THE TRIGONOMETRY OF ESCHER'S WOODCUT "CIRCLE LIMIT III"

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ABSTRACT. In M. C. Escher's circular woodcuts, replicas of a fish (or cross, or angel, or devil), diminishing in size as they recede from the centre, fit together so as to fill and cover a disc. *Circle Limits I, II,* and *IV* are based on Poincaré's circular model of the hyperbolic plane, whose lines appear as arcs of circles orthogonal to the circular boundary (representing the points at infinity). Suitable sets of such arcs decompose the disc into a theoretically infinite number of similar "triangles," representing congruent triangles filling the hyperbolic plane. Escher replaced these triangles by recognizable shapes. *Circle Limit III* is likewise based on circular arcs, but in this case, instead of being orthogonal to the boundary circle, they meet it at equal angles of almost precisely 80°. (Instead of a straight line of the hyperbolic plane, each arc represents one of the two branches of an "equidistant curve.") Consequently, his construction required an even more impressive display of his intuitive feeling for geometric perfection. The present article analyzes the structure, using the elements of trigonometry and the arithmetic of the biquadratic field $\mathbb{Q}(\sqrt{2}+\sqrt{3})$: subjects of which he steadfastly claimed to be entirely ignorant.

Introduction

Concerning his four Circle Limit woodcuts, M. C. Escher wrote:

> *Circle Limit I*, being a first attempt, displays all sorts of shortcomings . . . and leaves much to be desired . . . . There is no continuity, no "traffic flow" nor unity of colour in each row. . . . In the coloured woodcut *Circle Limit III*, the shortcomings of *Circle Limit I* are largely eliminated. We now have none but "through traffic" series, and all the fish belonging to one series have the same colour and swim after each other head to tail along a circular route from edge to edge. . . . Four colours are needed so that each row can be in complete contrast to its surroundings. As all these strings of fish shoot up like rockets . . . from the boundary and fall back again whence they came, not a single component reaches the edge. For beyond that there is "absolute nothingness." And yet this round world cannot exist without the emptiness around it . . . because it is out there in the "nothingness" that the centre points of the arcs that go to build up the framework are fixed with such geometric exactitude. ([2], p. 109)

The purpose of the present article is to demonstrate this "geometric exactitude" (see Fig. 1) by finding the radii and centres of the first three sets of four congruent circles that trace the backs of the "strings of fish." I naturally assume that the relevant arcs of these circles cross one another at equal angles of 60°, decompose the interior of the "boundary" into alternate triangular and quadrangular regions,

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and all cut the boundary at the same pair of supplementary angles

$$\omega,\ \pi - \omega.$$

The acute angle ω appears on the side of each arc where the regions are quadrangular.

An earlier article ([1], p. 24) used hyperbolic trigonometry to prove that

$$\begin{aligned}\cos\omega &= \sinh\left(\frac{1}{4}\log 2\right)\\ &\approx \sinh\ 0.1732868 \approx 0.1741553.\end{aligned}$$

Since $\cos(79°58') \approx 0.17424$, ω scarcely differs from the value 80° which can easily be measured in Escher's woodcut. Here I obtain this expression for ω by a more elementary procedure.

The Angle ω at the Boundary

Figure 2 is a sketch of the middle part of Escher's "framework," showing the centres O_ν, at distances

$$d_\nu = AO_\nu$$

from the centre A of the boundary circle, of radius 1, and showing the radii

$$r_\nu = O_\nu X_\nu.$$

From the triangle X_1AO_1, whose angle ω at X_1 is opposite to the side $AO_1 = d_1$, as in Figure 3, we have

$$d_1^2 = 1 + r_1^2 - xr_1, \tag{1}$$

where

$$x = 2\cos\omega. \tag{2}$$

Similarly, the triangle X_2AO_2, whose angle $\pi - \omega$ at X_2 is opposite to d_2, yields

$$d_2^2 = 1 + r_2^2 + xr_2. \tag{3}$$

Because the angle between two intersecting circles equals the angle between their radii to a common point, the triangle O_1AC has angles $2\pi/3$, $\pi/4$, and $\pi/12$ opposite to sides

$$AO_1 = d_1,\ CO_1 = r_1,\ CA = d_2 - r_2,$$

respectively, as in Figure 4. Hence, we have

$$\frac{d_1}{\sin(2\pi/3)} = \frac{r_1}{\sin(\pi/4)} = \frac{d_2 - r_2}{\sin(\pi/12)},$$

that is,

$$\frac{d_1}{\sqrt{3}} = \frac{r_1}{\sqrt{2}} = \frac{d_2 - r_2}{(\sqrt{3} - 1)/\sqrt{2}}. \tag{4.1}$$

The similar triangle O_2AB, with angles $2\pi/3$ and $\pi/4$ opposite to sides

$$AO_2 = d_2 \text{ and } BO_2 = r_2,$$

respectively, yields

$$\frac{d_2}{\sqrt{3}} = \frac{r_2}{\sqrt{2}} = \frac{AB}{(\sqrt{3} - 1)/\sqrt{2}}. \tag{4.2}$$

Thus,

$$d_\nu^2 = \frac{3}{2} r_\nu^2 \ (\nu = 1 \text{ or } 2) \tag{5}$$

and expressions (1) and (3) for d_ν^2 yield quadratic equations for r_ν:

$$r_1^2 + 2xr_1 - 2 = 0, \ r_2^2 - 2xr_2 - 2 = 0.$$

Solving these equations for the positive numbers r_ν, we find

$$r_1 = -x + \sqrt{x^2 + 2}, \ r_2 = x + \sqrt{x^2 + 2}. \tag{6}$$

From (4.1) we have

$$(\sqrt{3} - 1)r_1 = 2(d_2 - r_2) = (\sqrt{6} - 2)r_2,$$

and from (6),

$$r_1 r_2 = 2.$$

It follows that

$$\begin{aligned} r_1^2 &= (\sqrt{6} - 2)(\sqrt{3} + 1), \\ r_2^2 &= (\sqrt{6} + 2)(\sqrt{3} - 1), \\ 4x^2 &= (r_2 - r_1)^2 = r_1^2 + r_2^2 - 2r_1r_2 \\ &= 2\sqrt{2}(\sqrt{2} - 1)^2 \end{aligned} \tag{7}$$

and

$$\begin{aligned} x = 2^{-1/4}(2^{1/2} - 1) &= 2^{1/4} - 2^{-1/4} \\ &= 2\sinh(\frac{1}{4}\log 2). \end{aligned}$$

<u>The First Two Circles</u>

Since $\sqrt{x^2 + 2} = \sqrt{2^{1/2} + 2^{-1/2}} = 2^{-1/4}\sqrt{3}$, (6) yields

$$\begin{aligned} r_1 &= 2^{-1/4}(1 - \sqrt{2} + \sqrt{3}) \approx 1.1081646, \\ r_2 &= 2^{-1/4}(\sqrt{2} - 1 + \sqrt{3}) \approx 1.8047860, \end{aligned} \tag{8}$$

and, from (5),

$$\begin{aligned} d_1 &= 2^{-3/4}(\sqrt{3} - \sqrt{6} + 3) \approx 1.3572189, \\ d_2 &= 2^{-3/4}(\sqrt{6} - \sqrt{3} + 3) \approx 2.2104024. \end{aligned}$$

From (4.2) we have

$$\begin{aligned} AB &= \frac{1}{2}(\sqrt{3} - 1)r_2 \\ &= 2^{-3/4}(-1 + 2\sqrt{2} + \sqrt{3} - \sqrt{6}) \approx 0.6605975. \end{aligned}$$

<u>The Biquadratic Field $\mathbb{Q}(\sqrt{2} + \sqrt{3})$</u>

The numbers $(a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6})/q$, where a, b, c, and d are integers and q is a positive integer, are easily seen to constitute a *field* ([3], p. 230). This field is called $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ because it can be expressed as the set of all rational functions of the special number $\theta = \sqrt{2} + \sqrt{3}$, in terms of which

$$\sqrt{2} = \frac{1}{2}(\theta - \theta^{-1}),\ \sqrt{3} = \frac{1}{2}(\theta + \theta^{-1}),\ \sqrt{6} = \frac{1}{2}(\theta^2 - 5).$$

In this field, θ is called an *integer* because it satisfies a monic equation, namely

$$\theta^4 - 10\theta^2 + 1 = 0.$$

When we assert that "factorization is unique," we disregard, as factors, the *units*, which are divisors of 1; for if $st = 1$, any number

$$n = nst$$

has the trivial factorization $ns \times t$.

Computing (7) and (8), we obtain the apparently surprising identity

$$(1 - \sqrt{2} + \sqrt{3})^2 = 2(\sqrt{3} - \sqrt{2})\,(\sqrt{3} + 1).$$

This "factorization" loses its element of surprise when we face the obvious fact that $\sqrt{3} - \sqrt{2}$ is a unit:

$$(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = 1.$$

<u>The Third Circle</u>

Looking again at Figure 2, we see that

$$d_3^2 = 1 + r_3^2 - xr_3$$

and, since the third circle passes through B,

$$d_3 - r_3 = AB.$$

Thus,

$$\begin{aligned}
d_3 + r_3 &= \frac{1 - xr_3}{AB},\ 2r_3 = \frac{1 - xr_3}{AB} - AB,\\
r_3 &= \frac{(1/AB) - AB}{2 + x/AB} = \frac{1 - AB^2}{2AB + x}\\
&= \frac{1 - 2^{-3/2}(18 - 10\sqrt{2} - 10\sqrt{3} + 6\sqrt{6})}{2^{1/4}(-1 + 2\sqrt{2} + \sqrt{3} - \sqrt{6}) + 2^{-1/4}(\sqrt{2} - 1)}\\
&= \frac{-9 + 6\sqrt{2} + 5\sqrt{3} - 3\sqrt{6}}{2^{1/4}(3 - 2\sqrt{3} + \sqrt{6})}\\
&= 2^{-1/4}\frac{5 - 3\sqrt{2} - 3\sqrt{3} + 2\sqrt{6}}{-2 + \sqrt{2} + \sqrt{3}}\\
&= 2^{-1/4}\left(\frac{-1 + 2\sqrt{6}}{-2 + \sqrt{2} + \sqrt{3}} - 3\right)
\end{aligned}$$

$$= 2^{-1/4}\left(\frac{(-1 + 2\sqrt{6})(-2 + 5\sqrt{2} + 3\sqrt{3} + 4\sqrt{6})}{(-2 + \sqrt{2} + \sqrt{3})(-2 + 5\sqrt{2} + 3\sqrt{3} + 4\sqrt{6})} - 3\right)$$

$$
\begin{aligned}
&= \quad 2^{-1/4}\left(\frac{50 \;+\; 13\sqrt{2} \;+\; 17\sqrt{3} \;-\; 8\sqrt{6}}{23} \;-\; 3\right) \\
&= \quad 2^{-1/4}\left(\frac{-19 \;+\; 13\sqrt{2} \;+\; 17\sqrt{3} \;-\; 8\sqrt{6}}{23}\right) \\
&\approx \quad 0.3375915
\end{aligned}
$$

and

$$
\begin{aligned}
d_3 &= \; r_3 \;+\; AB \\
&= 2^{-3/4}\frac{3 \;+\; 27\sqrt{2} \;+\; 7\sqrt{3} \;-\; 6\sqrt{6}}{23} \;\approx\; 0.998189.
\end{aligned}
$$

Since Escher's bounding circle has diameter 41 cm, our results

$$
\begin{aligned}
r_1 \;&\approx\; 1.10816,\; d_1 \;\approx\; 1.3572, \\
r_2 \;&\approx\; 1.8048,\; d_2 \;\approx\; 2.2104, \\
r_3 \;&\approx\; 0.3376,\; d_3 \;\approx\; 0.9982
\end{aligned}
$$

should be multiplied by 20.5 to obtain the distances in centimetres:

$$
\begin{array}{ll}
22.7, & 27.8, \\
37.0, & 45.3, \\
6.92, & 20.46.
\end{array}
$$

These distances agree perfectly with actual measurements in the woodcut itself.

Acknowledgments

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References

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2. B. Ernst, *The Magic Mirror of M. C. Escher*, New York: Random House (1976).
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