}\ U^-) K_{-2\nu_{\alpha_\beta}}$.

Proof. Set $\nu = \nu_{\alpha_\beta}$. Let $G^-(2\nu)$ be the $(\mathrm{ad}_{2\nu} U_q(\mathfrak{g}))$-module defined in Section 3.2. Note that 1 corresponds to a highest weight vector of highest possible weight inside $G^-(2\nu)$. Hence the socle of $G^-(2\nu)$, which is isomorphic to $L(\nu)$, is equal to $(\mathrm{ad}_{2\nu} U^-)1$. By the discussion preceding the lemma, $(\mathrm{ad}_{2\nu} U^-)1$ is isomorphic to $(\mathrm{ad}\ U^-) K_{-2\nu}$ as $U_{\pi'}$-modules with respect to the ordinary adjoint action. Thus as $U_{\pi'}$-modules, $(\mathrm{ad}\ U^-) K_{-2\nu}$ and $L(\nu)$ are isomorphic. Hence, by Theorem 5.3, $(\mathrm{ad}\ U^-) K_{-2\nu}$ contains a nonzero element $Y K_{\beta-2\nu}$ of weight $-\beta$ that admits a trivial action with respect to $(\mathrm{ad}\ U_{\pi'})$. By the definition of the adjoint action, we have $Y \in U_{-\beta}^-$. Since $(\mathrm{ad}\ U_{\pi'})$ acts trivially on $K_{\beta-2\nu}$, we also have $(\mathrm{ad}\ m) Y = \epsilon(m) Y$ for all $m \in U_{\pi'}$. Multiplying Y by a (possibly negative) power of $(q-1)$ if necessary, we may assume that $Y \in \hat{U}$ but $Y \notin (q-1)\hat{U}$. Hence Y specializes to a nonzero element of weight $-\beta$ in $U(\mathfrak{n}^-)$ that commutes with all elements of $\mathfrak{g}_{\pi'}$. By Corollary 5.2, Y specializes to a nonzero scalar multiple of $f_{-\beta}$. Now suppose that Y' is another element in $U_{-\beta}^-$ satisfying $(\mathrm{ad}\ m) Y' = \epsilon(m) Y'$ for all $m \in U_{\pi'}$. Assume that Y' is not a scalar multiple of Y. Then some linear combination of Y and Y' must specialize to an element in $U(\mathfrak{n}^-)$ different from the specialization of Y. This contradicts Corollary 5.2, thus proving the desired uniqueness assertion. $\square$

Remark 5.5. In general, the lifts of root vectors $f_{-\beta}$ given in Corollary 5.4 do not agree with the corresponding weight $-\beta$ element of the quantum PBW basis defined by Lusztig (see [Ja, Chapter 8]). To get an idea of why this is true, assume that in addition $\beta = w\alpha_\beta = \alpha_\beta + w_\beta\alpha_\beta$ where $w = s_{\alpha_\beta} w_\beta$, $w_\beta = w(\pi')_0$, and $\pi' = \mathrm{Supp}(\beta) \setminus \{\alpha_\beta\}$ as happens for many of the weights appearing in the strongly orthogonal θ-systems of Theorem 2.7. By Lemma 3.1, $F = T_{w_\beta}^{-1}(F_{-\alpha_\beta} K_{\alpha_\beta})$ is $(\mathrm{ad}\ U_{\pi'}^-)$ invariant. Applying $T_{\alpha_\beta}^{-1}$ to F yields an element that is trivial with respect to the action of $(\mathrm{ad}\ F_i)$ for simple roots $\alpha_i \in \pi'$ that also satisfy $(\alpha_i, \alpha_\beta) = 0$. This is because $T_{\alpha_\beta}(F_i) = F_i$ under these conditions. However, if $(\alpha_i, \alpha_\beta) \neq 0$, then $T_{\alpha_\beta}^{-1}(F_i) \neq F_i$, and, as a result, we cannot expect $(\mathrm{ad}\ F_i)(T_{\alpha_\beta}^{-1}(F)) = 0$. Hence F is generally not a trivial element with respect to the action of $(\mathrm{ad}\ U_{\pi'})$. If instead, we express $\beta = w'\alpha_j$ with $\alpha_j \in \pi'$ and w' some element in the Weyl group $\mathcal{W}$, then we cannot expect $(\mathrm{ad}\ F_j)(T_{w'}^{-1}(F_j K_j)) = 0$. In contrast to the results in this section, we see in the next section that Lusztig's automorphisms play a key role in defining quantum analogs of root vectors for a different family of positive roots.

5.3. Quantum analogs of special root vectors, type A.
We establish a result similar to Corollary 5.4 for positive roots in type A.

Theorem 5.6. *Let* $\beta \in \Delta^+$ *and* $\alpha_\beta \in \mathrm{Supp}(\beta)$ *such that*

(i) $\beta = p(\alpha_\beta) + w_\beta\alpha_\beta = w\alpha_\beta$,
(ii) $\pi' \subseteq \mathrm{StrOrth}(\beta)$,

where $\pi' = \mathrm{Supp}(\beta)\setminus\{\alpha_\beta, p(\alpha_\beta)\}$, $w_\beta = w(\pi')_0$, *and* $w = w(\mathrm{Supp}(\beta)\setminus\{\alpha_\beta\})_0$. *Then there exists a unique nonzero element (up to scalar multiple)* $Y \in (U_{-\beta}^-)^{U_{\pi'}}$ *such that* $Y K_\beta K_{-2\nu_{\alpha_\beta}} \in (\mathrm{ad}\ U^-) K_{-2\nu_{\alpha_\beta}}$. *Moreover, a scalar multiple of* Y *specializes to* $f_{-\beta}$ *as* q *goes to* 1.

Proof. Set $\nu = \nu_{\alpha_\beta}$ and $\pi'' = \operatorname{Supp}(\beta) \setminus \{\alpha_\beta\}$. Choose k so that $\alpha_\beta = \alpha_k$. Set $Y = T_w^{-1}(F_k)$. By the definition of T_w and its inverse (see Section 3.3 and Section 3.4 of [Ko]), we see that Y is a nonzero scalar multiple of $T_w^{-1}(F_k K_k)K_{-\beta}$. By Lemma 3.1, $Y K_\beta$ is in $(\operatorname{ad} U_{\pi''}^-)(F_k K_k)$ and is a lowest weight vector of weight $-\beta$ in the $(\operatorname{ad} U_{\pi''})$-module generated by $F_k K_k$. Since $Y K_\beta$ has weight $-\beta$ and $(\beta, \alpha_i) = 0$ for all $\alpha_i \in \pi'$, it follows that $Y K_\beta$ is a trivial $(\operatorname{ad} U_{\pi'})$ vector in this module. Thus $(\operatorname{ad} m)Y K_\beta = (\operatorname{ad} m)Y = \epsilon(m)Y$ for all $m \in U_{\pi'}$. The facts that $F_k K_k K_{-2\nu}$ is a scalar multiple of $(\operatorname{ad} F_k)K_{-2\nu}$ and $(\operatorname{ad} U_{\pi''})$ acts trivially on $K_{-2\nu}$ ensure that $Y \in [(\operatorname{ad} U^-)K_{-2\nu}]K_{2\nu-\beta}$. Moreover, by the definition of the Lusztig automorphisms, Y specializes to the root vector $f_{-\beta}$ as q goes to 1.

Note that $(\operatorname{ad} U^-)K_{-2\nu} = (\operatorname{ad} U^-)(F_k K_k K_{-2\nu}) + \mathbb{C}(q)K_{-2\nu}$ since $(\operatorname{ad} F_i)K_{-2\nu} = 0$ for all $i \neq k$. Hence, any element in $(\operatorname{ad} U^-)K_{-2\nu}$ of weight $-\beta$ is contained in the $(\operatorname{ad} U_{\pi''})$-module $(\operatorname{ad} U_{\pi''})(F_k K_k K_{-2\nu})$. As stated in Lemma 3.1, this $(\operatorname{ad} U_{\pi''})$-module is isomorphic to the finite-dimensional $(\operatorname{ad} U_{\pi''})$-module with lowest weight $-\beta$. Hence $(\operatorname{ad} U_{\pi''})F_k K_k K_{-2\nu}$ contains a unique nonzero element (up to scalar multiple) of weight $-\beta$. This proves the uniqueness assertion. $\square$

Let φ be the $\mathbb{C}$ algebra automorphism of $U_q(\mathfrak{g})$ which fixes all E_i, F_i, sends K_i to K_i^{-1}, and sends q to q^{-1} and let φ' be the $\mathbb{C}$ algebra automorphism of $U_q(\mathfrak{g})$ which fixes all $K_i^{-1}E_i, F_i K_i$, sends K_i to K_i^{-1}, and sends q to q^{-1}. Note that $\kappa\varphi = \varphi'\kappa$ where κ is the quantum Chevalley antiautomorphism of Section 3.4.

The next lemma will be useful in the analysis of quantum root vectors and Cartan elements associated to a weight β satisfying the conditions of Theorem 5.6. Note that in these cases, $\Delta(\operatorname{Supp}(\beta))$ is a root system of type A. It turns out that this lemma is also useful in later sections for the analysis associated to weights β that satisfy condition (4) of Theorem 2.7. In this latter case, $\Delta(\operatorname{Supp}(\beta))$ is of type B.

Lemma 5.7. *Let $\alpha_{i_1}, \ldots, \alpha_{i_{m+1}}$ be a set of roots with $(\alpha_{i_j}, \alpha_{i_k}) = -1$ if $k = j \pm 1$ and 0 if $k \neq j \pm 1$ and $i_j \neq i_k$. Then*

(i) $((\operatorname{ad} E_{i_1} \cdots E_{i_m})E_{i_{m+1}}) = (-q)^m \varphi((\operatorname{ad} E_{i_{m+1}} \cdots E_{i_2})E_{i_1})$,

(ii) $((\operatorname{ad} F_{i_1} \cdots F_{i_m})F_{i_{m+1}}K_{i_{m+1}}) = q^m \varphi'((\operatorname{ad} F_{i_{m+1}} \cdots F_{i_2})F_{i_1}K_{i_1})$.

Proof. By the definition of the adjoint action and the assumptions of the lemma, we see that

$$\begin{aligned}
(\operatorname{ad} E_j)E_{j+1} &= E_j E_{j+1} - q E_{j+1} E_j \\
&= -q(E_{j+1}E_j - q^{-1}E_j E_{j+1}) = -q\varphi((\operatorname{ad} E_{j+1})E_j).
\end{aligned}$$

The first assertion follows from this equality and induction. The second assertion follows by applying the quantum Chevalley antiautomorphism κ to both sides and the similar equality given by (3.6). $\square$

We are interested in $\beta \in \Delta^+$ that satisfy (i) and (ii) of Theorem 5.6 and appear in one of the strongly orthogonal θ-systems of Theorem 2.7 and its proof. In particular, β fits into case (3) of Theorem 2.7 and by Remark 2.8, $\operatorname{Supp}(\beta)$ is of type A and θ restricts to an involution of type AIII/AIV on the Lie algebra $\mathfrak{g}_{\operatorname{Supp}(\beta)}$. One checks that this situation only arises when $\mathfrak{g}$ is of type A, D, or E. Hence Δ, and thus $\Delta(\operatorname{Supp}(\beta))$ are both simply-laced. Note that the simple root α_β corresponds to the first simple root of $\operatorname{Supp}(\beta)$ and $p(\alpha_\beta)$ corresponds to the last when the simple roots of $\operatorname{Supp}(\beta)$ are ordered in the standard way corresponding to the Dynkin

diagram of type A. Thus, $(\alpha_\beta, \beta) \neq 0$ and $(p(\alpha_\beta), \beta) \neq 0$. Moreover, since Δ is of simply-laced type, we have

$$(5.8) \qquad\qquad (\alpha_\beta, \beta) = -1 = (p(\alpha_\beta), \beta).$$

The next corollary establishes connections between a lift of the root vector e_β to G^+ to a lift of the root vector $f_{-\beta}$ to U^-.

Corollary 5.8. *Let $\beta \in \Delta^+$ and $\alpha_\beta \in \mathrm{Supp}(\beta)$ such that*

(i) $\beta = p(\alpha_\beta) + w_\beta \alpha_\beta = w\alpha_\beta$,
(ii) $\pi' \subseteq \mathrm{StrOrth}(\beta)$,

where $\pi' = \mathrm{Supp}(\beta) \setminus \{\alpha_\beta, p(\alpha_\beta)\}$, $w_\beta = w(\pi')_0$, $w = w(\mathrm{Supp}(\beta) \setminus \{\alpha_\beta\})_0$. Assume further that $\Delta(\mathrm{Supp}(\beta))$ is a root system of type A, the symmetric pair $\mathfrak{g}_{\mathrm{Supp}(\beta)}, (\mathfrak{g}_{\mathrm{Supp}(\beta)})^\theta$ is of type AIII/AIV, and neither α_β nor $p(\alpha_\beta)$ is in $\mathrm{Orth}(\beta)$. Let Y be the unique nonzero element (up to scalar multiple) in $(U^-_{-\beta})^{U_{\pi'}}$ such that $YK_\beta K_{-2\nu} \in (\mathrm{ad}\ U^-)K_{-2\nu_{\alpha_\beta}}$. Then

(a) *Y is the unique nonzero element (up to scalar multiple) in $(U^-_{-\beta})^{U_{\pi'}}$ such that $F_{-p(\alpha_\beta)}Y - q^{-1}YF_{-p(\alpha_\beta)} = 0$,*
(b) *Y is the unique nonzero element (up to scalar multiple) in $(U^-_{-\beta})^{U_{\pi'}}$ such that $F_{-\alpha_\beta}Y - qYF_{-\alpha_\beta} = 0$.*

Moreover, if $X \in (G^+_\beta)^{U_{\pi'}}$ and

$$(5.9) \qquad \left(\theta_q(F_{-\alpha_\beta}K_{\alpha_\beta})K_{\alpha_\beta}^{-1}\right)X - qX\left(\theta_q(F_{-\alpha_\beta}K_{\alpha_\beta})K_{\alpha_\beta}^{-1}\right) = 0,$$

then $\kappa(X)$ is a nonzero scalar multiple of Y.

Proof. Set $\nu = \nu_{\alpha_\beta}$. By Theorem 5.6, there exists a unique nonzero element (up to scalar multiple) $YK_{\beta-2\nu}$ in $(U^-_{-\beta})^{U_{\pi'}}$ that is also in $(\mathrm{ad}\ U^-)K_{-2\nu}$. Recall (Section 3.2) that $G^-K_{-2\nu}$ has a unique nonzero (up to scalar multiple) $(\mathrm{ad}\ U_q(\mathfrak{g}))$ lowest weight vector Y' of weight $-\nu + w\nu = -w\alpha_\beta = -\beta$ and this lowest weight vector is in $(\mathrm{ad}\ U^-)K_{-2\nu}$. Since $(\beta, \alpha_i) = 0$ for all $\alpha_i \in \pi'$, $Y' \in (U^-_{-\beta})^{U_{\pi'}}$. The uniqueness property ensures that $Y' = YK_{\beta-2\nu}$ and so $YK_{\beta-2\nu}$ is the unique (up to scalar multiple) $(\mathrm{ad}\ U_q(\mathfrak{g}))$ lowest weight vector of $G^-K_{\beta-2\nu}$. By (5.8) and the definition of the adjoint action in Section 3.1, we have

$$(5.10) \qquad (F_{-p(\alpha_\beta)}Y - q^{-1}YF_{-p(\alpha_\beta)}) = [(\mathrm{ad}\ F_{-p(\alpha_\beta)})YK_{\beta-2\nu}]K_{-\beta+2\nu} = 0.$$

In particular, Y is the unique nonzero element of $(U^-_{-\beta})^{U_{\pi'}}$ (up to scalar multiple) that satisfies (5.10) which proves (a).

By Lemma 5.7, $\varphi(Y)$ is the unique nonzero element of $(U^-_{-\beta})^{U_{\pi'}}$ (up to scalar multiple) that satisfies

$$(5.11) \qquad (F_{-\alpha_\beta}\varphi(Y) - q^{-1}\varphi(Y)F_{-\alpha_\beta}) = [(\mathrm{ad}\ F_{-\alpha_\beta})\varphi(Y)K_{\beta-2\nu}]K_{-\beta+2\nu} = 0.$$

Applying φ to each side of (5.11) yields

$$(5.12) \qquad\qquad (F_{-\alpha_\beta}Y - qYF_{-\alpha_\beta}) = 0.$$

Hence, we can also view Y as the unique nonzero element of $(U^-_{-\beta})^{U_{\pi'}}$ (up to scalar multiple), that satisfies (5.12). Thus (b) also holds.

Now suppose that $X \in (G_\beta^+)^{U_{\pi'}}$ and X satisfies (5.9). Note that (5.9) is equivalent to

$$(5.13) \qquad q^{-1}\left(\theta_q(F_{-\alpha_\beta}K_{\alpha_\beta})\right)X - qX\theta_q(F_{-\alpha_\beta}K_{\alpha_\beta}) = 0.$$

Since $\mathfrak{g}_{\mathrm{Supp}(\beta)}, (\mathfrak{g}_{\mathrm{Supp}(\beta)})^\theta$ is of type AIII/AIV, there are two possibilities for the intersection π_θ and $\mathrm{Supp}(\beta)$. The first is

$$\pi_\theta \cap \mathrm{Supp}(\beta) \subsetneq \mathrm{Supp}(\beta) \setminus \{\alpha_\beta, p(\alpha_\beta)\}$$

and the second is

$$\pi_\theta \cap \mathrm{Supp}(\beta) = \mathrm{Supp}(\beta) \setminus \{\alpha_\beta, p(\alpha_\beta)\}.$$

In the first case, we have $\theta_q(F_{-\alpha_\beta}K_{\alpha_\beta}) = E_{p(\alpha_\beta)}$ and so (5.13) becomes

$$(5.14) \qquad q^{-1}E_{p(\alpha_\beta)}X - qXE_{p(\alpha_\beta)} = 0$$

while in the second case, it follows from (4.1) that

$$\theta_q(F_{-\alpha_\beta}K_{\alpha_\beta}) = c[(\mathrm{ad}\ E_{i_1}E_{i_2}\cdots E_{i_m})E_{p(\alpha_\beta)}]$$

for some nonzero scalar c and choice of $i_1, \ldots, i_m$ in $\{1, \ldots, n\}$ with $\alpha_{i_j} \in \pi'$ for $j = 1, \ldots, m$. Since $\mathfrak{g}_{\mathrm{Supp}(\beta)}, (\mathfrak{g}_{\mathrm{Supp}(\beta)})^\theta$ is of type AIII/AIV, the sequence of roots

$$\alpha_{i_0} = \alpha_{p(\alpha_\beta)}, \alpha_{i_1}, \ldots, \alpha_{i_m}, \alpha_{i_{m+1}} = \alpha_\beta$$

form a set of simple roots for a root system of type A_{m+2}. Note that $\{\alpha_{i_1}, \ldots, \alpha_{i_m}\} = \pi'$. Moreover,

$$(\mathrm{ad}\ F_{i_m}\cdots F_{i_1}E_{i_1}E_{i_2}\cdots E_{i_m})E_{p(\alpha_\beta)}$$

is a nonzero scalar multiple of $E_{p(\alpha_\beta)}$. Since X commutes with all elements of $U_{\pi'}$, applying $(\mathrm{ad}\ F_{i_m}\cdots F_{i_1})$ to both sides of (5.13) yields (5.14) in this case as well.

Now (5.14) is equivalent to

$$(5.15) \qquad E_{p(\alpha_\beta)}K_{p(\alpha_\beta)}^{-1}X - qXE_{p(\alpha_\beta)}K_{p(\alpha_\beta)}^{-1} = 0.$$

Applying the quantum Chevalley antiautomorphism κ (see Section 3.4) to X and to both sides of (5.15) yields an element $\kappa(X) \in U_{-\beta}^-$ that commutes with all elements of $U_{\pi'}$ and satisfies

$$F_{-p(\alpha_\beta)}\kappa(X) - q^{-1}\kappa(X)F_{-p(\alpha_\beta)} = 0.$$

By (a), $\kappa(X)$ is a (nonzero) scalar multiple of Y. $\qquad\qquad\qquad\square$

5.4. **The lower triangular part.** Recall that $\mathcal{T}_\theta$ is equal to the group consisting of all elements K_β, $\beta \in Q(\pi)^\theta$. Write $\mathbb{C}(q)[\mathcal{T}_\theta]$ for the group algebra generated by $\mathcal{T}_\theta$. Note that $\mathbb{C}(q)[\mathcal{T}_\theta]$ is the Laurent polynomial ring with generators

- $K_i, \alpha_i \in \pi_\theta$,
- $K_iK_{\theta(\alpha_i)}, \alpha_i \notin \pi_\theta$ and $i < p(i)$.

Theorem 5.9. *Let θ be a maximally split involution and let $\Gamma_\theta = \{\beta_1, \ldots, \beta_m\}$ be a maximum strongly orthogonal θ-system as in Theorem 2.7. For each j, there exists a unique nonzero element Y_j (up to scalar multiple) in $U_{-\beta_j}^-$ such that*

  (i) *$Y_j \in [(\mathrm{ad}\,U^-)K_{-2\nu_j}]K_{2\nu_j - \beta_j}$ where ν_j is the fundamental weight corresponding to α_{β_j}.*
  (ii) *For all $\alpha_s \in \mathrm{StrOrth}(\beta_j)$, Y_j commutes with E_s, F_s, and $K_s^{\pm 1}$.*
  (iii) *A scalar multiple of Y_j specializes to $f_{-\beta_j}$ as q goes to 1.*
  (iv) *If β_j satisfies Theorem 2.7(5), then $Y_j \in [(\mathrm{ad}\,F_{-\alpha_\beta'})(\mathrm{ad}\,U^-)K_{-2\nu_j}]K_{2\nu_j - \beta_j}$.*

Moreover, $\mathbb{C}(q)[\mathcal{T}_\theta][Y_1,\ldots,Y_m]$ is a commutative polynomial ring over $\mathbb{C}(q)[\mathcal{T}_\theta]$ in m generators that specializes to $U(\mathfrak{h}^\theta \oplus \mathfrak{k})$ as q goes to 1 where $\mathfrak{k}$ is the commutative Lie algebra generated by $f_{-\beta_1},\ldots,f_{-\beta_m}$.

Proof. We use the notation of Theorem 2.7. Fix j with $1 \le j \le m$ and set $\beta = \beta_j$ and $\nu = \nu_j$. Set $\pi' = \mathrm{Supp}(\beta) \setminus \{\alpha'_\beta, \alpha_\beta\}$ and $w_\beta = w(\pi')_0$. Let k be the integer in $\{1,\ldots,n\}$ such that $\alpha_{\beta_j} = \alpha_k$. Note that $(\mathrm{ad}\, E_k)K_{-2\nu}$ is a scalar multiple of $E_k K_{-2\nu}$ while $(\mathrm{ad}\, E_s)K_{-2\nu} = 0$ for $s \ne k$.

Suppose that Y is a nonzero element of $U^-_{-\beta}$ satisfying conditions (i). Since Y has weight $-\beta$, it follows from (i) that

$$Y \in [(\mathrm{ad}\, U^-_{\mathrm{Supp}(\beta)})K_{-2\nu}]K_{2\nu-\beta}$$

and hence Y is an element of G^- of weight $-\beta$. It follows that $Y \in G^-_{\mathrm{Supp}(\beta)}$. Consider $\alpha_s \in \mathrm{StrOrth}(\beta)$ and assume that $\alpha_s \notin \mathrm{Supp}(\beta)$. Recall that $(\alpha_s, \alpha) \le 0$ for all $\alpha \in \pi \setminus \{\alpha_s\}$. Since β is a positive root and $(\beta, \alpha_s) = 0$, we must have $(\alpha_s, \alpha) = 0$ for all $\alpha \in \mathrm{Supp}(\beta)$. Moreover, since α_s is a simple root not equal to α, we have that $\alpha_s - \alpha$ is not a root. This ensures that $\alpha_s + \alpha$ is also not a root and so α_s, α are strongly orthogonal for all $\alpha \in \mathrm{Supp}(\beta)$. Using the defining relations of $U_q(\mathfrak{g})$, it is straightforward to see that $E_s, F_s, K_s^{\pm 1}$ commute with any element in $U_{\mathrm{Supp}(\beta)}$ and hence commute with Y. Thus, given $Y \in U^-_{-\beta}$ satisfying (i), we need only show (ii) holds for $\alpha_s \in \mathrm{StrOrth}(\beta) \cap \mathrm{Supp}(\beta)$.

Using Theorem 2.7, we break the argument up into five possibilities for β. Each of these five cases is handled separately. In the discussion below, we drop the subscript j from Y_j and simply write Y.

Case 1. $\beta = \alpha_\beta = \alpha'_\beta$. In this case, $Y = F_k$ and it is easy to see that Y is the unique nonzero element (up to scalar multiple) of weight $-\beta$ satisfying the three conditions (i), (ii), and (iii).

Case 2. $\beta = \alpha_\beta + w_\beta \alpha_\beta$ and $\alpha_\beta = \alpha'_\beta$. By Theorem 2.7(ii), we have $\pi' \subseteq \mathrm{StrOrth}(\beta)$. By Corollary 5.4, there exists a unique nonzero element (up to scalar multiple) $Y \in (U^-_{-\beta})^{U_{\pi'}}$. Moreover, $Y \in [(\mathrm{ad}\, U^-)K_{-2\nu}]K_{2\nu-\beta}$ and multiplying by a scalar if necessary, Y specializes to $f_{-\beta}$ as q goes to 1. Thus Y satisfies (i), (ii), and (iii).

Case 3. $\beta = p(\alpha_\beta) + w_\beta \alpha_\beta = w\alpha_\beta$ where $w = w(\mathrm{Supp}(\beta) \setminus \{\alpha_\beta\})_0$. By Theorem 5.6 and its proof, $Y = T_w^{-1}(F_k)$ satisfies the conclusions of the theorem.

Case 4. $\beta = \alpha'_\beta + w_\beta \alpha_\beta = w\alpha'_\beta$ where $w = w(\mathrm{Supp}(\beta) \setminus \{\alpha'_\beta\})_0$. Furthermore, we may assume that $\mathrm{Supp}(\beta) = \{\gamma_1,\ldots,\gamma_r\}$ generates a root system of type B_r where we are using the standard ordering, $\alpha'_\beta = \gamma_r$ the unique short root and $\alpha_\beta = \gamma_1$. Note that $w = s_1 \cdots s_r$ is a reduced expression for w where s_i is the reflection associated to the simple root γ_i for each i.

Set $Y' = T_w^{-1}(F_{-\gamma_s})$. Arguing as in the proof of Theorem 5.6, we see that Y' commutes with $E_s, F_s, K_s^{\pm 1}$ for all $\alpha_s \in \mathrm{StrOrth}(\beta)$ and Y' specializes to $f_{-\beta}$ as q goes to 1. Hence Y' satisfies (ii) and (iii). Set $\pi'' = \mathrm{Supp}(\beta) \setminus \{\alpha'_\beta\}$. By Lemma 3.1, $Y'K_\beta$ is a lowest weight vector in the $(\mathrm{ad}\, U_{\pi''})$- module generated by $F_{-\gamma_r} K_{\gamma_r}$. Arguing as in the proof of Theorem 5.6, we observe that Y' is the unique nonzero element (up to scalar multiple) element in $[(\mathrm{ad}\, U^-_{\pi''})F_{-\gamma_r}K_{\gamma_r}]K_{-\beta}$ of weight $-\beta$ satisfying (ii) and (iii). However, $\gamma_r = \alpha'_\beta$ and we want this result to hold for

$\gamma_1 = \alpha_\beta$. To achieve this, we apply the $\mathbb{C}$ algebra automorphism φ' defined in the previous section. Indeed, set $Y = \varphi'(Y'K_\beta)K_{-\beta}$. It follows from the definition of φ' that Y also satisfies (ii) and (iii).

To see that Y satisfies (i), we apply Lemma 5.7. In particular, using the reduced expression for w and the fact that $Y'K_\beta$ is a lowest weight vector in the $(\mathrm{ad}\, U_{\pi'})$-module generated by $F_{-\gamma_r}K_{\gamma_r}$ yields

$$Y'K_\beta = c(\mathrm{ad}\, F_{-\gamma_1}F_{-\gamma_2}\cdots F_{-\gamma_{s-1}})(F_{-\gamma_s}K_{\gamma_s})$$

for some nonzero scalar c. By Lemma 5.7, we see that

$$YK_\beta = q^{s-1}c(\mathrm{ad}\, F_{-\gamma_s}F_{-\gamma_{s-1}}\cdots F_{-\gamma_2})(F_{-\gamma_1}K_{\gamma_1})$$

and so $Y \in [(\mathrm{ad}\, U^-)K_{-2\nu}]K_{2\nu-\beta}$ as desired.

Case 5. $\beta = \alpha'_\beta + \alpha_\beta + w_\beta \alpha_\beta$ and $\alpha'_\beta \neq \alpha_\beta$. Furthermore, $\alpha'_\beta \in \pi_\theta$, α'_β is strongly orthogonal to all simple roots in $\mathrm{Supp}(\beta) \setminus \{\alpha'_\beta, \alpha_\beta\}$, and α'_β and β are orthogonal but not strongly orthogonal to each other. In particular, $\beta' = \beta - \alpha_{\beta'}$ is a positive root. Note also that $\beta' = \alpha_\beta + w\alpha_\beta$ where $w = w(\mathrm{Supp}(\beta') \setminus \{\alpha_\beta\})_0$. Moreover, properties of β ensure that $(\mathrm{Supp}(\beta) \setminus \{\alpha_\beta\}) \subseteq \mathrm{StrOrth}(\beta)$. Arguing as in Case 2 yields an element $Y' \in U^-_{-\beta'}$ satisfying (i), (ii), and (iii) with β replaced by β'. Set $Y = [(\mathrm{ad}\, F_{-\alpha'_\beta})Y'K_{\beta'}]K_{-\beta}$ and note that Y satisfies (i)–(iv).

Now suppose that Y'' is another element in $U^-_{-\beta}$ satisfying (i), (ii), (iii), and (iv). Since $Y'' \in [(\mathrm{ad}\, U^-)K_{-2\nu}]K_{2\nu-\beta}$, the multiplicity of α'_β in β is 1 and α'_β is strongly orthogonal to all $\alpha_i \in \pi'$, we have

$$[(\mathrm{ad}E_{\alpha'_\beta})(Y''K_{\beta-2\nu})]K_{2\nu-\beta} \in (U^-_{-\beta'})^{U_{\pi'}} \cap (\mathrm{ad}\, U^-)K_{-2\nu}.$$

By the uniqueness part of Corollary 5.4, $[(\mathrm{ad}E_{\alpha'_\beta})(Y''K_{\beta-2\nu})]K_{2\nu-\beta}$ is a scalar multiple of Y'. On the other hand, if

$$Y'' = [(\mathrm{ad}\, F_{-\alpha'_\beta})fK_{\beta'}]K_{-\beta'}$$

for some $f \in U^-_{-\beta'}$, then $[(\mathrm{ad}E_{\alpha'_\beta})(Y''K_{\beta-2\nu})]K_{2\nu-\beta}$ is a scalar multiple of f and so $f = Y'$ after suitable scalar adjustment. This completes the proof of the uniqueness assertion in this case.

We have shown that for each $\beta_j \in \Gamma_\theta$, there exists a unique nonzero element (up to scalar multiple) $Y_j \in U^-_{-\beta}$ satisfying (i)-(iv). Recall that each $\beta \in \Gamma_\theta$ satisfies $\theta(\beta) = -\beta$ and so $(\alpha, \beta) = 0$ for all $\alpha \in \pi_\theta$. Hence, the fact that Y_j has weight $-\beta_j$ ensures that Y_j commutes with all elements in $\mathcal{T}_\theta$. Now consider Y_j and Y_s with $s > j$. By construction, Y_s is in $U^-_{\mathrm{Supp}(\beta_s)}$. By Theorem 2.7, $\mathrm{Supp}(\beta_s) \subseteq \mathrm{StrOrth}(\beta_j)$. Hence assertion (ii) ensures that each F_r with $\alpha_r \in \mathrm{Supp}(\beta_s)$ commutes with Y_j. Thus Y_s commutes with Y_j for all $s > k$. It follows that $\mathbb{C}(q)[\mathcal{T}_\theta][Y_1, \ldots, Y_m]$ is a commutative ring.

By the discussion above, each Y_j specializes to the root vector $f_{-\beta_j}$ as q goes to 1. Recall that $\mathcal{B}_\theta$ specializes to $U(\mathfrak{g}^\theta)$ and, in particular, $\mathbb{C}(q)[\mathcal{T}_\theta]$ specializes to $U(\mathfrak{h}^\theta)$ as q goes to 1. Therefore, $\mathbb{C}(q)[\mathcal{T}_\theta][Y_1, \ldots, Y_m]$ specializes to $U(\mathfrak{h}^\theta \oplus \mathfrak{t})$ as q goes to 1. Note that $\mathfrak{h}^\theta \oplus \mathfrak{t}$ is a commutative Lie algebra. Thus, $U(\mathfrak{h}^\theta \oplus \mathfrak{t})$ is a polynomial ring in $\dim \mathfrak{h}^\theta + m$ variables. It follows that there are no additional relations among the Y_j other than the fact that they commute since such relations would specialize to extra relations for elements of $U(\mathfrak{h}^\theta \oplus \mathfrak{t})$. Hence $\mathbb{C}(q)[\mathcal{T}_\theta][Y_1, \ldots, Y_m]$ is a polynomial ring in m variables over $\mathbb{C}(q)[\mathcal{T}_\theta]$. $\qquad\square$

6. Conditions for commutativity

Given a weight $\beta \in Q^+(\pi)$ and a subset π' of $\mathrm{Supp}(\beta)$, we establish conditions for special elements in $U_q(\mathfrak{g})$ to commute with all elements in $U_{\pi'}$. These results will be used in Section 7 to show that the quantum Cartan subalgebra elements commute with each other.

6.1. **Generalized normalizers.** One of the key ideas in establishing commutativity is to use the simple fact that if $a \in \mathcal{B}_\theta$, then so is the commutator $ba - ab$ for any $b \in \mathcal{B}_\theta$. Many of the arguments we use break down a into sums of terms that still retain this commutator property for certain choices of b. In particular, these summands are elements of a generalized normalizer. By generalized normalizer, we mean the following: given three algebras C, D, U with $C \subseteq D \subseteq U$, the generalized normalizer of D inside U with respect to C, denoted $\mathcal{N}_U(D : C)$, is defined by

$$\mathcal{N}_U(D : C) = \{X \in U | \; bX - Xb \in D \text{ for all } b \in C\}.$$

Since we will only be considering $U = U_q(\mathfrak{g})$, we drop the U subscript and just write $\mathcal{N}(D : C)$.

We use standard commutator notation in the discussion below. In other words, $[b, a] = ba - ab$ for all $a, b \in U_q(\mathfrak{g})$. It follows from the definition of the adjoint action (see Section 3.1) that $[F_i, a] = ((\mathrm{ad}\, F_i)a) K_i^{-1}$. Hence in many cases it is straightforward to translate between the commutator and the adjoint action.

Recall the notion of l-weight defined in Section 3.1.

Lemma 6.1. *Let π' be a subset of π, let $\zeta \in Q^+(\pi)$ and $\xi \in Q(\pi)$, and let g, u be nonzero elements of $G^-_{-\zeta}, U^+$, respectively. Either there exists $\alpha_i \in \pi'$ such that*

$$(6.1) \qquad (\mathrm{ad}\, F_i)gK_\xi u \in g'K_\xi u + G^-_{-\zeta}U^0U^+,$$

where g' is a nonzero term in $G^-_{-\zeta-\alpha_i}$ or gK_ξ is a nonzero $(\mathrm{ad}\, U_{\pi'})$ lowest weight vector. Hence if

$$(6.2) \qquad X \in gK_\xi u + \sum_{\lambda \gneq \zeta} G^-_{-\lambda}U^0U^+,$$

where $g \in G^-_{-\zeta}$ and $X \in \mathcal{N}(\mathcal{B}_\theta : \mathcal{B}_\theta \cap U_{\pi'})$, then gK_ξ is an $(\mathrm{ad}\, U_{\pi'})$ lowest weight vector or $uK_{\xi+\zeta} \in \mathcal{M}^+\mathcal{T}_\theta$.

Proof. For each $\alpha_i \in \pi'$ we have

$$(\mathrm{ad}\, F_i)gK_\xi u = ((\mathrm{ad}\, F_i)gK_\xi) K_i^{-1}uK_i + gK_\xi ((\mathrm{ad}\, F_i)u).$$

Note that

$$((\mathrm{ad}\, F_i)gK_\xi) K_i^{-1}uK_i \in G^-_{-\zeta-\alpha_i}K_\xi U^+ \quad \text{and} \quad gK_\xi ((\mathrm{ad}\, F_i)u) \in G^-_{-\zeta}U^0U^+.$$

Hence if $(\mathrm{ad}\, F_i)gK_\xi \neq 0$ for some choice of i with $\alpha_i \in \pi'$, then (6.1) holds with $g' = ((\mathrm{ad}\, F_i)gK_\xi) K_{-\xi} \neq 0$. On the other hand, if $(\mathrm{ad}\, F_i)gK_\xi = 0$ for all i with $\alpha_i \in \pi'$, then gK_ξ is a nonzero $(\mathrm{ad}\, U_{\pi'})$ lowest weight vector.

Now assume that X is an element of $\mathcal{N}(\mathcal{B}_\theta : \mathcal{B}_\theta \cap U_{\pi'})$ that satisfies (6.2). Let $\alpha_i \in \pi'$ and assume that $(\mathrm{ad}\, F_i)g \neq 0$. It follows that $g'K_\xi uK_i^{-1}$ is a minimal l-weight summand of $[B_i, gK_\xi u]$ where $g' \in G^-_{-\zeta-\alpha_i}$ and $[F_i, gK_\xi u]K_i = [(\mathrm{ad}\, F_i)gK_\xi u]$ as in (6.1). Note that $g'K_\xi uK_i^{-1} \in U^-_{-\zeta-\alpha_i}uK_{\xi+\zeta}$. By Remark 4.2, we see that $uK_{\xi+\zeta} \in \mathcal{M}^+\mathcal{T}_\theta$. $\square$

6.2. **Lowest weight terms.** Recall the notion of height defined in Section 2.1. Here, we use a version of height with respect to a subset τ of π. In particular, given a weight $\beta = \sum_{\alpha_i \in \pi} m_i \alpha_i$, set $\mathrm{ht}_\tau(\beta) = \sum_{\alpha_i \in \tau} m_i$.

Consider a weight $\beta \in Q^+(\pi)$ and let $b = \sum_{\mathrm{wt}(I)=\beta} B_I a_I$ be an element of $\mathcal{B}_\theta$ where each $a_I \in \mathcal{M}^+ \mathcal{T}_\theta$. Set $\tilde{b}$ equal to the element of $\mathcal{B}_\theta$ so that $\mathcal{P}(\tilde{b}) = \sum_{\mathrm{wt}(I)=\beta} F_I a_I$ as constructed in Proposition 4.3. Assume that, in addition, there exists a subset π' of $\mathrm{Supp}(\beta)$ so that $\mathrm{ht}_\tau(\beta) = 2$ where $\tau = \mathrm{Supp}(\beta) \setminus \pi'$. By the discussion at the end of Section 4.3, $\tilde{b}$ is contained in a sum of spaces of the form

$$G^-_{-\beta+\lambda+\eta+\gamma} U^+_{\theta(-\lambda)-\gamma} K_{-\beta+2\gamma'},$$

where $\beta, \lambda, \eta, \gamma, \gamma'$ satisfy the conditions of Lemma 4.1. We can express $\tilde{b}$ as a sum of four terms a_0, a_1, a_2, and a_3 where

- $a_3 = \sum_{\mathrm{wt}(I)=\beta} F_I a_I$,
- $a_2 \in \sum_{\delta \in Q^+(\pi') \setminus \{0\}} G^-_{-\beta+\delta} U^0 U^+$,
- $a_1 \in \sum_{\mathrm{ht}_\tau(\delta)=1} G^-_{-\delta} U^0 U^+$,
- $a_0 \in \sum_{\delta \in Q^+(\pi')} G^-_{-\delta} U^0 U^+$.

We will be using terms of the form $\tilde{b}$ associated to weights β that live in special strongly orthogonal θ-systems in order to construct the quantum Cartan element at β. A crucial part of the construction relies on understanding when a summand of minimal l-weight of carefully chosen elements in $U_q(\mathfrak{g})$ is an $(\mathrm{ad}\ U_{\pi'})$ lowest weight vector. We present three lemmas that analyze properties of such $(\mathrm{ad}\ U_{\pi'})$ lowest weight vectors. These results will eventually be applied to a_0, a_1, and a_2 with respect to particular choices of $\tilde{b}$.

The first lemma addresses properties of $(\mathrm{ad}\ U_{\pi'})$ lowest weight vectors contained in $G^-_{-\beta+\delta} U^0 U^+$ where $\delta \in Q^+(\pi')$ and $0 \leq \delta < \beta$. Note that in the context of this lemma, $\delta = \lambda + \gamma$. Moreover, the weights λ, γ, and γ' in this lemma satisfy similar conditions to those of Lemma 4.1.

Lemma 6.2. *Let* $\beta \in Q^+(\pi)$ *and let* π' *be a subset of* $\mathrm{Orth}(\beta) \cap \mathrm{Supp}(\beta)$. *Let* λ, γ, γ' *be three elements in* $Q^+(\pi)$ *such that*

(i) $\lambda \leq \beta$.
(ii) $\gamma \leq \beta - \lambda$, $\gamma \leq \theta(-\lambda)$, *and* $\lambda + \gamma \in Q^+(\pi')$.
(iii) $0 \leq \gamma' \leq \gamma$.

If $G^-_{-\beta+\lambda+\gamma} K_{-\beta+2\gamma'}$ *contains a nonzero* $(\mathrm{ad}\ U_{\pi'})$ *lowest weight vector, then* $\lambda = \gamma = \gamma' = 0$ *and* $G^-_{-\beta+\lambda+\gamma} K_{-\beta+2\gamma'} = U^-_{-\beta}$.

Proof. Let g be a nonzero element of $G^-_{-\beta+\lambda+\gamma}$ so that $gK_{-\beta+2\gamma'}$ is an $(\mathrm{ad}\ U_{\pi'})$ lowest weight vector. Note that the assumption $\lambda + \gamma \in Q^+(\pi')$ combined with the fact that both λ and γ are in $Q^+(\pi)$, forces both λ, γ to be in $Q^+(\pi')$. Since $\gamma' \leq \gamma$, we also have $\gamma' \in Q^+(\pi')$.

Set $\xi = (\beta - 2\gamma')/2$. Recall that we can make G^- into a $U_q(\mathfrak{g})$-module, denoted $G^-(2\xi)$, using the twisted adjoint action $\mathrm{ad}_{2\xi}$ (Section 3.2). Moreover, $G^-(2\xi)$ and $G^- K_{-2\xi}$ are isomorphic as (U^-)-modules where the action on the latter space is via the ordinary adjoint action. Hence g is an $(\mathrm{ad}_{2\xi} U_{\pi'})$ lowest weight vector, and, by the discussion in Section 3.2, g generates a finite-dimensional $(\mathrm{ad}_{2\xi} U_{\pi'})$-submodule of $G^-(2\xi)$. The fact that this is a twisted action implies that the weight of g viewed

as a vector in the $(\mathrm{ad}_{2\xi}U_q(\mathfrak{g}))$-module $G^-(2\xi)$ is

$$\xi - \beta + \lambda + \gamma = -\beta/2 + \lambda + \gamma - \gamma'.$$

Upon restricting to the action of $U_{\pi'}$, the weight of g is $\lambda + \gamma - \gamma'$ since $\lambda, \gamma, \gamma' \in Q^+(\pi')$ and $\tilde{\beta} = 0$. Note that $\gamma - \gamma' \geq 0$ and $\lambda \geq 0$. Hence $\lambda + \gamma - \gamma' \geq 0$. On the other hand, g is a lowest weight vector, hence its weight must be nonpositive. This forces $\lambda + \gamma - \gamma' = 0$ and so $\lambda = \gamma - \gamma' = 0$. Since $\gamma' \leq \gamma \leq \theta(-\lambda) = 0$, we also have $\gamma = \gamma' = 0$. The final equality of weight spaces of $U_q(\mathfrak{g})$ follows upon replacing λ, γ, and γ' with 0. □

The second lemma focuses on understanding $(\mathrm{ad}\,U_{\pi'})$ lowest weight vectors contained in $G^-_{-\delta}U^0$ where $\delta \in Q^+(\pi')$. In the notation of this lemma, δ corresponds to $\beta - \lambda - \gamma$.

Lemma 6.3. *Let $\beta \in Q^+(\pi)$ and let π' be a subset of $\mathrm{Orth}(\beta) \cap \mathrm{Supp}(\beta)$. Let λ, γ, γ' be three elements in $Q^+(\pi)$ such that*

   (i) $0 < \lambda \leq \beta$.
   (ii) $\gamma \leq \beta - \lambda$, $\gamma \leq \theta(-\lambda)$ and $\beta - \lambda - \gamma \in Q^+(\pi')$.
   (iii) $\gamma' \leq \gamma$ and $\gamma' \in Q^+(\pi')$.

If $G^-_{-\beta+\lambda+\gamma}K_{-\beta+2\gamma'}$ contains a nonzero $(\mathrm{ad}\,U_{\pi'})$ lowest weight vector, then $\gamma' = 0$, $\beta = \lambda + \gamma$, and

$$G^-_{-\beta+\lambda+\gamma}K_{-\beta+2\gamma'} = \mathbb{C}(q)K_{-\beta}.$$

Proof. Let g be a nonzero element of $G^-_{-\beta+\lambda+\gamma}$ so that $gK_{-\beta+2\gamma'}$ is an $(\mathrm{ad}\,U_{\pi'})$ lowest weight vector. Assumption (ii) ensures that $G^-_{-\beta+\lambda+\gamma} \subset G^-_{\pi'}$.

Set $\xi = (\beta - 2\gamma')/2$. It follows from Section 3.2, that $\tilde{\xi} \in P^+(\pi')$ and the weight of g is $-\tilde{\xi} + w_{\pi'}\tilde{\xi} = -\xi + w_{\pi'}\xi$ where $w_{\pi'} = w(\pi')_0$. Hence, the assumptions on $gK_{-2\xi}$ ensure that

(6.3) $$\beta - \lambda - \gamma = \xi - w_{\pi'}\xi = \frac{1}{2}\left((\beta - 2\gamma') - w_{\pi'}(\beta - 2\gamma')\right).$$

Since $\tilde{\beta} = 0$, we must have $\tilde{\xi} - w_{\pi'}\tilde{\xi} = -2\tilde{\gamma}' + w_{\pi'}2\tilde{\gamma}'$. But $\tilde{\xi} \in P^+(\pi')$ and $\gamma' \in Q^+(\pi')$ yields a contradiction unless $\xi = \tilde{\xi} = -2\tilde{\gamma}' = -2\gamma' = 0$. Hence g has weight $\tilde{\xi} - w_{\pi'}\tilde{\xi}$ which equals zero. This in turn implies that $\beta - \lambda - \gamma = 0$, and $G^-_{-\beta+\lambda+\gamma}K_{-\beta+2\gamma'} = G^-_0 K_{-\beta} = \mathbb{C}(q)K_{-\beta}$. □

The third lemma in the series also analyzes properties of $(\mathrm{ad}\,U_{\pi'})$ lowest weight vectors contained in $G^-_{-\delta}U^0$ with $\delta \in Q^+(\pi')$ where again $\delta = \beta - \lambda - \gamma$. However, in contrast to Lemmas 6.2 and 6.3, we assume that $\gamma' \notin Q^+(\pi')$. In addition, we add assumptions to β that correspond to some of the properties of roots in the set Γ_θ of Theorem 2.7.

Lemma 6.4. *Let $\beta \in Q^+(\pi)$ and set $\pi' = \mathrm{Orth}(\beta) \cap \mathrm{Supp}(\beta)$. Assume that $\mathrm{ht}_{\pi \setminus \pi'}\beta = 2$ and θ restricts to an involution on $\Delta(\pi')$. Let λ, γ, γ' be three weights in $Q^+(\pi)$ satisfying*

   (i) $0 < \lambda \leq \beta$ and $\lambda \in Q^+(\pi \setminus \pi_\theta)$.
   (ii) $\gamma \leq \beta - \lambda$, $\gamma \leq \theta(-\lambda)$, and $\beta - \lambda - \gamma \in Q^+(\pi')$.
   (iii) $\gamma' \leq \gamma$ and $\gamma' \notin Q^+(\pi')$.
   (iv) $-w_{\pi'} = -w(\pi')_0$ restricts to a permutation on $\pi_\theta \cap \pi'$.

If $G^-_{-\beta+\lambda+\gamma}K_{-\beta+2\gamma'}$ contains a nonzero (ad $U_{\pi'}$) lowest weight vector, then $\theta(-\lambda)+\gamma-2\gamma' \in Q^+(\pi_\theta)$, $\lambda+\gamma-2\gamma' \in Q(\pi)^\theta$ and

$$(6.4) \qquad G^-_{-\beta+\lambda+\gamma}U^+_{\theta(-\lambda)-\gamma}K_{-\beta+2\gamma'} \subseteq U^-\mathcal{M}^+\mathcal{T}_\theta.$$

Proof. Let g be a nonzero element of $G^-_{-\beta+\lambda+\gamma}$ so that $gK_{-\beta+2\gamma'}$ is an (ad $U_{\pi'}$) lowest weight vector. We start the argument as in the proof of Lemma 6.3. Using the assumption $\beta - \lambda - \gamma \in Q^+(\pi')$, we see that $G^-_{-\beta+\lambda+\gamma} \subset G^-_{\pi'}$. The weight $\xi = (\beta-2\gamma')/2$ satisfies $\tilde{\xi} \in P^+(\pi')$. Also, the weight of g is $-\tilde{\xi}+w_{\pi'}\tilde{\xi} = -\xi+w_{\pi'}\xi$. In particular, just as in Lemma 6.3, the assumptions on $gK_{-2\xi}$ ensure that (6.3) is true.

We are assuming $\gamma' \notin Q^+(\pi')$ (see (iii)). This combined with the assumptions that both γ' and γ are in $Q^+(\pi)$ and $\gamma' \leq \gamma$, ensure that γ is also not in $Q^+(\pi')$. Suppose that $\lambda \in Q^+(\pi')$. The fact that $\lambda \in Q^+(\pi \setminus \pi_\theta)$ implies that $\theta(-\lambda) \in Q^+(\pi)$. Since θ restricts to an involution on $\Delta(\pi')$, it follows that $\theta(-\lambda)$ is in $Q^+(\pi')$. By (ii) and (iii), we have $\gamma' \leq \theta(-\lambda)$ which forces $\gamma' \in Q^+(\pi')$, a contradiction of (iii). Hence $\lambda \notin Q^+(\pi')$. Therefore, the assumption on the height of β combined with $\beta - \lambda - \gamma \in Q^+(\pi')$ (assumption (ii)) yields

$$\mathrm{ht}_{\mathrm{Supp}(\beta)\setminus\pi'}(\lambda) = \mathrm{ht}_{\mathrm{Supp}(\beta)\setminus\pi'}(\gamma) = \mathrm{ht}_{\mathrm{Supp}(\beta)\setminus\pi'}(\gamma') = 1.$$

Thus $\mathrm{ht}_{\mathrm{Supp}(\beta)\setminus\pi'}(\beta - 2\gamma') = 0$. Since $\gamma' \leq \gamma$ and $\gamma' \leq \beta - \lambda$, it follows that $\beta - 2\gamma' \geq 0$ and hence $\beta - 2\gamma' \in Q^+(\pi')$. Hence, by assumption (iv) and (6.3), we have

$$\mathrm{ht}_{\pi\setminus\pi_\theta}(\beta - \lambda - \gamma) = \mathrm{ht}_{\pi\setminus\pi_\theta}\left((\beta - 2\gamma')/2 - w_{\pi'}(\beta - 2\gamma')/2\right) = \mathrm{ht}_{\pi\setminus\pi_\theta}(\beta - 2\gamma').$$

Thus

$$\mathrm{ht}_{\pi\setminus\pi_\theta}(\lambda + \gamma - 2\gamma') = 0.$$

It follows from assumptions (ii) and (iii) that $\gamma' \leq \theta(-\lambda)$ and so $2\gamma' \leq \theta(-\lambda)+\gamma$. Therefore,

$$\theta(-\lambda) + \gamma - 2\gamma' \geq 0.$$

Since $\lambda \in Q^+(\pi \setminus \pi_\theta)$, we have that $\theta(-\lambda) - \lambda \in (p(\lambda) - \lambda) + Q^+(\pi_\theta)$. Recall that p permutes the roots of $\pi \setminus \pi_\theta$. Hence $\mathrm{ht}_{\pi\setminus\pi_\theta}(\theta(-\lambda) - \lambda) = 0$ and so

$$0 = \mathrm{ht}_{\pi\setminus\pi_\theta}(\lambda + \gamma - 2\gamma') = \mathrm{ht}_{\pi\setminus\pi_\theta}(\theta(-\lambda) + \gamma - 2\gamma').$$

This equality combined with the positivity of the weight $\theta(-\lambda)+\gamma-2\gamma'$ guarantees that $\alpha_i \notin \mathrm{Supp}(\theta(-\lambda) + \gamma - 2\gamma')$ for each $\alpha_i \in \pi \setminus \pi_\theta$. Thus

$$(6.5) \qquad \theta(-\lambda) + \gamma - 2\gamma' \in Q^+(\pi_\theta)$$

and so

$$\lambda + \gamma - 2\gamma' = (\theta(\lambda) + \lambda) + \theta(-\lambda) + \gamma - 2\gamma' \in Q(\pi)^\theta.$$

Therefore,

$$(6.6) \qquad G^-_{-\beta+\lambda+\gamma}K_{-\beta+2\gamma'} = U^-_{-\beta+\lambda+\gamma}K_{-\lambda-\gamma+2\gamma'} \subseteq U^-_{-\beta+\lambda+\gamma}\mathcal{T}_\theta.$$

Now $\gamma' \leq \gamma$ and $\gamma \leq \theta(-\lambda)$ ensure that $0 \leq \theta(-\lambda) - \gamma \leq \theta(-\lambda)+\gamma-2\gamma'$. Thus, (6.5) implies that $\theta(-\lambda) - \gamma$ is also in $Q^+(\pi_\theta)$. Hence $U^+_{\theta(-\lambda)-\gamma} \subseteq \mathcal{M}^+$. This result combined with (6.6) yields (6.4). $\square$

6.3. **Highest weight terms.** In this section, we analyze certain elements in subspaces of the form $U^+ K_\zeta$ that also belong to $\mathcal{N}(\mathcal{B}_\theta : \mathcal{B}_\theta \cap U_{\pi'})$. These results will be applied to the term $a_2 \in \sum_{\delta \in Q^+(\pi')} G_{-\delta}^- U^0 U^+$ of Section 6.2 for special choices of $\tilde{b}$.

Lemma 6.5. *Let* $\beta \in Q^+(\pi)$ *such that* $\beta - \alpha_i \notin Q^+(\pi_\theta)$ *and* $\beta + \alpha_i \notin Q^+(\pi_\theta)$ *for all* $\alpha_i \in \operatorname{Supp}(\beta)$. *Set* $\pi' = \operatorname{Orth}(\beta) \cap \operatorname{Supp}(\beta)$ *and assume that* θ *restricts to an involution on* $\Delta(\pi')$. *Let* $X \in U^+ K_{-\beta}$. *If* $X \in \mathcal{N}(\mathcal{B}_\theta : \mathcal{B}_\theta \cap U_{\pi'})$, *then* $X \in (U_q(\mathfrak{g}))^{\mathcal{B}_\theta \cap U_{\pi'}}$.

Proof. It follows from the defining relations (in particular, see (4.5)) of $U_q(\mathfrak{g})$ that

$$[F_i, X] \in U^+ K_{-\beta+\alpha_i} + U^+ K_{-\beta-\alpha_i}.$$

The assumptions on $\beta \pm \alpha_i$ ensure that neither $K_{-\beta+\alpha_i}$ nor $K_{-\beta-\alpha_i}$ is in $\mathcal{T}_\theta$ for all $\alpha_i \in \pi'$. It follows that $\mathcal{P}([F_i, X]) = 0$ for all $\alpha_i \in \pi'$. A similar argument shows that $\mathcal{P}([K_i^{-1}, X]) = 0$. Note that

$$[\theta_q(F_i K_i) K_i^{-1}, X] \in U^+ K_{-\beta-\alpha_i}$$

for all $\alpha_i \in \pi' \setminus \pi_\theta$. Hence we also have $\mathcal{P}([\theta_q(F_i K_i) K_i^{-1}, X]) = 0$ for these choices of α_i. It follows that $\mathcal{P}([B_i, X]) = 0$ for all $\alpha_i \in \pi'$.

If $\alpha_i \in \pi'$, then $B_i \in U_{\pi'}$ since θ restricts to an involution on $\Delta(\pi')$. Hence, by assumption, $[B_i, X] \in \mathcal{B}_\theta$ for all i such that $\alpha_i \in \pi' \setminus \pi_\theta$. It follows from Corollary 4.4 that

$$[B_i, X] = 0 \text{ for all } \alpha_i \in \pi' \setminus \pi_\theta.$$

Similar arguments show that $\mathcal{P}([E_i, X]) = \mathcal{P}([F_i, X]) = \mathcal{P}([K_i^{\pm 1}, X]) = 0$ for all $\alpha_i \in \pi_\theta \cap \pi'$. Hence $[E_i, X] = [F_i, X] = [K_i^{\pm 1}, X] = 0$ for all $\alpha_i \in \pi_\theta \cap \pi'$. It follows that $[b, X] = 0$ for all $b \in \mathcal{B}_\theta \cap U_{\pi'}$.   □

Recall that in the fifth case of Theorem 2.7, $\beta = \alpha'_\beta + \alpha_\beta + w_\beta \alpha_\beta$ where $\alpha'_\beta \in \pi_\theta$, $\alpha'_\beta \neq \alpha_\beta$, $w_\beta = w(\operatorname{Supp}(\beta) \setminus \{\alpha_\beta, \alpha'_\beta\})_0$ and α'_β is strongly orthogonal to all simple roots in $\operatorname{Supp}(\beta) \setminus \{\alpha_\beta, \alpha'_\beta\}$. It follows that $\alpha'_\beta \notin \operatorname{Supp}(w_\beta \alpha_\beta)$ and so α'_β is also not in $\operatorname{Supp}(\beta - \alpha'_\beta)$. Set $\beta' = \beta - \alpha'_\beta$. Since $\theta(\beta) = -\beta$ and $\theta(\alpha'_\beta) = \alpha'_\beta$, we have $\theta(\beta - \alpha'_\beta) = -\beta - \alpha'_\beta$ and so $\theta(\beta') = -\beta' - 2\alpha'_\beta$. Thus β' satisfies the following conditions:

- $\theta(\beta') \in -\beta' - Q^+(\pi_\theta)$.
- $\operatorname{Supp}(\theta(-\beta') - \beta') \cap \operatorname{Supp}(\beta') = \emptyset$.

Note that if $\theta(\beta) = -\beta$, then β also satisfies the above two properties (with β' replaced by β). We use these two properties as assumptions on β in the next lemma so that we can eventually apply it to both roots β satisfying $\theta(\beta) = -\beta$ and to roots of the form β' derived from case (5) of Theorem 2.7.

Lemma 6.6. *Let* $\beta \in Q^+(\pi)$ *such that* $\theta(-\beta) - \beta \in Q^+(\pi_\theta)$ *and* $\operatorname{Supp}(\theta(-\beta) - \beta) \cap \operatorname{Supp}(\beta) = \emptyset$. *Set* $\pi' = \operatorname{Orth}(\beta) \cap \operatorname{Supp}(\beta)$. *Assume that*

- $\pi' \subseteq \operatorname{Orth}(\theta(-\beta))$.
- θ *restricts to an involution on* $\Delta(\pi')$.
- $\operatorname{ht}_{\operatorname{Supp}(\beta) \setminus \pi'}(\beta) = 2$.
- $(\pi_\theta \cup \mathcal{S}) \cap \operatorname{Supp}(\beta) \subseteq \pi'$.

Let

$$X \in \sum_{\eta \in Q^+(\pi')} \sum_{0 \leq \gamma \leq \beta} U^+_{\theta(-\beta+\gamma)-\gamma-\eta} K_{-\beta}.$$

If $X \in \mathcal{N}(\mathcal{B}_\theta : \mathcal{B}_\theta \cap U_{\pi'})$, then $X \in U^+_{\theta(-\beta)} K_{-\beta} + \mathbb{C}(q) K_{-\beta}$ and $X \in (U_q(\mathfrak{g}))^{U_{\pi'}}$.

Proof. Assume $X \in \mathcal{N}(\mathcal{B}_\theta : \mathcal{B}_\theta \cap U_{\pi'})$. Since $\mathrm{ht}_{\mathrm{Supp}(\beta) \setminus \pi'} \beta = 2$ and $\pi_\theta \cap \mathrm{Supp}(\beta) \subseteq \pi'$, it follows that $\beta \pm \alpha_i \notin \pi_\theta$ for all $\alpha_i \in \pi$. Hence by Lemma 6.5, $X \in (U_q(\mathfrak{g}))^{\mathcal{B}_\theta \cap U_{\pi'}}$.

Let $\gamma \in Q^+(\pi)$ and $\eta \in Q^+(\pi')$ such that in its expression as a sum of weight vectors, X has a nonzero contribution from $U^+_{\theta(-\beta+\gamma)-\gamma-\eta} K_{-\beta}$. Note that this ensures that

$$\theta(-\beta+\gamma) - \gamma - \eta \in Q^+(\pi).$$

Write $\gamma = \gamma'_1 + \gamma'_2$ with $\gamma'_2 \in Q^+(\pi')$ and $\gamma'_1 \in Q^+(\mathrm{Supp}(\beta) \setminus \pi')$. Since θ restricts to an involution on $\Delta(\pi')$, we have $\theta(\gamma'_2) \in Q(\pi')$. The fact that $\pi_\theta \cap \mathrm{Supp}(\beta) \subseteq \pi'$ forces $\theta(-\gamma) \in p(\gamma'_1) + Q^+(\pi_\theta) \subseteq p(\gamma'_1) + Q^+(\pi')$. Hence

$$\mathrm{ht}_{\mathrm{Supp}(\beta) \setminus \pi'}(\gamma) = \mathrm{ht}_{\mathrm{Supp}(\beta) \setminus \pi'}(\theta(-\gamma)).$$

By the hypothesis of the lemma, $\mu = \theta(-\beta) - \beta$ satisfies $\mathrm{Supp}(\mu) \cap \mathrm{Supp}(\beta) = \emptyset$. The fact that θ restricts to an automorphism of $\Delta(\pi')$ combined with the fact that $\pi_\theta \cap \mathrm{Supp}(\beta) \subset \pi'$ guarantees that

$$\mathrm{ht}_{\mathrm{Supp}(\beta) \setminus \pi'}(\beta) = \mathrm{ht}_{\mathrm{Supp}(\beta) \setminus \pi'}(\theta(-\beta)).$$

Thus by our assumptions on the height of β and the results of the previous paragraph, we observe that

$$\mathrm{ht}_{\mathrm{Supp}(\beta) \setminus \pi'}(\gamma) \in \{0, 1\}$$

and so

$$\mathrm{ht}_{\mathrm{Supp}(\beta) \setminus \pi'}(\theta(-\gamma) + \gamma) \in \{0, 2\}.$$

It follows that either $\theta(-\beta+\gamma) - \gamma - \eta \in Q^+(\pi')$ or $\theta(-\gamma) + \gamma + \eta \in Q^+(\pi')$. Thus

$$X \in \sum_{\zeta \in Q^+(\pi'), \zeta < \beta} U^+_{\beta+\mu-\zeta} K_{-\beta} + \sum_{\zeta \in Q^+(\pi'), \zeta < \beta} U^+_\zeta K_{-\beta}.$$

Note that $[b, X] = 0$ for $b \in \mathcal{M}_{\pi'}$ ensures that each weight summand of X also commutes with all $b \in \mathcal{M}_{\pi'}$. Thus each weight summand of X generates a trivial (ad $\mathcal{M}_{\pi'}$)-module.

Recall that $\mathcal{B}_\theta$ is invariant under the action of κ_θ (see Section 4), which is conjugate to the Chevalley antiautomorphism κ, defined in Section 3.4. Applying κ_θ to X yields an element $Y \in (U_q(\mathfrak{g}))^{\mathcal{B}_\theta \cap U_{\pi'}}$ such that

$$Y \in \sum_{\zeta \in Q^+(\pi'), \zeta < \beta} G^-_{-\beta-\mu+\zeta} K_{-\beta} + \sum_{\zeta \in Q^+(\pi'), \zeta < \beta} G^-_{-\zeta} K_{-\beta}.$$

Since $X \in \mathcal{N}(\mathcal{B}_\theta : \mathcal{B}_\theta \cap U_{\pi'})$ so is Y and hence $[B_i, Y] = 0$ for all $\alpha_i \in \pi'$. By the previous paragraph, we also have $Y \in (U_q(\mathfrak{g}))^{\mathcal{M}_{\pi'}}$.

Suppose that $Y_{-\beta-\mu+\zeta} K_{-\beta} \in G^-_{-\beta-\mu+\zeta} K_{-\beta}$ is a nonzero (ad $U_{\pi'}$) lowest weight vector for some $\zeta \in Q^+(\pi')$ with $\zeta < \beta$. Since $(\mu, \alpha) = 0$ for all $\alpha \in \pi'$, it follows that $Y_{-\beta-\mu+\zeta} K_{-\beta-\mu}$ is also a nonzero (ad $U_{\pi'}$) lowest weight vector. Hence, we may apply Lemma 6.2 with $\theta(-\beta) = \beta + \mu$ playing the role of β, ζ playing the role

of λ and $\gamma = \gamma' = 0$. This forces $\zeta = 0$ and so $Y_{-\beta-\mu+\zeta}K_{-\beta} \in G^-_{-\beta-\mu}K_{-\beta}$. Now suppose that $Y_{-\zeta}K_{-\beta} \in G^-_{-\zeta}K_{-\beta}$ is a nonzero (ad $U_{\pi'}$) lowest weight vector for some $\zeta \in Q^+(\pi')$ with $\zeta < \beta$. Applying Lemma 6.3 with $\gamma = \gamma' = 0$ and $\lambda = \beta - \zeta$ yields $\zeta = 0$ and so $Y_{-\zeta}K_{-\beta} \in \mathbb{C}(q)K_{-\beta}$.

Now choose δ maximal so that

$$(6.7) \qquad Y \in Y_{-\delta}K_{-\beta} + \sum_{\xi \gneq \delta} G^-_{-\xi}K_{-\beta}.$$

By Lemma 6.1, $Y_{-\delta}K_{-\beta}$ is an (ad $U_{\pi'}$) lowest weight vector. Hence by the previous paragraph, either $\delta = \beta + \mu$ or $\delta = 0$. If $\delta = 0$, then $Y \in \mathbb{C}(q)K_{-\beta}$ and thus so is X and the lemma follows. Thus, we assume that $\delta = \beta + \mu = \theta(-\beta)$.

Since $\delta = \beta + \mu$, (6.7) can be rewritten as

$$Y \in Y_{-\beta-\mu}K_{-\beta} + Y_{-\delta'}K_{-\beta} + \sum_{\xi \gneq \delta'} G^-_{-\xi}K_{-\beta},$$

where $Y_{-\delta} = Y_{-\beta-\mu} \in G^-_{-\beta-\mu}$ is an (ad $U_{\pi'}$) lowest weight vector and $Y_{-\delta'} \in G^-_{-\delta'}$ with $\delta' \ngeq \beta+\mu$. By Section 3.2, $Y_{-\beta-\mu}K_{-\beta}$ generates a finite-dimensional $(\mathrm{ad}_\beta U_{\pi'})$-module. Since $\beta + \mu$ restricts to 0 with respect to π', this module is trivial and so

$$(6.8) \qquad (\mathrm{ad}_\beta E_i)(Y_{-\beta-\mu}K_{-\beta}) = 0 \text{ for each } \alpha_i \in \pi'.$$

We also have $(\mathrm{ad}_\beta E_i)K_{-\beta} = (\mathrm{ad}\, E_i)K_{-\beta} = 0$ and, thus

$$[E_i K_i^{-1}, Y_{-\beta-\mu}K_{-\beta}] = ((\mathrm{ad}\, E_i)Y_{-\beta-\mu}K_{-\beta})K_i^{-1}$$

for all $\alpha_i \in \pi'$.

It follows from (6.8), Section 3.2 (in particular, the inequality (3.4) and the related discussion), and the definition of the adjoint action in Section 3.1 that

$$\deg_{\mathcal{F}}([E_i, Y_{-\beta-\mu}K_{-\beta}]) = \deg_{\mathcal{F}}(((\mathrm{ad}\, E_i)Y_{-\beta-\mu}K_{-\beta})) < \deg_{\mathcal{F}} K_{-\beta}$$

for all $\alpha_i \in \pi' \setminus \pi_\theta$. Since θ restricts to an involution on $\Delta(\pi')$, it follows that $\theta_q(F_i K_i) \in (\mathrm{ad}\, \mathcal{M}^+_{\pi'})E_{p(i)}$ for all $\alpha_i \in \pi' \setminus \pi_\theta$. Recall that $Y_{-\beta-\mu}K_{-\beta}$ is (ad $\mathcal{M}_{\pi'}$) invariant. Hence, we also have

$$\deg_{\mathcal{F}}[\theta_q(F_i K_i), Y_{-\beta-\mu}K_{-\beta}] < \deg_{\mathcal{F}} K_{-\beta}$$

for all $\alpha_i \in \pi' \setminus \pi_\theta$. On the other hand, $[F_i K_i, G^- K_{-\beta}] \subseteq G^- K_{-\beta}$ and so

$$\deg_{\mathcal{F}}[F_i K_i, G^- K_{-\beta}] = \deg_{\mathcal{F}} K_{-\beta}.$$

Hence, if $[F_i, Y_{-\delta'}K_{-\beta}] \neq 0$, then

$$[B_i, Y] = [F_i, Y_{-\delta'}K_{-\beta}] + \text{ terms of lower degree with respect to } \mathcal{F}.$$

But $[B_i, Y] = 0$, and so, we must have $[F_i, Y_{-\delta'}K_{-\beta}] = 0$ for all $\alpha_i \in \pi' \setminus \pi_\theta$.

Recall that each weight summand of Y generates a trivial (ad $\mathcal{M}_{\pi'}$)-module. This fact combined with the above paragraph yields $[F_i, Y_{-\delta'}K_{-\beta}] = 0$ for all $\alpha_i \in \pi'$. Thus, $Y_{-\delta'}K_{-\beta}$ must be an (ad $U_{\pi'}$) lowest weight vector. Since $\delta' < \beta + \mu$, it follows from the arguments above that $\delta' = 0$. Therefore,

$$Y \in Y_{-\beta-\mu}K_{-\beta} + \mathbb{C}(q)K_{-\beta}.$$

Applying κ_θ to every term in the above expression yields the desired analogous assertion for X.

Note that $X \in (U_q(\mathfrak{g}))^{U_{\pi'}}$ if and only if $Y \in (U_q(\mathfrak{g}))^{U_{\pi'}}$. By assumptions on π' and β, it follows that $K_{-\beta} \in (U_q(\mathfrak{g}))^{U_{\pi'}}$ and $Y_{-\beta-\mu}K_{-\beta}$ commutes with all elements of $\mathcal{T} \cap U_{\pi'}$. Hence it is sufficient to show that $Y_{-\beta-\mu}K_{-\beta} \in (U_q(\mathfrak{g}))^{U_{\pi'}}$. We have already shown that $[a, Y_{-\beta-\mu}K_{-\beta}] = 0$ for all $a \in \mathcal{M}_{\pi'}$ and all $a = F_i$ with $\alpha_i \in \pi'$. Consider $\alpha_i \in \pi' \setminus \pi_\theta$. Now $[B_i, Y_{-\beta-\mu}K_{-\beta}]$ can be written as a sum of terms of weights

$$-\beta - \mu - \alpha_i, \ -\beta - \mu, \ \text{and} \ -\beta - \mu - \theta(\alpha_i).$$

But $[B_i, Y_{-\beta-\mu}] = [B_i, Y] = 0$ and so each of these weight terms must be zero. In particular, the term of weight $-\beta - \mu - \theta(\alpha_i)$, which is $[\theta_q(F_iK_i)K_i^{-1}, Y_{-\beta-\mu}K_{-\beta}]$, equals zero for all $\alpha_i \in \pi' \setminus \pi_\theta$. The lemma now follows from the fact that $U_{\pi'}$ is generated by $\mathcal{M}_{\pi'}$, $F_i, \alpha_i \in \pi'$, $U^0 \cap U_{\pi'}$, and $\theta_q(F_iK_i)K_i^{-1}, \alpha_i \in \pi' \setminus \pi_\theta$. □

6.4. **Determining commutativity.** Let τ be a subset of π. For each $m \geq 0$, set

$$G^-_{(\tau,m)} = \sum_{\mathrm{ht}_\tau(\delta)=m} G^-_{-\delta} \quad \text{and} \quad U^+_{(\tau,m)} = \sum_{\mathrm{ht}_\tau(\delta)=m} U^+_\delta.$$

Note that

$$G^-_{(\tau,m)} \cap G^-_{(\tau,j)} = 0$$

whenever $j \neq m$. We also have

$$(6.9) \quad [K_i^{\pm 1}, G^-_{(\tau,j)}] \subseteq G^-_{(\tau,j)}, \ [F_i, G^-_{(\tau,j)}] \subseteq G^-_{(\tau,j)} \ \text{and} \ [E_i, G^-_{(\tau,j)}] \subseteq G^-_{(\tau,j)} U^0 U^+$$

for all $j \geq 0$ and $\alpha_i \in \pi \setminus \tau$. It follows that for each j, the space $G^-_{(\tau,j)} U^0 U^+$ is preserved by the action of $(\mathrm{ad}\, U_{\pi \setminus \tau})$. Thus, if θ restricts to an involution on $\Delta(\pi')$ where π' is a subset of $\pi \setminus \tau$, then

$$[B_i, G^-_{(\tau,j)}] \subseteq G^-_{(\tau,j)} U^0 U^+$$

for all $j \geq 0$ and $\alpha_i \in \pi'$. Analogous statements hold for $G^-_{(\tau,j)}$ replaced by $U^+_{(\tau,j)}$. Moreover, we have

$$[B_i, G^-_{(\tau,j)} U^0 U^+_{(\tau,m)}] \subseteq G^-_{(\tau,j)} U^0 U^+_{(\tau,m)}$$

for all nonnegative integers j, m and all $\alpha_i \in \pi'$.

Now suppose that β is a positive weight, π' a subset of $\mathrm{Supp}(\beta)$, and $\tau = \mathrm{Supp}(\beta) \setminus \pi'$. Assume further that $\mathrm{ht}_\tau(\beta) = 2$. Note that

$$\sum_{\delta \in Q(\pi')} G^-_{-\beta+\delta} = G^-_{(\tau,2)} \quad \text{and} \quad \sum_{\delta \in Q(\pi')} G^-_{-\delta} = G^-_{(\tau,0)}.$$

Proposition 6.7. *Let $\beta \in Q^+(\pi)$ such that $\theta(-\beta)-\beta \in Q^+(\pi_\theta)$ and $\mathrm{Supp}(\theta(-\beta)-\beta) \cap \mathrm{Supp}(\beta) = \emptyset$. Set $\pi' = \mathrm{Orth}(\beta) \cap \mathrm{Supp}(\beta)$ and assume that $\pi' \subseteq \mathrm{Orth}(\theta(-\beta))$. Assume that*

- *θ restricts to an involution on $\Delta(\pi')$.*
- *$\mathrm{ht}_{\mathrm{Supp}(\beta) \setminus \pi'}(\beta) = 2$.*
- *$(\pi_\theta \cup \mathcal{S}) \cap \mathrm{Supp}(\beta) \subseteq \pi'$.*
- *$-w_{\pi'} = -w(\pi')_0$ restricts to a permutation on $\pi_\theta \cap \pi'$.*

Let $a = \sum_{I,\mathrm{wt}(I) \leq \beta} B_I a_I$ be an element of $\mathcal{B}_\theta$ where each $a_I \in \mathcal{M}_\theta^+$ and a_I is a scalar whenever $\mathrm{wt}(I) = \beta$, such that the following two conditions hold:

(i) *$\mathcal{P}(a) = \sum_{\mathrm{wt}(I)=\beta} F_I a_I$.*

(ii) *$\mathcal{P}(a) \in (U_q(\mathfrak{g}))^{U_{\pi'}}$.*

Then

$$a \in \mathcal{P}(a) + G^-_{(\tau,1)} U^0 U^+_{(\tau,1)} + \mathbb{C}(q) K_{-\beta} + X_{\theta(-\beta)} K_{\theta(-\beta)-\beta},$$

where $\tau = \mathrm{Supp}(\beta) \setminus \pi'$ *and* $X_{\theta(-\beta)} \in G^+_{\theta(-\beta)}$. *Moreover,*

$$a \in (U_q(\mathfrak{g}))^{\mathcal{B}_\theta \cap U_{\pi'}}, \quad X_{\theta(-\beta)} K_{\theta(-\beta)-\beta} \in (U_q(\mathfrak{g}))^{U_{\pi'}},$$

the lowest weight summand of a *is* $\mathcal{P}(a)$ *(which is also the lowest l-weight summand of* a*), and the highest weight summand of* a *is the element* $X_{\theta(-\beta)} K_{\theta(-\beta)-\beta}$ *in* $G^+_{\theta(-\beta)} K_{\theta(-\beta)-\beta}$.

Proof. The fact that θ restricts to an involution on $\Delta(\pi')$ ensures that $B_i \in U_{\pi'}$ for all i with $\alpha_i \in \pi'$. Hence $[B_i, \mathcal{P}(a)] = 0$ for all $\alpha_i \in \pi'$.

We may assume that $a = \tilde{b}$ where $b = \sum_{I, \mathrm{wt}(I)=\beta} B_I a_I$ as in Proposition 4.3. By the discussion at the end of Section 4.3, we may write

$$(6.10) \qquad a = \sum_{\mathrm{wt}(I)=\beta} F_I a_I + \sum_{\lambda', \lambda, \gamma, \gamma'} b_{-\beta+\lambda'} u_{\lambda,\gamma} K_{-\beta+2\gamma'},$$

where each index of four weights $\lambda, \eta, \gamma, \gamma'$ with $\eta = \lambda' - \lambda - \gamma$ satisfy the conditions of Lemma 4.1 with respect to β and $u_{\lambda,\gamma} \in U^+_{\theta(-\lambda)-\gamma}$. Set $\tau = \mathrm{Supp}(\beta) \setminus \pi'$. As in Section 6.2, a can be expressed as a sum $a = a_0 + a_1 + a_2 + a_3$ where

$$a_3 = \sum_{\mathrm{wt}(I)=\beta} F_I a_I,$$

$$a_2 = \sum_{\lambda' \in Q^+(\pi') \setminus \{0\}} \sum_{\lambda, \gamma, \gamma'} b_{-\beta+\lambda'} u_{\lambda,\gamma} K_{-\beta+2\gamma'} \in G^-_{(\tau,2)} U^0 U^+,$$

$$a_1 = \sum_{\mathrm{ht}_\tau(\beta-\lambda')=1} \sum_{\lambda, \gamma, \gamma'} b_{-\beta+\lambda'} u_{\lambda,\gamma} K_{-\beta+2\gamma'} \in G^-_{(\tau,1)} U^0 U^+,$$

$$a_0 = \sum_{\beta-\lambda' \in Q^+(\pi')} \sum_{\lambda, \gamma, \gamma'} b_{-\beta+\lambda'} u_{\lambda,\gamma} K_{-\beta+2\gamma'} \in G^-_{(\tau,0)} U^0 U^+.$$

Given $\alpha_j \in \pi'$, we have $[B_j, a] = [B_j, a_2 + a_1 + a_0] \in \mathcal{B}_\theta$ and so, by (6.9), $[B_j, a] = [B_j, a_2 + a_1 + a_0]$ is an element of $G^-_{(\tau,2)} U^0 U^+ + G^-_{(\tau,1)} U^0 U^+ + G^-_{(\tau,0)} U^0 U^+$. Since $a \in \mathcal{B}_\theta$, we also have $a_2 + a_1 + a_0 \in \mathcal{N}(\mathcal{B}_\theta : \mathcal{B}_\theta \cap U_{\pi'})$.

The fact that $\mathcal{P}(a) = \mathcal{P}(a_3) = a_3$ (i.e., assumption (i)) ensures that $u_{\lambda,\gamma} K_{\lambda'+2\gamma'} \notin \mathcal{M}^+ \mathcal{T}_\theta$ for any choice of $(\lambda', \lambda, \gamma, \gamma')$ with $b_{-\beta+\lambda'} u_{\lambda,\gamma} K_{-\beta+2\gamma'} \neq 0$. Thus

$$\mathcal{P}(a_2) = \mathcal{P}(a_1) = \mathcal{P}(a_0) = 0.$$

Let $(\lambda'_0, \lambda_0, \gamma_0, \gamma'_0)$ be chosen so that $b_{-\beta+\lambda'_0} u_{\lambda_0,\gamma_0} K_{-\beta+2\gamma'_0} \neq 0$ and $\beta - \lambda'_0$ is maximal. It follows that

$$(6.11) \qquad a_2 + a_1 + a_0 \in b_{-\beta+\lambda'_0} u_{\lambda_0,\gamma_0} K_{-\beta+2\gamma'_0} + \sum_{\delta \not\geq \beta-\lambda'_0} G^-_{-\delta} U^0 U^+.$$

By Lemma 6.1, $b_{-\beta+\lambda'_0} K_{-\beta+2\gamma'_0}$ is an (ad $U_{\pi'}$) lowest weight vector.

Suppose that λ' and $\bar{\lambda}'$ are two weights such that $\mathrm{ht}_\tau(\beta-\lambda') > \mathrm{ht}_\tau(\beta-\bar{\lambda}')$. It follows that $\beta - \bar{\lambda}' \not\geq \beta - \lambda'$. Hence if $a_2 \neq 0$, we can always choose $(\lambda'_0, \lambda_0, \gamma_0, \gamma'_0)$

as in (6.11) so that $b_{-\beta+\lambda_0'} u_{\lambda_0,\gamma_0} K_{-\beta+2\gamma_0'}$ is a nonzero biweight summand of a_2. In other words, $\lambda_0' > 0$ and $\lambda_0' \in Q^+(\pi')\backslash\{0\}$. Recall that the four-tuple $(\lambda_0, \eta_0, \gamma_0, \gamma_0')$ satisfies the conditions of Lemma 4.1 where $\eta_0 = \lambda_0' - \lambda_0 - \gamma_0$. Applying Lemma 6.2 to $\lambda_0 + \eta_0$ (instead of λ_0), γ_0, γ_0', and β, we see that $\lambda_0 + \eta_0 = \gamma_0 = \gamma_0' = 0$. But then $\lambda_0' = 0$, a contradiction. Therefore, $a_2 = 0$. It follows that

$$a = a_3 + a_1 + a_0 \in \mathcal{P}(a) + G_{(\tau,1)}^- U^0 U^+ + G_{(\tau,0)}^- U^0 U^+.$$

We next argue that $a_1 \in G_{(\tau,1)}^- U^0 U_{(\tau,1)}^+$. Consider a biweight summand d of a where $d = b_{-\beta+\lambda'} u_{\lambda,\gamma} K_{-\beta+2\gamma'}$. By the definition of a_1, the assumption on the height of β and the fact that $\eta \in \mathcal{S} \cap \mathrm{Supp}(\beta) \subseteq \pi'$, we see that

$$(6.12) \qquad\qquad \mathrm{ht}_{\mathrm{Supp}(\beta)\backslash\pi'}(\lambda + \gamma) = 1.$$

We can write $\lambda = \lambda_1 + \lambda_2$ where $\lambda_1 \in Q^+(\mathrm{Supp}(\beta) \backslash \pi')$ and $\lambda_2 \in Q^+(\pi')$. Since $\theta(-\beta)-\beta \in Q^+(\pi_\theta)$, $\mathrm{Supp}(\theta(-\beta)-\beta)\cap\mathrm{Supp}(\beta) = \emptyset$ and θ restricts to an involution on $\Delta(\pi')$, it follows that

$$\theta(-\lambda) \in p(\lambda_1) + Q^+(\pi') + Q^+(\mathrm{Supp}(\theta(-\beta) - \beta)).$$

Hence

$$\mathrm{ht}_{\mathrm{Supp}(\beta)\backslash\pi'}(\lambda) = \mathrm{ht}_{\mathrm{Supp}(\beta)\backslash\pi'}(\theta(-\lambda)).$$

Now $\gamma \leq \theta(-\lambda)$. Since γ, λ, and $\theta(-\lambda)$ are all positive weights, we see from (6.12) that

$$\mathrm{ht}_{\mathrm{Supp}(\beta)\backslash\pi'}(\lambda) = \mathrm{ht}_{\mathrm{Supp}(\beta)\backslash\pi'}(\theta(-\lambda)) = \mathrm{ht}_{\mathrm{Supp}(\beta)\backslash\pi'}(\theta(-\lambda) - \gamma) = 1.$$

Thus $U_{\theta(-\lambda)-\gamma}^+ \subseteq U_{(\tau,1)}^+$. Hence $u_{\lambda,\gamma} \in U_{(\tau,1)}^+$ and so $a_1 \in G_{(\tau,1)}^- U^0 U_{(\tau,1)}^+$, as claimed.

Since $\pi_\theta \cap \mathrm{Supp}(\beta) \subseteq \pi'$, we have $U_{(\tau,1)}^+ \cap \mathcal{M}^+ = 0$. Consider $b \in \mathcal{B}_\theta \cap U_{\pi'}$ and recall that $[b, a_3] = 0$. It follows that $[b, a_1 + a_0] \in \mathcal{B}_\theta$. By the discussion preceding the proposition, $[b, a_1]$ is also in $G_{(\tau,1)}^- U^0 U_{(\tau,1)}^+$. On the other hand, $[b, a_0] \in G_{\pi'}^- U^0 U^+$. Hence, if $[b, a_1] \neq 0$, then there exists ζ and ξ with $\mathrm{ht}_\tau(\zeta) = \mathrm{ht}_\tau(\xi) = 1$ such that

$$[b, a_1 + a_0] = b' + \sum_{\delta \not\geq \zeta} G_{-\delta}^- U^0 U^+,$$

where b' is a nonzero element of $G_{-\zeta}^- U^0 U_\xi^+$. This contradicts Remark 4.2 since $[b, a_1 + a_0] \in \mathcal{B}_\theta$. Hence $[b, a_1] = 0$.

We have shown that $a_2 = 0$ and $a_1 \in (U_q(\mathfrak{g}))^{\mathcal{B}_\theta \cap U_{\pi'}}$. Hence $a_0 \in \mathcal{N}(\mathcal{B}_\theta : \mathcal{B}_\theta \cap U_{\pi'})$. In analogy to (6.11), a_0 satisfies

$$a_0 \in b_{-\beta+\lambda_1'} u_{\lambda_1,\gamma_1} K_{-\beta+2\gamma_1'} + \sum_{\delta \not\geq \beta-\lambda_1'} G_{-\delta}^- U^0 U^+$$

for an appropriate choice of $\lambda_1', \lambda_1, \gamma_1, \gamma_1'$. Using Lemma 4.1, we obtain that $b_{-\beta+\lambda_1'} K_{-\beta+2\gamma_1'}$ is an (ad $U_{\pi'}$) lowest weight vector. By definition of a_0, $\beta - \lambda_1' \in Q^+(\pi')$. If $\gamma' \notin Q^+(\pi')$, then by Lemma 6.4, $b_{-\beta+\lambda_1'} u_{\lambda_1,\gamma_1} K_{-\beta+2\gamma_1'} \in U^- \mathcal{M}^+ \mathcal{T}_\theta$ contradicting the fact that $\mathcal{P}(a_0) = 0$. Therefore, $\gamma' \in Q^+(\pi')$. Applying Lemma 6.3 (with $\lambda_1 + \eta_1$ in the role of λ of the lemma) yields

$$b_{-\beta+\lambda_1'} K_{-\beta+2\gamma_1'} \in \mathbb{C}(q) K_{-\beta}.$$

Hence by the maximality of $\beta - \lambda'_1$, we see that $a_0 \in U^+ K_{-\beta}$. By Lemma 6.6,

$$a_0 \in U^+_{\theta(-\beta)} K_{-\beta} + \mathbb{C}(q) K_{-\beta} \quad \text{and} \quad a_0 \in (U_q(\mathfrak{g}))^{U_{\pi'}}.$$

Note that $K_{-\beta}$ and $K_{\theta(-\beta)}$ are both elements of $(U_q(\mathfrak{g}))^{U_{\pi'}}$. Thus there exists an element $X_{\theta(-\beta)}$ in $G^+_{\theta(-\beta)} \cap (U_q(\mathfrak{g}))^{U_{\pi'}}$ such that

$$a_0 \in X_{\theta(-\beta)} K_{\theta(-\beta)-\beta} + \mathbb{C}(q) K_{-\beta}.$$

Since a_3 and a_0 are in $(U_q(\mathfrak{g}))^{U_{\pi'}}$ and $a_1 \in (U_q(\mathfrak{g}))^{\mathcal{B}_\theta \cap U_{\pi'}}$, it follows that

$$a = a_3 + a_1 + a_0 \in (U_q(\mathfrak{g}))^{\mathcal{B}_\theta \cap U_{\pi'}}.$$

From (6.10) we see that $\mathcal{P}(a)$ is both the lowest weight term in the expansion of a as a sum of weight vectors and the lowest l-weight summand of a. Applying κ_θ, the version of the Chevalley antiautomorphism that preserves $\mathcal{B}_\theta$, to a yields a nonzero element $\kappa_\theta(a)$ in $\mathcal{B}_\theta$ with

$$\mathcal{P}(\kappa_\theta(a)) = \mathcal{P}(\kappa_\theta(X_{\theta(-\beta)})),$$

where $X_{\theta(-\beta)} K_{\theta(-\beta)-\beta}$ is the contribution to a from $U^+_{\theta(-\beta)} K_{-\beta}$. By Corollary 4.4, $\mathcal{P}(\kappa_\theta(a))$ is nonzero. Hence $X_{\theta(-\beta)} K_{\theta(-\beta)-\beta} \neq 0$ and is the term of highest weight in the expansion of a as a sum of weight vectors. $\qquad\square$

7. Constructing Cartan subalgebras

We present the main result of the paper here, namely, the construction of quantum analogs of Cartan subalgebras (or, more precisely, their enveloping algebras) of $\mathfrak{g}^\theta$ where $\mathfrak{g}, \mathfrak{g}^\theta$ is a maximally split symmetric pair.

7.1. Specialization. Recall that $\mathcal{B}_\theta$ specializes to $U(\mathfrak{g}^\theta)$ as q goes to 1 (Section 4.1). The next lemma sets up the basic tools needed to verify quantum Cartan subalgebras specialize to their classical counterparts as q goes to 1.

Lemma 7.1. *Let $b = b_1 + C$ be an element of $\mathcal{B}_\theta$ such that*

- *$b_1 \in \mathcal{B}_\theta \cap \hat{U}$ and b_1 specializes to a nonzero element of $U(\mathfrak{g}^\theta)$ as q goes to 1.*
- *$C \in \sum_{\zeta,\xi} G^-_{-\zeta} U^0 U^+_\xi$ where each $\xi \notin Q^+(\pi_\theta)$ and $\zeta \notin Q^+(\pi_\theta)$.*

Then $C \in (q-1)\hat{U}$ and b specializes to $\bar{b}_1$ as q goes to 1. Moreover, if $\kappa_\theta(b_1) = ab_1 + b_2$ where $b_2 \in \sum_{\xi \notin Q^+(\pi_\theta)} G^- U^0 U^+_\xi$ and a is a scalar, then a evaluates to a nonzero scalar in $\mathbb{C}$ at $q = 1$ and $\kappa_\theta(b) = ab$.

Proof. The fact that $\xi \notin Q^+(\pi_\theta)$ ensures that $U^+_\xi \cap \mathcal{M}^+ = 0$. Repeated applications of Lemma 4.1 show that $C \in \sum_{\xi \notin Q^+(\pi_\theta)} \mathcal{B}_\theta \hat{U}^0 U^+_\xi$. Thus, the specialization of $(q-1)^s C$ cannot be a nonzero element of $U(\mathfrak{g}^\theta)$ for any s.

Choose s minimal such that $(q-1)^s b \in \mathcal{B}_\theta \cap \hat{U}$. The assumptions on b_1 force $s \geq 0$. Moreover, the minimality of s ensures that $(q-1)^s b$ specializes to a nonzero element of $U(\mathfrak{g})$. If $s > 0$, then $(q-1)^s b_1 \in (q-1)\hat{U}$ and so the specialization of $(q-1)^s b$ as q goes to 1 is the same as the specialization of $(q-1)^s C$. This contradicts the fact that $b \in \mathcal{B}_\theta$ and $\mathcal{B}_\theta$ specializes to $U(\mathfrak{g}^\theta)$. Hence $s = 0$, $b \in \hat{U}$, b specializes to b_1 and so $C \in (q-1)\hat{U}$.

Note that $\kappa_\theta(b_1)$ specializes to the image of $\bar{b}_1$ under the classical Chevalley antiautomorphism of $\mathfrak{g}$. Hence $ab_1 + b_2 \in \hat{U}$ forces a to evaluate to a nonzero

element of $\mathbb{C}$ at $q = 1$. Since κ_θ preserves $\mathcal{B}_\theta$, $\kappa_\theta(b) - ab = b_2 - a\kappa_\theta(C)$ is an element of $\mathcal{B}_\theta$. Now $\kappa_\theta(C) \in \sum_{\zeta \notin Q^+(\pi_\theta)} G^- U^0 U^+_\zeta$, hence by the same arguments as above, $(q-1)^s \kappa_\theta(C)$ cannot specialize to a nonzero element of $\mathcal{B}_\theta$ for any s. The assumptions of the lemma ensure the same is true for b_2. Hence $b_2 - a\kappa_\theta(C) = 0$ and $\kappa_\theta(b) = ab$. $\square$

7.2. **Quantum symmetric pair Cartan subalgebras.** In the next result, we put together the upper and lower triangular parts associated to positive roots β using Section 6.4 in order to identify a quantum Cartan subalgebra of $\mathcal{B}_\theta$. We also show that this Cartan subalgebra satisfies a uniqueness property with respect to the action of the quantum Chevalley antiautomorphism.

Note that in general, we cannot expect to lift a term of the form $e_\beta + f_{-\beta}$ with $\theta(\beta) = -\beta$ to an element in $U^+_\beta + G^-_{-\beta}$ that is also in $\mathcal{B}_\theta$. We see in the next theorem that for most cases, we get an intermediate term (referred to as C_j or just C) that specializes to 0 as q goes to 1.

Theorem 7.2. *Let $\mathfrak{g}, \mathfrak{g}^\theta$ be a maximally split pair. Let $\Gamma_\theta = \{\beta_1, \ldots, \beta_m\}$ be a maximum strongly orthogonal θ-system and let $\{\alpha_{\beta_1}, \ldots, \alpha_{\beta_m}\}$ be a set of simple roots satisfying the conditions of Theorem 2.7 and Remark 2.8. For each j, there exists a unique nonzero element H_j (up to scalar multiple) in $\mathcal{B}_\theta$ such that $H_j = X_{\beta_j} + C_j + s_{\beta_j}(K_{-\beta_j} - 1) + Y_{-\beta_j}$ where s_{β_j} is a (possibly zero) scalar and*

- (i) *$X_{\beta_j} \in G^+_{\beta_j}$, $Y_{-\beta_j} \in U^-_{-\beta_j}$, and $C_j \in U_{\mathrm{Supp}(\beta_j)} \cap G^-_{(\tau,1)} U^0 U^+_{(\tau,1)}$ where $\tau = \{\alpha_{\beta_j}, p(\alpha_{\beta_j})\}$.*
- (ii) *$X_{\beta_j} \in [(\mathrm{ad}\, U^+) K_{-2\nu_j}] K_{-\beta_j + 2\nu_j}$ and $Y_{-\beta_j} \in [(\mathrm{ad}\, U^-) K_{-2\nu_j}] K_{-\beta_j + 2\nu_j}$ where ν_j is the fundamental weight associated to the simple root α_{β_j}.*
- (iii) *$\{X_{\beta_j}, Y_{-\beta_j}\} \subset (U_q(\mathfrak{g}))^{U_{\pi_j} \mathcal{T}_\theta}$ and $C_j \in (U_q(\mathfrak{g}))^{(\mathcal{B}_\theta \cap U_{\pi_j}) \mathcal{T}_\theta}$ where $\pi_j = \mathrm{StrOrth}(\beta_j)$.*
- (iv) *If β_j satisfies Theorem 2.7(5), then $Y_{-\beta_j} \in [(\mathrm{ad}\, F_{-\alpha'_\beta})(\mathrm{ad}\, U^-) K_{-2\nu_j}] K_{2\nu_j - \beta_j}$.*
- (v) *$\kappa_\theta(H_j) = H_j$ for each j.*

Moreover, $\mathbb{C}(q)[\mathcal{T}_\theta][H_1, \ldots, H_m]$ is a commutative polynomial ring over $\mathbb{C}(q)[\mathcal{T}_\theta]$ in m generators that specializes to the enveloping algebra of a Cartan subalgebra of $\mathfrak{g}^\theta$ as q goes to 1.

Proof. Consider $\gamma \in Q^+(\pi)$ and $\alpha_i \in \pi \setminus \mathrm{Supp}(\gamma)$. It follows that $(\alpha_i, \gamma) \leq 0$ with equality if and only if $(\alpha_i, \alpha) = 0$ for all simple roots $\alpha \in \mathrm{Supp}(\gamma)$. Therefore, E_i, F_i, K_i^{-1} commute with all elements of G^+_γ and all elements of $U^-_{-\gamma}$ for all $\alpha_i \in \mathrm{Orth}(\gamma) \setminus \mathrm{Supp}(\beta)$. Hence, in proving (iii), we need only consider those $\alpha_i \in \mathrm{StrOrth}(\beta_j) \cap \mathrm{Supp}(\beta_j)$.

Suppose that for each $\beta_j \in \Gamma_\theta$, we have identified an element $H_j = X_{\beta_j} + C_j + s_{\beta_j}(K_{-\beta_j} - 1) + Y_{-\beta_j}$ so that $X_{\beta_j}, Y_{-\beta_j}$, and C_j each satisfy the relevant parts of (i), (ii), (iii), (iv), and, moreover, the following two properties hold:

- (a) $Y_{-\beta_j}$ specializes to $f_{-\beta_j}$ as q goes to 1.
- (b) $\kappa_\theta(Y_{-\beta_j})$ is a nonzero scalar multiple of X_{β_j}.

Note that these two properties ensure X_{β_j} specializes to e_{β_j}. Hence, by Lemma 7.1, H_j specializes to $e_{\beta_j} + f_{-\beta_j}$ and $\kappa_\theta(H_j) = H_j$ after a suitable scalar adjustment. Thus the assertion concerning specialization follows from these conditions.

Note that by construction of Γ_θ, we have $(\gamma, \beta_j) = 0$ for all $\gamma \in Q(\pi)^\theta$ and all $j = 1, \ldots, m$. Hence for each $j = 1, \ldots, m$ and each $K_\gamma \in \mathcal{T}_\theta$, $[K_\gamma, H_j]$,

which is an element of $\mathcal{B}_\theta$, equals $[K_\gamma, C_j]$. By assumption on C_j, we also have $[K_\gamma, C_j] \in G^-_{(\tau,1)} U^0 U^+_{(\tau,1)}$. Since $\tau \cap \pi_\theta = \emptyset$, it follows that $\mathcal{M}^+ \cap U^+_{(\tau,1)} = 0$ and so $\mathcal{P}([K_\gamma, C_j]) = 0$. By Corollary 4.4, $[K_\gamma, C_j] = 0$. Thus each H_j commutes with every element of $\mathcal{T}_\theta$. By Theorem 2.7(i), $\operatorname{Supp}(\beta_i) \subseteq \operatorname{StrOrth}(\beta_j)$ for all $i > j$. Thus (i) and (iii) ensure that H_j commutes with H_i for $i > j$ and so the elements $\{H_1, \ldots, H_m\}$ pairwise commute. Hence $\mathbb{C}(q)[\mathcal{T}_\theta][H_1, \ldots, H_m]$ is a commutative ring. The property that each H_j specializes to $e_{\beta_j} + f_{-\beta_j}$ implies that $\mathbb{C}(q)[\mathcal{T}_\theta][H_1, \ldots, H_m]$ specializes to the enveloping algebra of the Cartan subalgebra of $\mathfrak{g}^\theta$, which is a polynomial ring over $\mathbb{C}$. This guarantees that there are no additional relations among the H_j and so $\mathbb{C}(q)[\mathcal{T}_\theta][H_1, \ldots, H_m]$ is a polynomial ring in m generators over $\mathbb{C}(q)[\mathcal{T}_\theta]$.

For the uniqueness assertion, suppose that H'_j also satisfies the conclusions of the theorem. Note that conditions (i), (ii), (iii), (iv) and the specialization assertion are enough to ensure that each Y_{β_j} corresponds to Y_j of Theorem 5.9. Hence, rescaling if necessary, we get that $H_j - H'_j \in G^-_{(\tau,1)} U^0 U^+_{(\tau,1)} + G^+_{\beta_j} + \mathbb{C}(q)(K_{-\beta_j} - 1)$. Hence $\mathcal{P}(H_j - H'_j)$ is a scalar, say s in $\mathbb{C}(q)$. Since $H_j - H'_j - s \in \mathcal{B}_\theta$, it follows from Corollary 4.4 that $H_j = H'_j + s$. The fact that both H_j and H'_j are in $U_{-\beta_j} + G^-_{(\tau,1)} U^0 U^+_{(\tau,1)} + G^+_{\beta_j} + \mathbb{C}(q)(K_{-\beta_j} - 1)$ forces $s = 0$ and hence $H_j = H'_j$.

We complete the proof by identifying the H_j whose summands satisfy (i), (ii), (iii), (iv) and the two properties (a) and (b) highlighted above. The proof follows the same breakdown into cases of possible roots in Γ_θ as in the proof of Theorem 5.9. Note that (iv) only comes into play in Case 5. We drop the subscript j, writing β for β_j, H for H_j, etc. Set $w_\beta = w(\operatorname{Supp}(\beta) \setminus \{\alpha_\beta, \alpha'_\beta\})_0$.

Case 1. $\beta = \alpha_\beta = \alpha'_\beta$. Choose k so that $\alpha_k = \alpha_\beta = \beta$. Since $\theta(\alpha_k) = \theta(\beta) = -\beta = -\alpha_k$, we see that $H = B_k - s_k = F_k + E_k K_k^{-1} + s_k(K_k^{-1} - 1)$ is in $\mathcal{B}_\theta$. Moreover, this element specializes to $f_k + e_k$ as q goes to 1. It is straightforward to check that H satisfies (i)–(iii), (a), and (b).

Case 2. $\beta = \alpha_\beta + w_\beta \alpha_\beta$ and $\alpha_\beta = \alpha'_\beta$. By Theorem 5.9, there exists $Y_{-\beta}$ that satisfies (i), (ii), and (a). Write

$$Y_{-\beta} = \sum_{\operatorname{wt}(I) = \beta} F_I a_I$$

for appropriate scalars a_I. By Proposition 6.7, there exists $a \in \mathcal{B}_\theta$ such that

$$\mathcal{P}(a) = \sum_{\operatorname{wt}(I) = \beta} F_I a_I,$$

the lowest weight summand of a is $Y_{-\beta}$, the highest weight summand of a is a term $X_\beta \in G^+_\beta$ (since $\theta(-\beta) = \beta$), and X_β commutes with all elements of $U_{\pi'}$. Hence $\kappa_\theta(X_\beta)$ is an element of U^-_β that commutes with all elements of $U_{\pi'}$. It follows from Corollary 5.4, that $\kappa_\theta(X_\beta)$ is a scalar multiple of $Y_{-\beta}$ and so (b) holds. By (3.6) and the second half of assertion (ii), X_β satisfies (ii). Hence X_β and $Y_{-\beta}$ satisfy the assertions in (i)–(iii), (a), and (b).

By Proposition 6.7, $a = X_\beta + C + sK_{-\beta} + Y_{-\beta}$ for some nonzero scalar s and element C in $U_q(\mathfrak{g})$ satisfying (i) and (iii). Thus, the desired Cartan element is $H = a - s = X_\beta + C + s(K_{-\beta} - 1) + Y_{-\beta}$ in this case.

Case 3. $\beta = p(\alpha_\beta) + w_\beta\alpha_\beta = w\alpha_\beta$ where $w = w(\text{Supp}(\beta) \setminus \{\alpha_\beta\}_0$. Arguing as in Case 2 yields an element $H = Y_{-\beta} + C + X_\beta + s(K_{-\beta} - 1)$ so that $Y_{-\beta}$ satisfies the relevant parts of (i), (ii), (iii), (a) and C satisfies the relevant parts of (i) and (iii).

Choose i so that $\alpha_\beta = \alpha_i$. By Theorem 2.7(3) and Remark 2.8, we may assume $\Delta(\text{Supp}(\beta))$ is a root system of type A and the symmetric pair $\mathfrak{g}_{\text{Supp}(\beta)}, (\mathfrak{g}_{\text{Supp}(\beta)})^\theta$ is of type AIII/AIV. Corollary 5.8(b) further ensures that

$$F_i Y_{-\beta} - q Y_{-\beta} F_i = 0.$$

On the other hand, it follows from properties of C that

(7.1) $$F_i(H - Y_{-\beta}) - q(H - Y_{-\beta})F_i \in \sum_{m \leq 2} G^-_{(\tau,m)} U^0 U^+.$$

It follows from the discussion concerning $\mathcal{S}$ in Section 4.1 that $\alpha_i \notin \mathcal{S}$ and so $B_i = F_i + c_i\theta_q(F_i K_i)K_i^{-1}$ for a suitable scalar c_i (i.e., $s_i = 0$ and there is no K_i^{-1} term in B_i). Since $\theta(-\beta) = \beta$, we see from (7.1) and Remark 4.2 that

$$B_i H - q H B_i \in \sum_{m \leq 2} \sum_{r \leq 2} G^-_{(\tau,m)} U^0 U^+_{(\tau,r)}.$$

On the other hand,

$$\left(\theta_q(F_i K_i)K_i^{-1}\right) X_\beta - q X_\beta \left(\theta_q(F_i K_i)K_i^{-1}\right)$$

either equals 0 or has weight $\beta + \theta(-\alpha_i)$. Since $\text{ht}_\tau(\beta) + \theta(-\alpha_i)) = 3$, this term must be 0. By Corollary 5.8, $\kappa_\theta(X_\beta)$ is a nonzero scalar multiple of $Y_{-\beta}$, thus (b) holds. Hence, H satisfies the conclusions of the lemma.

Case 4. $\beta = \alpha'_\beta + w_\beta\alpha_\beta = w\alpha'_\beta$ where $w = w(\text{Supp}(\beta) \setminus \{\alpha'_\beta\})_0$. More precisely, we assume that β satisfies the conditions of Theorem 2.7(4) as further clarified by Remark 2.8. Hence, $\beta = w\alpha'_\beta$. Moreover, we may assume that $\text{Supp}(\beta) = \{\gamma_1, \ldots, \gamma_s\}$ generates a root system of type B_s and is ordered in the standard way so that $\alpha'_\beta = \gamma_s$ is the unique shortest simple root and $\alpha_\beta = \gamma_1$.

Let m be chosen so that the mth simple root of π, namely α_m, equals the first root γ_1 of $\text{Supp}(\beta)$. It follows from the description of the generators for $\mathcal{B}_\theta$ in Section 4.1 that

$$B_m = F_{-\gamma_1} + c[(\text{ad } E_{\gamma_2} E_{\gamma_3} \cdots E_{\gamma_{s-1}} E^2_{\gamma_s} E_{\gamma_{s-1}} \cdots E_{\gamma_2}) E_{\gamma_1}] K_{\gamma_1}^{-1}$$

for some nonzero scalar c. Note that if $\alpha_i \in \pi_\theta$ and $b \in \mathcal{B}_\theta$, then

$$[(\text{ad} F_i)b K_j] K_j^{-1} = [F_i b K_j K_i - b K_j F_i K_i] K_j^{-1} = F_i b K_i - q^{-(\alpha_i, \alpha_j)} b F_i K_i$$

is also an element of $\mathcal{B}_\theta$ for all choices of j. Hence, since $\gamma_j \in \pi_\theta$ for $j = 2, \ldots, s$, the element

$$H' = [(\text{ad } F_{-\gamma_s} F_{-\gamma_{s-1}} \cdots F_{-\gamma_2}) B_m K_{\gamma_1}] K_{\gamma_1}^{-1}$$

$$= [(\text{ad } F_{-\gamma_s} F_{-\gamma_{s-1}} \cdots F_{-\gamma_2}) F_{-\gamma_1} K_{\gamma_1}] K_{\gamma_1}^{-1} + c'[(\text{ad } E_{\gamma_s} E_{\gamma_{s-1}} \cdots E_{\gamma_2}) E_{\gamma_1}] K_{\gamma_1}^{-1}$$

is also in $\mathcal{B}_\theta$ where c' is a suitably chosen nonzero scalar. Note that

$$K_{-\beta} = K_{\gamma_s}^{-1} K_{\gamma_{s-1}}^{-1} \cdots K_{\gamma_2}^{-1} K_{\gamma_1}^{-1} \in \mathcal{T}_\theta K_{\gamma_1}^{-1}.$$

Hence $K_{\gamma_1} K_{-\beta}$ is in $\mathcal{B}_\theta$. Set

(7.2) $$X_\beta = c'[(\text{ad } E_{\gamma_s} E_{\gamma_{s-1}} \cdots E_{\gamma_2}) E_{\gamma_1}] K_{-\beta} \text{ and}$$

$$Y_{-\beta} = [(\text{ad } F_{-\gamma_s} F_{-\gamma_{s-1}} \cdots F_{-\gamma_2}) F_{-\gamma_1} K_{\gamma_1}] K_{-\beta}$$

and note that $H = H'K_{-\beta+\gamma_1} = X_\beta + Y_{-\beta}$ is in $\mathcal{B}_\theta$. We show that H satisfied (i), (ii), (iii), (a), and (b) with $C = 0$.

Note that $E_{\gamma_1} = d[(\mathrm{ad}\, E_{\gamma_1})K_{-2\nu_1}]K_{-2\nu_1}$ and $F_{-\gamma_1}K_{\gamma_1} = d'[(\mathrm{ad}\, F_{-\gamma_1})K_{-2\nu_1}]K_{-2\nu_1}$ for nonzero scalars d and d'. Hence by (7.2), X_β and $Y_{-\beta}$ both satisfy assertion (ii). Assertions (i) and (a) follow from the construction of X_β and $Y_{-\beta}$ and the definition of the adjoint action (Section 3.1). Assertion (b) follows from (3.6) of Section 3.4. Assertion (iii) for X_β follows from the fact that X_β is a highest weight vector with respect to the action of (ad $U_{\mathrm{Supp}(\beta)\setminus\{\gamma_1,\gamma_2\}}$) and has weight β with $(\beta,\gamma_j) = 0$ for $j = 2,\ldots,s-1$. Assertion (iii) for $Y_{-\beta}$ now follows from (b).

Case 5. $\beta = \alpha'_\beta + \alpha_\beta + w_\beta\alpha_\beta$ and $\alpha_\beta \neq \alpha'_\beta$. Set $\beta' = \beta - \alpha'_\beta$. Recall from Theorem 2.7 that $\alpha'_\beta \in \pi_\theta$, α'_β is strongly orthogonal to all roots in $\mathrm{Supp}(\beta') \setminus \{\alpha_\beta\}$, and $(\alpha'_\beta,\beta) = 0$. Note also that $(\alpha'_\beta,\beta') = -(\alpha'_\beta,\alpha'_\beta)$ and $\theta(-\beta') = \beta' + 2\alpha'_\beta$.

By the proof of Theorem 5.9 Case 5, there exists $Y_{-\beta'}$ that satisfies (i), (ii), and (a) with respect to β' instead of β. Write

$$Y_{-\beta'} = \sum_{\mathrm{wt}(I)=\beta'} F_I a_I$$

for appropriate scalars a_I. By Proposition 6.7, there exists

$$H' = Y_{-\beta'} + C' + X_{\theta(-\beta')}K_{2\alpha'_\beta} + sK_{-\beta'}$$

in $\mathcal{B}_\theta$ such that $Y_{-\beta'} \in U^-_{-\beta'}$, $X_{\theta(-\beta')} \in G^+_{\theta(-\beta')}$, $C' \in U_{\mathrm{Supp}(\beta')} \cap G^-_{(\tau,1)}U^+_{(\tau,1)}U^0$, and s is a scalar. Moreover, each of these terms commute with all elements of $U_{\pi'}$.

Let k be the index so that $\alpha'_\beta = \alpha_k$ and recall that $(\alpha_k,\nu) = 0$ where ν is the fundamental weight associated to α_β. It follows that (ad F_k)$(uK_{-2\nu})K_{2\nu} = $ (ad F_k)u for all u. Thus in much of the discussion below, we ignore the $K_{-2\nu}$ term which can always be added in later. Note that $F_k \in \mathcal{B}_\theta$ since $\alpha_k \in \pi_\theta$. Note that

$$[F_kK_{-\beta'} - q^{-(\beta',\alpha_k)}K_{-\beta'}F_k] = 0.$$

Set

$$H = [F_kH' - q^{-(\beta',\alpha_k)}H'F_k] = Y_{-\beta} + C + X_\beta,$$

where $C = [F_kC' - q^{-(\beta',\alpha_k)}C'F_k]K_k$,

(7.3) $$Y_{-\beta} = (F_kY_{-\beta'} - q^{-(\beta',\alpha_k)}Y_{-\beta'}F_k) = [(\mathrm{ad}\, F_k)(Y_{-\beta'}K_{\beta'})]K_{-\beta},$$

and

$$X_\beta = F_k(X_{\theta(-\beta')}K_{\theta(\beta')}K_{-\beta'}) - q^{-(\beta',\alpha_k)}(X_{\theta(-\beta')}K_{\theta(\beta')}K_{-\beta'})F_k$$
$$= [(\mathrm{ad}\, F_k)(X_{\theta(-\beta')}K_{\theta(\beta')})]K_{-\beta}.$$

We argue that H and its summands satisfy (i)–(iv), (a), and (b). We see from (7.3) (as in the proof of Case 5 of Theorem 5.9) that $Y_{-\beta}$ satisfies the relevant properties of (i), (ii), and (a) since $Y_{-\beta'}$ does. Note that (7.3) also ensures that $Y_{-\beta}$ satisfies (iv). By Proposition 6.7,

$$\{X_{\theta(-\beta')}, Y_{-\beta'}\} \subseteq (U_q(\mathfrak{g}))^{U_{\pi_j}} \text{ and } C' \in (U_q(\mathfrak{g}))^{\mathcal{B}_\theta \cap U_{\pi'}},$$

where $\pi' = \mathrm{Supp}(\beta') \setminus \{\alpha_\beta\}$. Hence, since F_k commutes with all elements in $U_{\pi'}$, it follows from these inclusions and the discussion at the beginning of the proof (before the cases) that $X_\beta, Y_{-\beta}$, and C satisfy (iii). So we only need to show that X_β satisfies (i) and (ii) and $X_\beta, Y_{-\beta}$ are related as in (b).

Since C' is an element of $G^-_{(\tau,1)}U^0 U^+_{(\tau,1)}$, so is C. Thus $\mathcal{P}(C) = 0$. Note that we also have $X_\beta \in G^- U^0 U^+_{(\tau,2)}$ and so $\mathcal{P}(X_\beta) = 0$. Set

$$b = F_k \left(\sum_{\mathrm{wt}(I)=\beta'} B_I a_I \right) - q^{-(\beta',\alpha_k)} \left(\sum_{\mathrm{wt}(I)=\beta'} B_I a_I \right) F_k$$

and define $\tilde{b}$ as in Proposition 4.3. Note that $\mathcal{P}(\tilde{b}) = \mathcal{P}(H) = Y_{-\beta}$. Hence by Corollary 4.4, $H = \tilde{b}$. Note that in addition to β', β also satisfies the conditions of Proposition 6.7. This is because $\alpha_k \in \mathrm{Orth}(\beta)$ even if it is not strongly orthogonal to β and so β satisfies the required assumption on height. Note further that we may choose $a = H$ since H satisfies Proposition 6.7 (i) and (ii). Hence by Proposition 6.7 we get

$$X_\beta \in U^+_\beta K_{-\beta} = G^+_\beta$$

and so X_β satisfies assertion (i) of this theorem. Hence X_β satisfies both (i) and (iii).

We next show that

(7.4) $$(\mathrm{ad}\ E_k F_k^2)(X_{\theta(-\beta')}K_{\theta(-\beta')}) = (\mathrm{ad}\ F_k)(X_{\theta(-\beta')}K_{\theta(-\beta')})$$

up to a nonzero scalar. The fact that $\alpha_k \in \pi_\theta$ ensures that $E_k, F_k, K_k^{\pm 1} \in \mathcal{B}_\theta$. Hence

$$[(\mathrm{ad}\ E_k)(H'K_{\beta'})]K_{-\beta'}$$
$$= [(\mathrm{ad}\ E_k)\left(Y_{-\beta'}K_{\beta'} + C'K_{\beta'} + X_{\theta(-\beta')}K_{\theta(-\beta')} + s\right)]K_{-\beta'} \in \mathcal{B}_\theta.$$

Note that $(\mathrm{ad}\ E_k)(Y_{-\beta'}K_{\beta'}) = 0$ since $\alpha_k \notin \mathrm{Supp}(\beta')$. It is straightforward to check that $(\mathrm{ad}\ E_k)s = 0$ and $(\mathrm{ad}\ E_k)(X_{\theta(-\beta')}K_{\theta(-\beta')}) \in U^+_{\theta(-\beta')+\alpha_k}$. Also, $[(\mathrm{ad}\ E_k)(C'K_{\beta'})]K_{-\beta'} \in G^-_{(\tau,1)}U^0 U^+_{(\tau,1)}$ since the same is true for C'. It follows that

$$[(\mathrm{ad}\ E_k)(H'K_{\beta'}]K_{-\beta'} \in G^-_{(\tau,1)}U^0 U^+_{(\tau,1)} + U^+_{\theta(-\beta')+\alpha_k} \subseteq \sum_{m \geq 1} G^- U^0 U^+_{(\tau,m)}.$$

By Remark 4.2, we get $[(\mathrm{ad}\ E_k)(H'K_{\beta'})]K_{-\beta'} = 0$. Since $(G^-_{(\tau,1)}U^0 U^+_{(\tau,1)}) \cap (U^0 G^+_{\theta(-\beta')+\alpha_k}) = 0$, we must have $(\mathrm{ad}\ E_k)(X_{\theta(-\beta')}K_{\theta(-\beta')}) = 0$. Equality (7.4) now follows from rewriting $E_k F_k^2$ as a linear combination of $F_k^2 E_k$ and $F_k u$ for some $u \in U^0$ using the defining relation (4.5) for $U_q(\mathfrak{g})$.

Equality (7.4) and the definition of X_β implies that

(7.5) $$(\mathrm{ad}\ E_k F_k)(X_\beta K_\beta) = X_\beta K_\beta$$

up to a nonzero scalar. Using (4.5) we see that

(7.6) $$(\mathrm{ad}\ F_k)(X_\beta K_\beta) \in U^+ + U^+ K_k^2 + U^+ F_k K_k.$$

Note that $(\mathrm{ad}\ E_k)U^+ K_k^2 \subseteq U^+ K_k^2$ while $(\mathrm{ad}\ E_k)u F_k K_k \in U^+ F_k K_k + U^+(K_k^2 - 1)$ for any choice of $u \in U^+$. Since $X_\beta K_\beta \in U^+$, it follows from (7.5), (7.6) and the fact that X_β satisfies (iii) that

$$(\mathrm{ad}\ F_k)(X_\beta K_\beta) \in (U^+_{\beta-\alpha_k})^{U_{\pi'}} = (U^+_{\beta'})^{U_{\pi'}}.$$

Applying κ to $((\mathrm{ad}\ F_k)(X_\beta K_\beta))K_{-\beta'}$ yields an element $Y'_{-\beta'} \in (U^-_{-\beta'})^{U_{\pi'}}$. By the uniqueness assertion of Corollary 5.4, it follows that $Y'_{-\beta'} = Y_{-\beta'}$ up to nonzero scalar. By identity (3.6), (7.5), and (7.3), we see that

$$\kappa(Y_{-\beta}K_\beta) = \kappa((\mathrm{ad}\ F_k)Y_{-\beta'}K_{\beta'})$$
$$= (\mathrm{ad}\ E_k)\kappa(Y_{-\beta'}K_{\beta'}) = (\mathrm{ad}\ E_k F_k)(X_\beta K_\beta) = X_\beta K_\beta$$

up to nonzero scalars and so (b) holds. Since $Y_{-\beta'}$ satisfies (ii) with β replaced by β', it now follows from the identity (3.6) that $X_\beta K_\beta \in [(\mathrm{ad}\ U^+)K_{-2\nu}]K_{-\beta+2\nu}$ where recall that ν is the fundamental weight associated to α_β. Hence, we see that X_β satisfies (ii). $\qquad\square$

We refer to the algebra $\mathcal{H} = \mathbb{C}(q)[\mathcal{T}_\theta][H_1, \ldots, H_m]$ of Theorem 7.2 as a quantum Cartan subalgebra of $\mathcal{B}_\theta$. The next result is the first step in understanding finite-dimensional $\mathcal{B}_\theta$-modules with respect to the action of $\mathcal{H}$. It is an immediate consequence of Corollaries 3.3 and 3.5.

Corollary 7.3. *Let $\mathcal{H}$ be a quantum Cartan subalgebra of the coideal subalgebra $\mathcal{B}_\theta$ associated to the symmetric pair $\mathfrak{g}, \mathfrak{g}^\theta$. Any finite-dimensional unitary $\mathcal{B}_\theta$-module can be written as a direct sum of eigenspaces with respect to the action of $\mathcal{H}$. Moreover, any finite-dimensional $U_q(\mathfrak{g})$-module can be written as a direct sum of eigenspaces with respect to the action of $\mathcal{H}$.*

Remark 7.4. Assume that $\mathfrak{g}, \mathfrak{g}^\theta$ is a symmetric pair of type AI. The algebra $\mathcal{B}_\theta$ in this case is referred to in the literature as the (nonstandard) q-deformed algebra $U'_q(\mathfrak{so}_n)$ and its finite-dimensional modules have been analyzed and classified by Klimyk, Iorgov, and Gavrilik (see for example [GK] and [IK]) using quantum versions of Gel'fand-Tsetlin basis. Rowell and Wenzl ([RW]) revisit the representation theory of this coideal subalgebra. They take a different approach that involves developing a highest weight module theory using a Cartan subalgebra $\mathcal{H}$. This Cartan subalgebra $\mathcal{H}$ agrees with the one produced by the above theorem using Theorem 2.7. In particular, in this case, $\mathfrak{h}_\theta = 0$, $\Gamma_\theta = \{\alpha_1, \alpha_3, \cdots, \alpha_{\lfloor (n+1)/2 \rfloor}\}$, and $\mathcal{H} = \mathbb{C}(q)[B_1, B_3, \ldots, B_{\lfloor (n+1)/2 \rfloor}]$. For more details on the representation theory of $\mathcal{B}_\theta$ developed using $\mathcal{H}$, the reader is referred to [RW].

Remark 7.5. Suppose that $\beta = p(\alpha_\beta) + w_\beta \alpha_\beta$ with $\alpha_\beta \neq p(\alpha_\beta)$ as in Case 3 of the proof of the above theorem. Then θ restricts to an involution of type AIII/AIV. Moreover, the element α_β can be switched with $p(\alpha_\beta)$ in the construction of the corresponding Cartan element associated to $\beta \in \Gamma_\theta$. The end result is still a commutative polynomial ring specializing to the enveloping algebra of the same Cartan subalgebra of $\mathfrak{g}^\theta$. However, the Cartan element corresponding to β is different for the two constructions. We take a closer look at such roots in the analysis of a large family of symmetric pairs of type AIII/AIV in Section 8 below.

8. A family of examples: Type AIII/AIV

In this section, we consider the following family of examples: symmetric pairs of type AIII/AIV with $\pi_\theta = \emptyset$. Note that the discussion presented here has applications to other symmetric pairs as well. In particular, consider for a moment the case where $\mathfrak{g}, \mathfrak{g}^\theta$ is an arbitrary symmetric pair with corresponding maximum strongly orthogonal θ-system Γ_θ. Suppose that there is a root $\beta \in \Gamma_\theta$ such that $\beta = p(\alpha_\beta) + w_\beta \alpha_\beta$ with $\alpha_\beta \neq p(\alpha_\beta)$ and $\Delta(\mathrm{Supp}(\beta))$ is a root system of type A as

in Case 3 of the proof of Theorem 2.7. By Remark 2.8, the involution θ restricts to an involution of type AIII/AIV. If, in addition, $\pi_\theta = \emptyset$, then the discussion in this section applies to the construction of the Cartan element associated to β.

The presentation in this section was in part inspired by conversations with Stroppel ([S]) who has analyzed the finite-dimensional simple modules of the quantum symmetric pair coideal subalgebras in type AIII/AIV. In [Wa], Watanabe also looks at symmetric pairs of type AIII with $\pi_\theta = 0$ and restricts to the case when $\mathfrak{g}$ is of type A_n where n is even. He obtains a triangular decomposition for $\mathcal{B}_\theta$ for this subfamily where the components are each described as subspaces (see [Wa, Remark 2.2.10]). We see below that the Cartan part of the triangularization in [Wa] agrees with the Cartan subalgebra of this paper, and thus this subspace is actually a subalgebra of $\mathcal{B}_\theta$.

8.1. **Overview.** Assume that $\mathfrak{g}, \mathfrak{g}^\theta$ is a symmetric pair of type AIII/AIV with $r = \lfloor (n+1)/2 \rfloor$ where $n \geq 2$ and r is as in the proof of Theorem 2.7 for Type AIII/AIV. The involution θ on $\mathfrak{g}$ is defined by $\theta(\alpha_i) = -\alpha_{n-i+1}$ for $i = 1, \ldots, n$. By Theorems 2.5 and 2.7, the set

$$\mathfrak{h}_\theta \oplus \left(\bigoplus_{\beta \in \Gamma_\theta} \mathbb{C}(e_\beta + f_{-\beta}) \right)$$

is a Cartan subalgebra for $\mathfrak{g}^\theta$ where $\mathfrak{h}_\theta = \mathrm{span}_\mathbb{C}\{h_i - h_{n-i+1} \mid i = 1, \ldots, r\}$, $\Gamma_\theta = \{\beta_1, \cdots, \beta_r\}$, and

$$\beta_j = \alpha_j + \alpha_{j+1} + \cdots + \alpha_{n-j+1}$$

for $j = 1, \cdots, r$.

The right coideal subalgebra $\mathcal{B}_\theta$ is generated over $\mathbb{C}(q)$ by

$$B_i = F_i + E_{n-i+1} K_i^{-1}$$

and

$$K_i K_{n-i+1}^{-1}$$

for $i = 1, \ldots, n$. (In the notation of Section 4.1, we are taking $c_i = -1$ and $s_i = 0$ for each i. Other choices of parameters yield analogous results. Note that $\mathcal{S} = \{\alpha_r\}$ if r is odd and is empty otherwise, so that the only time s_i could take on nonzero values is when r is odd and $i = r$.) The B_i, $i = 1, \ldots, n$ satisfy the following relations (see for example [Ko], Theorem 7.4):

- For $i, j \in \{1, \ldots, n\}$ with $a_{ij} = 0$, we have

$$B_i B_j - B_j B_i = \delta_{n-i+1,j}(q - q^{-1})^{-1}(K_{n-i+1} K_i^{-1} - K_{n-i+1}^{-1} K_i).$$

- For $i, j \in \{1, \ldots, n\}$ with $a_{ij} = -1$, we have

$$B_i^2 B_j - (q + q^{-1}) B_i B_j B_i + B_j B_i^2 = -\delta_{i,n-i+1} q B_j$$
$$+ \delta_{i,n-j+1}(q + q^{-1})(q K_{n-i+1} K_i^{-1} + q^{-2} K_{n-i+1}^{-1} K_i) B_i.$$

Note that $a_{ij} = -1$ and $i = n - i + 1$ if and only if n is odd, $i = (n+1)/2 = r$, and $j = i \pm 1$. Also, $a_{ij} = -1$ and $i = n - j + 1$ if and only if n is even and $\{i, j\} = \{\frac{n}{2}, \frac{n}{2} + 1\} = \{r, r+1\}$.

We also have the following relations:

$$(K_{n-i+1} K_i^{-1}) B_j (K_{n-i+1} K_i^{-1})^{-1} = q^{(\alpha_{n-i+1} - \alpha_i, -\alpha_j)} B_j$$

for all i, j. It follows that $B_1, \ldots, B_{r-1}, B_{r+1}, \ldots, B_n$ generates an algebra isomorphic to $U_q(\mathfrak{sl}_{r-1})$ if n is odd and $B_1, \ldots, B_{r-1}, B_{r+2}, \cdots B_n$ generates an algebra isomorphic to $U_q(\mathfrak{sl}_{r-1})$ if n is even.

8.2. **Case 1: $n = 2$.** In this case, $\Gamma_\theta = \{\beta\}$ where $\beta = \alpha_1 + \alpha_2$. Note that $\alpha_\beta = \alpha_1$ and

$$(1 - q^2)^{-1}[(\text{ad } F_2 F_1) K_{-2\nu}] K_{-\beta + 2\nu} = (F_2 F_1 - q F_1 F_2)$$

where $\nu = \nu_1$ is the fundamental weight associated to α_1. Also, $\kappa(F_2 F_1 - q F_1 F_2)$ is a nonzero scalar multiple of $(E_1 E_2 - q E_2 E_1) K_1^{-1} K_2^{-1}$. Set

(8.1) $H = B_2 B_1 - q B_1 B_2 - (q - q^{-1})^{-1}[q^{-1} K - K^{-1}] - (1 + q)^{-1}$

where $K = K_1 K_2^{-1}$. A straightforward computation shows that

$$H = (F_2 F_1 - q F_1 F_2) + q^{-2}(E_1 E_2 - q E_2 E_1) K_1^{-1} K_2^{-1}$$
$$+ (1 + q)^{-1}(K_1^{-1} K_2^{-1} - 1) + (q^{-1} - q) F_1 E_1 K_2^{-1}$$

and hence H satisfies the conclusions of Theorem 7.2. The Cartan subalgebra for $\mathcal{B}_\theta$ in this case is $\mathcal{H} = \mathbb{C}(q)[K, K^{-1}][H]$. By (8.1), we can use $B_2 B_1 - q B_1 B_2$ instead of H as a generator for $\mathcal{H}$.

It follows from (3.3) of [AKR] that $CK^{1/3}$ is in the center of the simply connected version $\check{\mathcal{B}}_\theta$ (see Section 4.1) where

$$C = B_2 B_1 - q B_1 B_2 + \frac{1}{(q - q^{-1})} K^{-1} - \frac{(q + q^{-1})}{(q - q^{-1})} K$$
$$= H + \frac{q}{(q - q^{-1})} K + (1 + q)^{-1}.$$

Thus we have the following equivalent descriptions of $\mathcal{H}$ as a polynomial ring in one variable over the commutative ring $\mathbb{C}(q)[K, K^{-1}]$:

$$\mathcal{H} = \mathbb{C}(q)[K, K^{-1}][H] = \mathbb{C}(q)[K, K^{-1}][B_2 B_1 - q B_1 B_2] = \mathbb{C}(q)[K, K^{-1}][C].$$

Note that $\mathcal{B}_\theta$ admits a second quantum Cartan subalgebra that satisfies the conclusions of Theorem 7.2. This second subalgebra is constructed using the same maximum orthogonal system $\Gamma_\theta = \{\beta\}$, but instead of choosing $\alpha_\beta = \alpha_1$, we pick $\alpha_\beta = \alpha_2$. This second Cartan subalgebra is $\mathcal{H}' = \mathbb{C}(q)[K, K^{-1}][H']$ where H' takes the same form as H with B_1 and B_2 interchanged. In analogy to $\mathcal{H}$ above, we see that

$$\mathcal{H}' = \mathbb{C}(q)[K, K^{-1}][H'] = \mathbb{C}(q)[K, K^{-1}][B_1 B_2 - q B_2 B_1] = \mathbb{C}(q)[K, K^{-1}][C'],$$

where $C' = H' + (q + 2q^{-1})/(q + q^{-1}) K^{-1} + (1 + q)$ and $C' K^{-1/3}$ is a central element.

In [AKR], Aldenhoven, Koelink, and Roman analyze the finite-dimensional modules for $\mathcal{B}_\theta$ in this case. They use the algebra generated by $\mathcal{H}$ and $\mathcal{H}'$ together as the Cartan subalgebra for $\mathcal{B}_\theta$. Using the relationship between C, C' and the central elements $CK^{1/3}$, $C' K^{-1/3}$, it is straightforward to see that this larger algebra is commutative. Aldenhoven, Koelink, and Roman show that any finite-dimensional simple $\mathcal{B}_\theta$-module is generated by a vector v, referred to as a highest weight vector, that is, an eigenvector with respect to the action of K and satisfies $B_1 v = 0$. They characterize finite-dimensional simple modules based on the eigenvalue of the highest weight vector and the dimension of the module (see [AKR, Proposition 3.1]).

Given a finite-dimensional simple module for $\mathcal{B}_\theta$, they explicitly describe the action of this large Cartan subalgebra on a basis (see [AKR, Corollary 3.3]).

8.3. **Case 2:** $n = 3$. In this case $\Gamma_\theta = \{\beta_1, \beta_2\}$ where $\beta_1 = \alpha_2$ and $\beta_2 = \alpha_1 + \alpha_2 + \alpha_3$. The Cartan element associated to β_1 is simply $H_1 = B_2 = F_2 + E_2 K_2^{-1}$. Set $\beta = \beta_2$ and let $\nu = \nu_1 = (\alpha_3 + 2\alpha_2 + 3\alpha_1)/4$, the fundamental weight associated to α_1. Set

$$Y_{-\beta} = [F_3(F_2 F_1 - q F_1 F_2) - q(F_2 F_1 - q F_1 F_2) F_3]$$

and note that

$$Y_{-\beta} = [(\text{ad } F_3 F_2 F_1) K_{-2\nu_1}] K_{2\nu_1 - \beta}$$

up to a nonzero scalar multiple which we ignore. Set

$$X_\beta = [E_1(E_2 E_3 - q E_3 E_2) - q(E_2 E_3 - q E_3 E_2) E_1] K_{-\beta}.$$

Note that $\kappa(Y_{-\beta}) = X_\beta$ (up to multiplication by a nonzero scalar).
    Set

(8.2) $\qquad H_2 = [B_3(B_2 B_1 - q B_1 B_2) - q(B_2 B_1 - q B_1 B_2) B_3] + B_2 K_1 K_3^{-1}.$

A straightforward (lengthy) computation shows that

$$H_2 = Y_{-\beta} + X_\beta - (q - q^{-1}) F E_1 K_3^{-1} - q(q - q^{-1})(F_1 E K_3^{-1} K_2^{-1})$$

where $F = (F_2 F_1 - q F_1 F_2)$ and $E = (E_1 E_2 - q E_2 E_1)$. Thus H_2 satisfies the conclusions of Theorem 7.2. The Cartan subalgebra for $\mathcal{B}_\theta$ in this case is $\mathcal{H} = \mathbb{C}(q)[K, K^{-1}][H_1, H_2]$ where $K = K_1 K_3^{-1}$. Since $H_1 = B_2$ and H_2 commute with each other, we see that B_2 commutes with

(8.3) $\qquad (B_3(B_2 B_1 - q B_1 B_2) - q(B_2 B_1 - q B_1 B_2) B_3).$

Moreover, it follows from the explicit formulas for H_1 and H_2 that

$$\mathcal{H} = \mathbb{C}(q)[K, K^{-1}][B_2, (B_3(B_2 B_1 - q B_1 B_2) - q(B_2 B_1 - q B_1 B_2) B_3)].$$

8.4. **The general case.** We show that the general case for this family of examples is very similar to the low rank cases $n = 2, 3$. Given two elements A, B in $U_q(\mathfrak{g})$ and a nonzero scalar $a \in \mathbb{C}(q)$, we define the a-commutator $[A, B]_a$ by

$$[A, B]_a = AB - aBA.$$

Define elements H_j' in $\mathcal{B}_\theta$ by

$$H_j' = [B_{n-j+1}, [B_{n-j}, \cdots, [B_{j+1}, B_j]_q]_q \cdots]_q$$

for all $j = 1, \ldots, r$. Note that the elements H_j' satisfy the recursive relations

(8.4) $\qquad\qquad H_j' = [B_{n-j+1}, [H_{j+1}', B_j]_q]_q$

for $j = 1, \ldots, r - 1$. In Theorem 8.2 below, we prove that the Cartan subalgebra $\mathcal{H}$ of $\mathcal{B}_\theta$ defined by Theorem 7.2 based on the choice of Γ_θ of Theorem 2.7 satisfies

$$\mathcal{H} = \mathbb{C}(q)[\mathcal{T}_\theta][H_1', \ldots, H_r'].$$

For n even, the elements H_j' are the same as the h_j' defined by (5) of [Wa] (up to reordering of the subscripts). An immediate consequence of Theorem 8.2 is that the Cartan part in [Wa] is the same as the Cartan subalgebra presented here for the family of coideal subalgebras considered in [Wa].

Note that the lowest weight term of H'_j is

$$W_j = [F_{n-j+1}, [F_{n-j}, \cdots, [F_{j+1}, F_j]_q \cdots]_q = c[(\text{ad } F_{n-j+1} \cdots F_{j+1}) F_j K_j] K_{-\beta_j}$$

for some nonzero scalar c where $-\beta_j = -\alpha_j - \alpha_{j+1} - \cdots - \alpha_{n-j+1}$. In particular, the lowest weight term of H'_j agrees with the lowest weight term of the Cartan element of $\mathcal{H}$ associated to weight β_j as defined in Theorem 7.2.

Lemma 8.1. *The elements $H'_1, \ldots, H'_r$ pairwise commute with each other.*

Proof. Assume first that n is odd. Note that H'_k is in the subalgebra of $U_q(\mathfrak{g})$ generated by the set $\{B_i \mid i = k, k+1, \ldots, n-k+1\}$, We argue that B_i commutes with H'_j for all $i = j+1, \ldots, n-j$, and so H'_k commutes with H'_j for all $k > j$. This will prove the lemma when n is odd.

Note that $H'_r = B_r$ and

$$H'_{r-1} = (B_{r+1}(B_r B_{r-1} - q B_{r-1} B_r) - q(B_r B_{r-1} - q B_{r-1} B_r) B_{r+1}).$$

This is just (8.3) with $1, 2, 3$ replaced by $r-1, r, r+1$, respectively. In particular, the results of Section 8.3 ensure that $H'_r = B_r$ commutes with H'_{r-1}. Now assume that $j < r-1$ and B_i commutes with H'_{j+1} for all $i = j+2, \ldots, n-j-1$. Since B_i commutes with B_j and B_{n-j+1} for all $j+2 \le i \le n-j-1$, it follows from (8.4) that B_i also commutes with H'_j. Thus it is sufficient to show that B_{j+1} and B_{n-j} commutes with H'_j.

Let $\mathcal{A}$ be the algebra generated by $U_q(\mathfrak{sl}_{r-1})$ and a new element X subject only to the following additional relations:

- $F_i X - X F_i = E_i X - X E_i = 0$ for all $i \in \{1, \ldots, r-2\}$.
- $K_i X K_i^{-1} = X$ for all $i = 1, \ldots, r-1$.

Since $U_q(\mathfrak{sl}_{r-1})$ is a subalgebra of $\mathcal{A}$, $\mathcal{A}$ admits an $(\text{ad } U_q(\mathfrak{sl}_{r-1}))$ action. The above relations ensure that X has weight 0 with respect to this action and $(\text{ad } F_i)X = (\text{ad } E_i)X = 0$ for all $i = 1, \ldots, r-2$. Recall that $j \le r-2$. Set

$$V_1 = (\text{ad } E_{r-1} \cdots E_{j+1}) E_j,$$
$$V_2 = [(\text{ad } F_j \cdots F_{r-2}) F_{r-1} K_{r-1}] K_j^{-1} \cdots K_{r-1}^{-1},$$
$$V_3 = [V_2, [X, V_1]_{q^{-1}}]_{q^{-1}}.$$

Note that V_1 is the lowest weight vector in the $(\text{ad } U_{\{\alpha_{r-1}, \cdots, \alpha_{j+1}\}})$-module generated by E_j. Indeed, this follows from Lemma 3.1 with $\pi' = \{\alpha_{r-1}, \cdots, \alpha_{j+1}\}$ and $i = j$. Since $(\alpha_{j+1}, \alpha_{r-1} + \alpha_{r-2} + \cdots + \alpha_j) = 0$, V_1 is also a trivial vector with respect to the action of $(\text{ad } U_{\{\alpha_{j+1}\}})$. Hence $(\text{ad } E_{j+1}) V_1 = (\text{ad } F_{j+1}) V_1 = 0$. Similar arguments yield $(\text{ad } F_{j+1}) V_2 = (\text{ad } E_{j+1}) V_2 = 0$. It follows that $(\text{ad } F_{j+1}) V_3 = (\text{ad } E_{j+1}) V_3 = 0$. Since V_3 has weight 0 with respect to the action of $(\text{ad } U_q(\mathfrak{sl}_{r-1}))$, we have

(8.5) $[F_{j+1}, V_3] = 0 \quad \text{and} \quad [E_{j+1}, V_3] = 0.$

It follows from the relations for $\mathcal{B}_\theta$ that the map defined by

$$E_i \mapsto B_i, \quad F_i \mapsto B_{n-i+1}, \quad K_i \mapsto K_{n-i+1}^{-1} K_i,$$
$$K_{n-i+1} \mapsto K_{n-i+1} K_i^{-1}, \quad X \mapsto B_r, \quad q \mapsto q^{-1}$$

for $i = 1, \ldots, r-1$ defines a $\mathbb{C}$ algebra homomorphism. Moreover, this map sends V_3 to H'_j. Applying this homomorphism to (8.5) yields

$$[B_{n-j}, H'_j] = 0 \quad \text{and} \quad [B_{j+1}, H'_j] = 0,$$

as desired.

Now assume that n is even. We have $H'_r = [B_{r+1}, B_r]_q$ and
$$H'_{r-1} = [B_{r+2}, [B_{r+1}, [B_r, B_{r-1}]_q]_q]_q.$$
Note that both elements commute with all elements in $\mathcal{T}_\theta$. Using the relations of $\mathcal{B}_\theta$, one can show that Cartan element H_{r-1} associated to the root $\beta_{r-1} = \alpha_{r+2} + \alpha_{r+1} + \alpha_r + \alpha_{r-1}$ satisfies
$$(8.6) \qquad\qquad H_{r-1} = H'_{r-1} + uH'_r + v$$
for appropriate elements $u, v \in \mathbb{C}(q)[\mathcal{T}_\theta]$. This is very similar to the result for H_2 when $n = 3$ given in Section 8.3. Since B_r, B_{r+1} both commute with H_{r-1}, we must have $H'_r = [B_{r+1}, B_r]_q$ commutes with H_{r-1}. Hence the expression for H_{r-1} given in (8.6) ensures that $[B_{r+1}, B_r]_q$ commutes with H'_{r-1}. Arguing using induction as in the n is odd case, we see that B_i for all $i \in \{j+1, j+2, \ldots, r-1, r+2, \ldots, n-j\}$ and $[B_{r+1}, B_r]_q$ commute with H'_j for all j. Note that for all $k > j$, the element H'_k is in the subalgebra generated by B_i for $i \in \{j+1, j+2, \ldots, r-1, r+2, \ldots, n-j\}$ and $[B_{r+1}, B_r]_q$. Hence H'_k commutes with H'_j for all $k > j$ and the lemma follows. $\qquad\square$

The next result shows that $\mathcal{H}' = \mathcal{H}$, thus establishing a nice formula for a set of generators of $\mathcal{H}$ for this family of symmetric pairs.

Theorem 8.2. *Let $\mathfrak{g}, \mathfrak{g}^\theta$ be a symmetric pair of type AIII/AIV such that $r = \lfloor (n+1)/2 \rfloor$. Let $\Gamma_\theta = \{\beta_1, \ldots, \beta_r\}$ where $\beta_j = \alpha_j + \alpha_{j+1} + \cdots + \alpha_{n-j+1}$ and set $\alpha_{\beta_j} = \alpha_j$ for each j. Then*
$$(8.7) \qquad\qquad \mathcal{H} = \mathbb{C}[\mathcal{T}_\theta][H'_1, \ldots, H'_r],$$
where $\mathcal{H}$ is the quantum Cartan subalgebra as defined in Theorem 7.2 and
$$H'_j = [B_{n-j+1}, [B_{n-j}, \ldots, [B_{j+1}, B_j]_q \cdots]_q$$
for all $j = 1, \ldots, r$.

Proof. If n is odd, then $H_r = B_r = H'_r$ which is clearly in the right hand side of (8.7). If n is even, then by (8.1) of Section 8.2, we see that H_r is in the right hand side of (8.7).

Now consider $j < r$. Recall that the construction of the quantum Cartan element H_j in Theorem 7.2 is based on Proposition 6.7 which in turn uses Proposition 4.3. In the notation of Proposition 4.3, $H_j = \tilde{b}$ where $H'_j = b$. Thus, we can write
$$H_j = H'_j + \sum_{\mathrm{wt}(I) < \beta_j} B_I a_I$$
where each $a_I \in \mathcal{T}_\theta$. Moreover, by the discussion at the end of Section 4.3 and Lemma 4.1, we see that $H_j, H'_j, H_j - H'_j$ are all elements in the set
$$(8.8) \qquad\qquad \sum_{\lambda, \gamma, \gamma'} G^-_{-\beta_j + \lambda + \gamma} U^+_{\theta(-\lambda) - \gamma} K_{-\beta_j + 2\gamma'},$$
where the sum runs over λ, γ, γ' in $Q^+(\pi)$ such that $0 < \lambda \le \beta_j$, $\gamma \le \theta(-\lambda)$, and $\gamma' \le \gamma$. Set $\pi_j = \{\alpha_{j+1}, \ldots, \alpha_{n-j}\}$.

Let $\lambda + \gamma$ be minimal such that $H_j - H'_j$ admits a nonzero biweight summand of l-weight $-\beta_j + \lambda + \gamma$, say $guK_{-\beta_j + 2\gamma'} \in G^-_{-\beta_j + \lambda + \gamma} U^+_{\theta(-\lambda) - \gamma} K_{-\beta_j + 2\gamma'}$ where $g \in G^-$ and $u \in U^+$. As explained above, H_j and H'_j have the same lowest weight term W_j and so $\lambda + \gamma > 0$. Since $H_j - H'_j \in \mathcal{B}_\theta$, it follows from Remark 4.2 that $guK_{-\beta_j + 2\gamma'} \in U^- \mathcal{M}^+ \mathcal{T}_\theta$. Note that $\mathcal{M}^+$ is just the identity element

because $\pi_\theta = \emptyset$ and so $u = 1$. By Theorem 7.2, $\mathcal{P}(H_j) = W_j$. On the other hand, $\mathcal{P}(gK_{-\beta_j+2\gamma'}) = gK_{-\beta_j+2\gamma'}$ and so $gK_{-\beta_j+2\gamma'}$ does not appear as a biweight summand of H_j with respect to the expansion along the lines of (8.8). Hence $gK_{-\beta_j+2\gamma'}$ must appear as a biweight summand of H_j'.

Note that $\mathcal{M}^+$ is just the identity element because $\pi_\theta = \emptyset$ and so $u = 1$. Since

$$gK_{-\beta_j+2\gamma'} \in U_{-\beta+\lambda+\gamma}^- K_{-\beta_j+2\gamma'} = G_{-\beta_j+\lambda+\gamma}^- K_{2\gamma'-\lambda-\gamma},$$

we must have $2\gamma' - \lambda - \gamma \in Q(\pi)^\theta$. Also, the fact that H_j and H_j' commutes with all elements in $\mathcal{T}_\theta = \langle K_i K_{n-i+1}^{-1}| \ i = 1, \ldots, r \rangle$ ensures that $p(\lambda+\gamma) = \lambda+\gamma$. By the definition of β_j and π_j it follows that either $\beta_j - \lambda - \gamma \in Q^+(\pi_j)$ or $\lambda+\gamma \in Q^+(\pi_j)$. If $\lambda+\gamma \in Q^+(\pi_j)$, then it follows from Lemma 6.2 that $\lambda = \gamma = 0$ which contradicts the fact that $\lambda + \gamma > 0$. Hence $\beta_j - \lambda - \gamma \in Q^+(\pi_j)$ and $g \in U_{\pi_j}$.

Assume first that n is odd. By Theorem 7.2 and the proof of Lemma 8.1, both H_j and H_j' commute with all elements of $U_{\pi_j} \cap \mathcal{B}_\theta$. By Lemma 6.1, $gK_{-\beta_j+2\gamma'}$ is an (ad U_{π_j}) lowest weight vector. By Section 3.2, the restriction of $-\beta_j + 2\gamma'$ to π_j is -2ξ for some $\xi \in P^+(\pi_j)$ and $\beta_j - \lambda - \gamma$ restricts to $\xi - w_j\xi$ where $w_j = w(\pi_j)_0$. Since $\beta_j - \lambda - \gamma \in Q^+(\pi_j)$, these two weights are equal, namely $\beta_j - \lambda - \gamma = \xi - w_j\xi$. Hence $\beta_j - \lambda - \gamma \in Q^+(\pi_j) \cap P^+(\pi_j)$. The only weight less than β_j in $Q^+(\pi_j)$ which is also in $P^+(\pi_j)$ is β_{j+1}. Hence the weight of g is

$$(8.9) \qquad \xi - w_j\xi = \beta_j - \lambda - \gamma = \beta_{j+1} = \alpha_{j+1} + \cdots + \alpha_{n-j}.$$

It follows that $\lambda+\gamma = \alpha_j + \alpha_{n-j+1}$. Since $\gamma' \leq \gamma$, we must have $\gamma' \in \{0, \alpha_j, \alpha_{n-j+1}, \alpha_j + \alpha_{n-j+1}\}$. Using the fact that $\beta_j - 2\gamma'$ restricts to 2ξ and ξ satisfies (8.9), it is straightforward to check that $\gamma' = \alpha_j$ or $\gamma' = \alpha_{n-j+1}$.

As explained above, $gK_{-\beta_j+2\gamma'}$ appears as a biweight summand in the expansion of H_j' using (8.8). From the definition of H_j' and the fact that g has weight $-\beta_{j+1}$, $gK_{-\beta_j+2\gamma'}$ must appear in the weight vector expansion of the sum of the following two terms:

$$[E_j K_{n-j+1}^{-1}, [F_{n-j}, \ldots, [F_{j+1}, F_j]_q \cdots]_q$$

and

$$[F_{n-j+1}, [F_{n-j}, \ldots, [F_{j+1}, E_{n-j+1}K_j^{-1}]_q]_q \cdots]_q.$$

It follows that $g = [F_{n-j}, [F_{n-j-1}, \ldots, [F_{j+2}, F_{j+1}]_q \cdots]_q]_q K_{\beta_{j+1}}$ up to some nonzero scalar. Since $\gamma' = \alpha_j$ or α_{n-j+1}, we have

$$gK_{-\beta_j+2\gamma'} = gK_{-\beta_{j+1}}(K_j K_{n-j+1})^{-1}K_{2\gamma'}$$
$$= [F_{n-j}, [F_{n-j-1}, \ldots, [F_{j+2}, F_{j+1}]_q \cdots]_q]_q t,$$

where $t = K_j K_{n-j+1}^{-1}$ or $t = K_j^{-1}K_{n-j+1}$. Thus

$$H_j \in H_j' + H_{j+1}'t + \sum_{\mathrm{wt}(I)<\beta_{j+1}} B_I a_I.$$

Note that all three elements H_j, H_j', H_{j+1}' commute with all elements in $U_{\pi_{j+2}} \cap \mathcal{B}_\theta$ and all elements in $\mathcal{T}_\theta$. So, using a similar argument as above yields

$$(8.10) \qquad H_j = H_j' + H_{j+1}'t + H_{j+2}'s + \sum_{\mathrm{wt}(I)<\beta_{j+2}} B_I a_I$$

with $s \in \mathcal{T}_\theta$. Hence, repeated applications of this argument yield

$$H_j \in \mathbb{C}(q)[\mathcal{T}_\theta][H_j', H_{j+1}', \ldots, H_r'].$$

Since the coefficient of H_j' in H_j is 1, we get

(8.11) $$\mathbb{C}(q)[\mathcal{T}_\theta][H_j, H_{j+1}, \ldots, H_r] = \mathbb{C}(q)[\mathcal{T}_\theta][H_j', H_{j+1}', \ldots, H_r'].$$

This proves (8.7) when n is odd.

Now assume that n is even. Let V^- be the subalgebra of U^- generated by

$$F_1, \ldots, F_{r-1}, [F_{r+1}, F_r]_q, F_{r+2}, \ldots, F_n.$$

Note that V^- is isomorphic to $U_q^-(\mathfrak{sl}_{n-1})$ (the algebra generated by $F_1, \ldots, F_{n-1}$) via the map defined by

- $F_i \mapsto F_i$ for $i = 1, \ldots, r-1$,
- $[F_{r+1}, F_r]_q \mapsto F_r$,
- $F_i \mapsto F_{i-1}$ for $i = r+2, \ldots, n$.

Hence V^- inherits an adjoint action from $U_q^-(\mathfrak{sl}_{n-1})$. Now $H_j - H_j'$ commutes with all elements in $\mathcal{T}_\theta$ and all elements in the algebra generated by

$$B_{j+1}, \ldots, B_{r-1}, [B_{r+1}, B_r]_q, B_{r+2}, \ldots, B_{n-j}.$$

The theorem follows using an argument very similar to the odd n case. The key ingredient is an analysis of the lowest weight summand of $H_j - H_j'$ with respect to the adjoint action of V^-. This term must lie in $U^- \mathcal{T}_\theta$ and be invariant under the action of (ad $\mathcal{T}_\theta$). This forces the lowest weight summand to be in the algebra generated by V^- and $\mathcal{T}_\theta$. Arguing as above yields an analog of (8.10). Repeated applications of this type of argument yield (8.11) for n even which completes the proof of the theorem. □

Remark 8.3. There is a close relationship between the quantum Cartan subalgebra in this case and the center of $\mathcal{B}_\theta$. In particular, by [KoL], there exists z_j in $\mathcal{B}_\theta$ such that

- (i) $z_j \in \left[(\text{ad } U_{\pi_j})K_{-2\nu_j}\right]K_{2\nu_j - \beta_j}$,
- (ii) $z_j = [(\text{ad} F_{n-j+1} \cdots F_{j+1})F_j K_j]K_{-\beta_j} +$ higher weight terms,
- (iii) $z_j K_{\beta_j - 2\nu_j}$ is in the center of $(U_{\pi_j} \cap \mathcal{B}_\theta)\check{\mathcal{T}}_\theta$.

Note that (ii) implies that

$$z_j = H_j' + \sum_{\text{wt}(I) < \beta_j} B_I \mathcal{T}_\theta.$$

One can expand (ad $U_{\pi_j})K_{-2\nu_j}$ as a sum of weight spaces in a manner analogous to the expansion of the B_I in Lemma 4.1. (See for example [KoL, Section 1.5 (10)].) An inductive argument similar to the one used in the above theorem yields $\mathcal{H} = \mathbb{C}(q)[\mathcal{T}_\theta][z_1, \ldots, z_r]$.

Remark 8.4. In [Wa], Watanabe uses the braid group automorphisms of [KoP] in order to determine the action of the Cartan on highest weight generating vectors. A natural question is: Can we use these automorphisms to construct quantum Cartan subalgebras? We see that the answer is no for examples considered in this section where here we take the most obvious choice of braid group automorphism to pass from one Cartan element to another. In particular, consider the case where

$n = 3$ as in Section 8.3. Applying the braid group automorphism τ_1^- of [KoP] to the Cartan element $H_1 = B_2$ we get the element

$$\tau_1^{-1}(H_1) = (iq^{1/2})([B_1, [B_3, B_2]_q]_q - qB_2K_1K_3^{-1})$$

of $\mathcal{B}_\theta$ where here we have adjusted the outcome to reflect a slightly different choice of generators for $\mathcal{B}_\theta$. On the surface, this term looks similar to H_2 as defined in (8.2). However, $[B_1, [B_3, B_2]_q]_q$ is not a scalar multiple of $[B_3[B_2, B_1]_q]_q$ and so H_2 is not equal to a scalar multiple of $\tau_1^-(H_1)$. Moreover, $[B_1, [B_3, B_2]_q]_q$ does not commute with B_2 and so H_1 does not commute with $\tau_1^{-1}(H_1)$. Thus, in contrast to the Cartan subalgebras studied here, the subalgebra generated by $H_1, \tau_1^{-1}(H_1), K_1K_3^{-1}, K_3K_1^{-1}$ is not commutative.

Remark 8.5. Note that the set $\Gamma = \{\beta_1, \ldots, \beta_r\}$ where $r = \lfloor (n+1)/2 \rfloor$ and

$$\beta_j = \alpha_j + \alpha_{j+1} + \cdots + \alpha_{n-j+1}$$

for $j = 1, \ldots, r$ is also a strongly orthogonal θ-system for the symmetric pair $\mathfrak{g}, \mathfrak{g}^\theta$ of type AI. Choosing $\alpha_{\beta_j} = \alpha_j$ and $\alpha'_\beta = \alpha_{\beta_{n-j+1}}$ for each j, we see that Γ satisfies conditions Theorem 2.7 (i)–(iv) but does not fall into one of the five cases of this theorem. A natural question is: Can we still identify a quantum Cartan subalgebra of $\mathcal{B}_\theta$ defined by this strongly orthogonal θ-system? In the case when $n = 3$, a straightforward computation shows that $H'_{\beta_2} = B_2$ commutes with $H'_{\beta_1} = [B_3, [B_2, B_1]_q]_q$ where $B_i = F_i + E_iK_i^{-1}$ for each i. More generally, in analogy to the type AIII case, the algebra $\mathcal{H}' = \mathbb{C}(q)[H'_1, \ldots, H'_r]$ is a commutative polynomial ring where

$$H'_j = [B_{n-j+1}, [B_{n-j}, \cdots, [B_{j+1}, B_j]_q]_q \cdots]_q$$

for $j = 1, \ldots, r$. Furthermore, $\mathcal{H}'$ specializes to the enveloping algebra of a Cartan subalgebra of $\mathfrak{g}^\theta$ as q goes to 1. However, $\mathcal{H}$ is not κ_θ invariant. This is because $\kappa_\theta(B_j) = B_j$ for $j - 1, \ldots, n$ and so

$$\kappa_\theta(H'_j) = [B_j, [B_{j+1}, \cdots, [B_{n-j}, B_{n-j+1}]_q]_q \cdots]_q \neq H'_j,$$

and, moreover, $\kappa_\theta(H'_j) \notin \mathcal{H}'$ for all $j < (n+1)/2$. Hence Corollary 3.3 does not apply to $\mathcal{H}'$. Nevertheless, it is likely that one can use a twist of κ_θ obtained by composing it with an automorphism of $U_q(\mathfrak{g})$ induced by the nontrivial diagram automorphism for type A_n in order to establish the semisimplicity of unitary $\mathcal{B}_\theta$-modules with respect to the action of $\mathcal{H}'$. Thus $\mathcal{H}'$ could potentially serve as an alternative quantum Cartan subalgebra for $\mathcal{B}_\theta$ distinct from the quantum Cartan subalgebra in type AI defined by Theorems 2.7 and 7.2 (see Remark 7.4).

9. Appendix: Commonly used notation

Section 2: $\mathbb{C}, \mathbb{R}, \mathbb{N}, \mathfrak{g}, \theta, \mathfrak{g}^\theta$

Section 2.1: $\mathfrak{n}^-, \mathfrak{h}, \mathfrak{n}^+, \Delta, \Delta^+, h_i, e_\alpha, f_{-\alpha}, \pi = \{\alpha_1, \ldots, \alpha_n\}, \mathcal{W}, w_0, e_i, f_i, h_\alpha, (\ ,\ ),$ $\mathfrak{g}_{\pi'}, Q(\pi'), Q^+(\pi'), \Delta(\pi'), \mathcal{W}_{\pi'}, w(\pi')_0, P(\pi), P^+(\pi), P(\pi'), P^+(\pi'), \text{mult}_\alpha(\beta), \text{Supp}(\beta),$ $\text{ht}(\beta)$

Section 2.2: $\mathfrak{g}_\mathbb{R}, \mathfrak{h}_\mathbb{R}, \mathfrak{t}_\mathbb{R}, \mathfrak{p}_\mathbb{R}, \mathfrak{a}_\mathbb{R}, Q(\pi)^\theta, P(\pi)^\theta, \Delta_\theta, \text{Orth}(\beta), \text{StrOrth}(\beta), \mathfrak{h}_\theta, \mathfrak{a}$

Section 2.4: $\pi_\theta, p, \Gamma_\theta, \alpha_\beta, \alpha'_\beta$

Section 3.1: $U_q(\mathfrak{g}), E_i, F_i, K_i^{\pm 1}, \epsilon, U^+, U^-, \mathcal{T}, U^0, G^+, G^-, \hat{U}, V_\lambda,$ biweight, l-weight, $K_\xi, \check{\mathcal{T}}, \check{U}, M(\lambda), L(\lambda), U_{\pi'}, L_{\pi'}(\lambda), S^{U_{\pi'}}$

Section 3.2: $\text{ad}_\lambda, G^-(\lambda), \delta M(\lambda/2), \deg_\mathcal{F}, \tilde{\lambda}, F(\check{U}), F_{\pi'}(\check{U})$

Section 3.3: T_i, T_w, σ

Section 3.4: κ

Section 4.1: $\mathcal{M}$, θ_q, $\mathcal{T}_\theta$, $\mathcal{B}_\theta$, B_i ($\alpha_i \notin \pi_\theta$), $\mathbf{c} = (\mathbf{c_1}, \dots, \mathbf{c_n})$, $\mathbf{s} = (\mathbf{s_1}, \dots, \mathbf{s_n})$, $\mathcal{S}$, κ_θ, $\check{\mathcal{T}}_\theta$, $\check{\mathcal{B}}_\theta$

Section 4.2: $\mathcal{M}^+$, B_i ($\alpha_i \in \pi_\theta$), F_J , B_J, $\mathrm{wt}(j_1, \dots, j_m)$

Section 4.3: $\mathcal{P}$

Section 5.1: $\bar{L}(\lambda)$, $\bar{L}_{\pi'}(\lambda)$, $[\bar{L}(\lambda) : \bar{L}_{\pi'}(\gamma)]$, ν_α

Section 5.2: $[L(\lambda) : L_{\pi'}(\gamma)]$

Section 5.3: φ, φ'

Section 6.1: $\mathcal{N}(D : C)$, $[b, a]$

Section 6.2: $\mathrm{ht}_\tau(\beta)$

Section 6.4: $G^-_{(\tau, m)}$, $U^+_{(\tau, m)}$

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