

NONNEGATIVE SOLUTIONS OF THE INITIAL-DIRICHLET PROBLEM FOR GENERALIZED POROUS MEDIUM EQUATIONS IN CYLINDERS

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In this paper we study nonnegative solutions u of a class of nonlinear evolutions $\partial u/\partial t = \Delta\varphi(u)$, whose best known example is the porous medium equation $\varphi(u) = u^m$, $m > 1$ (see [9] for a recent survey on these equations). We study our solutions u , subject to Dirichlet boundary conditions in $D \times (0, T)$, where D is a bounded smooth domain in \mathbf{R}^n , i.e., $u|_{\partial D \times (0, T)} \equiv 0$. The corresponding theory for $D = \mathbf{R}^n$ was studied by the authors in [3]. We find that there is one special solution $\beta(x, t)$ (the 'friendly giant') which tends to infinity as $t \downarrow 0$, and that all other solutions $u(x, t)$ are in one-to-one correspondence with suitable pairs of measures μ on D and λ on ∂D . We also show that any solution on $D \times (0, T)$ extends to a solution in $D \times (0, \infty)$, and that the friendly giant $\beta(x, t)$ governs the asymptotic behavior as $t \uparrow +\infty$. The nonlinearity is assumed to be continuous, increasing, with $\varphi(0) = 0$, and also verifies the growth conditions

$$0 < a \leq \frac{u\varphi'(u)}{\varphi(u)} \leq \frac{1}{a}, \quad 0 < u < +\infty,$$

$$1 + a \leq \frac{u\varphi'(u)}{\varphi(u)}, \quad \text{for } u \geq u_0$$

for some constant a , $0 < a < 1$, and some $u_0 > 0$. We will denote by Γ_a the class of φ 's which verify the above conditions, together with the normalization conditions $u_0 = 1$, $\varphi(1) = 1$.

We say that u is a strong solution of the initial Dirichlet problem (IDP) for the equation $\partial u/\partial t = \Delta\varphi(u)$ in $D \times (0, \infty)$ if u is a continuous, nonnegative function in $\overline{D} \times (0, \infty)$, $u \equiv 0$ on $\partial D \times (0, \infty)$, and for all smooth functions ψ on $\overline{D} \times [0, \infty)$, which vanish on $\partial D \times [\tau_1, \tau_2]$, $\tau_1 > 0$, we have

$$\iint_{D \times [\tau_1, \tau_2]} \left[\varphi(u)\Delta\psi + u \frac{\partial\psi}{\partial t} \right] dx dt$$

$$= \int_D u(x, \tau_2)\psi(x, \tau_2) dx - \int_D u(x, \tau_1)\psi(x, \tau_1) dx.$$

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Note that our definition of strong solution is not the usual one, since we do not require the existence of derivatives in a sense stronger than the distributional one. Our solutions are often referred to in the literature as continuous weak solutions. We are using this definition to emphasize the distinction from mere weak solutions, which we will discuss in future works.

The results of this paper give a complete description of all strong solutions of the (IDP) for $\varphi \in \Gamma_a$. It turns out that there is an exceptional strong solution (the ‘friendly giant’) β , which is ‘identically $+\infty$ at $t = 0$ ’. This β is the supremum of all strong solutions (see Lemma 4). (In the case of $\varphi(u) = u^m$, the existence of β is easy and well known; for φ ’s verifying $\varphi'' \geq 0$, $\int_a^\infty du/\varphi(u) < \infty$, P. Sacks [12] has independently established the existence of β .) Moreover, β is the only solution which is identically $+\infty$ at $t = 0$ (Lemma 5). All other strong solutions have a uniform growth as $t \downarrow 0$. In fact (Lemma 6) if u is a strong solution of the (IDP), $u \neq \beta$ if and only if

$$\sup_{t>0} \int_D u(x, t)\delta(x) dx < +\infty, \quad \text{where } \delta(x) = \text{dist}(x, \partial D).$$

One can then conclude (Theorem 7) that if u is a strong solution of the (IDP), $u \neq \beta$, u has a trace as $t \downarrow 0$, i.e., there are two measures μ on D and λ on ∂D , with

$$\int_D \delta(x) d\mu(x) < \infty, \quad \int_{\partial D} d\lambda < +\infty,$$

such that if $\eta \in C^\infty(\mathbf{R}^n)$, $\eta|_{\partial D} \equiv 0$,

$$\lim_{t \downarrow 0} \int_D u(x, t)\eta(x) dx = \int_D \eta d\mu + \int_{\partial D} \frac{\partial \eta}{\partial N} d\lambda,$$

where $\partial/\partial N$ is differentiation in the direction of the outward normal N . Moreover (Theorem 8), the trace (μ, λ) determines the solution uniquely. Also (Theorem 9), this is the correct class of traces, i.e., if (μ, λ) verifies the bounds above, there exists a (unique) strong solution u with (μ, λ) as trace.

Notice that the growth condition $\sup_{t>0} \int_D u(x, t)\delta(x) dx < \infty$ is independent of the nonlinearity φ , in contrast with the growth conditions in the case of $\mathbf{R}^n \times (0, \infty)$ (see [3]). Moreover, note that the trace of u as $t \downarrow 0$ includes a piece λ on ∂D . The existence and uniqueness results show that the λ ’s are necessary.

Finally, we show that strong solutions of the (IDP) in $D \times (t_1, t_2)$ actually extend to be strong solutions in $D \times (t_1, +\infty)$. We also show that under some additional assumptions on φ ($1 + a \leq u\varphi'(u)/\varphi(u)$ for all $u > 0$, and $\lim_{h \downarrow 0} (\varphi(hu)/\varphi(h))$ exists for each u) the ‘friendly giant’ β gives the asymptotic behavior of all strong solutions as $t \uparrow \infty$, i.e.,

$$\lim_{t \uparrow +\infty} \frac{u(x, t)}{\beta(x, t)} = 1$$

for all strong solutions $u \neq 0$. (See also [14] for the case of the porous medium equation.)

One can also define a concept of weak solution of the initial Dirichlet problem for $\partial u/\partial t = \Delta\varphi(u)$. We say that u is a weak solution of the initial Dirichlet problem for $\partial u/\partial t = \Delta\varphi(u)$ if $u, \varphi(u) \in L^1(D \times I)$ for all intervals I compactly contained in $(0, \infty)$, $u \geq 0$ for a.e. (x, t) , and for all $\psi \in C^\infty(\mathbf{R}^n \times \mathbf{R})$, with $\eta(x, t) = 0$ for $t \notin I$, $\eta(x, t) \equiv 0$ for $x \in \partial D$, we have

$$\iint \left[\frac{\partial \psi}{\partial t} u(x, t) + \Delta \psi \varphi(u)(x, t) \right] dx dt = 0.$$

In a forthcoming paper we will show that, when φ is also convex, every weak solution of the initial Dirichlet problem coincides a.e. with a strong solution of the initial Dirichlet problem.

It is easy to see that all of our results (with the exception of the existence of the 'friendly giant' β) also hold in the case when $\varphi(u) = u$, i.e., when we are dealing with the ordinary heat equation. Even in this case, our results seem to be new. (However, some work has been done previously for the heat equation in one space dimension, i.e., when $D \subset \mathbf{R}^1$, see Hartman and Wintner [6], and Fulks [4, 5]).

It is also of interest to study weak and strong solutions of the initial Dirichlet problem when $\varphi(u) = u^m$, $m < 1$. In a forthcoming paper we will show that when $(n-2)/n < m < 1$ we have a complete description of all strong solutions of the initial Dirichlet problem. The description is the same as the one given in this paper for the case $m > 1$, except that the 'friendly giant' β does not exist. We will also show that when $(n-2)/n < m < 1$ all weak solutions of the initial Dirichlet problem are strong solutions. It is known (see Ni and Sacks [8]) that for $(n-2)/(n+2) < m \leq (n-2)/n$ there are weak solutions of the initial Dirichlet problem which are not strong solutions. We suspect that this is true for $0 < m \leq (n-2)/n$. Before passing to the body of the paper, we would like to make a comment on our methods. The key idea is to study $w(x, t) = Gu(x, t)$, the Green's potential of u . This idea originates in the work of M. Pierre [10], who established a maximum principle for the w 's (Lemma 3), which is crucial to our work.

The following compactness principle for strong solutions of the initial Dirichlet problem will be used frequently in the sequel.

Lemma 1. *Let $\{u_j\}_{j=1}^\infty$ be a sequence of strong solutions of the (IDP) in $D \times (0, \infty)$. Suppose that given $t > 0$, there exists a constant $M_t > 0$ such that $\sup_{x \in D} u_j(x, t) \leq M_t$. Then, there exists a subsequence $\{v_j\}$ and a strong solution u of the (IDP) in $D \times (0, \infty)$ such that $\{u_{v_j}\}$ converges uniformly to u in $\bar{D} \times [s, \tau]$ for all $0 < s < \tau < +\infty$.*

Proof. By the maximum principle (see for example Lemma 2.3 in [3]), we have that

$$\sup_{x \in D \times [s/2, 2\tau]} u_j(x, t) \leq M_{s/2}.$$

We then invoke Corollary 2.10 of [3] and Theorem 1.1 of [11] to obtain the equicontinuity of $\{u_j(x, t)\}$ on $K \times [s, \tau]$, where $K \subset \subset D$. To obtain equicontinuity on $\bar{D} \times [s, \tau]$, one needs to observe that a slight modification of Sacks' proof of Theorem 1.1 of [11] gives a modulus of continuity up to the lateral part of the boundary, when the lateral values are 0.

Lemma 2. *Let u_1, u_2 be strong solutions of the (IDP) in $D \times (0, \infty)$. Let $\eta \geq 0$, $\eta \in C_0^\infty(D)$, and let $T < +\infty$, $\{\tau_j\} \downarrow 0$, $\tau_0 = T$. Then, there exist nonnegative measures $\{\lambda_j\}$ on D , with $\int_D d\lambda_j \leq \int_D \eta dx$, and*

$$\int_D [w_1(x, T) - w_2(x, T)]\eta(x) dx = \int_D [w_1(x, \tau_j) - w_2(x, \tau_j)]d\lambda_j(x),$$

where $w_i = Gu_i$, i.e., the Green's potential of u_i (the solution of $\Delta Gu_i = -u_i$, $Gu_i|_{\partial D} \equiv 0$).

This lemma originates in the work of M. Pierre [10], where it is proved for the case when φ is Lipschitz near the origin. One can also show that $0 \leq G\lambda_{j+1} \leq G\lambda_j \leq G\eta$, although we will not need this fact here. (This is needed for unbounded domains, see [3] or [10].) The proof for our class of nonlinearities φ is contained in the proof of Lemma 2.12 of [3].

Let u be a strong solution of the (IDP), and let $w = Gu$. It is easy to see that $\partial w/\partial t = -\varphi(u) \leq 0$ (see for example (2.25) in [3]), and hence, $\lim_{t \downarrow 0} w(x, t)$ exists for each $x \in D$. Lemma 2 is used in order to establish the following maximum principle.

Lemma 3 (Pierre's maximum principle). *Let u_1, u_2 be two strong solutions of the (IDP) in $D \times (0, \infty)$. Suppose that $u_2 \in C(\bar{D} \times [0, \infty))$ and that $w_i = Gu_i$ verify*

$$\lim_{t \downarrow 0} w_1(x, t) \geq \lim_{t \downarrow 0} w_2(x, t).$$

Then $w_1(x, t) \geq w_2(x, t)$.

Proof. The argument is due to M. Pierre [10]. Let $\eta(x) \in C_0^\infty(D)$, $\eta \geq 0$, and let $\{\lambda_j\}$ be measures as in Lemma 2. By extracting a subsequence (which we still call $\{\lambda_j\}$), we can assume that there exists a finite measure λ_∞ such that $\lim_{j \rightarrow \infty} \int_D \theta d\lambda_j = \int \theta d\lambda_\infty$ for all continuous functions θ on \bar{D} , that vanish on ∂D . Let

$$E = \int_D [w_1(x, T) - w_2(x, T)]\eta(x) dx = \int_D w_1(x, \tau_j) d\lambda_j - \int_D w_2(x, \tau_j) d\lambda_j.$$

If $j \geq k$, since $\partial w_1/\partial t \leq 0$, we have

$$E \geq \int_D w_1(x, \tau_k) d\lambda_j - \int_D w_2(x, \tau_j) d\lambda_j.$$

Since $w_1(x, \tau_k)$ belongs to $C(\bar{D})$ and is 0 on ∂D ,

$$\lim_{j \rightarrow \infty} \int_D w_1(x, \tau_k) d\lambda_j = \int_D w_1(x, \tau_k) d\lambda_\infty.$$

Since u_2 , and hence w_2 , are continuous up to $t = 0$, if $h_2(x) = \lim_{t \downarrow 0} w_2(x, t)$, then

$$\lim_{j \rightarrow \infty} \int_D w_2(x, \tau_j) d\lambda_j = \int_D h_2(x) d\lambda_\infty.$$

Thus,

$$E \geq \int_D w_1(x, \tau_k) d\lambda_\infty - \int_D h_1(x) d\lambda_\infty,$$

and thus, by monotone convergence

$$E \geq \int_D \left[\lim_{t \downarrow 0} w_1(x, t) - h_2(x) \right] d\lambda_\infty.$$

Hence, $E \geq 0$ and the lemma follows.

We can now proceed to study all strong solutions of the (IDP) in $D \times (0, \infty)$. We start out by studying an exceptional solution.

Lemma 4 (Existence of the ‘friendly giant’). *For $f \in C(\bar{D})$, $f|_{\partial D} = 0$, $f \geq 0$, let u_f be the unique strong solution of the initial Dirichlet problem $\partial u/\partial t = \Delta\phi(u)$ in $D \times (0, \infty)$, $u \equiv 0$ on $\partial D \times (0, \infty)$, $u(x, 0) = f(x)$ for $x \in D$. (See [1] for example, for the existence and uniqueness of u_f .) Let $\beta(x, t) = \sup u_f(x, t)$, where the supremum is taken over all such f ’s. Then, $\beta(x, t)$ is a strong solution of the (IDP) in $D \times (0, \infty)$.*

Proof. For $\varphi \in \Gamma_a$, let γ be its inverse function. Put $\psi(z) = \int_0^z \gamma(s) ds$. Since $\varphi(s) \geq s^{1+\alpha_0}$ for $s \geq 1$ and some $\alpha_0 > 0$, we have $\psi(z) \leq Cz^{1+\alpha_1}$, $z \geq 1$, for some $\alpha_1 \in (0, 1)$; also note for future reference that $\gamma(s) \leq s^\theta$, $\theta \in (0, 1)$, $s \geq 1$. For Ω a bounded smooth domain, we define $z = z_\Omega$ as the solution of

$$(*_\Omega) \quad \sup \left\{ \int_\Omega \psi(z), z \in H_0^1(\Omega), \int_\Omega |\nabla z|^2 = 1 \right\}.$$

Here $H_0^1(\Omega) = \text{cl } C_0^\infty(\Omega)$ under the norm $\int_\Omega |\nabla h|^2$. $(*_\Omega)$ has a solution since Ω is a bounded smooth domain, $\psi(z) \leq cz^2$, and $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$. Moreover (see [2]) z is smooth, vanishes on $\partial\Omega$, and $z > 0$ in Ω . Also z solves the equation $\Delta z = -\mu\gamma(z)$ in Ω for some $\mu \in \mathbf{R}$. Since $\int_\Omega \Delta z < 0$, it follows that $\mu > 0$.

We now define $u(x, t)$ implicitly by $\varphi(u(x, t)) = z(x)/t^\alpha$, where we have chosen Ω so that $\bar{D} \subset \Omega$. Let $m = \inf_\Omega z > 0$. We claim that we can choose $\alpha > 0$, and $T_0 > 0$ so that u is a supersolution of $\partial u/\partial t = \Delta\phi(u)$ in $D \times (0, T_0)$. We compute

$$\Delta\phi(u) - \frac{\partial u}{\partial t} = -\frac{\mu\gamma(z)}{t^\alpha} + \frac{\alpha z}{t^{\alpha+1}} \gamma' \left(\frac{z}{t^\alpha} \right).$$

Choose T_1 so small that for $0 < t < T_1$, $z/t^\alpha \geq 1$ for $(x, t) \in D \times (0, T_1)$, and consider from now on $0 < t < T_1$. Since $\gamma'(s) \leq C\gamma(s)/s$, $s > 0$, we have

$$\begin{aligned} \Delta\varphi(u) - \frac{\partial u}{\partial t} &\leq -\frac{\mu\gamma(z)}{t^\alpha} + C\frac{\alpha z}{t^{\alpha+1}}\frac{\gamma(z/t^\alpha)}{(z/t^\alpha)} = -\frac{\mu\gamma(z)}{t^\alpha} + \frac{C\alpha}{t}\gamma(z/t^\alpha) \\ &\leq -\frac{c_1}{t^\alpha} + \frac{c_2\alpha}{t^{1+\theta\alpha}}, \quad 0 < \theta < 1. \end{aligned}$$

This shows that we can choose α and T_0 so that $\Delta\varphi(u) - \partial u/\partial t \leq 0$ in $D \times (0, T_0)$.

Let now $f \in C(\overline{D})$, $f|_{\partial D} \equiv 0$, $f \geq 0$, and let u_f be as before. By the maximum principle (see Lemma 2.3 in [3]) and the above supersolution, we see that, for all such f 's

$$u_f(x, t) \leq C/t^\alpha, \quad 0 < t < T_0,$$

where C is independent of f . The maximum principle also shows that

$$u_f(x, t) \leq C_1 \quad \text{for } t \geq T_0.$$

Now pick $K_n \subset K_{n+1} \subset\subset D$, $\bigcup K_n = D$, and choose $f_n \equiv n$ on K_n , $\text{supp } f_n \subset K_{n+1}$, $f_n \in C_0(D)$. By the maximum principle, Lemma 1 and the above bounds, $f_n \rightarrow \alpha(x, t)$ uniformly in $\overline{D} \times [s, \tau]$, where $\alpha(x, t)$ is a strong solution of the (IDP). The maximum principle now shows that $\alpha(x, t) \equiv \gamma(x, t)$, where $\gamma(x, t) = \sup u_f(x, t)$, $f \in C_0(D)$. The fact that $\gamma(x, t) = \beta(x, t)$ is simple, and the proof is concluded.

Lemma 5 (Uniqueness of the friendly giant). *Let $u(x, t)$ be a strong solution of the (IDP). Let $w(x, t) = Gu(x, t)$. Suppose that $\lim_{t \downarrow 0} w(x, t) = +\infty$ for all $x \in D$. Then $u(x, t) \equiv \beta(x, t)$, where β is the solution constructed in Lemma 4.*

Proof. First note that if β is the solution constructed in Lemma 4, and $B(x, t) = G\beta(x, t)$, then $\lim_{t \downarrow 0} B(x, t) \equiv +\infty$. Thus, the lemma provides a characterization of β . Let now $\{u_n\}$ be a sequence of solutions of the (IDP) with $u_n \in C(\overline{D} \times [0, \infty))$, $0 \leq u_n \leq u_{n+1}$, $u_n \equiv 0$ on $\partial D \times [0, \infty)$, $u_n \uparrow \beta$. (For the existence of such u_n 's, see the proof of Lemma 4.) Apply now Lemma 3 to w and w_n to conclude that $w(x, t) \geq w_n(x, t)$. Hence, since $w_n(x, t) \uparrow B(x, t)$, we see that $w(x, t) \geq B(x, t)$. Since, by definition of β , $u(x, t+1/n) \leq \beta(x, t)$, we have $u(x, t) \leq \beta(x, t)$ and hence $w(x, t) \leq B(x, t)$. Thus $w(x, t) = B(x, t)$, and therefore $u(x, t) = \beta(x, t)$.

Lemma 6. *Let u be a strong solution of the (IDP). Then, $u \neq \beta$ if and only if*

$$\sup_{t>0} \int_D u(x, t)\delta(x) dx < \infty,$$

where $\delta(x) = \text{dist}(x, \partial D)$.

Proof. By definition of β , $u(x, t + 1/n) \leq \beta(x, t)$ for each n , and hence $u(x, t) \leq \beta(x, t)$. Thus, by the boundedness of β on $D \times [1, \infty)$,

$$\sup_{t \geq 1} \int_D u(x, t) \delta(x) dx < +\infty.$$

Assume that $u \not\equiv \beta$. Then, by Lemma 5, there exists $x_0 \in D$ such that $\lim_{t \downarrow 0} w(x_0, t) < \infty$. Hence, by monotonicity,

$$\sup_{0 < t < +\infty} \int_D G(x_0, y) u(y, t) dy < \infty.$$

But, $G(x_0, y) \geq C\delta(y)$, and one implication follows.

For the other implication, let $a(x) = \int_D G(x, y) dy$, and notice that $c_1 \delta(x) \leq a(x) \leq c_2 \delta(x)$. Moreover, by the symmetry of the Green's function,

$$\int_D w(x, t) dx = \int_D a(x) u(x, t) dx.$$

Hence, if $\sup_{t > 0} \int_D \delta(x) u(x, t) dx < \infty$, then $\sup_{t > 0} \int_D w(x, t) dx < \infty$, and hence, by monotone convergence $\lim_{t \downarrow 0} w(x, t)$ is integrable. Thus, $u \equiv \beta$.

Theorem 7 (Existence of initial trace). *Let u be a strong solution of the (IDP). Assume that $u \not\equiv \beta$. Then there is a positive Borel measure μ on D , with $\int_D \delta(x) d\mu(x) < +\infty$, and a positive Borel measure λ on ∂D , with $\int_{\partial D} d\lambda < +\infty$, such that whenever $\eta \in C^\infty(\mathbf{R}^n)$, $\eta|_{\partial D} \equiv 0$ we have*

$$\lim_{t \downarrow 0} \int_D u(x, t) \eta(x) dx = \int_D \eta d\mu + \int_{\partial D} \frac{\partial \eta}{\partial N} d\lambda,$$

where $\partial \eta / \partial N$ is the derivative in the direction of the outward normal N .

Proof. Let $h(x) = \lim_{t \downarrow 0} w(x, t)$. By the proof of Lemma 6, $\int_D h(x) dx < \infty$. Hence h is superharmonic in D , $h \geq 0$, and $\|w(x, t) - h(x)\|_{L^1(D)} \rightarrow 0$ as $t \downarrow 0$. Thus,

$$\int_D u(x, t) \eta(x) dx = - \int_D w(x, t) \Delta \eta(x) dx \rightarrow - \int_D h(x) \Delta \eta(x) dx \text{ as } t \downarrow 0.$$

Hence, by the Riesz decomposition theorem (see [7]), $h(x) = G\mu(x) + \alpha(x)$, where $\alpha(x)$ is a nonnegative harmonic function in D , and $G\mu(x) = \int_D G(x, y) d\mu(y)$, where μ is a positive Borel measure on D , with $\int_D \delta(y) d\mu(y) < +\infty$. Since $\alpha(x)$ is a nonnegative harmonic function, by the Martin representation theorem (see [7]) there is a positive measure λ and ∂D such that $\alpha(x) = \int_{\partial D} K(x, Q) d\lambda(Q)$, where $K(x, Q)$ is the kernel function for D . Now,

$$\int_D \left(\int_D G(x, y) d\mu(y) \right) \Delta \eta(x) dx = - \int_D \eta(y) d\mu(y),$$

while

$$\int_D \left(\int_{\partial D} K(x, Q) d\lambda(Q) \right) \Delta \eta(x) dx = - \int_{\partial D} \frac{\partial \eta}{\partial N}(Q) d\lambda(Q).$$

To justify these last two equalities, we proceed as follows: For the first one, note that our assumption on μ implies that $\int_D \left| \int_D G(x, y) d\mu(y) \right| dx < \infty$. Hence, it is enough to verify the first formula in the case when $d\mu(y) = f(y) dy$, with $f \in C_0^\infty(D)$. In this case it follows from Green's formula. For the second formula, note that $\int_D \int_{\partial D} K(x, Q) d\lambda(Q) dx < \infty$. (This follows by well-known estimates for the kernel function $K(x, Q)$; see for example [13].) Hence, it suffices to verify the formula for $d\lambda(Q) = f(Q) d\omega(Q)$, $f \in C^\infty(\partial D)$, where $d\omega(Q)$ is the harmonic measure for D . In this case our identity follows again by Green's formula.

Finally, we would like to remark that μ and λ are uniquely determined by u .

Theorem 8 (Uniqueness). *Suppose that u_1 and u_2 are strong solutions of the (IDP) and that*

$$\lim_{t \downarrow 0} \int_D u_1(x, t) \eta(x) dx = \lim_{t \downarrow 0} \int_D u_2(x, t) \eta(x) dx$$

for all $\eta \in C^\infty(\mathbf{R}^n)$, $\eta \equiv 0$ on ∂D . Then $u_1 \equiv u_2$.

Proof. Assume first that there exists one such η for which

$$\lim_{t \downarrow 0} \int_D u_1(x, t) \eta(x) dx = +\infty.$$

Then, by Theorem 7, $u_1 \equiv \beta$ and $u_2 \equiv \beta$. If, on the other hand, the above limit is finite for all such η 's, by Lemma 6, $u_1 \not\equiv \beta$, $u_2 \not\equiv \beta$. Let $\theta(x) \in C_0^\infty(D)$, $\theta(x) \geq 0$, $h_i(x) = \lim_{t \downarrow 0} w_i(x, t)$. Then

$$\int_D h_i(x) \theta(x) dx = \lim_{t \downarrow 0} \int_D u_i(x, t) \eta(x) dx,$$

and hence $h_1(x) \equiv h_2(x)$. For $s > 0$, let $w(x, t) = w_2(x, t + s)$. Since $h_2(x) \geq w_2(x, s)$, Lemma 3 implies that $w_1(x, t) \geq w(x, t)$, and hence $w_1(x, t) \geq w_2(x, t)$. Thus, $w_1(x, t) \equiv w_2(x, t)$ and the theorem follows.

Theorem 9 (Existence). *Given a pair of measures, μ on D and λ on ∂D , with $\mu \geq 0$, $\lambda \geq 0$, $\int_D \delta(x) d\mu(x) < \infty$, $\int_{\partial D} d\lambda < \infty$, there is a (unique) strong solution u of the (IDP) such that, for $\eta \in C^\infty(\mathbf{R}^n)$, $\eta|_{\partial D} \equiv 0$, we have*

$$\lim_{t \downarrow 0} \int_D u(x, t) \eta(x) dx = \int_D \eta d\mu + \int_{\partial D} \frac{\partial \eta}{\partial N} d\lambda.$$

Proof. The proof will be carried out in two steps. For the first step, we will assume that $\lambda \equiv 0$ and that $\text{supp } \mu \subset K \subset\subset D$. As in Lemma 4, for $f \in C_0^\infty(D)$, $f \geq 0$, let u_f be the unique strong solution of the initial Dirichlet problem $\partial u / \partial t = \Delta \varphi(u)$ in $D \times (0, \infty)$, $u \equiv 0$ on $\partial D \times (0, \infty)$, $u(x, 0) = f(x)$ for $x \in D$. (See [1] for instance, for the existence of u_f .) We will need to establish the following two estimates for u_f :

$$(*) \quad \sup_{t > 0} \int_D u_f(x, t) dx \leq \int_D f(x) dx,$$

and

(**) If $\eta \in C^\infty(\mathbf{R}^n)$, $\eta \equiv 0$ on ∂D , and $M = \sup_{t>0} \int_D u_f(x, t) dx$, then there exists $\alpha > 0$ such that, for $0 < t < 1$, we have

$$\left| \int_D [u_f(x, t) - f(x)] \eta(x) dx \right| \leq C_{M, \eta} t^\alpha.$$

To establish (*), note that when $\phi'(0) > 0$, and ϕ is smooth, then $u_f(x, t)$ is smooth for $t > 0$, and

$$\frac{\partial}{\partial t} \int_D u_f(x, t) dx = \int_D \Delta \phi(u_f) dx = \int_{\partial D} \frac{\partial}{\partial N} \phi(u_f) \leq 0,$$

and hence (*) follows by a passage to the limit.

To establish (**), note that

$$\int_D [u_f(x, t) - f(x)] \eta(x) dx = \int_0^t \int_D \phi(u_f(x, s)) \Delta \eta(x) dx ds.$$

Next, extend u_f to $\mathbf{R}^n \times (0, \infty)$ by setting it to be equal to 0 outside D . The resulting function u is a subsolution in $\mathbf{R}^n \times (0, \infty)$ of $\partial u / \partial t = \Delta \phi(u)$. We next need to observe that in the proof of (4.5) in [3], we only used the fact that u is a subsolution of $\partial u / \partial t = \Delta \phi(u)$, which in addition, is a uniform limit of smooth subsolutions to $\partial u / \partial t = \Delta \psi(u)$ for $\psi \in \Gamma_a$. This last fact follows for our u , by approximating u_f . (**) follows.

Now let μ verify $\text{supp } \mu \subset K \subset\subset D$. We seek a strong solution u to the (IDP) such that

$$\lim_{t \downarrow 0} \int_D u(x, t) \eta(x) dx = \int_D \eta(x) d\mu(x)$$

for all $\eta \in C^\infty(\mathbf{R}^n)$, $\eta|_{\partial D} \equiv 0$. Pick $f_j \in C_0^\infty(D)$, $f_j \geq 0$, such that $\int_D f_j \leq \int_D d\mu$, $\text{supp } f_j \subset \tilde{K} \subset\subset D$, and such that $\int_D f_j(x) dx \rightarrow \int_D d\mu$ weakly. Let $u_j = u_{f_j}$. Since $u_j \leq \beta$, by Lemma 1, there exists a subsequence (which we still denote $\{u_j\}$) and a strong solution u of the (IDP), such that $\{u_j\}$ converges uniformly to u in $\bar{D} \times [s, \tau]$ for all $0 < s < \tau < \infty$. By (**)

$$\lim_{t \downarrow 0} \int_D u(x, t) \eta(x) dx = \int_D \eta(x) d\mu(x).$$

We next pass to the general case. Let

$$h(x) = \int_D G(x, y) d\mu(y) + \int_{\partial D} K(x, Q) d\lambda(Q).$$

(See the proof of Theorem 7 for the relevant notation.) Again, h is superharmonic in D , $h \geq 0$, and $\int_D h(x) dx < +\infty$. Choose smooth domains $D_j, \bar{D}_j \subset\subset D_{j+1}$, $\bigcup_{j=1}^\infty D_j = D$, and define $h_j(x)$ be requiring that $h_j = h$ in \bar{D}_j , $h_j = 0$ on ∂D , and that h_j be harmonic in $D \setminus \bar{D}_j$. The h_j are superharmonic in D , $h_j \uparrow h$, and $h_j(x) = \int_D G(x, y) d\mu_j(y)$, with $\text{supp } \mu_j \subset \bar{D}_j$.

Let u_j be the solution corresponding to μ_j , constructed in the first step of the proof. As in the proof of Theorem 8, if

$$w_j(x, t) = \int_D G(x, y)u_j(y, t) dy, \quad \lim_{t \downarrow 0} w_j(x, t) = h_j(x).$$

Moreover, $u_j(x, t) \leq \beta(x, t)$, and hence Lemma 1 implies that a subsequence (which we still call $\{u_j\}$) converges uniformly in $\bar{D} \times [s, \tau]$ to a strong solution u of the (IDP). Let $w(x, t) = \int_D G(x, y)u(y, t) dy$. We claim that $w(x, t) \leq h(x)$. In fact, $w(x, t) = \lim_{j \rightarrow \infty} w_j(x, t)$, while $w_j(x, t) \leq h_j(x) \leq h(x)$. Since $\int_D h(x) dx < \infty$, $u \neq \beta$.

We next claim that $w(x, t) \geq w_j(x, t)$. We know that $w(x, t) = \lim_{k \rightarrow \infty} w_k(x, t)$. Choose $k > j$, and $s > 0$, and apply Lemma 3 to $w_k(x, t)$ and $w_j(x, t + s)$. This yields $w_k(x, t) \geq w_j(x, t + s)$ which implies our claim. Thus,

$$h(x) \geq \lim_{t \downarrow 0} w(x, t) \geq \lim_{t \downarrow 0} w_j(x, t) = h_j(x),$$

and so $\lim_{t \downarrow 0} w(x, t) = h(x)$. The proof of Theorem 7 now yields the proof.

Remarks. (1) Suppose that u is a strong solution of the (IDP) defined only in $D \times (t_2, t_2)$. Then, there exists a strong solution \tilde{u} of the (IDP) in $D \times (t_1, +\infty)$, which coincides with u in $D \times (t_1, t_2)$. In order to see this, assume without loss of generality that $t_1 = 0$. Let $w(x, t) = Gu(x, t)$. Then if $\lim_{t \downarrow 0} w(x, t) \equiv +\infty$, the proofs of Lemmas 3 and 5 show that $u(x, t) \equiv \beta(x, t)$, $t \in (t_1, t_2)$. If on the other hand, there exists x_0 such that $\lim_{t \downarrow 0} w(x_0, t) < +\infty$, the proof of Lemma 3, Lemma 6 and Theorem 7 give us a pair of initial traces (μ, λ) . Our existence result (Theorem 9) constructs \tilde{u} .

(2) (The friendly giant as a universal attractor). Suppose that $1 + a \leq u\phi'(u)/\phi(u)$ for all $u > 0$, and that $\lim_{h \rightarrow 0} (\phi(hu)/\phi(h)) = \psi(u)$ exists uniformly for u in compact subsets of $[0, \infty)$. (For example, if $\phi(u) = u^m$, $\psi(u) = u^m$.) Let $\beta_\phi(x, t)$ be the solution constructed in Lemma 4, corresponding to ϕ . Then, if u is any nonzero strong solution of the (IDP) corresponding to ϕ ,

$$\lim_{t \uparrow +\infty} \frac{u(x, t)}{\beta_\phi(x, t)} = 1,$$

uniformly on compact subsets of D .

To establish the remark, we may assume that $u \neq \beta$. For $s > 0$ let

$$\psi_s(u) = \frac{\phi(\Lambda_\phi(1/s)u)}{\phi(\Lambda_\phi(1/s))},$$

where Λ_ϕ is the inverse function on $(0, \infty)$ of $\phi(u)/u$. ($\phi(u)/u$ is increasing on $(0, \infty)$ by our first assumption.) Note that $\Lambda_\phi(r) \rightarrow 0$ as $r \rightarrow 0$. Let $v_s(x, t) = u(x, st)/\Lambda_\phi(1/s)$, so that v_s is a strong solution of the initial Dirichlet problem for $\partial v_s/\partial t = \Delta \psi_s(v_s)$. We claim that $\lim_{s \rightarrow \infty} v_s(x, t) = \beta_\psi(x, t)$. In fact, note that $\psi_s \in \Gamma_a$, and that the proof of Lemma 4 shows that, for all

$\theta \in \Gamma_a$, $\beta_\theta(x, t) \leq C/t^\alpha$, where C and α depend only on a . Note also that (see Lemma 1) strong solutions of the (IDP) for $\theta \in \Gamma_a$ are equicontinuous, and that $\psi_s \rightarrow \psi$ as $s \rightarrow \infty$ uniformly on compact sets by assumption. Thus, if $\{s_j\}$ is any sequence going to $+\infty$, there exists a subsequence $\{s_{\nu_j}\}$ so that $v_{s_{\nu_j}}(x, t) \rightarrow v(x, t)$ uniformly on compact subsets of $\bar{D} \times (0, \infty)$, where $v(x, t)$ is a strong solution of the (IDP) for $\partial v/\partial t = \Delta\psi(v)$. It remains to show that $v(x, t) = \beta_\psi(x, t)$. Let $w(x, t) = Gu(x, t)$ and $h(x) = \lim_{t \downarrow 0} w(x, t)$. Since $u(x, t)$ is not identically 0, and $u \neq \beta_\phi$ we know that

$$h(x) = \int_D G(x, y) d\mu(y) + \int_{\partial D} K(x, Q) d\lambda(Q),$$

where either μ or λ is not 0 (see Theorem 7). It is then an easy consequence of the Hopf maximum principle that $h(x) \geq c\delta(x)$. Let now $f \in C_0(D)$, and let $u_{j,f}$ be the strong solution of the (IDP) for $\psi_{s_{\nu_j}}$, with initial value f (see Lemma 4). Also, let $w_{j,f} = Gu_{j,f}$. We claim that, for j large, $Gv_{s_{\nu_j}} \geq w_{j,f}$. Since $u_{j,f}$ is continuous up to $t = 0$, $w_{j,f}(x, 0) = Gf(x)$, while

$$\lim_{t \downarrow 0} Gv_{s_{\nu_j}}(x, t) = \lim_{t \downarrow 0} \frac{w(x, s_{\nu_j} t)}{\Lambda_\phi(1/s_{\nu_j})} = \frac{h(x)}{\Lambda_\phi(1/s_{\nu_j})}.$$

But, $Gf(x) \leq c\delta(x)$, and so Lemma 3 establishes the claim. As $j \rightarrow \infty$, $\psi_{s_{\nu_j}} \rightarrow \psi$, and hence $u_{j,f} \rightarrow u_f$, uniformly on compact subsets of $\bar{D} \times [0, +\infty)$, where u_f is the strong solution of the (IDP) for ψ , with initial value f . Hence, if $w_f = Gu_f$, $Gv \geq w_f$. But then, $Gv \geq G\beta_\psi$. Since also $Gv \leq G\beta_\psi$, $v = \beta_\psi$, as claimed.

Next, note that if β_{ψ_s} is the solution constructed in Lemma 4 corresponding to ψ_s , then $\beta_{\psi_s}(x, t) = \beta_\phi(x, st)/\Lambda_\phi(1/s)$. This follows because of Lemma 5 and the fact that as functions of t both are strong solutions of $\partial u/\partial t = \Delta\psi_s(u)$. Hence,

$$\begin{aligned} \lim_{s \rightarrow \infty} \frac{u(x, st)}{\Lambda_\phi(1/s)\beta_\psi(x, t)} &= 1 = \lim_{s \rightarrow \infty} \frac{u(x, st)\beta(x, st)}{\Lambda_\phi(1/s)\beta(x, st)\beta_\psi(x, t)} \\ &= \lim_{s \rightarrow \infty} \frac{u(x, st)}{\beta_\phi(x, st)} \frac{\beta_{\psi_s}(x, t)}{\beta_\psi(x, t)}. \end{aligned}$$

However, we claim that $\lim_{s \rightarrow \infty} \beta_{\psi_s}(x, t) = \beta_\psi(x, t)$, which would finish the proof of the remark. Once more, $\beta_{\psi_s}(x, t) = \beta_\phi(x, st)/\Lambda_\phi(1/s)$, and the proof that $\lim_{s \rightarrow \infty} v_s(x, t) = \beta_\psi(x, t)$ applies to establish the claim.

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