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ON DISCONTINUOUS ACTION OF MONODROMY GROUPS ON THE COMPLEX n-BALL

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1. Introduction

In [DM, 3.10, 3.15] there is defined for each $(n+3)$-tuple $(\mu_1, \ldots, \mu_{n+3})$ of nonintegral real numbers with $\sum_i \mu_i$ integral a subgroup Γ_μ acting on a complex $(n+1)$-dimensional vector space V_μ, which preserves a hermitian form of signature $((\sum_i \langle \mu_i \rangle) - 1, (\sum_i \langle 1 - \mu_i \rangle) - 1)$ [DM, 2.21], where for any real number r, $\langle r \rangle$ denotes its *fractional part*, i.e., $0 \leq \langle r \rangle < 1$ and $r - \langle r \rangle$ is an integer. Let V^+ denote the set $\{v \in V_\mu ; \langle v, v \rangle > 0\}$. We will be concerned principally with the case that the signature of the hermitian form is (1 positive, n negative) in which case the image of V^+ in the projective space $\mathbf{P}^n$ associated to V_μ is the complex n-ball $\mathbf{B}^n$ (also called the n-disc). We deal almost exclusively with the action of Γ_μ on $\mathbf{B}^n$ and $\mathbf{P}^n$. We call an $(n+3)$-tuple μ satisfying $0 < \mu_i < 1$ for $i = 1, \ldots, n+3$ and $\sum_i \mu_i = 2$ a disc $(n+3)$-tuple.

In [DM, Theorem 10.19] it is proved: Let μ be a disc $(n+3)$-tuple satisfying condition INT: for all $i \neq j$ such that $\mu_i + \mu_j < 1$,

$$(1 - \mu_i - \mu_j)^{-1} \text{ is an integer.}$$

Then Γ_μ is a lattice in the projective unitary group $\mathrm{PU}(1, n)$; that is, Γ_μ is discrete in $\mathrm{PU}(1, n)$ and $\mathrm{PU}(1, n)/\Gamma_\mu$ has finite Haar measure.

Subsequently in [M1], the hypothesis INT was weakened to condition ΣINT: there is a subset $S_1 \subset \{1, \ldots, n+3\}$ such that $\mu_i = \mu_j$ for all $i, j \in S_1$ and moreover, for all $i \neq j$ such that $\mu_i + \mu_j < 1$,

$$(1 - \mu_i - \mu_j)^{-1} \in \begin{cases} \frac{1}{2}\mathbf{Z} & \text{if } i, j \in S_1, \\ \mathbf{Z} & \text{otherwise}, \end{cases}$$

where $\mathbf{Z}$ denotes the integers.

One of the main results of this paper, Theorem 4.13, asserts that if Γ_μ is discrete in $\mathrm{Aut}\,\mathbf{B}^n$ and $n > 3$, then μ satisfies condition ΣINT. In [M1], the

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set of all disc $(n+3)$-tuples satisfying condition $\Sigma\mathrm{INT}$ is determined; these exist only for $n \leq 9$. As a corollary, there exist lattices Γ_μ in $\mathrm{Aut}\,\mathbf{B}^n$ only for $n \leq 9$.

The strategy for proving Theorem 4.13 is first to solve the

Problem. Find all disc $(n+3)$*-tuples* μ *such that* Γ_μ *is discrete in* $\mathrm{Aut}\,\mathbf{B}^n$.

For $n = 1$, this problem could have been solved by H. A. Schwarz, for he solved the analogous problem for 4-tuples μ whose Γ_μ preserve a definitive hermitian form [S]. The problem for $n = 1$ is solved in §3. Thereafter, we use in §4 an inductive procedure which is formulated in terms of the graph that one associates to the disc tuple μ. This yields a set $\mathscr{D}$ of ten disc $(n+3)$-tuples, $n > 1$, containing all μ with Γ_μ discrete and with μ violating condition $\Sigma\mathrm{INT}$.

The assertion that Γ_μ is a lattice in $\mathrm{Aut}\,\mathbf{B}^n$ for *all* $\mu \in \mathscr{D}$ is Theorem 5.8, whose proof runs as follows. First we prove Proposition 5.3 which asserts that Γ_μ is a lattice in $\mathrm{Aut}\,\mathbf{B}^n$ if it is discrete there, by adapting a method used to prove one of the main theorems of [DM]. Thereafter, we prove an arithmeticity criterion Proposition 5.4, which comes from dropping a hypothesis in the arithmeticity criterion of [DM, Proposition 12.7]. Proposition 5.4 and the $\Sigma\mathrm{INT}$ criterion can be used to prove that Γ_μ is discrete in $\mathrm{Aut}\,\mathbf{B}^n$ for all but four μ in $\mathscr{D}$. The proof of discreteness of Γ_μ for three of the remaining μ in $\mathscr{D}$ depends on Kurt Sauter's Theorem 5.6, which settles a conjecture of mine and depends on Theorem 5.8 for the fourth.

2. Preliminaries

2.1. Let V be an $(n+1)$-dimensional vector space over the field $\mathbf{C}$ of complex numbers, and let V be endowed with a hermitian-form $(\ ,\ )$ of signature (1 positive, $n-1$ negative). Set $V^+ = \{v \in V; (v, v) > 0\}$ and set $\mathbf{B}^n = V^+/\mathbf{C}^*$. the projective unitary group of the hermitian form is isomorphic to $\mathrm{PU}(1, n)$ and may be identified with $\mathrm{Aut}\,\mathbf{B}^n$; $\mathrm{Aut}\,\mathbf{B}^n$ keeps invariant the metric on $\mathbf{B}^n$ given by

$$ds^2 = \frac{1}{\langle v, v\rangle}\begin{vmatrix} \langle dv, dv\rangle & \langle dv, v\rangle \\ \langle v, dv\rangle & \langle v, v\rangle \end{vmatrix}.$$

This is the *hyperbolic metric* of the complex n-ball. If $n = 1$, this metric yields the constant negative curvature metric on the disc $\mathbf{B}^1$; in particular the metric is invariant under the Schwarz reflections in geodesic lines of $\mathbf{B}^1$. For $n > 1$, there are no real $(2n-1)$-dimensional geodesic subspaces of $\mathbf{B}^n$. There do exist complex $(n-1)$-dimensional geodesic subspaces, and an element of finite order in $\mathrm{Aut}\,\mathbf{B}^n$ which fixes each point of such a $\mathrm{codim}_{\mathbf{C}}\,1$ geodesic subspace is called a *complex reflection.*

If μ is a disc $(n+3)$-tuple, we fix a hermitian form of signature $(1, n)$ associated to μ, and hereafter we usually identity Γ_μ of §1 with its image in $\mathrm{Aut}\,\mathbf{B}^n$, except when there is explicit reference to its action on the vector space V_μ.

By [DM, Proposition 9.2], the group Γ_μ is generated by complex reflections.

An $(n+3)$-tuple of real numbers $\mu = (\mu_1, \dots, \mu_{n+3})$ is called *monotone* if $\mu_1 \le \mu_j$, whenever $i < j$.

2.2. In his seminal paper [S], H. A. Schwarz solved the problem which may be formulated as: For which quadruples μ is the action of Γ_μ on the vector space V_μ finite? By [DM] 2.21, cited in the introduction, we may restrict ourselves to 4-tuples with $0 < \mu_i < 1$ and $\sum_i \mu_i = 1$, $i = 0, 1, 2, 3$; for such 4-tuples, Γ_μ preserves a negative definite hermitian form.

The following list of all such monotone μ with Γ_μ discrete (and *therefore finite*) is a transposition, in our parameters μ, of Schwarz's list of all solutions. Any quadruple on this list is called a *Schwarz quadruple.*

TABLE 2.3. Schwarz quadruples (ordered as in [S, p. 323]).

1.	a	a	$\frac{1}{2}-a$	$\frac{1}{2}-a$	$0 < a < 1$, a rational
2.	$\frac{1}{12}$	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{5}{12}$	
3.	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	
4.	$\frac{1}{24}$	$\frac{5}{24}$	$\frac{7}{24}$	$\frac{11}{24}$	
5.	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{7}{12}$	
6.	$\frac{1}{60}$	$\frac{11}{60}$	$\frac{19}{60}$	$\frac{29}{60}$	
7.	$\frac{1}{30}$	$\frac{9}{30}$	$\frac{9}{30}$	$\frac{11}{30}$	
8.	$\frac{1}{30}$	$\frac{5}{30}$	$\frac{5}{30}$	$\frac{19}{30}$	
9.	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{7}{20}$	$\frac{9}{20}$	
10.	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{8}{15}$	
11.	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{3}{10}$	
12.	$\frac{3}{30}$	$\frac{3}{30}$	$\frac{7}{30}$	$\frac{17}{30}$	
13.	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{7}{10}$	
14.	$\frac{7}{60}$	$\frac{13}{60}$	$\frac{17}{60}$	$\frac{23}{60}$	
15.	$\frac{5}{30}$	$\frac{5}{30}$	$\frac{7}{30}$	$\frac{13}{30}$	

An unordered subtriple of a Schwarz quadruple is called a *Schwarz triple.*

We next prove, using the definitions and notation of [DM], two lemmas that will be used repeatedly in §4.

Lemma 2.4. *Let μ be a disc N-tuple such that Γ_μ is discrete in* $\mathrm{PU}(1, N-3)$. *Set $S = \{1, 2, \dots, N\}$ and let T be a partition of S into k disjoint nonempty cosets $C_1, \dots, C_k$ such that $\nu_j < 1$ for each j, where $\nu_j = \sum_{s \in C_j} \mu_s$, $j = 1, \dots, k$. Set $\nu = (\nu_1, \dots, \nu_k)$. Then Γ_μ is discrete in* $\mathrm{PU}(1, k-3)$.

Proof. Let M denote the complement in $(\mathbf{P}^1)^S$ of all its subdiagonals and let $Q = \mathrm{Aut}\,\mathbf{P}^1 \backslash M$. Let Q_{sst} denote the μ-completion of Q [DM, §4.1] and $\pi\colon \widetilde{Q}_{st} \to Q_{st}$ the spread defined in [DM, 8.6].

By [DM, 8.7] we have a commutative diagram

$$\begin{array}{ccc} \widetilde{Q}_{st} & \xrightarrow{\tilde{w}} & B \\ \pi\downarrow & & \downarrow \\ Q_{st} & \longrightarrow & \Gamma_\mu\backslash B \end{array}$$

where B is the complex $N-3$ ball and $\tilde{w}$ is the Γ_μ-equivariant Schwarz map.

The partition T is a *stable* partition in the sense of [DM, 6.9] and to it corresponds the subset Q_T of Q_{st} corresponding to the subdiagonal of $(\mathbf{P}^1)^S$ of type T. By [DM, 8.8.1], the group Γ_ν can be identified with the stabilizer in Γ_μ of the subball of B which is the image under $\tilde{w}$ of a connected component of $\pi^{-1}(Q_T)$. By [DM, 2.21], the subball is the projective image of the cone of positive elements of a vector space with a hermitian form of signature $((\sum_i^k \nu_j)-1, (\sum_i^k(1-\nu_j))-1)$, i.e., $(1, k-3)$. Hence Γ_γ is a discrete subgroup of $\mathrm{PU}(1, k-3)$.

Terminology. The k-tuple ν above is called a *contraction* of μ.

Lemma 2.5. *Let μ be a disc N-tuple such that Γ_μ is discrete in* $\mathrm{PU}(1, N-3)$. *Assume that for three distinct indices $i_1 < i_2 < i_3$, $\mu_{i_1} + \mu_{i_2} + \mu_{i_3} < 1$. Then $(\mu_{i_1}, \mu_{i_2}, \mu_{i_3})$ is a Schwarz triple.*

Proof. We continue the foregoing notation. Let y be a point of Q_{st} corresponding to a point in $(\mathbf{P}^1)^S$ on the subdiagonal whose i_1, i_2, i_3 coordinates coincide and lying on no smaller subdiagonal. Then the decomposition group of y (cf. [DM, 8.2]) is generated by suitable monodromies with common base point of i_j around i_k, $j, k \in \{1, 2, 3\}$. Choose $\tilde{y} \in \pi^{-1}(y)$ and set $x = \tilde{w}(\tilde{y})$. Since $\tilde{w}$ is Γ_μ-equivariant, $(\Gamma_\mu)_{\tilde{y}} \subset (\Gamma_\mu)_x$. By hypothesis, Γ_μ is discrete in $\mathrm{PU}(1, N-3)$ and $\mathrm{PU}(1, N-3) = \mathrm{Aut}\, B$ by [DM, 2.21]. Since the stabilizer of a point in $\mathrm{Aut}\, B$ is compact, $(\Gamma_\mu)_x$ is finite. Consequently the decomposition group of y in $\pi_1(Q, 0)$, which is isomorphic to $(\Gamma_\mu)_{\tilde{y}}$, is a finite group.

Set $S = \{1, \ldots, N\}$. $S_1 = \{i_1, i_2, i_3\}$, $S_2 = S - S_1$. Let T_1, T_2 be trees with the vertices of T_i in S_i $(i = 1, 2)$, and let $\beta\colon T_1 \coprod T_2 \to \mathbf{P}^1$ be an embedding with $\beta|S = 0$, a base point for the space M (cf. [DM, 2.5]); for any choice of β one gets a base $\{w_1, \ldots, w_{N-2}\}$ of linear functions for $H_1^{lf}(P_0, L_0^\vee)$. We can select the embedding β and the base point 0 so that $\beta(T_1)$ and $\beta(T_2)$ lie in disjoint discs of $\mathbf{P}^1$ and with $\infty \in 0(S_2)$.

The same computation as in [DM, §12.3] shows that the action of $(\Gamma_\mu)_{\tilde{y}}$ depends, up to isomorphism, only on the complex numbers $\exp 2\pi\sqrt{-1}\mu_j$, $j = i_1, i_2, i_3$ and that $\mathbf{C}w_1 + \mathbf{C}w_2$ is stable under the indicated monodromies of i_1, i_2, i_3 around each other described in Figures (a) and (b) of [DM, §12.3]; moreover these monodromies generate the decomposition group of y if $\mu_{i_1} + \mu_{i_2} + \mu_{i_3} < 1$. Consequently (cf. [DM, §§12.4 and 13.1]), the group $(\Gamma_\mu)_{\tilde{y}}$

is isomorphic to the subgroup of $U(2,\mathbf{C})$ generated by complex reflections associated to the quadruple $\sigma = (\mu_{i_1}, \mu_{i_2}, \mu_{i_3}, 2 - \mu_{i_1} - \mu_{i_2} - \mu_{i_3})$ induced by the indicated monodromies of i_1, i_2, i_3 around each other. Since $(\Gamma_\mu)_{\tilde{y}}$ is a finite group, the monodromy group Γ_σ is finite. It follows from definitions that $(\mu_{i_1}, \mu_{i_2}, \mu_{i_3}, 1 - \mu_{i_1} - \mu_{i_2} - \mu_{i_3})$, rearranged to be monotone, is a Schwarz quadruple and our lemma is proved.

Remark. The 2-dimensional subspace $W = \mathbf{C}w_1 + \mathbf{C}w_2$ plays a key role in the contraction of the N-tuple μ to the 4-tuple ν. Adopting the notation used in [DM], set $V = H^1(P_0, L_0)$ and $V^* = H_1^{lf}(P_0, L_0^\vee)$, the dual space of V. The Γ'_μ invariant hermitian form H on V has signature (1 (positive), $N-3$ (negative)). H determines a semilinear isomorphism $\Theta\colon V^* \to V$ and a corresponding hermitian form on V^*. Let Δ' denote the subgroup of Γ_μ generated by the monodromies of i_j around i_k, $1 \le j < k \le 3$ indicated in [DM, §12.3, Figures (a) and (b)]. Let $\mathscr{S}$ denote the signature of H restricted to W. Assume that $\sum_{j=1}^3 \mu_{i_j} \neq 1$; then either $\mathscr{S} = (0,2)$ or $\mathscr{S} = (1,1)$.

Set $V = \{v \in V;\ H(v,v) > 0\}$, and let $P\colon V - 0 \to P(V)$ denote the canonical projection onto the projective space of 1-dimensional subspaces of V. Let $W^\perp$ denote the annihilator of W in V. One sees easily that Δ' fixes each element of $W^\perp$.

If $\mathscr{S} = (0,2)$, then Δ' fixes each point of the $(N-5)$-subball $p(V^+ \cap W^\perp)$, $i = B^{N-5}$, and W may be identified with the normal to B^{N-5} at one of its generic points. If on the other hand $\mathscr{S} = (1,1)$, then Δ' stabilizes the 1-ball $P(\Theta(W) \cap V^+)$. The group Γ_ν may be identified with the action of Δ' on $P(W)$ in the first case and on $P(\Theta(W) \cap V^+)$ in the second case.

3. Discrete Γ_μ in $\operatorname{Aut}\mathbf{B}^1$

Let P_1, P_2, P_3 be the three distinct vertices of a geodesic triangle in the disc $\mathbf{B}^1$, and let R_i be the rotation of $\mathbf{B}^1$ in the positive sense with center P_i and angle $2 \measuredangle P_i P_{i+1}$ (where we take each i modulo 3). Then $R_1 R_2 R_3 = 1$. Moreover, if $\measuredangle P_i = n_i \pi / d_i$, where n_i and d_i are relatively prime integers, then $R_i^{d_i} = 1$ $(i = 1,2,3)$.

Let κ_i denote Schwarz reflection in the geodesic side $P_{i-1}P_{i+1}$. Then for each i

$$\kappa_i^2 = 1, \qquad \kappa_i \kappa_{i-1} = R_{i-2}$$

so that

$$\begin{aligned}
\kappa_2 R_1 \kappa_2 &= \kappa_2 \kappa_3 \kappa_2 \kappa_2 = \kappa_2 \kappa_3 = R_1^{-1}, \\
\kappa_2 R_2 \kappa_2 &= \kappa_2 \kappa_1 \kappa_3 \kappa_2 = \kappa_3 \kappa_1 = R_2^{-1}, \\
\kappa_2 R_3 \kappa_2 &= \kappa_2 \kappa_2 \kappa_1 \kappa_2 = \kappa_1 \kappa_2 = R_3^{-1}.
\end{aligned}$$

Set $\Gamma = \langle R_1, R_2, R_3 \rangle$, the group generated by R_1, R_2, R_3, and set $\Gamma^* = \langle \kappa_2, \Gamma \rangle$. Then Γ is a normal subgroup of index 2 in Γ^*, and we note that

$\Gamma^* = \langle \kappa_1, \Gamma \rangle = \langle \kappa_2, \Gamma \rangle = \langle \kappa_1, \kappa_2, \kappa_3 \rangle$. Γ is a subgroup of $\operatorname{Aut}\mathbf{B}^1$, the group of biholomorphic maps $\mathbf{B}^1 \to \mathbf{B}^1$ and $\Gamma^* \subset \operatorname{Isom}\mathbf{B}^1$, the group of isometries of $\mathbf{B}^1$ in its hyperbolic metric.

Notation. $\Gamma(P_1, P_2, P_3)$ and $\Gamma^*(P_1, P_2, P_3)$ denote the above Γ and Γ^* respectively.

We shall determine necessary and sufficient conditions on the angles of the geodesic triangle $\triangle P_1P_2P_3$ that the group $\Gamma(P_1, P_2, P_3)$ be discrete in $\operatorname{Aut}\mathbf{B}^1$—or equivalently, that $\Gamma^*(P_1, P_2, P_3)$ be discrete.

We begin with some preliminaries. If the angles of the $\triangle P_1P_2P_3$ are all integral parts of π (including possibly $\pi/\infty = 0$ when a vertex is on the boundary of $\mathbf{B}^1$), then Γ is called a *triangle group* and Γ^* a *triangle reflection group.* It is well known under these *integrality hypotheses* that the triangle $P_1P_2P_3$ is a fundamental domain for Γ^*.

Let G denote a subgroup of $\operatorname{Isom}\mathbf{B}^1$ generated by Schwarz reflections. We denote by $\mathscr{M}(G)$ the set of all fixed point sets of Schwarz reflections in G; if κ is a Schwarz reflection, we call its fixed point set $M(\kappa)$ its *mirror.* It is well known that

D1. G is discrete in $\operatorname{Isom}\mathbf{B}^1$ if and only if $\mathbf{B}^1 - \bigcup\{M; M \in \mathscr{M}(G)\}$ has a nonempty interior.

D2. If G is discrete, it operates simply transitively on the connected components of $\mathbf{B}^1 - \bigcup\{M; M \in \mathscr{M}(G)]$, and each connected component F is the intersection of all mirror-bounded open half-spaces which contain F. In particular, F is convex. If G is a triangle reflection group, the interior of the $\Delta P_1P_2P_3$ is a connected component of the complement of the mirrors in $\mathscr{M}(G)$.

D3. Let $G = \langle \kappa_1, \kappa_2 \rangle$ where κ_i is a Schwarz reflection with mirror M_i $(i = 1, 2)$, and assume that $M_1 \cap M_2$ is nonempty. Then G is discrete if and only if it is a finite dihedral group of order $2d$, where d is the order of $\kappa_1\kappa_2$, and the angle between M_1 and M_2 is $n\pi/d$ with $\gcd(n, d) = 1$; in this case $\mathscr{M}(G)$ consists of all the geodesic lines through the point $M_1 \cap M_2$ forming the angle $k\pi/d$ with M_1, $k \in \mathbf{Z}$.

D4. If Γ^* is a discrete subgroup of $\operatorname{Isom}\mathbf{B}^1$ containing the group G of D_3, then the isotropy subgroup $\Gamma^*_{M_1 \cap M_2}$ of the point $M_1 \cap M_2$ is $\langle \kappa, \kappa' \rangle$, where κ, κ' are Schwarz reflections. This follows readily from D_2 and D_3.

Notation. Let $d_i \in \mathbf{Z}^+ \cup \infty$, $i = 1, 2, 3$, and let P_1, P_2, P_3 be the vertices of a geodesic triangle in $\mathbf{B}^1$ with $\measuredangle P_i = \pi/d_i$ $(i = 1, 2, 3)$. We denote any conjugate in $\operatorname{Aut}\mathbf{B}^1$ of $\Gamma(P_1, P_2, P_3)$ and $\Gamma^*(P_1, P_2, P_3)$ by $[d_1, d_2, d_3]$ and $[d_1, d_2, d_3]^*$ respectively.

By definition, $[d_1, d_2, d_3]$ is a triangle group. We will prove in Lemma 3.3 below that if $\Gamma(P_1, P_2, P_3)$ is discrete in $\operatorname{Aut}\mathbf{B}^1$, then it has a presentation as a triangle group.

One of the principal ingredients in the proof of Theorem 3.7 is the following procedure, called the *triangulation algorithm.*

Let P_1, P_2, P_3 be the vertices of a geodesic triangle in $\mathbf{B}^1$. Assume that

(1) $\Gamma(P_1, P_2, P_3)$ is discrete in $\operatorname{Aut}\mathbf{B}^1$;
(2) some angle of $\triangle P_1$, P_2, P_3 is *not* an integral part of π.

Set $\Gamma^* = \Gamma^*(P_1, P_2, P_3)$. Relabelling the indices if necessary, we may assume that $\sphericalangle P_1 = n_1/d_1$, $d_1 \in \mathbf{Z}^+$, $\gcd(n_1, d_1) = 1$, $n_1 > 1$. By D4, the isotropy group $\Gamma^*_{p_1}$ is a dihedral group of order $2d$ with $d|d_1$ since $\mathscr{M}(\Gamma^*_{p_1})$ contains the geodesic lines P_1P_2 and P_1P_3. Consequently, $\mathscr{M}(\Gamma^*_{p_1})$ contains a mirror making an angle π/d_1 with P_1P_2, by D3. We can therefore choose a point P_4 on the side P_2P_3 so that $\sphericalangle P_2P_1P_4 = \pi/d_1$.

Next choose the obtuse angle formed at P_4; that is, choose $i = 2$ or 3 so that $\sphericalangle P_1P_4P_i \geq \pi/2$. If $\sphericalangle P_1P_4P_i = \pi/2$, then $\triangle P_1P_4P_i$ is a right triangle; otherwise, the isotropy subgroup $\Gamma^*_{P_4}$ has order $2d_4$ with $d_4 > 2$, $\sphericalangle P_1P_4P_i = n_4\pi/d_4$ with $d_4 \in \mathbf{Z}^+$, $\gcd(n_4, d_4) = 1$, $n_4 > 1$. We then repeat the above procedure on $\triangle P_1P_4P_i$, choosing a point P_5 on the side P_1P_i and the obtuse angle formed at P_5. Repeating this procedure, one obtains successively smaller geodesic subtriangles of $\triangle P_1P_2P_3$, whose sides lie on mirrors in $\mathscr{M}(\Gamma^*)$. Inasmuch as Γ^* is discrete, the mirrors in $\mathscr{M}(\Gamma^*)$ cannot accumulate at any point of $\mathbf{B}^1$. Thus after a finite number of steps, one arrives at a right triangle whose sides lie on mirrors in $\mathscr{M}(\Gamma^*)$. We label the vertices of the right triangle $Q_1Q_2Q_3$ with $\sphericalangle Q_2 = \pi/2$ and $\sphericalangle Q_i = m_i\pi/q_i$, $\gcd(m_i, q_i) = 1$ if $q_i \neq \infty$. If not all angles of $\triangle Q_1Q_2Q_3$ are integral parts of π, then one can proceed to form successively smaller right triangles whose sides lie on mirrors on $\mathscr{M}(\Gamma^*)$. For example, if $m_1 > 1$, choose Q_4 on the side P_2P_3 so that $\sphericalangle Q_1Q_4Q_2 = \pi/q_1$, then repeat this procedure in $\triangle Q_1Q_2Q_4, \ldots$. Finally, one arrives after a finite number of steps at a right triangle whose sides lie on mirrors in $M(\Gamma^*)$.

We summarize the result of using the above triangulation algorithm in the following lemma.

Lemma 3.1. *Let P_1, P_2, P_3 be the vertices of a geodesic triangle in $\mathbf{B}^1$ not all of whose angles are integral parts of π. Assume that $\Gamma^*(P_1, P_2, P_3)$ is discrete in* $\operatorname{Isom}\mathbf{B}^1$. *Then $\Gamma^* \supset [2, s, t]^*$ for some s, t.*

The triangulation algorithm for a triangle P_1, P_2, P_3 with $\Gamma^*(P_1, P_2, P_3)$ discrete can be prolonged as follows.

Let $\mathscr{M}_k$ denote the subset of mirrors in $\mathscr{M}(\Gamma^*)$ meeting $\triangle P_1P_2P_3$ that has been accumulated at the kth stage. If the convex regions into which $\triangle P_1P_2P_3$ is partitioned by $\mathscr{M}_k(G)$ have as angles only integral parts of π, stop. Otherwise, for any two mirrors M_1, M_2 in $\mathscr{M}_k$ which intersect in a point of $\triangle P_1P_2P_3$, adjoin to $\mathscr{M}_k$ the set of all mirrors in $\mathscr{M}(\langle \kappa_1, \kappa_2 \rangle)$, where κ_i is the Schwarz reflection in M_i. This procedure is called the *prolonged* triangulation algorithm.

Lemma 3.2. *The triangle reflection group* $[2,s,t]^*$ *with* $s \le t$ *is maximal among discrete isometry groups of* $\mathbf{B}^1$ *generated by Schwarz reflections except when* $t = 2s > 7$, *in which case it is contained in* $[2,3,2s]^*$.

Proof. Suppose $\Gamma^* \supset [2,s,t]^*$. By D2, Γ^* has as fundamental domain a convex polygon F with N sides whose area is an integral part r^{-1} of the area of the right triangle with angles $\pi/2$, $\pi/3$, π/t, at vertices V_2, V_s', V_t respectively. By D3 the angles of F are at most $\pi/2$, and if V denotes the vertex of F with $V_t \in \Gamma^* V$, then $< V = \pi/nt$ with $n \in \mathbf{Z}^+$. Consequently,

$$\begin{aligned} r \cdot \operatorname{area} F &= r \cdot \pi \left(N - 2 - \sum^N \alpha_i \right) \\ &= \operatorname{area} \triangle V_2 V_s' V_t = \pi(1 - (\tfrac{1}{2} + \tfrac{1}{s} + \tfrac{1}{t})) \end{aligned}$$

where $\pi\alpha_1, \ldots, \pi\alpha_N$ denote the angles of F. Assume $r > 1$. Then

$$\begin{aligned} 2\left(N - 2 - \frac{N-1}{2} - \frac{1}{nt}\right) &\le \frac{1}{2} - \frac{1}{s} - \frac{1}{t}, \\ N - 3 - \frac{2}{nt} &\le \frac{1}{2} - \frac{1}{s} - \frac{1}{t}. \end{aligned}$$

Since this is impossible for $N > 3$, we infer that $N = 3$. Thus F is a triangle and by D3 its angles are the integral parts of π: $\alpha_1 = 1/nt$, $\alpha_2 = 1/d_2$, $\alpha_3 = 1/d_3$, $n \ge 1$, $2 \le d_2 \le d_3$. Returning to the inequality above

$$\begin{aligned} 2\left(1 - \frac{1}{nt} - \frac{1}{d_2} - \frac{1}{d_3}\right) &\le \frac{1}{2} - \frac{1}{s} - \frac{1}{t}, \\ 1 - \frac{1}{nt} &\le 2\left(\frac{1}{d_2} + \frac{1}{d_3}\right) - \frac{1}{2} - \frac{1}{s} + \frac{1}{nt} - \frac{1}{t}, \end{aligned}$$

$d_2 > 2$ is impossible otherwise

$$\begin{aligned} 1 - \frac{1}{nt} &\le \frac{4}{3} - \frac{1}{2} - \frac{1}{s}, \\ \frac{1}{6} \le \frac{1}{nt} - \frac{1}{s} &\le 0. \end{aligned}$$

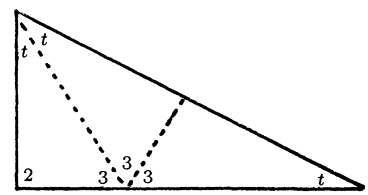
Consequently, $d_2 = 2$ and we have

$$\begin{aligned} 1 - \frac{1}{nt} &\le 1 + \frac{2}{d_3} - \frac{1}{2} - \frac{1}{s}, \\ \frac{1}{2} + \frac{1}{s} &\le \frac{2}{d_3} + \frac{1}{nt}, \end{aligned}$$

whence $d_3 = 3$, if $r > 1$, and

$$\begin{aligned} r\left(1 - \frac{1}{nt} - \frac{1}{2} - \frac{1}{3}\right) &= \frac{1}{2} - \frac{1}{s} - \frac{1}{t}, \\ \frac{r}{6} - \frac{r}{nt} &= \frac{1}{2} - \frac{1}{s} - \frac{1}{t}. \end{aligned} \tag{3.2.1}$$

Geometrically, we can see that $r = 2$ is impossible because $\overline{F} \cup \kappa\overline{F}$ is not a right triangle congruent to $\triangle V_2 V_s' V_t$ for κ a Schwarz reflection in the sides of F. On the other hand, from (3.2.1) we see that the integer $r \leq 3$ and $r = 3$ only for $n = 1$, $t = 2s$, in which case we have the diagram



where $t \in \{2, 3, \ldots, \infty\}$. Finally if $r = 1$, then $\Gamma^* = [2, s, t]^*$. The proof of Lemma 3.2 is now complete.

Lemma 3.3. *Let P_1, P_2, P_3 be the vertices of a geodesic triangle in $\mathbf{B}^1$ not all of whose angles are integral parts of π. Assume that the group $\Gamma(P_1, P_2, P_3)$ is discrete in* $\operatorname{Aut}\mathbf{B}^1$. *Then $\Gamma(P_1, P_2, P_3) = [2, s, t]$ for integers s, t.*

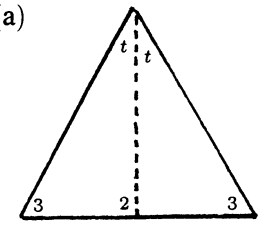
Proof. Set $\Gamma^* = \Gamma^*(P_1, P_2, P_3)$. Let $G = [2, s, t]^*$ and $G' = [2, s', t']^*$ be two triangle reflection groups in the sides of right triangles obtained via the triangulation algorithm (cf. Lemma 3.1), and let $\langle G, G' \rangle$ denote the group generated by G and G'. Since Γ^* is discrete in $\operatorname{Isom}\mathbf{B}^1$, we infer from Lemma 3.2 that $G = G' = \Gamma^*$ except when $\Gamma^* = [2, 3, 2s]^*$ and G or G' is $[2, s, 2s]^*$. The group $\Gamma(P_1, P_2, P_3)$ is the unique index 2 normal subgroup $\Gamma^* \cap \operatorname{Aut}\mathbf{B}^1$ of Γ^* and thus coincides with $[2, s, t]$ when $[2, s, t]^*$ is the only group arising from the triangulation algorithm, otherwise $\Gamma = [2, 3, 2s]$.

As an immediate consequence of Lemma 3.3, we get

Lemma 3.4. *Let P_1, P_2, P_3 be the vertices of a geodesic triangle whose angles are not all integral parts of π, and such that $\Gamma(P_1, P_2, P_3)$ is discrete. Set $r =$ the ratio of areas of $\triangle P_1 P_2 P_3$ and of a fundamental triangle of $[2, s, t]^* = \Gamma^*$. Let $\mathscr{M}_\infty$ denote the subset of mirrors of $\mathscr{M}(\Gamma^*)$ obtained by the prolonged triangulation algorithm. Then the number of connected components of $\triangle P_1 P_2 P_3 - \bigcup\{M ; M \in \mathscr{M}_\infty\}$ is at most r.*

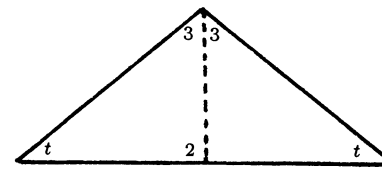
Lemma 3.5. *Set $\Gamma^* = \Gamma^*(P_1, P_2, P_3)$ where not all angles of $\triangle P_1 P_2 P_3$ are integral parts of π. Assume that $\Gamma^* = [2, 3, t]^*$. Then the triangle $\triangle P_1 P_2 P_3$ is one of the following five types.*

(a)

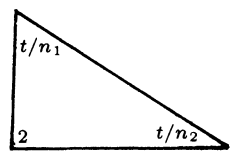


t odd,

(b)

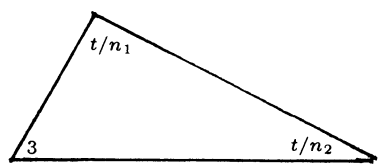


(c)



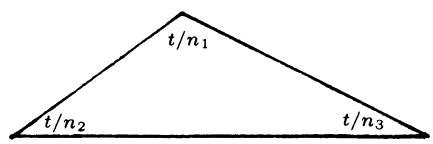
with $n_1 = 2,\ n_2 = 1$

(d)



with $n_1 = 3,\ n_2 = 1$

(e)



with $n_1 = 4,\ n_2 = n_3 = 1$ *or* $n_1 = n_2 = n_3 = 2$

Proof. We can choose a geodesic triangle as a fundamental domain for $[2,3,t]^*$ with vertices $V_2V_3V_t$, angles $\sphericalangle V_i = \pi/i$, and isotropy groups $\Gamma^*_{V_i}$ a dihedral group of order $2i$, $i = 2,3,t$. By the hypothesis $\Gamma^* = [2,3,t]^*$, each P_i is in the orbit of one of points V_j $(j = 2,3,t)$. Thus, there are a number of possibilities to consider.

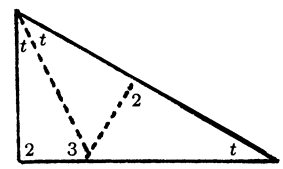
If two of the angles of the triangle $P_1P_2P_3$ are $\pi/3$, and the third angle, say $\sphericalangle P_1 = n\pi/t$ with $n > 1$, then the ratio of the area of the triangle $\triangle P_1P_2P_3$ to the area of the fundamental domain for $[2,3,t]^*$ is

$$r = \frac{1 - \left(\frac{1}{3} + \frac{1}{3} + \frac{n}{t}\right)}{1 - \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{t}\right)} = \frac{\frac{1}{3} - \frac{n}{t}}{\frac{1}{6} - \frac{1}{t}} \leq 2$$

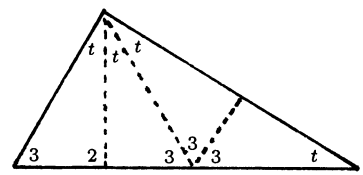
and $r = 2$ only if $n = 2$.

Let $M = \mathscr{M}(\Gamma^*)$ denote the set of fixed point sets of all the Schwarz reflections in Γ^*.

By D2 above, the group Γ^* operates simply transitively on the connected components of $\mathbf{B}^1 - \bigcup_{M \in \mathscr{M}} M$ and each such component is a geodesic triangle with angles $\pi/2$, $\pi/3$, π/t. Consequently r is an integer and $n = 2$, yielding (a). A similar argument works if one of the angles of the triangle $P_1P_2P_3$ is $2\pi/3$, yielding (b), or if one of the angles is $\pi/2$ or $\pi/3$ yielding
Case (c)



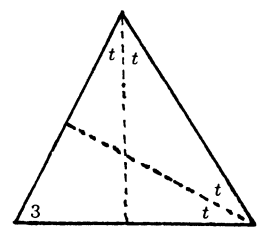
Case (d)



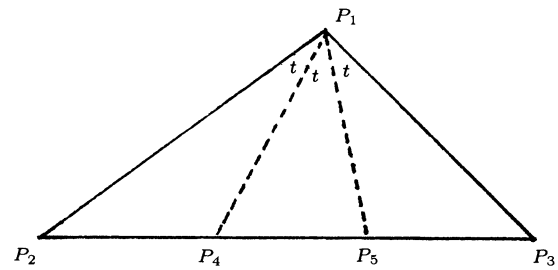
The argument in Case (d) is

$$r = \frac{1 - \left(\frac{1}{3} + \frac{n_1}{t} + \frac{n_2}{t}\right)}{1 - \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{t}\right)} = \frac{\frac{2}{3} - \frac{n_1 + n_2}{t}}{\frac{1}{6} - \frac{1}{t}} \leq 4$$

if $n_1 + n_2 \geq 4$. We can assume that $n_1 \geq n_2$ and $n_1 > 1$.

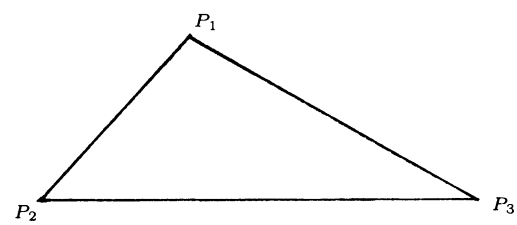


If $n_2 \geq 2$, then $n_1 + n_2 \geq 4$ so that $r \leq 4$; that is, the prolonged triangulation algorithm must yield, by Lemma 3.4, no more than four $[2, 3, t]$ triangles, which is contradicted by our diagram. Consequently, $n_2 = 1$, $n_1 = 3$.



Choose P_4 and P_5 on line P_2P_3 so that the successive lines P_1P_2, P_1P_4, P_1P_5, P_1P_3 each form an angle of π/t at P_1. If the angles at neither P_4 nor P_5 are right angles, then the prolonged triangulation algorithm will yield at least two triangles at each obtuse angle—a total of at least 5—this violating Lemma 3.4 since $r = 4$. From this, (d) follows.

Consider finally the case (e) in which $\sphericalangle P_i = n_i\pi/t \quad (i = 1, 2, 3)$,

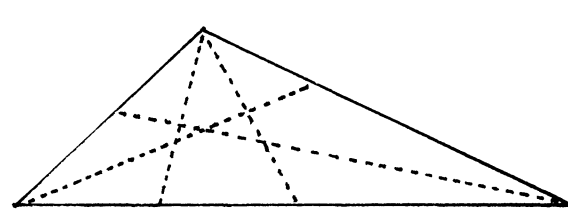


$n_1 \geq n_2 \geq n_3$, $n_1 > 1$, $t \geq 7$. As above, let r denote the ratio of areas

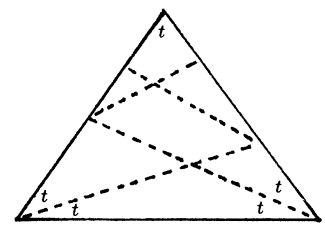
$$r = \frac{1 - \left(\frac{n_1}{t} + \frac{n_2}{t} + \frac{n_3}{t}\right)}{\frac{1}{6} - \frac{1}{t}} = \frac{1 - \frac{n_1 + n_2 + n_3}{t}}{\frac{1}{6} - \frac{1}{t}}.$$

Then $r \leq 6$ if $n_1 + n_2 + n_3 \geq 6$, and $r = 6$ only if $n_1 + n_2 + n_3 = 6$. $r \leq 12$ if $n_1 + n_2 + n_3 = 5$ and $r \leq 18$ if $n_1 + n_2 + n_3 = 4$.

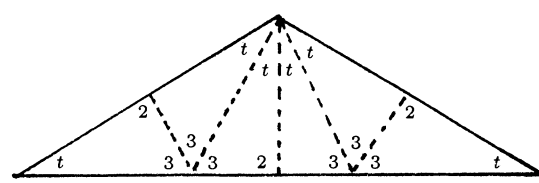
If $n_3 > 1$, then $n_1+n_2+n_3 \geq 6$ and the subdivision of the triangle $P_1P_2P_3$ by the mirrors of $\mathscr{M}(\Gamma^*)$ meeting the set of three vertices P_1, P_2, P_3 partitions the triangle $\triangle P_1P_2P_3$ into more than six regions unless $n_1 = n_2 = n_3 = 2$;



in this latter case, all the pieces are Γ^* transforms of $\triangle V_2 V_3 V_t$. Thus we can restrict ourselves to the case that $n_3 = 1$. Again, if $n_2 > 1$ we find that the prolonged partition algorithm generates more than six triangles of type $[2, 3, t]$.

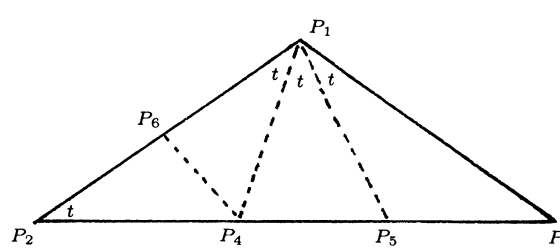


Hence we have $n_2 = n_3 = 1$, and $n_1 = 4$.



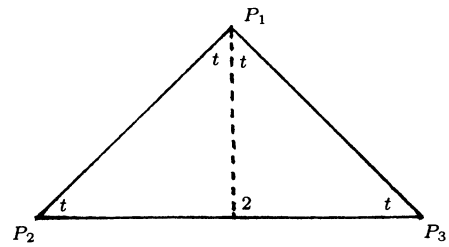
The only possibility is the one pictured.

Suppose next that $n_1 + n_2 + n_3 = 5$. Then $n_3 = 1$. If $n_2 = 1$, then $n_1 = 3$ and we consider

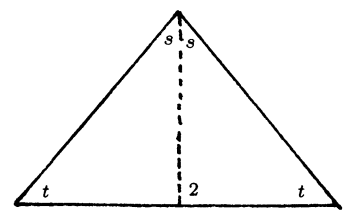


The points P_4 and P_5 are each in the Γ^* orbit of either V_3 of V_t. The angle $< P_1 P_4 P_2 \neq 2\pi/3$, otherwise drawing the angle bisector $P_4 P_6$ of $P_1 P_4 P_2$, we would find $\triangle P_1 P_4 P_6$ congruent to $\triangle P_1 P_4 P_5$—an impossibility. Hence $P_4 \in \Gamma^* V_t$. Since $t \geq 7$, the partition formed by the mirrors through P_4 and P_5 then partition $\triangle P_1 P_4 P_2$ into at least thirteen triangles, contradicting Lemma 3.4 since $r \leq 12$. Thus we can eliminate the possibility $n_1 = 3$, $n_2 = 1$, $n_3 = 1$.

We are reduced to considering the remaining case: $n_1 = 2$, $n_2 = 1$, $n_3 = 1$. We can assume that t is odd. Drawing the angle bisector $P_1 P_4$ of $< P_2 P_1 P_3$, we find that $[2, t, t]^* \subset \Gamma^* = [2, 3, t]^*$, which is impossible by Lemma 3.2. The proof of Lemma 3.5 is now complete.



Lemma 3.6. *If* $\Gamma^*(P_1, P_2, P_3) = [2, s, t]^*$, $s \neq 3 \neq t$, *and not all angles of the triangle* $P_1P_2P_3$ *are integral parts of* π, *then the triangle* $P_1P_2P_3$ *is of the type,*



(*s odd*)

Proof. Let $V_2V_sV_t$ denote the three distinct vertices of a fundamental domain for Γ^* with $\#\Gamma^*_{V_j} = 2j$, $j = 2, s, t$ (if $s = t$, distinguish between V_s and V_t). Clearly $P_i \in \Gamma^*\{V_2, V_s, V_t\}$, $i = 1, 2, 3$. But in fact $P_i \in \Gamma^*\{V_s, V_t\}$, $i = 1, 2, 3$. Otherwise the ratio of the areas of $\triangle P_1P_2P_3$ to $\triangle V_2V_sV_t$ would be the integer

$$r = \frac{1 - \left(\frac{1}{2} + \frac{n_1}{s} + \frac{n_2}{t}\right)}{1 - \left(\frac{1}{2} + \frac{1}{s} + \frac{1}{t}\right)} = \frac{\frac{1}{2} - \frac{n_1}{s} - \frac{n_2}{t}}{\frac{1}{2} - \frac{1}{s} - \frac{1}{t}}, \quad n_1 + n_2 \geq 3.$$

Then $r < 1$ unless $s < t$, $n_1 = 0$. But in that case

$$\frac{1}{2} - \frac{3}{t} \geq 2\left(\frac{1}{2} - \frac{1}{s} - \frac{1}{t}\right),$$
$$\frac{2}{s} \geq \frac{1}{2} + \frac{1}{t}$$

and this implies $s = 3$ or $s = 4$, $t = \infty$. The hypothesis of Lemma 3.6 rules out $s = 3$, and also $t = \infty$, since the angles of $\triangle P_1P_2P_3$ would be $(\pi/2, 0, 0)$, contrary to the hypothesis on not all being integral parts of π.

Consequently $P_i \notin \Gamma^*(V_2)$ for $i = 1, 2, 3$. The sum of the angles of $\triangle P_1P_2P_3$ is therefore $\pi(n_1/s + n_2/t)$ with $n_1 + n_2 \geq 4$. The ratio of areas of $\triangle P_1P_2P_3$ and $\triangle V_2V_sV_t$ is

$$r = \frac{1 - \left(\frac{n_1}{s} + \frac{n_2}{t}\right)}{\frac{1}{2} - \left(\frac{1}{s} + \frac{1}{t}\right)}.$$

Clearly $r \leq 2$ unless $s < t$, $n_1 = 0$, $n_2 = 4$, in which case

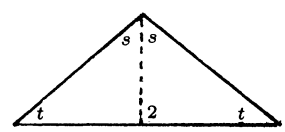
$$1 - \frac{4}{t} \geq 3\left(\frac{1}{2} - \frac{1}{s} - \frac{1}{t}\right),$$
$$\frac{3}{s} \geq \frac{1}{2} + \frac{1}{t},$$

i.e., $(s,t) = (4,4),(5,10)$, or $(6,\infty)$. The angles of $\triangle P_1P_2P_3$ being $(\pi/t, \pi/t, 2\pi/t)$ we see that the hypothesis that not all angles are integral parts of π is violated. We are left with the conclusion that $r = 2$, $n_1 = 2 = n_2$ and the angles of $\triangle P_1P_2P_3$ are either $(\pi/s, \pi/s, 2\pi/t)$ or $(\pi/t, \pi/t, 2\pi/s)$.

Collecting all the above lemmas, we get

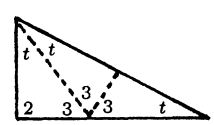
Theorem 3.7. *Let $\triangle P_1P_2P_3$ be a geodesic triangle not all of whose angles are integral parts of π. If $\Gamma(P_1, P_2, P_3)$ is discrete in* $\operatorname{Aut}\mathbf{B}^1$, *then $\triangle P_1P_2P_3$ is one of the following types:*

$D_{[s,t]}$

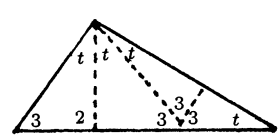


s odd, t integer or $\frac{s}{2}$

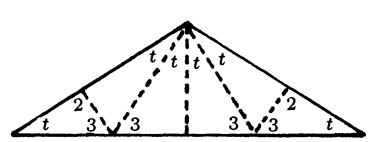
$D_{2,t}$



$D_{3,t}$



$D_{4,t}$



Next we reformulate Theorem 3.7 in terms of disc 4-tuples μ for which the groups Γ_μ coincide with the triangle groups $\Gamma(P_1, P_2, P_3)$ of the theorem.

We begin with some preliminary remarks. As pointed out in 2.1 for any disc $(n+3)$-tuple μ, the group Γ_μ is generated by complex reflections. Inasmuch as a complex reflection on $\mathbf{B}^1$ is nothing but a rotation, in the hyperbolic metric, about a point of $\mathbf{B}^1$, one can interpret Γ_μ as a triangle group on $\mathbf{B}^1$ for any disc quadruple μ.

Each geodesic triangle is uniquely determined by its orientation and three geodesic angles up to a transformation in $\operatorname{Aut}\mathbf{B}^1$. On the other hand, by [DM, 14.3.3] to each geodesic triangle $P_1P_2P_3$ with angles πa_1, πa_2, πa_3, there correspond exactly two 4-tuples μ and μ' such that Γ_μ and $\Gamma_{\mu'}$ are conjugate in $\operatorname{Aut}\mathbf{B}^1$ to the triangle group $\Gamma(P_1, P_2, P_3)$. The 4-tuple μ is obtained by solving the system of equations

$$\begin{cases} 1 - \mu_4 - \mu_i = a_i, & i = 1, 2, 3, \\ \sum_{i=1}^{4} \mu_i = 2. \end{cases}$$

The solution for μ is

$$\mu_i = \tfrac{1}{2}(1 + a_{i+1} + a_{i-1} - a_i), \qquad i = 1, 2, 3 \bmod 3,$$
$$\mu_4 = \tfrac{1}{2}(1 - a_1 - a_2 - a_3).$$

The second 4-tuple is given by

$$\mu' = (1 - \mu_1, 1 - \mu_2, 1 - \mu_3, 1 - \mu_4).$$

The existence of two $(n+3)$-tuples with equivalent Γ_μ and $\Gamma_{\mu'}$ is special for $n = 1$ and comes from the *projective* equivalence of the standard representation of $\mathrm{GL}(2)$ and its contragredient (cf. [DM, 14.3]). Solving for μ and μ' (and inverting the order of μ) yields the following result.

Theorem 3.8. *Let μ be a disc 4-tuple with Γ_μ discrete in $\operatorname{Aut}\mathbf{B}^1$. Assume that μ does not satisfy condition* INT. *Then up to a permutation, μ is one of the following*

$D_{[p,q]}$	$(\frac{1}{2} - \frac{1}{p} - \frac{1}{q}, \frac{1}{2} + \frac{1}{p} - \frac{1}{q}, \frac{1}{2} + \frac{1}{q}, \frac{1}{2} + \frac{1}{q})$	$\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$
$D'_{[p,q]}$	$(\frac{1}{2} - \frac{1}{q}, \frac{1}{2} - \frac{1}{q}, \frac{1}{2} - \frac{1}{p} + \frac{1}{q}, \frac{1}{2} + \frac{1}{p} + \frac{1}{q})$	$q < \infty$, *q odd integer*
		p integer or $q/2$
$D_{2,q}$	$(\frac{1}{2}(\frac{1}{2} - \frac{3}{q}), \frac{1}{2}(\frac{1}{2} + \frac{3}{q}), \frac{1}{2}(\frac{3}{2} - \frac{1}{q}), \frac{1}{2}(\frac{3}{2} + \frac{1}{q}))$	*q odd*
$D'_{2,q}$	$(\frac{1}{2}(\frac{1}{2} - \frac{1}{q}), \frac{1}{2}(\frac{1}{2} + \frac{1}{q}), \frac{3}{2}(\frac{1}{2} + \frac{1}{q}), \frac{3}{2}(\frac{1}{2} + \frac{1}{q}))$	$7 \le q < \infty$
$D_{3,q}$	$(\frac{1}{3} - \frac{2}{q}, \frac{1}{3} + \frac{2}{q}, \frac{2}{3} - \frac{1}{q}, \frac{2}{3} + \frac{1}{q})$	$3 \nmid q$
$D'_{3,q}$	$(\frac{1}{3} - \frac{1}{q}, \frac{1}{3} + \frac{1}{q}, 2(\frac{1}{3} - \frac{1}{q}), 2(\frac{1}{3} + \frac{1}{q}))$	$7 \le q < \infty$
$D_{4,q}$	$(\frac{1}{2} - \frac{3}{q}, \frac{1}{2} - \frac{1}{q}, \frac{1}{2} + \frac{2}{q}, \frac{1}{2} + \frac{2}{q})$	*q odd*
$D'_{4,q}$	$(\frac{1}{2} - \frac{2}{q}, \frac{1}{2} - \frac{2}{q}, \frac{1}{2} + \frac{1}{q}, \frac{1}{2} + \frac{3}{q})$	$7 \le q < \infty$.

3.9. Let $\mathscr{D}^{(4)}$ denote the set of monotone disc 4-tuples μ such that Γ_μ is discrete and μ does not satisfy condition INT. For any $\mu \in \mathscr{D}^{(4)}$, let E^μ denote the set of unordered pairs of distinct elements $\{i, j\}$ such that $(1-\mu_i-\mu_j)^{-1}$ is positive and not an integer. For any odd $q \geq 7$, set

$$[q] = (\tfrac{1}{2}-\tfrac{1}{q}, \tfrac{1}{2}-\tfrac{1}{q}, \tfrac{1}{2}-\tfrac{1}{q}, \tfrac{1}{2}+\tfrac{3}{q}).$$

For later purposes, we make some observations about the quadruples in $\mathscr{D}^{(4)}$, which can be verified by checking the list of Theorem 3.8.

3.9.1. For all $\mu \in \mathscr{D}^{(4)}$ with $\mu \neq [q]$, E^μ has only one element $\{i_0, j_0\}$. Moreover if the pair $(\mu_{i_0}, \mu_{j_0}) \neq (\frac{1}{2}-\frac{1}{q}, \frac{1}{2}-\frac{1}{q})$ with q odd then it determines μ uniquely. If $(\mu_{i_0}, \mu_{j_0}) = (\frac{1}{2}-\frac{1}{q}, \frac{1}{2}-\frac{1}{q})$, then μ is of type $D'_{[p,q]}$ or $[q]$.

3.9.2. Suppose $\mu \in \mathscr{D}^{(4)}$ and $\mu_l = 2\mu_k$ where $\{i_0, j_0\} \in E^\mu$ and $\{i_0, j_0, k, l\}$ is a permutation of $\{1, 2, 3, 4\}$. Then μ is of type $D'_{3,q}$ with $3 \nmid q$ and $q \geq 7$.

4. Discrete Γ_μ in $\operatorname{Aut}\mathbf{B}^n$, $n > 1$

4.1. By definition, a *graph* is a 1-dimensional simplicial complex; its 0-simplices are called *vertices*, and its 1-simplices are called *edges*. A subgraph H of a graph G is *full* if H contains any edge of G whose vertices are in H. By *abuse of language*, we identify a subset S of vertices of G with the full subgraph having S as its vertices.

Caution. If v is a vertex of G, the graph $G - \{v\}$ is the simplicial complex $G - v^*$, where v^* denotes the open star of v.

Notation. For any set S, let $|S| =$ the number of elements in S. If S is a graph, $|S| =$ the number of vertices in S.

Lemma 4.2. *Let G be a finite graph with $|G| = k$. Then we can find points $v_1, v_2, \ldots, v_s$ such that the graph with vertices $G - \{v_1, \ldots, v_s\}$ [i.e. $G - (v_1^* \cup v_2^* \cup \cdots \cup v_s^*)$] is a connected graph for any $s < k$.*

Proof. We use the well-known fact: Any maximal tree in a connected graph G contains all the vertices of G. Accordingly, no generality is lost in replacing the graph G by one of its maximal trees; that is, we lose no generality in adding the hypothesis that the graph G is a tree. Choose v_1 to be an extreme point of the tree G. Then the graph $G - v_1^*$ is a tree. Choose v_2 to be an extreme point of $G - v_1^*$, etc.

Let G be a graph, and let v be a vertex of G. We denote by G_v the connected component of v in G. Set

$$G^+ = \{v \in G; |G_v| > 1\}.$$

Then G^+ is a subgraph of G all of whose connected components have at least two elements.

Lemma 4.3. *Let G be a graph with $|G| \geq 5$. Then one can choose vertices v_1, v_2 in G such that $|(G - \{v_1, v_2\})^+| \geq |G^+| - 2$.*

Proof. If $|G - G^+| \geq 2$, then choose v_1, v_2 in $G - G^+$. If $|G - G^+| < 2$, then $|G^+| \geq 4$ and either (i) G^+ contains a connected component G_1 with exactly two vertices or at least four vertices, or (ii) G^+ contains at least two connected components G_1, G_2 each having exactly three vertices. In case (i), we can choose v_1, v_2 in G_1 by Lemma 4.2. In case (ii), choose v_i in G_i so that $G_i - v_i$ is connected $(i = 1, 2)$. In the latter two cases, $|(G - \{v_1, v_2\})^+| = |G^+| - 2$; if $|G - G^+| \geq 2$, then $|(G - \{v_1, v_2\})^+| = |G^+|$.

Given any finite set S, we consider real valued functions $\mu: s \to \mu_s$ satisfying $0 < \mu_s < 1$ for all $s \in S$ and $\sum_{s \in S} \mu_s = 2$. We call such a μ a *disc* $|S|$-tuple, or simply a *disc tuple.*

Definition. The graph S^μ is the simplicial complex whose vertices are the elements of S and whose edges are unordered pairs of distinct elements $\{s, t\}$ satisfying

$$(1 - \mu_s - \mu_t)^{-1} \text{ is positive and not in } \mathbf{Z}.$$

Given any $|S|$-tuple μ, and $s, t \in S$ with $\mu_s + \mu_t < 1$ and $s \neq t$, we form the partition ${}_{s,t}S$ into $|S| - 1$ cosets, one of which is the pair $\{s, t\}$ and the other cosets consisting of the single elements $\{k\}$ for $k \neq s$ or t. Let ${}_{s,t}\mu$ denote the function ν on ${}_{s,t}S$ given by $\nu_{\{k\}} = \mu_k$ for $k \notin \{s, t\}$, $\nu_{\{s,t\}} = \mu_s + \mu_t$. Then ${}_{s,t}\mu$ is a disc $(|S| - 1)$-tuple.

Terminology. A disc $(n+3)$-tuple μ is called *discrete* if Γ_μ is a discrete subgroup of $\operatorname{Aut} \mathbf{B}^n$.

4.4. Let μ be a discrete disc tuple defined on the set S. Assume that $\{i_0, j_0\}$ is a connected component of $(S^\mu)^+$. If $\mu_{i_0} + \mu_{j_0} + \mu_j < 1$ with $j \in S - \{i_0, j_0\}$, then $(\mu_{i_0}, \mu_{j_0}, \mu_j)$ is a Schwarz triple by Lemma 2.5. The following list gives all possible such triples satisfying the additional condition that $(\mu_{i_0}, \mu_{j_0}) \neq (\frac{1}{2} - \frac{1}{q}, \frac{1}{2} - \frac{1}{q})$ with $q = 3$ or 5 (cf 2.2).

μ_i	a	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{60}$	$\frac{1}{60}$	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{2}{15}$	$\frac{3}{30}$	$\frac{3}{30}$	$\frac{1}{10}$
μ_j	a	$\frac{3}{12}$	$\frac{5}{24}$	$\frac{7}{24}$	$\frac{2}{12}$	$\frac{11}{60}$	$\frac{19}{60}$	$\frac{5}{30}$	$\frac{3}{20}$	$\frac{7}{20}$	$\frac{4}{15}$	$\frac{3}{30}$	$\frac{7}{30}$	$\frac{1}{10}$
μ_j	$\frac{1}{2} - a$	$\frac{5}{12}$	$\frac{11}{24}$	$\frac{11}{24}$	$\frac{7}{12}$	$\frac{29}{60}$	$\frac{29}{60}$	$\frac{19}{30}$	$\frac{9}{20}$	$\frac{9}{20}$	$\frac{8}{15}$	$\frac{17}{30}$	$\frac{17}{30}$	$\frac{7}{10}$

We see by inspection that

(4.4.1) μ_j is uniquely determined by the pair (μ_{i_0}, μ_{j_0}) except when $(\mu_{i_0}, \mu_{j_0}) = (\frac{1}{10}, \frac{1}{10})$, in which case $\mu_j \in \{\frac{4}{10}, \frac{17}{30}, \frac{7}{10}\}$.

If $(\mu_{i_0}, \mu_{j_0}) \neq (\frac{1}{10}, \frac{1}{10})$, we denote the μ_j by μ_σ.

(4.4.2) $\mu_{i_0} + \mu_{j_0} < \inf(\frac{1}{2}, \mu_\sigma)$ if $(\mu_{i_0}, \mu_{j_0}, \mu_\sigma) \neq (a, a, \frac{1}{2} - a)$.

(4.4.3) $\mu_\sigma < \frac{1}{2}$ except for $(\mu_{i_0}, \mu_{j_0}, \mu_\sigma) = (\frac{1}{12}, \frac{2}{12}, \frac{7}{12})$, $(\frac{2}{15}, \frac{4}{15}, \frac{8}{15})$, $(\frac{1}{30}, \frac{5}{30}, \frac{19}{30})$ and $(\frac{3}{30}, \frac{7}{30}, \frac{17}{30})$.

(4.4.4) $\mu_\sigma \neq \frac{1}{2} - \frac{1}{q}$ with q odd if $(\mu_{i_0}, \mu_{j_0}) \neq (\frac{1}{q}, \frac{1}{q})$.

(4.4.5) Any Schwarz triple of the form $(a, \frac{1}{2} - a, \frac{1}{2} - \frac{1}{q})$ is $(\frac{3}{12}, \frac{3}{12}, \frac{5}{12})$, $(\frac{5}{24}, \frac{7}{24}, \frac{11}{24})$, or $(\frac{3}{20}, \frac{7}{20}, \frac{9}{20})$. In particular, $a \leq \frac{1}{4}$.

(4.4.6) Any Schwarz triple $(\frac{1}{6}, \frac{1}{6}, \mu_j)$ has $\mu_j \in \{\frac{1}{30}, \frac{7}{30}, \frac{1}{3}, \frac{13}{30}, \frac{1}{2}, \frac{19}{30}\}$.

Lemma 4.5. *Let μ be a discrete disc monotone 5-tuple with $(S^\mu)^+ = \{i_0, j_0\}$ and $\mu_k = \mu_l = \mu_m < \frac{1}{2}$. Then $\mu_{i_0} = \frac{1}{3} - \frac{1}{q}$, $\mu_{j_0} = 2(\frac{1}{3} - \frac{1}{q})$ with $q = 7, 8$, or 10.*

Proof. Consider the disc quadruple

$$\delta = (\mu_{i_0}, \mu_{j_0}, \mu_k, \mu_l + \mu_m) = (\mu_{i_0}, \mu_{j_0}, \mu_k, 2\mu_k).$$

By Lemma 2.4, δ is a discrete quadruple and by 3.9.2, δ is of type $D_{3,q}$ where $7 \leq q < \infty$ and $3 \nmid q$, $\mu_{i_0} = \frac{1}{3} - \frac{1}{q}$, $\mu_{j_0} = 2(\frac{1}{3} - \frac{1}{q})$, $\mu_k = \mu_l = \mu_m = \frac{1}{3} + \frac{1}{q}$. By hypothesis, k and l are not in $(S^\mu)^+$ and $\mu_k + \mu_l < 1$. By definition of $(S^\mu)^+$, we infer that $(1 - \mu_k - \mu_l)^{-1} \in \mathbf{Z}$; that is $(\frac{1}{3} - \frac{2}{q})^{-1}$ is an integer. This occurs only for $q = 7, 8$, or 10.

Lemma 4.6. *Let μ be a discrete disc monotone 5-tuple. Assume that $|(S^\mu)^+| = 2$ and $(\mu_{i_0}, \mu_{j_0}) \neq (\frac{1}{2} - \frac{1}{q}, \frac{1}{2} - \frac{1}{q})$, where $\{i_0, j_0\} = (S^\mu)^+$. Then μ is one of*

Case 1. $(\frac{1}{12}, \frac{2}{12}, \frac{7}{12}, \frac{7}{12}, \frac{7}{12})$ $(\frac{2}{15}, \frac{4}{15}, \frac{8}{15}, \frac{8}{15}, \frac{8}{15})$ $(\frac{1}{10}, \frac{1}{10}, \frac{4}{10}, \frac{7}{10}, \frac{7}{10})$.

Case 2. $(\frac{4}{21}, \frac{8}{21}, \frac{10}{21}, \frac{10}{21}, \frac{10}{21})$ $(\frac{5}{24}, \frac{10}{24}, \frac{11}{24}, \frac{11}{24}, \frac{11}{24})$ $(\frac{7}{30}, \frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{14}{30})$.

Case 3. $(\frac{3}{14}, \frac{3}{14}, \frac{4}{14}, \frac{9}{14}, \frac{9}{14})$ $(\frac{4}{18}, \frac{5}{18}, \frac{5}{18}, \frac{11}{18}, \frac{11}{18})$.

Case 4. $(\frac{1}{12}, \frac{3}{12}, \frac{5}{12}, \frac{5}{12}, \frac{10}{12})$.

Proof. We recall our notation: $S = \{1, 2, 3, 4, 5\}$, $S - (S^\mu)^+ = \{k, l, m\}$ with $k < l < m$, and $\mu_i \leq \mu_j$ for $i < j$.

Assume first that $(\mu_{i_0}, \mu_{j_0}, \mu_p) \neq (a, a, \frac{1}{2} - a)$ for $p \in \{k, l, m\}$, $a \neq \frac{1}{10}$. We break up the possibilities into five cases:

(1) $\mu_k + \mu_l > 1$. This case is split into two subcases:

Case (a). $(\mu_{i_0}, \mu_{j_0}) \neq (\frac{1}{10}, \frac{1}{10})$. By observation (4.4.1), μ_k, μ_l, and μ_m coincide with μ_σ of the Schwarz tuple $(\mu_{i_0}, \mu_{j_0}, \mu_\sigma)$. Here $\mu_{i_0} + \mu_{j_0} + 3\mu_\sigma = 2$; since $\mu_{i_0} + \mu_{j_0} < \frac{1}{2}$ by (4.4.2) it is necessary, though not sufficient, that $\mu_\sigma > \frac{1}{2}$. The possible μ are $(\frac{1}{12}, \frac{2}{12}, \frac{7}{12}, \frac{7}{12}, \frac{7}{12})$ and $(\frac{2}{15}, \frac{4}{15}, \frac{8}{15}, \frac{8}{15}, \frac{8}{15})$ by (4.4.3).

Case (b). $\mu_{i_0} = \mu_{j_0} = \frac{1}{10}$. Here μ_k, μ_l, and μ_m each belong to $\{\frac{4}{10}, \frac{7}{10}, \frac{17}{30}\}$. The only choice satisfying $\sum \mu_i = 2$ is $\mu = (\frac{1}{10}, \frac{1}{10}, \frac{4}{10}, \frac{7}{10}, \frac{7}{10})$.

(2) $\mu_l + \mu_m < 1$. Here we can apply 3.9.1 to the three disc quadruples $(\mu_{i_0}, \mu_{j_0}, \mu_k, \mu_l + \mu_m)$, $(\mu_{i_0}, \mu_{j_0}, \mu_l, \mu_k + \mu_m)$, $(\mu_{i_0}, \mu_{j_0}, \mu_m, \mu_k + \mu_l)$ to conclude that $\mu_k = \mu_l = \mu_m$. By Lemma 4.5, the disc quadruple $(\mu_{i_0}, \mu_{j_0}, \mu_k, 2\mu_k)$

can be only of type $D'_{3,q}$, $q = 7, 8, 10$. The resulting monotone μ are $(\frac{4}{21}, \frac{8}{21}, \frac{10}{21}, \frac{10}{21}, \frac{10}{21})$, $(\frac{5}{24}, \frac{10}{24}, \frac{11}{24}, \frac{11}{24}, \frac{11}{24})$, and $(\frac{7}{30}, \frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{14}{30})$.

(3) $\mu_k + \mu_m < 1$. Here $\mu_k + \mu_l < 1$ and we can apply 3.9.1 to the disc quadruples $(\mu_{i_0}, \mu_{j_0}, \mu_l, \mu_k + \mu_m)$ and $(\mu_{i_0}, \mu_{j_0}, \mu_m, \mu_k + \mu_l)$ to conclude that $\mu_l = \mu_m$.

In view of Case 2, we need only investigate

Case 3(a). $\mu_l + \mu_m > 1$.

Case 3(b). $\mu_l + \mu_m = 1$.

In Case 3(a), $\mu_{i_0} + \mu_{j_0} + \mu_k < 1$. Hence by (4.4.1), $\mu_k = \mu_\sigma$, and $\mu_\sigma = \mu_m = \frac{1}{2}(2 - \mu_{i_0} + \mu_{j_0} - \mu_\sigma)$. We must have

$$(1 - \mu_{i_0} - \mu_l)^{-1} = (\tfrac{1}{2}(\mu_{j_0} + \mu_\sigma - \mu_{i_0}))^{-1} \in \mathbf{Z}$$

and similarly $[\frac{1}{2}(\mu_{i_0} + \mu_j - \mu_{j_0})]^{-1} \in \mathbf{Z}$ as a result of $l \notin (S^\mu)^+$. This yields from the list of 4.4 the only possibility $\mu = (\frac{1}{12}, \frac{2}{12}, \frac{7}{12}, \frac{7}{12}, \frac{7}{12})$, which falls into Case 1. In Case 3(b), $\mu_{i_0} + \mu_{j_0} + \mu_k = 1$. By observation (4.4.2), $\mu_{i_0} + \mu_{j_0} < \frac{1}{2}$. Hence $\mu_k > \frac{1}{2}$, and $\mu_l + \mu_m > 2\mu_k > 1$. This contradiction shows there are no μ in this case.

(4) $\mu_k + \mu_m > 1$. Hence $\mu_l + \mu_m > 1$ and we find by applying observation (4.4.1) to the triples $(\mu_{i_0}, \mu_{j_0}, \mu_k)$ and $(\mu_{i_0}, \mu_{j_0}, \mu_l)$ that $\mu_k = \mu_\sigma = \mu_l$. Therefore $\mu_m = 2 - \mu_{i_0}$, $\mu_{j_0} - 2\mu_\sigma$. The resulting 5-tuples with k, l, m not in $(S^\mu)^+$ can only be

(i) $(\frac{1}{12}, \frac{3}{12}, \frac{5}{12}, \frac{5}{12}, \frac{10}{12})$,

(ii) $(\frac{1}{12}, \frac{2}{12}, \frac{7}{12}, \frac{7}{12}, \frac{7}{12})$,

(iii) $(\frac{1}{20}, \frac{7}{20}, \frac{9}{20}, \frac{9}{20}, \frac{14}{20})$,

(iv) $(\frac{2}{15}, \frac{4}{15}, \frac{8}{15}, \frac{8}{15}, \frac{8}{15})$,

(v) $(\frac{3}{30}, \frac{7}{30}, \frac{17}{15}, \frac{17}{30}, \frac{16}{30})$,

(vi) $(\frac{1}{10}, \frac{1}{10}, \frac{7}{10}, \frac{7}{10}, \frac{4}{10})$.

(v) and (vi) violate $\mu_l \leq \mu_m$. Of the other possibilities, (ii) and (iv) occur in Case 1. Possibility (iii) contracts to the disc quadruple $\delta = (\mu_1 + \mu_2, \mu_3, \mu_4, \mu_5) = (\frac{8}{20}, \frac{9}{20}, \frac{9}{20}, \frac{14}{20})$ for which $|(S^\delta)^+| = 3$ in violation of 3.9. Thus the only new 5-tuple in Case 4 is (i).

(5) None of the above five cases hold. Hence $\mu_k + \mu_m = 1$, $\mu_k + \mu_l \leq 1$, and $\mu_l + \mu_m \geq 1$. We break this up into two subcases.

Case 5(a). $\mu_l + \mu_m > 1$. Here $\mu_{i_0} + \mu_{j_0} + \mu_k < 1$. By observation (4.4.1) $\mu_k = \mu_\sigma$. Hence $\mu_m = 1 - \mu_\sigma$ and $\mu_l = 1 - \mu_{i_0} - \mu_{j_0}$.

We have $\mu_k + \mu_l = 1 + \mu_\sigma - \mu_{i_0} - \mu_{j_0} > 1$ by observation (4.4.2). Hence there are no μ in Case 5(a).

Case 5(b). $\mu_l + \mu_m = 1$. Here $\mu_k = \mu_l$, and $\mu_{i_0} + \mu_{j_0} + \mu_k = 1$. Thus $\mu_k = \mu_l = 1-\mu_{i_0}-\mu_{j_0}$, and $\mu_m = \mu_{i_0}+\mu_{j_0}$. By observation (4.4.2), $\mu_{i_0}+\mu_{j_0} < \frac{1}{2}$. Hence $\mu_k + \mu_m \leq 2\mu_m < 1$—a contradiction. Consequently, there are no μ in Case 5.

Finally we take up the previously excluded possibility and we assume:

(4.6.1) $(\mu_{i_0}, \mu_{j_0}, \mu_p) = (a, a, \frac{1}{2} - a)$ for some $p \in \{k, l, m\}$, with $a \neq \frac{1}{10}$, $\frac{1}{2} - \frac{1}{q}$ with q odd. We must have $\mu_l > \frac{1}{2} - a$ otherwise $\mu_{i_0} + \mu_{j_0} + \mu_k + \mu_l \leq 2a + 2(\frac{1}{2} - a) = 1$, which is impossible. Hence $\mu_k = \frac{1}{2} - a$. Furthermore, $2a + \mu_l \geq 1$, otherwise (a, a, μ_l) is a Schwarz triple, forcing $\mu_l - \frac{1}{2} - a$ by (4.4.1). Thus we have

(i) $\mu_k = \frac{1}{2} - a$,
(ii) $2a + \mu_m \geq 2a + \mu_l \geq 1$,
(iii) $\mu_l + \mu_m = 2 - (\frac{1}{2} + a) > 1$,
(iv) $\mu_k + \mu_l = 2 - (2a + \mu_m) \leq 1$.

Now we reconsider the five foregoing cases under assumption (4.6.1). The possibilities in Cases 1, 2, and 3(b) are excluded by (iv) and (iii) respectively. In Case 3(a), we have $\mu_l = \mu_m = \frac{1}{2}(2 - 2a - (\frac{1}{2} - a)) = \frac{1}{2}(\frac{3}{2} - a)$. From the fact $(S^k)^+ = \{i_0, j_0\}$ we get

$$M = (1 - \mu_{i_0} - \mu_l)^{-1} = [\tfrac{1}{2}(\tfrac{1}{2} - a)]^{-1}, \qquad \mu \in \mathbf{Z}.$$

Hence $a = \frac{1}{2} - \frac{2}{M}$ and under our hypotheses, μ is a positive odd integer distinct from 5, and

$$\mu = (\tfrac{1}{2} - \tfrac{2}{M}, \tfrac{1}{2} - \tfrac{2}{M}, \tfrac{2}{M}, \tfrac{1}{2} + \tfrac{1}{M}, \tfrac{1}{2} + \tfrac{1}{M})$$

possibly permuted with $[1 - (\frac{2}{M}, \frac{1}{2} + \frac{1}{M})]^{-1} \in \mathbf{Z}$; that is

$$(\tfrac{1}{2} - \tfrac{3}{M})^{-1} \in \mathbf{Z}.$$

These conditions imply $M = 7$ or 9. Hence μ is

$$(\tfrac{3}{14}, \tfrac{3}{14}, \tfrac{4}{14}, \tfrac{9}{14}, \tfrac{9}{14}) \quad \text{or} \quad (\tfrac{4}{18}, \tfrac{5}{18}, \tfrac{5}{18}, \tfrac{11}{18}, \tfrac{11}{18}).$$

Case 4. $\mu_k + \mu_m > 1$ is excluded, for otherwise $2a + \mu_l < 1$, contrary to (ii).

Case 5. $\mu_k + \mu_m = 1$ is also seen to be impossible. For $\mu = (a, a, \frac{1}{2} - a, 1-2a, \frac{1}{2}+a)$ with $1-2a \leq \frac{1}{2}+a$, i.e., $\frac{1}{6} < a$. Consequently $\mu_k+\mu_l = \frac{3}{2}-3a < 1$ and $\delta = (a, a, \frac{3}{2} - 3a, \frac{1}{2} + a)$ is a non-INT disc quadruple. By (3.9.1) and the hypothesis that $a \neq \frac{1}{2} - \frac{1}{q}$ with q odd, we see that

$$\delta = (\tfrac{1}{2} - \tfrac{2}{q}, \tfrac{1}{2} - \tfrac{2}{q}, \tfrac{1}{2} + \tfrac{1}{q}, \tfrac{1}{2} + \tfrac{3}{q}), \quad q \text{ odd}, 7 \leq q < \infty.$$

Thus $a = \frac{1}{2} - \frac{2}{q}$ and

$$\tfrac{1}{2} + a = \begin{cases} \delta_3 = \frac{1}{2} + \frac{1}{q}, & \text{or} \\ \delta_4 = \frac{1}{2} + \frac{3}{q}. \end{cases}$$

Accordingly,

$$\tfrac{1}{2} - \tfrac{2}{q} = a = 1/q \text{ or } 3/q$$

and $\frac{1}{2} = \frac{3}{q}$ or $\frac{5}{q}$, i.e., $\frac{2}{q} = \frac{1}{3}$ or $\frac{1}{5}$, contrary to the hypothesis $a \neq \frac{1}{2} - \frac{1}{q}$ with q odd.

The proof of Lemma 4.6 is now complete.

Definition. A graph G is simplex-like if each pair of its vertices belong to an edge of G.

Lemma 4.7. *Let μ be a discrete disc tuple defined on the set S.*

(a) *Each connected component of $(S^\mu)^+$ is simplex-like.*

(b) *If a connected component S_1 of $(S^\mu)^+$ has at least three vertices, then there is an odd q such that $\mu_s = \frac{1}{2} - \frac{1}{q}$ for all $s \in S_1$.*

Proof. By induction on $|S|$.

The lemma is true for $|S| = 4$ by 3.9.1.

Assume it is true for $|S| < N$. Let $N = |S| \geq 5$.

Let S_1 be a connected component of $(S^\mu)^+$. Without loss of generality, we can assume that $|S_1| \geq 3$. If $|(S^\mu)^+ - S_1| \geq 2$, choose distinct elements s, t in $(S^\mu)^+ - S_1$, set $T = {}_{s,t}S$, $\nu = {}_{s,t}\mu$. Then clearly S_1 injects into $(T^\nu - \{\{s,t\}\})^+$ which in turn injects into $(T^\nu)^+$ and thus S_1 injects into a connected component T_1 of $(T^\nu)^+$. By the induction hypothesis, T_1 is simplex-like with $\nu_{\{k\}} = \frac{1}{2} - \frac{1}{q}$, q odd for each $k \in S_1$. Since $\nu_{\{k\}} = \mu_k$ for each $k \in S_1$, the result follows for S.

If $|(S^\mu)^+ - S_1| < 2$, then $(S^\mu)^+ = S_1$ since each connected component in $(S^\mu)^+$ has at least two elements; that is, $(S^\mu)^+$ is connected. Applying Lemma 4.2, it is possible to choose elements $s, t \in S$ so that $(S^\mu - \{s,t\})^+$ is connected and has at least three elements. Set $T = {}_{s,t}S$, $\nu = {}_{s,t}\mu$. Inasmuch as $(S^\mu - \{s,t\})^+$ injects into $(T^\nu)^+$, we get by the induction hypothesis that there is an odd integer q with $\mu_j = \frac{1}{2} - \frac{1}{q}$ for all $j \in (S^\mu)^+ - \{s,t\}$. Since the latter is simplex-like, any element of $\{s,t\}$ that is connected to one element of $(S^\mu)^+ - \{s,t\}$ is connected to all. Consequently one can replace $\{s,t\}$ by other elements of $(S^\mu)^+$ so as to obtain $\mu_s = \frac{1}{2} - \frac{1}{q}$ for all $s \in (S^\mu)^+$. The proof of the lemma is now complete.

Lemma 4.8. *Let μ be a discrete disc tuple defined on the set S. Then $(S^\mu)^+$ has at most one connected component with more than two elements. Moreover, if $(S^\mu)^+$ has connected components S_1 and S_2 with $|S_1| > 2$ and $S_2 = \{i_0, j_0\}$, then $\{\mu_{i_0}, \mu_{j_0}\} \neq (\frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p})$ with p odd.*

Proof. Suppose v, w are in $(S^\mu)^+$ with $|(S^\mu)^+_v| \geq 3$, $|(S^\mu)^+_w| \geq 3$, and $(S^\mu)^+_v \neq (S^\mu)^+_w$. By Lemma 4.7, there are odd integers p, q such that $\mu_s = \frac{1}{2} - \frac{1}{p}$, $\mu_t = \frac{1}{2} - \frac{1}{q}$ for all $s \in (S)^{\mu^+}_v$, $t \in (S^\mu)^+_w$. Clearly no edge of $(S^\mu)^+$ connects v to w. Hence $\frac{1}{p} + \frac{1}{q} = 1 - \mu_v - \mu_w$ is the reciprocal of an integer, which is impossible since $(p+q)/pq$ has its numerator divisible by 2 but not its

denominator. Hence $(S^\mu)_v^+ = (S^\mu)_w^+$. The rest of the lemma follows by the same argument.

Lemma 4.9. *If $|S| \geq 6$ and $(S^\mu)^+$ has a connected component with at least three elements, then $(S^\mu)^+$ is connected.*

Proof. Suppose not. Then $(S^\mu)^+$ contains connected components S_1 and S_2 with $|S_2| = 2$, $|S_1| > 2$, by the preceding lemma. By Lemma 4.7, there is an odd integer q such that $\mu_s = \frac{1}{2} - \frac{1}{q}$ for all $s \in S_1$. Set $S_2 = \{i_0, j_0\}$ with $\mu_{i_0} \leq \mu_{j_0}$. By Lemma 4.8, $(\mu_{i_0}, \mu_{j_0}) \neq (\frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p})$ with p odd.

If $\mu_{i_0} + \mu_{j_0} + \mu_s < 1$ for some $s \in S$, then (μ_{i_0}, μ_{j_0}) is on the list of 4.4. It follows that $\mu_{i_0} + \mu_{j_0} \leq \frac{1}{2}$ by (4.4.2), and (4.4.5) applied to $(a, \frac{1}{2} - a, \frac{1}{2} - q)$ if $(\mu_{i_0}, \mu_{j_0}) = (a, a)$. Therefore $\mu_{i_0} + \mu_{j_0} + \frac{1}{2} - \frac{1}{q} < 1$. Hence by (4.4.1) and (4.4.4) either

(a) $(\mu_{i_0}, \mu_{j_0}, \frac{1}{2} - \frac{1}{q}) = (\frac{1}{10}, \frac{1}{10} \frac{4}{10})$ or

(b) $(\mu_{i_0}, \mu_{j_0}, \frac{1}{2} - \frac{1}{q}) = (\frac{1}{q}, \frac{1}{q}, \frac{1}{2} - \frac{1}{q})$, q odd.

Since q is odd, (a) is excluded. In (b), $\mu = (\frac{1}{q}, \frac{1}{q}, \frac{1}{2} - \frac{1}{q}, \frac{1}{2} - \frac{1}{q}, \frac{1}{2} - \frac{1}{q}, \dots)$ and by Lemma 2.4, μ can be contracted to the discrete disc quadruple $\delta = (\frac{1}{q}, \frac{1}{q}, 1 - \frac{2}{q}, 1 - \frac{2}{q})$. By Theorem 3.8 and Lemma 4.8 this is possible only if δ is of type $D'_{4,p}$ with p odd, i.e., $\frac{1}{q} = 1 - \frac{2}{p}$ or $\frac{2}{p} + \frac{1}{q} = 1$ with p, q odd. This implies $p = q = 3$, which is impossible, since $(1 - \mu_{i_0} - \mu_{j_0})^{-1} \notin \mathbf{Z}$. Consequently $\mu_{i_0} + \mu_{j_0} + \mu_s \geq 1$ for all $s \in S$. Since $|S| \geq 6$, $3(\frac{1}{2} - \frac{1}{q}) \leq 1$; that is $\frac{1}{2} - \frac{1}{q} \leq \frac{1}{3}$. We can have $q = 3$ or 5.

If $q = 3$, $\mu = (\mu_{i_0}, \mu_{j_0}, \mu_k, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \dots)$ then $\mu_s + \frac{1}{6} + \frac{1}{6} < 1$ implies $\mu_s \in \{\frac{1}{30}, \frac{7}{30}, \frac{1}{3}, \frac{13}{30}, \frac{1}{2}, \frac{19}{30}\}$ by (4.4.6). We have $\mu_{i_0} + \mu_{j_0} + \frac{1}{6} \geq 1$ and $\mu_{i_0} < \frac{1}{2}$, hence $\mu_{i_0} + \frac{1}{6} + \frac{1}{6} < 1$. It follows that $\mu_{i_0} \in \{\frac{1}{30}, \frac{7}{30}, \frac{1}{3}, \frac{17}{30}\}$. Moreover, $(1 - \mu_{i_0} - \frac{1}{6})^{-1} \in \mathbf{Z}$ implies that $\mu_{i_0} = \frac{1}{3}$. If $\mu_k \in S - (S_1 \cup S_2)$, then $\mu_k + \frac{1}{6} + \frac{1}{6} < 1$ and therefore $\mu_k = \frac{2}{3}$ or $\mu_k \in \{\frac{1}{30}, \frac{7}{30}, \frac{1}{3}, \frac{13}{30}, \frac{1}{2}, \frac{19}{30}\}$. $\sum \mu_s = 2$ implies that μ_{j_0} can be expressed with denominator 30 if $S \neq S_1 \cup S_2$, and this is obviously true if $S = S_1 \cup S_2$. From above we see that $1 - \mu_{i_0} - \frac{1}{6} \leq \mu_{j_0} \leq 1 - \mu_{i_0}$; this implies that $\mu_{j_0} = \frac{m}{30}$ with m an integer, $15 \leq m \leq 19$. Since $\{i_0, j_0\} = S_2$ and $\frac{1}{6} = \mu_s$ with $s \in S_1$, we have $((20 - m)/30)^{-1} = (1 - \mu_{i_0} - \mu_{j_0})^{-1} \notin \mathbf{Z}$ and $((25 - m)/30)^{-1} = (1 - \mu_{j_0} - \frac{1}{6})^{-1} \in \mathbf{Z}$, conditions that can be satisfied by no integer m between 15 and 19. This contradiction proves that S is connected in the case $q = 3$.

If $q = 5$, then $\mu = (\mu_{i_0}, \mu_{j_0}, \mu_k, \frac{3}{10}, \frac{3}{10}, \frac{3}{10}, \dots)$. Here $|S_1| = 3$, for otherwise, $\mu_{i_0} + \mu_{j_0} + \mu_k < 1$. For at least one $s \in S - S_1$, we must have $\mu_s + \frac{3}{10} + \frac{3}{10} < 1$, otherwise $\mu_s \geq \frac{4}{10}$ for all $s \in S - S_1$ and $\sum_{s \in S} \mu_s \geq 3\frac{3}{10} + 3\frac{4}{10} > 2$. Hence $(\frac{3}{10}, \frac{3}{10}, \mu_s)$ is a Schwarz triple so that $\mu_s = \frac{1}{10}$ by 2.3.11. Since $s \in S - S_2$, we have $(1 - \mu_s - \frac{3}{10})^{-1} \in \mathbf{Z}$—which is absurd. This contradiction establishes that

if $(S^\mu)^+$ has a connected component with at least three elements, then $(S^\mu)^+$ is connected. As a consequence of Lemmas 4.7, 4.8, and 4.9, we have

Corollary 4.10. *If $|S| \geq 6$, then $(S^\mu)^+$ is connected and either μ satisfies condition $\Sigma\mathrm{INT}$ or $|(S^\mu)^+| = 2$.*

We show next that the latter possibility can occur if $|S| = 6$.

Lemma 4.11. *Let μ be a discrete 6-tuple. Then μ satisfies condition $\Sigma\mathrm{INT}$ except for $(\frac{1}{12}, \frac{3}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$.*

Proof. Let S be a set on which μ is defined.

By Corollary 4.10, $(S^\mu)^+$ is a connected graph. If there is an odd integer q such that $\mu_s = \frac{1}{2} - \frac{1}{q}$ for all $s \in (S^\mu)^+$, then μ satisfies condition $\Sigma\mathrm{INT}$. By Lemma 4.7, that is the case if $|(S^\mu)^+| > 2$.

Suppose now that μ does not satisfy condition $\Sigma\mathrm{INT}$. Then $|(S^\mu)^+| = 2$ and $(\mu_v, \mu_w) \neq (\frac{1}{2} - \frac{1}{q}, \frac{1}{2} - \frac{1}{q})$ with q odd for $v, w \in (S^\mu)^+$. Choose $s, t \in S - (S^\mu)^+$, set $T = {}_{s,t}S$, and $\nu = {}_{s,t}\mu$. Then $(S^\mu)^+$ injects into $(T^\nu)^+$. By Lemma 4.6, ν is one of

I. $(\frac{1}{12}, \frac{3}{12}, \frac{5}{12}, \frac{5}{12}, \frac{10}{12})$; or

II. $(\frac{1}{10}, \frac{1}{10}, \frac{4}{10}, \frac{7}{10}, \frac{7}{10})$, $(\frac{1}{12}, \frac{2}{12}, \frac{7}{12}, \frac{7}{12}, \frac{7}{12})$,

$(\frac{3}{14}, \frac{3}{14}, \frac{4}{14}, \frac{9}{14}, \frac{9}{14})$, $(\frac{2}{15}, \frac{4}{15}, \frac{8}{15}, \frac{8}{15}, \frac{8}{15})$,

$(\frac{4}{18}, \frac{5}{18}, \frac{5}{18}, \frac{11}{18}, \frac{11}{18})$, $(\frac{4}{21}, \frac{8}{21}, \frac{10}{21}, \frac{10}{21}, \frac{10}{21})$,

$(\frac{5}{24}, \frac{10}{24}, \frac{11}{24}, \frac{11}{24}, \frac{11}{24})$, $(\frac{7}{30}, \frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{14}{30})$.

In the first case, $(\mu_v, \mu_w) = (\frac{1}{12}, \frac{3}{12})$ and $\mu_s = \mu_t = \frac{5}{12}$. In all the other cases, the ν cannot be obtained as a contraction of a μ for that would produce non-Schwarz triples (μ_i, μ_j, μ_k) with $\mu_i + \mu_j + \mu_k < 1$, in violation of Lemma 2.5. Thus, the only disc 6-tuple which may be discrete is $\mu = (\frac{1}{12}, \frac{3}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$. We shall see by the arithmeticity criterion of Proposition 5.4 below that Γ_μ is arithmetic and hence μ is discrete.

Lemma 4.12. *Let μ be a discrete disc tuple on S. If $|S| \geq 6$, then $(S^\mu)^+$ is connected.*

Proof. We use induction on $|S|$. The result is true for $|S| = 6$ by Lemmas 4.11 and 4.7. Assume the result for N-tuples with $N < |S|$.

Suppose $(S^\mu)^+$ is not connected. By Lemma 4.9, all connected components of $(S^\mu)^+$ have exactly two elements. Let $S_1 = \{v, w\}$ be a two element component of $(S^\mu)^+$. Set $T = {}_{v,w}S$, $\nu = {}_{v,w}\mu$. Then the graph $(S^\mu)^+ - \{v, w\}$ injects into $(T^\nu)^+$. By induction $(T^\nu)^+$ is connected. Consequently, $(S^\mu)^+$ has at most two connected components. By Lemma 4.9, each of these connected

components has exactly two elements. Hence $|(S^\mu)^+| = 4$. Since $|S| \geq 6$, we can choose distinct elements $s, t \in S - (S^\mu)^+$. Set $U = {}_{s,t}S, \pi = {}_{s,t}\mu$. Then the graph $(S^\mu)^+$ injects into (U^π). If $|(S^\mu)^+| > 2$, then $|(U^\pi)^+| > 2$. By the induction hypothesis, $(U^\pi)^+$ is connected. Hence, by Lemma 4.7, there is an odd q with $\pi_u = \frac{1}{2} - \frac{1}{q}$ for all $u \in (U^\pi)^+$. It follows at once that $\mu_u = \frac{1}{2} - \frac{1}{q}$ for all $u \in (S^\mu)^+$ and that $(S^\mu)^+$ is connected—contrary to our supposition. This contradiction establishes the lemma.

Theorem 4.13. *Let μ be a discrete disc N-tuple. If $N > 6$, then μ satisfies condition* $\Sigma\mathrm{INT}$.

Proof. Let S denote the set on which μ is defined. We know by Lemma 4.11 that $(S^\mu)^+$ is connected. If $|(S^\mu)^+| > 2$, then μ satisfies condition $\Sigma\mathrm{INT}$ by Lemma 4.7.

Consider first the case that $|S| = 7$. We need only eliminate the case that $|(S^\mu)^+| = 2$ with $(\mu_s, \mu_t) \neq (\frac{1}{2} - \frac{1}{q}, \frac{1}{2} - \frac{1}{q})$ for q odd, $s, t \in (S^\mu)^+$, and set $T = {}_{v,w}S$, $\nu = {}_{v,w}\mu$.

By Lemma 4.11, $\nu = (\frac{1}{12}, \frac{3}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$. But this 6-tuple cannot be the contraction of a 7-tuple respecting (4.4.1). Hence for $|S| = 7$, μ satisfies condition $\Sigma\mathrm{INT}$—or equivalently, $(S^\mu)^+$ is simplex-like with $\mu_s = \frac{1}{2} - \frac{1}{q}$ for all $s \in (S^\mu)^+$. Adopting this latter assertion as induction hypothesis, one deduces the result by induction on $|S|$.

5. Lattices Γ_μ in $\mathrm{Aut}\,\mathbf{B}^n$

5.1. Let $\mathscr{D}$ denote the following set of ten disc tuples:

$$(\tfrac{1}{12}, \tfrac{3}{12}, \tfrac{5}{12}, \tfrac{5}{12}, \tfrac{5}{12}, \tfrac{5}{12}),\ (\tfrac{1}{12}, \tfrac{3}{12}, \tfrac{5}{12}, \tfrac{5}{12}, \tfrac{10}{12}),\ (\tfrac{1}{10}, \tfrac{1}{10}, \tfrac{4}{10}, \tfrac{7}{10}, \tfrac{7}{10}),$$
$$(\tfrac{1}{12}, \tfrac{2}{12}, \tfrac{7}{12}, \tfrac{7}{12}, \tfrac{7}{12}),\ (\tfrac{3}{14}, \tfrac{3}{14}, \tfrac{4}{14}, \tfrac{9}{14}, \tfrac{9}{14}),\ (\tfrac{2}{15}, \tfrac{4}{15}, \tfrac{8}{15}, \tfrac{8}{15}, \tfrac{8}{15}),$$
$$(\tfrac{4}{18}, \tfrac{5}{18}, \tfrac{5}{18}, \tfrac{11}{18}, \tfrac{11}{18}),\ (\tfrac{4}{21}, \tfrac{8}{21}, \tfrac{10}{21}, \tfrac{10}{21}, \tfrac{10}{21}),\ (\tfrac{5}{24}, \tfrac{10}{24}, \tfrac{11}{24}, \tfrac{11}{24}, \tfrac{11}{24}),$$
$$(\tfrac{7}{30}, \tfrac{13}{30}, \tfrac{13}{30}, \tfrac{13}{30}, \tfrac{14}{30}).$$

By Lemmas 4.6, 4.11 and Theorem 4.13, *any monotone $(n+3)$-tuple μ with Γ_μ discrete in* $\mathrm{Aut}\,\mathbf{B}^n$, *$n > 1$, and $\mu \notin \mathscr{D}$ satisfies condition* $\Sigma\mathrm{INT}$.

In this section we answer the question: For which μ is Γ_μ a lattice in $\mathrm{Aut}\,\mathbf{B}^n$? We shall need the following variation of a known result.

Lemma 5.2. *Let G be a locally compact group satisfying the second axiom of countability, Γ a discrete subgroup in G, and H a closed subgroup of G. Assume that $\Gamma \cap H$ is a lattice in H. Then ΓH is a closed subset in G.*

Proof. In case Γ is a lattice in G, the result is known (cf. Raghunathan [R, Theorem 1.13]). The proof in [R] in fact applies under the weaker hypothesis that $\Gamma \cap H$ is a lattice in H. The argument is as follows. By Theorem 1.12 of [R], a sequence $(\Gamma \cap H)x_n$ in $\Gamma \cap H \backslash H$ has no convergent subsequence in

$\Gamma \cap H \backslash H$ if and only if there exists a sequence $\{\gamma_n ; 1 \leq n < \infty\}$ in $\Gamma \cap H$ such that

$(*)$ $\quad \gamma_n \neq 1$ *for any* n *, and* $x_n^{-1}\gamma_n x_n$ *converges to* 1 *as* $n \to \infty$.

Consider the projection $\pi\colon G \to \Gamma\backslash G$. If $\pi(H)$ is not closed, then there exists a sequence $\{x_n ; 1 \leq n < \infty\}$ in H such that $\lim_{n\to\infty} \pi(x_n) = \pi(z)$, $z \notin \Gamma H$. Hence $\{x_n\}$ has no convergent subsequence in $\pi(H)$. Hence $(\Gamma \cap H)x_n$ has no convergent subsequence in $\Gamma \cap H \backslash H$. Therefore there is a sequence $\{\gamma_n\}$ in $\Gamma \cap H$ satisfying $(*)$.

By passing to a subsequence if necessary we can assume that we can find $\theta_n \in \Gamma$ such that $\theta_n x_n$ converges to z. Then

$$\begin{aligned} w_n = x_n \gamma_n x_n^{-1} &= x_n^{-1}\theta_n^{-1}\theta_n\gamma_n\theta_n^{-1}\theta_n x_n \\ &= x_n^{-1}\theta_n^{-1}\delta_n\theta_n x_n, \qquad \delta_n \in \Gamma \end{aligned}$$

tends to 1 and $\theta_n x_n$ tends to z. Therefore $\delta_n = -\theta_n x_n w_n x_n^{-1}\theta_n^{-1}$ converges to 1. Since Γ is discrete, $\delta_n = 1$ for large n and therefore $\gamma_n = 1$—a contradiction.

Proposition 5.3. *Let* μ *be a disc* $(n+3)$*-tuple on the set* S *such that* Γ_μ *is discrete in* $\operatorname{Aut}\mathbf{B}^n$. *Then* Γ_μ *is a lattice in* $\operatorname{Aut}\mathbf{B}^n$.

Proof. We use the terminology and notation of [DM] with $\mathbf{B}^n = B(\alpha)^+$. Our proof is parallel to the proof of [DM, Theorems 10.18.2 and 11.4] in which it is shown, under hypotheses INT for μ, that $\tilde{\omega}_\mu$ descends to a homeomorphism of Q_{sst} into $\Gamma_\mu\backslash\overline{B(\alpha)^+}$ carrying Q_{st} onto $\Gamma_\mu\backslash B(\alpha)^+$. In place of hypothesis INT we have the hypothesis that Γ_μ is discrete in $\operatorname{Aut}\mathbf{B}^n$. We exploit the covering map properties of $\tilde{\omega}_\mu$.

For any stable partition T of S, we denote by $\mu(T)$ the corresponding disc $|T|$-tuple on T with $\mu(T)_C = \sum_{s\in C}\mu_s$ for any coset $C \in T$. As in [DM, 6.9], the corresponding moduli space $Q(T)$ can be identified with the subspace Q_T of Q_{st} and the map $\tilde{\omega}_{\mu(T)}$ may be identified with the restriction of the map $\tilde{\omega}_\mu$ to a connected component $\widetilde{Q(T)}^C$ of $\rho^{-1}(Q_T)$ where $\rho\colon \tilde{Q}_{st} \to Q_{st}$. In such an identification, the monodromy group $\Gamma_{\mu(T)}$ becomes identified with the subgroup of Γ generated by the monodromies around the cosets of T. The image of $\tilde{\omega}_{\mu(T)}$ lies in the subball $B(\alpha(T))^+$ of $B(\alpha)^+$ whose points are fixed under the conjugate of the (local) decomposition group of Q_T corresponding to $\widetilde{Q(T)}$.

We shall proceed by induction on n. Set $\Gamma = \Gamma_\mu$. Proposition 5.3 is clearly true if $n = 1$. Assume it is true in dimensions less than n.

Let T be a stable partition with $|T| = |S| - 1$. Let δ_T be a generator for the decomposition group of Q_T. Set $Y = B(\alpha)_0^+$, let Y^{δ_T} denote the fixed point set of δ_T, and set $Y_T = \Gamma Y^{\delta_T}$. The stabilizer of Y^{δ_T} in Γ contains the

subgroup identified with $\Gamma_{\mu(T)}$. Inasmuch as Γ is discrete in $\operatorname{Aut}\mathbf{B}^n$, $\Gamma_{\mu(T)}$ is discrete in $\operatorname{Aut}\mathbf{B}^n$. By the induction hypothesis, $\Gamma_{\mu(T)}$ is a lattice in $\operatorname{Aut}\mathbf{B}^{n-1}$. Consequently, by Lemma 5.2 Y^{δ_T} is closed in Y; that is, Y_T is closed in Y.

Set $Y_1 = \coprod_{\operatorname{card} T = \operatorname{card} S - 1} Y_T$. Then Y_1 is closed in Y and has $\mathbf{C}$-codimension 1. Set $X_1 = \tilde{\omega}_\mu(Y)_1$, $X^1 = \tilde{Q}_{st} - X_1$, and let φ denote the restriction of $\tilde{\omega}_\mu$ to X^1. We prove next

(5.3.1) (i) X^1 is connected nonempty and $Y - Y_1$ is connected.

(ii) $\varphi\colon X^1 \to Y - Y_1$ is a proper covering map.

(i) Clearly $X_1 \subset (X_1 \cap \tilde{Q}) \cup \rho^{-1}(Q_{st} - Q)$ and in fact there is equality since $\tilde{\omega}_\mu(\rho^{-1}(Q)_T) \subset Y_1$ for any stable partition T of S. At points of $X_1 \cap \tilde{Q}$, the map $\tilde{\omega}_\mu$ is etale and hence at such points $X_1 \cap \tilde{Q}$ has $\mathbf{C}$-codimension 1, i.e., $\mathbf{R}$-codimension 2 in $\tilde{Q}$. Consequently, X^1 is nonempty and connected. Similarly, $Y - Y_1$ is connected.

(ii) Let C be a compact subset of $Y - Y_1$. We first claim that $\Gamma\backslash\Gamma\varphi^{-1}(C)$ is compact. For consider the map $\omega_\mu\colon \tilde{Q}_{sst} \to \overline{B}(\alpha)_0^+$; it descends to a map $\overline{\varphi}\colon Q_{sst} \to \Gamma\backslash\overline{B}(\alpha)_0^+$ which is proper since Q_{sst} is compact. By Proposition 8.7 of [DM], $\tilde{\omega}_\mu$ sends $\tilde{Q}_{sst} - \tilde{Q}_{st}$ to the boundary of the ball $B(\alpha)_0^+$. Consequently, $\overline{\varphi}^{-1}(\Gamma\backslash Y) \subset Q_{st}$ and $\Gamma\backslash\Gamma\varphi^{-1}(C) \simeq \overline{\varphi}^{-1}(\Gamma\backslash\Gamma C)$. Our claim now follows from the fact that $\Gamma\backslash\Gamma C$ is closed in $\Gamma\backslash B(\alpha)_0^+$.

Next, we assert that $\varphi^{-1}(C)$ is compact. For we can choose a compact subset K such that $\varphi^{-1}(C) \subset \Gamma K$. Then for any $c \in C$, $\varphi^{-1}(c) = \gamma k$ with $\gamma \in \Gamma$, $k \in K$ implies that $c = \gamma\varphi(k)$ so that $\gamma^{-1} \in \varphi(K)C \cap \Gamma$, a finite set since Γ is discrete. It follows that $\varphi^{-1}(C)$ lies in a finite union of translates of K and is therefore compact. Thus φ is a proper map.

By Proposition 3.9 of [DM], the map φ is etale. Define the function f on $\tilde{Q}$ by

$$f(x) = \sup\{r;\ \varphi \text{ is a homeomorphism of a neighborhood of } x \text{ onto } B_r(\varphi(x))\}.$$

The function f is continuous and therefore has a positive minimum on $\varphi^{-1}(C)$ for any compact set C in $Y - Y_1$. Let $y \in Y - Y_1$ and let $r = \inf_{x \in \varphi^{-1}(y)} f(x)$. Then each point in $\varphi^{-1}(y)$ has a neighborhood mapping homeomorphically onto $B_r(y)$. By Proposition 10.11 of [DM], φ is a covering map.

Inasmuch as $Y - Y_1$ is connected and X' is nonempty, the covering map φ is surjective. Consequently $\Gamma\backslash\tilde{\omega}_\mu(\tilde{Q}_{sst})$, being compact on $\Gamma\backslash\overline{B(\alpha)}^+$ and containing a dense subset of $\Gamma\backslash Y$, contains $\Gamma\backslash Y$. Hence $\tilde{\omega}_\mu(\tilde{Q}_{st}) = Y$.

If one uses the remark following (11.3.1) of [DM], then the proof of Theorem 11.4 of [DM] proves

If Γ is discrete in $\operatorname{Aut}\mathbf{B}^n$, then $\Gamma\backslash\tilde{\omega}_\mu(\tilde{Q}_{st})$ has finite measure.

Since $\widetilde{\omega}_\mu(\widetilde{Q}_{st}) = Y$, Γ is a lattice in $\operatorname{Aut}\mathbf{B}^n$. The proof of Proposition 5.3 is now complete.

Next we prove an arithmeticity criterion

Proposition 5.4. *Let μ be a disc $(n+3)$-tuple. Let d denote the least common denominator of $\{\mu_1\}$. Then Γ_μ is an arithmetic lattice in* $\operatorname{Aut}\mathbf{B}^n$ *if and only if*

$$\text{(5.4.1)} \qquad \begin{array}{l} \textit{for each integer } A \textit{ relatively prime to } d \textit{ with } 1 < Q < d-1, \\ \sum_i \langle A\mu_i \rangle = 1 \textit{ or } n+2. \end{array}$$

Proof. Proposition 12.7 of [DM] is precisely the above proposition but with the additional hypothesis that μ satisfies condition INT. The proof of Proposition 5.4 is in effect exactly like the proof of [DM, Proposition 12.7] except that we use Proposition 5.3 in place of hypothesis INT.

Let F denote the cyclotomic field $\mathbf{Q}(\sqrt[d]{1})$ and let $E = F \cap \mathbf{R}$. Let $\mathscr{O}$ denote the ring of algebraic integers in F. By [DM, 12.1], the vector space V_μ is defined over the ring $\mathscr{O}$ and has a hermitian form ψ_0 defined over $\mathscr{O}$ such that $\Gamma_\mu \subset \mathrm{PU}(V_\mu, \psi_0)$, where Γ_μ is the image of a linear group Γ'_μ in $U(V_\mu, \psi_0)$. The projective unitary group $\mathrm{PU}(V_\mu, \psi_0)$ is the set of $\mathbf{R}$-rational points of an algebraic group G defined over the field $F \cap \mathbf{R}$ since it is isomorphic to the adjoint group of $U(V_\mu, \psi_0)$ and the adjoint representation of a unitary group is equivalent to the tensor product of its standard representation and its complex conjugate. In particular

$$\text{(5.4.2)} \qquad \Gamma_\mu \subset G(\mathscr{O} \cap \mathbf{R}).$$

The hypothesis (5.4.1) implies that for each $\sigma \in \operatorname{Gal} F/\mathbf{Q}$ with $\sigma \neq$ identity on E, ${}^\sigma\psi_0$ is either positive definite or negative definite; or equivalently, that $({}^\sigma G)(\mathbf{R})$ is compact for $\sigma \in \operatorname{Gal}(F/\mathbf{Q})$ with σ not the identity on E. It follows at once that the set of $\mathbf{R}$-rational points in

$$\operatorname{Restr}_{E|\mathbf{Q}} G := \prod_{\sigma \in \operatorname{Gal}(E/\mathbf{Q})} {}^\sigma G$$

can be factored into $G(\mathbf{R}) \times K$ with K a compact group. The subgroup $G(\mathscr{O} \cap \mathbf{R})$ of $G(\mathbf{R})$ may be taken as the projection onto the first factor $G(\mathbf{R})$ of the subgroup $(\operatorname{Restr}_{E|\mathbf{Q}} G)(\mathbf{Z})$. The latter is discrete in $G(\mathbf{R}) \times K$ and hence projects onto a discrete subgroup of the first factor. This proves that $G(\mathscr{O} \cap \mathbf{R})$ is a discrete subgroup of $G(\mathbf{R})$. By (5.4.2) it follows that Γ_μ is discrete in $G(\mathbf{R})$. Since $G(\mathbf{R}) = \operatorname{Aut}\mathbf{B}^n$, we can apply Proposition 5.3 to conclude that Γ_μ is alattice in $\operatorname{Aut}\mathbf{B}^n$. It follows at once that Γ_μ is an arithmetic lattice in $\operatorname{Aut}\mathbf{B}^n$.

5.5. By means of the arithmeticity criterion (5.4.1), one can verify that all the disc tuples of the set $\mathscr{D}$ of 5.1 yield arithmetic lattices except for the last four

which have the form

$$(5.5.1)\quad \mu = (\tfrac{1}{2}-\tfrac{1}{p}, \tfrac{1}{2}-\tfrac{1}{p}, \tfrac{1}{2}-\tfrac{1}{p}, \tfrac{1}{6}+\tfrac{1}{p}, 2(\tfrac{1}{6}+\tfrac{1}{p})), \qquad p = 42, 24, 15,$$

$$(5.5.1)' \qquad (\tfrac{4}{18}, \tfrac{5}{18}, \tfrac{5}{18}, \tfrac{11}{18}, \tfrac{11}{18}).$$

In [M2, §4] we study the relation of Γ_μ with μ of the form

$$(5.5.2)\qquad \mu = (\tfrac{1}{2}-\tfrac{1}{p}, \tfrac{1}{2}-\tfrac{1}{p}, \tfrac{1}{2}-\tfrac{1}{p}, \tfrac{1}{4}+\tfrac{3}{2p}-\tfrac{1}{2}t, \tfrac{1}{4}+\tfrac{3}{2p}+\tfrac{1}{2}t)$$

to the family of groups $\Gamma_{(p,t)}$ generated by complex reflections defined in [M1]. When μ, p, t are related as in (5.5.2), which can be inverted to

$$(5.5.2)' \qquad p = (\tfrac{1}{2}-\mu_1)^{-1}, \qquad t = \mu_4 - \mu_3$$

then Γ_μ and $\Gamma_{(p,t)}$ are commensurable. A study of the fundamental domains in $\mathbf{B}^2$ of the groups $\Gamma_{(p,t)}$ led to conjecturing Theorem 5.6 below.

Given integers π, ρ, σ, set

$$(5.5.3)\qquad \mu(\pi, \rho, \sigma) = (\tfrac{1}{2}-\tfrac{1}{\pi}, \tfrac{1}{2}-\tfrac{1}{\pi}, \tfrac{1}{2}-\tfrac{1}{\pi}, \tfrac{1}{2}+\tfrac{1}{\pi}-\tfrac{1}{\rho}, \tfrac{1}{2}+\tfrac{1}{\pi}-\tfrac{1}{\sigma})$$

whenever $\frac{1}{\pi}+\frac{1}{\rho}+\frac{1}{\sigma} = \frac{1}{2}$.

Clearly, such a μ satisfies condition $\Sigma\mathrm{INT}$ if and only if $(1-\mu_4-\mu_5)^{-1} = (\frac{1}{2}-\frac{3}{\pi})^{-1}$ is an integer whenever it is positive, or equivalently,

$$(5.5.4)\qquad \pi \in \Sigma := \{3, 4, 5, 6, 7, 8, 9, 10, 12, 18, \infty\}.$$

Theorem 5.6. *Assume that ρ and σ are integers satisfying $\frac{1}{\rho}+\frac{1}{\sigma} = \frac{1}{6}$. Then $\Gamma_{\mu(3,\rho,\sigma)}$ contains a conjugate of $\Gamma_{\mu(\rho,3,\sigma)}$ in* $\mathrm{Aut}\,\mathbf{B}^n$.

The proof of Theorem 5.6 will appear in the Ph.D. dissertation of my student, Kurt Sauter. The proof consists of an explicit construction of the requisite isomorphisms.

As an immediate consequence of Theorem 5.6 one gets

Corollary 5.7. *Let ρ, σ be integers satisfying $\frac{1}{\rho}+\frac{1}{\sigma} = \frac{1}{6}$. Then $\Gamma_{\mu(\rho,3,\sigma)}$ is commensurable to a conjugate of $\Gamma_{\mu(\sigma,3,\rho)}$ in* $\mathrm{Aut}\,\mathbf{B}^n$.

The disk 5-tuple μ of (5.5.1) clearly corresponds to the (p, t) of (5.5.2) with $t = \frac{1}{6}+\frac{1}{p}$ or equivalently to the $\mu(p, 3, q)$ of (5.5.3) with $\frac{1}{p}+\frac{1}{q} = \frac{1}{6}$. By Theorem 5.6, $\Gamma_{(p,3,q)}$ is commensurable to a conjugate of $\Gamma_{(q,3,p)}$ in $\mathrm{Aut}\,\mathbf{B}^n$. If $p = 42, 24, 15$, then $q = 7, 8, 10$ respectively and by (5.5.4) $\mu(q, 3, p)$ satisfies condition $\Sigma\mathrm{INT}$. By the main theorem of [M2], $\Gamma_{\mu(q,3,p)}$ is discrete. Consequently $\Gamma_{\mu(p,3,q)}$ is discrete for $p = 42, 24, 15$.

The following theorem, due to appear in a forthcoming paper by Deligne and Mostow, is a generalization of a result in Sauter's dissertation.

Theorem 5.8. *Let*

$$\mu = (\alpha, \alpha, \alpha, \alpha, 2 - 4\alpha),$$
$$\nu = (1 - 2\alpha, 2\alpha - \tfrac{1}{2}, 2\alpha - \tfrac{1}{2}, 1 - \alpha, 1 - \alpha),$$

where $0 < \alpha < 1$. *Then* $\Gamma_\nu \cap \Gamma_\mu$ *is of index at most* 2 *in* Γ_μ, *where* Γ_ν *denotes a suitable conjugate of the monodromy group corresponding to* ν.

We apply this result to the 5-tuple $(5.5.1)'$, which equals ν when $\alpha = \frac{7}{18}$. For $\alpha = \frac{1}{2} - \frac{1}{p}$, p an integer ≥ 3, μ satisfies condition ΣINT; and Γ_μ is discrete; hence Γ_ν is discrete in $\mathrm{PU}(1,2)$. Sauter has verified Theorem 5.8 for $(5.5.1)'$ by explicitly embedding Γ_ν into Γ_μ.

One can finally assert

Theorem 5.9. *Let* $\mathscr{D}$ *be the set of ten* 5*-tuples and* 6*-tuples listed in* 5.1. *Let* μ *be a monotone disc* $(n+3)$*-tuple with* Γ_μ *discrete in* $\mathrm{Aut}\,\mathbf{B}^n$ *and* $n > 1$. *Then either* μ *satisfies condition* ΣINT *or* μ *is any one of the tuples in* $\mathscr{D}$.

Appendix

We present here a complete list of all disc N-tuples satisfying condition ΣINT for $N \geq 5$. This list, obtained by Thurston via computer in [T], corrects the incomplete list that was hand-calculated in [M2].

	N	denominator	numerators												nonarithmetic
1.	12	6	1	1	1	1	1	1	1	1	1	1	1	1	
2.	11	6	1	1	1	1	1	1	1	1	1	1	2		
3.	10	6	1	1	1	1	1	1	1	1	1	3			
4.	10	6	1	1	1	1	1	1	1	1	2	2			
5.	9	6	1	1	1	1	1	1	1	1	4				
6.	9	6	1	1	1	1	1	1	1	2	3				
7.	9	6	1	1	1	1	1	1	2	2	2				
8.	8	4	1	1	1	1	1	1	1	1					
9.	8	6	1	1	1	1	1	1	1	5					
10.	8	6	1	1	1	1	1	1	2	4					
11.	8	6	1	1	1	1	1	2	2	3					
12.	8	6	1	1	1	1	1	1	3	3					
13.	8	6	1	1	1	1	2	2	2	2					
14.	7	4	1	1	1	1	1	1	2						
15.	7	6	1	1	1	1	1	2	5						
16.	7	6	1	1	1	1	1	3	4						
17.	7	6	1	1	1	1	2	2	4						
18.	7	6	1	1	1	1	2	3	3						
19.	7	6	1	1	1	2	2	2	3						
20.	7	6	1	1	2	2	2	2	2						
21.	7	10	2	3	3	3	3	3	3						
22.	7	12	2	2	2	2	2	7	7						
23.	6	3	1	1	1	1	1	1							
24.	6	4	1	1	1	1	1	3							
25.	6	4	1	1	1	1	2	2							

	N	denominator	numerators						nonarithmetic
26.	6	6	1	1	1	1	3	5	
27.	6	6	1	1	1	1	4	4	
28.	6	6	1	1	1	2	2	5	
29.	6	6	1	1	1	2	3	4	
30.	6	6	1	1	1	3	3	3	
31.	6	6	1	1	2	2	2	4	
32.	6	6	1	1	2	2	3	3	
33.	6	6	1	2	2	2	2	3	
34.	6	8	1	3	3	3	3	3	
35.	6	10	2	3	3	3	3	6	
36.	6	10	3	3	3	3	3	5	
37.	6	12	2	2	2	2	7	9	
38.	6	12	2	2	2	4	7	7	
39.	6	12	3	3	3	3	5	7	NA
40.	6	12	3	3	3	5	5	5	
41.	5	3	1	1	1	1	2		
42.	5	4	1	1	1	2	3		
43.	5	4	1	1	2	2	2		
44.	5	5	2	2	2	2	2		
45.	5	6	1	1	1	4	5		
46.	5	6	1	1	2	3	5		
47.	5	5	1	1	2	4	4		
48.	5	6	1	1	3	3	4		
49.	5	6	1	2	2	2	5		
50.	5	6	1	2	2	3	4		
51.	5	6	1	2	3	3	3		
52.	5	6	2	2	2	3	3		
53.	5	8	1	3	3	3	6		
54.	5	8	2	2	2	5	5		
55.	5	8	3	3	3	3	4		
56.	5	9	2	4	4	4	4		
57.	5	10	1	4	4	4	7		
58.	5	10	2	3	3	3	9		
59.	5	10	2	3	3	6	6		
60.	5	10	3	3	3	3	8		
61.	5	10	3	3	3	5	6		
62.	5	12	1	5	5	5	8		
63.	5	12	2	2	2	7	11		
64.	5	12	2	2	2	9	9		
65.	5	12	2	2	4	7	9		
66.	5	12	2	2	6	7	7		
67.	5	12	2	4	4	7	7		
68.	5	12	3	3	3	5	10		
69.	5	12	3	3	3	7	8		NA
70.	5	12	3	3	5	5	8		
71.	5	12	3	3	5	6	7		NA
72.	5	12	3	5	5	5	6		
73.	5	12	4	4	4	5	7		NA
74.	5	12	4	4	5	5	6		NA
75.	5	12	4	5	5	5	5		
76.	5	14	2	5	5	5	11		
77.	5	14	5	5	5	5	8		
78.	5	15	4	5	5	5	8		NA
79.	5	18	1	8	8	8	11		
80.	5	18	2	7	7	7	13		NA

	N	denominator	numerators					nonarithmetic
81.	5	18	2	7	7	10	10	
82.	5	18	3	3	3	13	10	
83.	5	18	5	7	7	7	10	
84.	5	18	7	7	7	7	8	NA
85.	5	20	5	5	5	11	14	NA
86.	5	20	6	6	6	9	13	
87.	5	20	6	6	9	9	10	NA
88.	5	24	4	4	4	17	19	NA
89.	5	24	7	9	9	9	14	NA
90.	5	30	5	5	5	19	26	
91.	5	30	5	5	5	22	23	NA
92.	5	30	9	9	9	11	22	
93.	5	42	7	7	7	29	34	NA
94.	5	42	13	15	15	15	26	NA

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