ANALYTIC TORSION AND R-TORSION
FOR UNIMODULAR REPRESENTATIONS

WERNER MÜLLER

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0. INTRODUCTION

Let M be a closed C^∞-manifold of dimension n. Both R-torsion and analytic torsion are smooth invariants of acyclic orthogonal (or unitary) representations ρ of the fundamental group $\pi_1(M)$. The Reidemeister-Franz torsion (or R-torsion) $\tau_M(\rho)$ of ρ is defined in terms of the combinatorial structure of M given by its smooth triangulations. The analytic torsion $T_M(\rho)$ was introduced by Ray and Singer [RS] as an analytic counterpart of R-torsion. In order to define the analytic torsion one has to choose a Riemannian metric on M. Then $T_M(\rho)$ is a certain weighted alternating product of regularized determinants of the Laplacians on differential q- forms of M with values in the flat bundle E_ρ defined by ρ. It was conjectured by Ray and Singer [RS] that $T_M(\rho) = \tau_M(\rho)$ for all acyclic orthogonal (or unitary) representations ρ. This conjecture was proved independently by Cheeger [C] and the author [Mü].

The restriction to orthogonal (or unitary) representations is certainly a limitation of the applicability of this result if $\pi_1(M)$ is infinite because an infinite discrete group will have, in general, many nonorthogonal finite-dimensional representations. It is the purpose of the present paper to remove this limitation.

We call a representation $\rho : \pi_1(M) \to GL(E)$ on a finite-dimensional real or complex vector space E *unimodular* if $|\det \rho(\gamma)| = 1$ for all $\gamma \in \pi_1(M)$. Then we define R-torsion and analytic torsion for unimodular representations, and the main result is that for odd-dimensional manifolds M the equality of the two torsions extends to all unimodular representations.

Now we shall explain this in more detail. Let $\rho : \pi_1(M) \to \mathrm{GL}(E)$ be an acyclic unimodular representation. Then the definition of R-torsion also makes sense for ρ. Nothing has to be changed. The problem is how to define the analytic torsion. If ρ is an orthogonal (or unitary) representation, then the flat bundle E_ρ over M defined by ρ can be equipped with a natural metric which is compatible with the flat connection. Associated to the metrics on E_ρ and M is the Laplacian Δ_q acting on the space $\Lambda^q(M; E_\rho)$ of E_ρ-valued differential q-forms on M. The zeta function of Δ_q is defined in the usual way by

$$\zeta_q(s; \rho) = \sum_{\lambda_j > 0} \lambda_j^{-s}, \qquad \mathrm{Re}(s) > n/2,$$

where the λ_j run through the eigenvalues of Δ_q. It is well known that this function has a meromorphic continuation to $\mathbf{C}$ which is holomorphic at $s = 0$ [Se]. Then the definition of analytic torsion given by Ray and Singer is

$$(0.1) \qquad T_M(\rho) = \exp\left(\frac{1}{2} \sum_{q=0}^{n} (-1)^q q \frac{d}{ds} \zeta_q(s; \rho)\big|_{s=0} \right).$$

For acyclic representations, $T_M(\rho)$ is independent of the metric on M.

For an arbitrary finite-dimensional representation $\rho : \pi_1(M) \to \mathrm{GL}(E)$, there is no metric on E_ρ which is compatible with the flat connection. In order to define the analytic torsion in this case we proceed as follows. We pick a metric h on E_ρ. Using the metrics on M and E_ρ, we define an inner product on $\Lambda^*(M; E_\rho)$ which in turn gives rise to the Laplacian Δ_h. Then we define the analytic torsion $T_M(\rho; h)$ by formula (0.1) where $\zeta_q(s; \rho)$ is now the zeta function of the q-component of the new Laplacian Δ_h.

We note that this approach was used by Schwarz [S] in his treatment of abelian Chern-Simons theory and that it is actually the origin for our definition of analytic torsion for nonorthogonal representations. We also note that a similar approach has recently been used by Bar-Natan and Witten [B-NW] to deal with the perturbative expansion of nonabelian Chern-Simons gauge theory with noncompact gauge group.

The first important result about analytic torsion in the present context is that, for an odd-dimensional manifold M and an acyclic representation ρ, $T_M(\rho; h)$ is independent of the choice of h on E_ρ. Of course, it is also independent of the Riemannian metric on M and we denote its common value, for any choice of h and g, by $T_M(\rho)$. In general, the variation of $T_M(\rho; h)$ with respect to h and g is given by an explicit formula. If $\dim M$ is even, the variational formula contains additional terms which are locally computable, that is, they are obtained by integrating densities which in any coordinate system are given by universal polynomials in the components $g_{ij}(x)$ and $h_{ij}(x)$ of the metrics and a finite number of their partial derivatives. There are examples showing that these terms may not vanish even if the representation is acyclic.

Using the inner product induced on the space of harmonic forms, we can also define the R-torsion $\tau_M(\rho; h)$ for nonacyclic representations ρ.

On the first sight, the choice of an arbitrary metric on E_ρ seems to be very artificial, but it is no more artificial than the arbitrary choice of the Riemannian

metric on M. In general, there is no distinguished choice of a Riemannian metric on M. Only for special manifolds, like locally symmetric manifolds, do there exist distinguished metrics on M which make analytic torsion for orthogonal representations accessible to computations. In this case, however, there exist also distinguished metrics on E_ρ for many unimodular representations which makes analytic torsion for these representations also accessible to computations.

The main result of the present paper is

Theorem 1. *Let M be a closed smooth Riemannian manifold of odd dimension, and let $\rho : \pi_1(M) \to \mathrm{GL}(E)$ be a unimodular representation on a finite-dimensional real (or complex) vector space. Let E_ρ be the associated flat bundle. Then for any choice of a metric on E_ρ we have $T_M(\rho\,;h) = \tau_M(\rho\,;h)$.*

We remark that, in general, Theorem 1 does not hold in even dimensions.

To prove Theorem 1 we follow essentially Cheeger's proof [C]. First we show that $\log T_M(\rho\,;h) - \log \tau_M(\rho\,;h)$ is independent of the metrics h on E_ρ and g on M. The main idea is then to keep track of

$$e_M(\rho) = \log T_M(\rho\,;h) - \log \tau_M(\rho\,;h)$$

if one does surgery on an embedded k-sphere in M and finally to reduce everything to the case where one is able to show the equality explicitly. Cheeger's proof is well suited to work for nonorthogonal representations, because the decisive part of the proof consists of local analysis near a given handle of M and the handles considered are such that E_ρ restricted to any of them is trivial as a flat bundle. Since we are free to choose the metrics h and g according to our purpose, we choose h and g to be the standard product metrics near the given handle. Then all the local analysis done by Cheeger in [C] extends without change to the present situation.

Now we describe briefly the content of the paper. In §1 we review Reidemeister torsion for unimodular representations and we establish some of its properties. In §2 we introduce analytic torsion for finite-dimensional representations and we prove some of its properties. We also consider analytic torsion for manifolds with boundary and prove some results related to the variation of analytic torsion and R-torsion in this case. This is needed in §3 where we establish the equality of the two torsions for unimodular representations. We explain the main steps of the proof and refer to Cheeger's paper [C] for all details not discussed in this section. Finally, in §4 we consider two examples where nonorthogonal representations occur in nature and the analytic torsion as defined in §2 arises naturally in this context. The first example are compact locally symmetric manifolds. The results of Borel and Wallach [BW] can be used to obtain numerous examples of acyclic unimodular representations. The corresponding flat bundles can be equipped with canonical metrics which are locally homogeneous so that methods of harmonic analysis can be applied. The second example is Chern-Simons gauge theory. Witten [W1] has shown that, for a compact gauge group, the analytic torsion of flat connections occurs in the perturbative expansion of the path integral defined by the Chern-Simons action.

This has been recently extended to noncompact gauge groups by Bar-Natan and Witten [B-NW], and we explain how it is related to Theorem 1.

1. REIDEMEISTER TORSION

In this section we study Reidemeister torsion for unimodular representations and collect some of its basic properties. Most of it is similar to the case of an orthogonal representation and we refer to [Mi1] for details.

Given a real vector space E of dimension n, we set

$$\det E = \Lambda^n(E).$$

If $E = \{0\}$, then $\det E = \mathbf{R}$. Furthermore, if L is a one-dimensional vector space, each nonzero element $l \in L$ determines a unique element $l^{-1} \in L^*$ defined by the equation $l^{-1}(l) = 1$ and we shall use the notation $L^{-1} = L^*$.

A volume on E will be a nonzero element $\omega \in \det E$. Any volume determines an isomorphism $\det E \cong \mathbf{R}$. Note also that a volume ω can be written as $e_1 \wedge \cdots \wedge e_n$ for some basis $e_1, \ldots, e_n$ of E.

Let

$$(1.1) \qquad C_\bullet : 0 \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \to 0$$

be a chain complex of finite-dimensional real vector spaces, and let

$$H_q(C_\bullet) = (\ker \partial_q)/\operatorname{Im} \partial_{q+1}$$

be the qth homology group of $C_\bullet$. The *determinant line* of the complex (1.1) is the one-dimensional vector space

$$(1.2) \qquad \det(C_\bullet) = \bigotimes_{q=0}^{n} \left(\det C_q \right)^{(-1)^q}.$$

We also set

$$\det H_*(C_\bullet) = \bigotimes_{q=0}^{n} \left(\det H_q(C_\bullet) \right)^{(-1)^q}.$$

Let $b_q = \dim \partial(C_q)$ and $h_q = \dim H_q(C_\bullet)$. For each q, $1 \le q \le n$, we choose $\theta_q \in \Lambda^{b_q}(C_q)$ such that $\partial \theta_q \neq 0$. Furthermore, let $0 \neq \mu_q \in \det H_q(C_\bullet)$ and $\nu_q \in \Lambda^{h_q}(\ker \partial_q)$ be such that $\pi(\nu_q) = \mu_q$ where $\pi : \ker \partial_q \to H_q(C_\bullet)$ is the canonical projection. Let $i : \ker \partial_q \to C_q$ be the natural embedding. Then $\partial \theta_{q+1} \wedge \theta_q \wedge i(\nu_q)$ is a nonzero element of $\det C_q$. Set $\mu = \bigotimes_{q=0}^{n} (\mu_q)^{(-1)^{q+1}}$. Then we define the *torsion*

$$T(C_\bullet) \in (\det C_\bullet) \otimes (\det H_*(C_\bullet))^{-1}$$

of the complex (1.1) by

$$(1.3) \qquad \begin{aligned} T(C_\bullet) = & \left(\partial \theta_1 \wedge i(\nu_0) \right) \otimes \left(\partial \theta_2 \wedge \theta_1 \wedge i(\nu_1) \right)^{-1} \otimes \cdots \\ & \otimes \left(\partial \theta_n \wedge \theta_{n-1} \wedge i(\nu_{n-1}) \right)^{(-1)^{n-1}} \otimes \left(\theta_n \wedge i(\nu_n) \right)^{(-1)^n} \otimes \mu. \end{aligned}$$

It is easy to see that $T(C_\bullet)$ is independent of the particular choices of θ_q, μ_q, and ν_q.

Now assume that we have chosen $0 \neq \omega \in \det C_\bullet$. Then ω defines an isomorphism $\det C_\bullet \cong \mathbf{R}$ and, therefore, also an isomorphism

$$\det C_\bullet \otimes \left(\det H_*(C_\bullet)\right)^{-1} \cong \left(\det H_*(C_\bullet)\right)^{-1}.$$

The image of $T(C_\bullet)$ with respect to this isomorphism will be denoted by

$$T(C_\bullet, \omega) \in \left(\det H_*(C_\bullet)\right)^{-1}.$$

Note that for a 2-term complex $C_\bullet : 0 \to C_0 \xrightarrow{A} C_1 \to 0$ with $C_0 = C_1 = \mathbf{R}^N$ and $\omega_q \in \det C_q$ being the canonical volumes, $T(C_\bullet, \omega) = \det A$.

Definition 1.4. The element $T(C_\bullet, \omega) \in \left(\det H_*(C_\bullet)\right)^{-1}$ defined above is called the *Reidemeister torsion* (or *R-torsion*) of $C_\bullet$ with respect to the volume ω.

If $C_\bullet$ is acyclic, i.e., $H_*(C_\bullet) = \{0\}$, then $\left(\det H_*(C_\bullet)\right)^{-1} = \mathbf{R}$ and $T(C_\bullet, \omega)$ is a real number. More generally, any choice of a volume $\mu \in \det H_*(C_\bullet)$ induces a natural isomorphism $\left(\det H_*(C_\bullet)\right)^{-1} \cong \mathbf{R}$ by sending $\lambda \mu^{-1}$ to $\lambda \in \mathbf{R}$. Thus $T(C_\bullet, \omega)$ can be identified with a real number $T(C_\bullet, \omega, \mu) \in \mathbf{R}$ and we set

(1.5) $$\tau(C_\bullet, \omega, \mu) = |T(C_\bullet, \omega, \mu)|.$$

Next we discuss some of the properties satisfied by R-torsion.

Let $C_\bullet$ and $C'_\bullet$ be two chain complexes with volumes $\omega \in \det C_\bullet$, $\omega' \in \det C'_\bullet$, $\mu \in \det H_*(C_\bullet)$, and $\mu' \in \det H^*(C'_\bullet)$. Let $C_\bullet \otimes C'_\bullet$ denote the tensor product complex with its standard differential, i.e.,

$$(C_\bullet \otimes C'_\bullet)_q = \bigoplus_{r+s=q} (C_r \otimes C'_s).$$

Then we have a natural isomorphism

$$\det(C_\bullet \otimes C'_\bullet) \cong (\det C_\bullet)^{\otimes \chi(C'_\bullet)} \otimes (\det C'_\bullet)^{\otimes \chi(C_\bullet)}.$$

Let $\omega \otimes \omega'$ denote the element of $\det(C_\bullet \otimes C'_\bullet)$ which corresponds to $\omega^{\otimes \chi(C'_\bullet)} \otimes (\omega')^{\otimes \chi(C_\bullet)}$ under this isomorphism. The Kunneth formula induces another natural isomorphism

$$\det H_*(C_\bullet \otimes C'_\bullet) \cong \det H_*(C_\bullet)^{\otimes \chi(C'_\bullet)} \otimes \det H_*(C'_\bullet)^{\otimes \chi(C_\bullet)}.$$

We let $\mu \otimes \mu'$ denote the element of $\det H_*(C_\bullet \otimes C'_\bullet)$ which corresponds to $\mu^{\otimes \chi(C'_\bullet)} \otimes (\mu')^{\otimes \chi(C_\bullet)}$ under this isomorphism.

Proposition 1.6. *The following equality holds*:

$$\tau(C_\bullet \otimes C'_\bullet, \omega \otimes \omega', \mu \otimes \mu') = \tau(C_\bullet, \omega, \mu)^{\chi(C'_\bullet)} \tau(C'_\bullet, \omega', \mu')^{\chi(C_\bullet)}$$

where $\chi(E_\bullet)$ denotes the Euler characteristic of the complex $E_\bullet$.

Proof. We introduce inner products on $C_\bullet$, $C'_\bullet$, $H_*(C_\bullet)$, and $H_*(C'_\bullet)$ inducing the corresponding volumes. These inner products define inner products on

$C_\bullet \otimes C'_\bullet$ and $H_*(C_\bullet \otimes C'_\bullet)$ in the canonical way. It is easy to see that the induced volumes are $\omega \otimes \omega' \in \det(C_\bullet \otimes C'_\bullet)$ and $\mu \otimes \mu' \in \det H_*(C_\bullet \otimes C'_\bullet)$. Now we can proceed as in Proposition 1.16 in [C].   Q.E.D.

Let

$$C^*_\bullet : 0 \leftarrow C^*_n \xleftarrow{\partial^*_n} C^*_{n-1} \xleftarrow{\partial^*_{n-1}} \cdots \xleftarrow{\partial^*_1} C^*_0 \leftarrow 0$$

be the dual complex to (1.1). Note that there are natural isomorphisms

$$\det C^*_q \cong (\det C_q)^{-1} \quad \text{and} \quad \det H_q(C^*_\bullet) \cong (\det H_q(C_\bullet))^{-1}.$$

Hence, we get a natural isomorphism

$$(1.7) \qquad \left(\det C_\bullet \otimes (\det H_*(C_\bullet))^{-1} \right) \otimes \left(\det C^*_\bullet \otimes (\det H_*(C^*_\bullet))^{-1} \right)^{(-1)^n} \cong \mathbf{R}.$$

Proposition 1.8. *With respect to the isomorphism* (1.7), *we have*

$$T(C_\bullet) \otimes T(C^*_\bullet)^{(-1)^n} = \pm 1.$$

The proof is straightforward and follows by an easy generalization of the argument used on p. 141 of [Mi2] to prove equality (3).

Let $\omega \in \det C_\bullet$ and $\mu \in \det H_*(C_\bullet)$ be volumes. Denote by $\omega^* \in \det C^*_\bullet \cong (\det C_\bullet)^{-1}$ and $\mu^* \in \det H_*(C^*_\bullet) \cong (\det H_*(C_\bullet))^*$ the dual volumes determined by ω and μ, respectively.

Corollary 1.9. *We have the equality*

$$\tau(C_\bullet, \omega, \mu)\, \tau(C^*_\bullet, \omega^*, \mu^*)^{(-1)^n} = 1.$$

The R-torsion arises in the following context. Let K be a finite cell complex and $\widetilde{K}$ the universal covering space of K with the fundamental group $\pi_1 = \pi_1(K)$ acting as deck transformations on $\widetilde{K}$. We think of K as being embedded as a fundamental domain in $\widetilde{K}$, so that $\widetilde{K}$ is the union of the translates of K under π_1. Let $C_q(\widetilde{K})$ be the real chain group generated by the q-cells of $\widetilde{K}$. Then $C_q(\widetilde{K})$ is a module over the real group algebra $\mathbf{R}(\pi_1)$. The q-cells of K form a preferred base for $C_q(\widetilde{K})$ as $\mathbf{R}(\pi_1)$-module. Let $\rho : \pi_1 \to \mathrm{GL}(E)$ be a representation of π_1 on a real vector space E of dimension N. It defines a flat bundle E_ρ over K and we define the chain group $C_q(K; E)$ of chains with values in the local system E_ρ by $C_q(K; E) = C_q(\widetilde{K}) \otimes_{\mathbf{R}(\pi_1)} E$. The boundary operator $\widetilde{\partial}_q : C_q(\widetilde{K}) \to C_{q-1}(\widetilde{K})$ induces $\partial_q : C_q(K; E) \to C_{q-1}(K; E)$ and we get a real chain complex

$$C_\bullet(K; E) : 0 \to C_n(K; E) \xrightarrow{\partial_n} C_{n-1}(K; E) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0(K; E) \to 0;$$

its homology will be denoted by $H_*(K; E)$. Similarly, we have the cochain complex with coefficients in E_ρ:

$$C^\bullet(K; E) : 0 \to C^0(K; E) \xrightarrow{\delta_1} C^1(K; E) \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_n} C^n(K; E) \to 0$$

where $C^q(K; E) = \mathrm{Hom}_{\mathbf{R}(\pi_1)}(C_q(\widetilde{K}), E)$ and $\delta = \partial^*$. Note that

$$C^q(K; E) \cong C^q(\widetilde{K}) \underset{\mathbf{R}(\pi_1)}{\otimes} E.$$

We shall denote the cohomology of this complex by $H^*(K; E)$.

Now assume that a volume $\theta \in \det E$ is given. Let e_j^q, $j = 1, \ldots, r_q$, be the oriented q-cells of K considered as a preferred base of the $\mathbf{R}(\pi_1)$-module $C_q(\widetilde{K})$, and let $x_1, \ldots, x_N$ be a base of E such that $\theta = \pm x_1 \wedge \cdots \wedge x_N$. Then $(e_j^q \otimes x_k)$ is a preferred base of $C_q(K; E)$ and it defines a volume $\omega_q \in \det C_q(K; E)$.

At this point, the volumes depend on several choices, namely,

    (a) the choice of the embedding of K in the covering space $\widetilde{K}$;
    (b) the orientation and ordering of the cells of K,
    (c) the choice of the base $x_1, \ldots, x_N$ of E.

To deal with (a), we make the following

Definition 1.10. A representation $\rho : \pi_1(K) \to \mathrm{GL}(E)$ is called *unimodular* if $|\det \rho(\gamma)| = 1$ for all $\gamma \in \pi_1(K)$.

If we assume that ρ is unimodular, then a different choice of the embedding of K into $\widetilde{K}$ corresponds to a change of base in $C_q(K; E)$ by a matrix whose determinant is of absolute value one; therefore, the volume ω_q changes at most by sign. Similarly, (b) and (c) cause only a change of sign. Set $\omega = \bigotimes_{q=0}^n (\omega_q)^{(-1)^q}$.

Remark. The volume ω depends, of course, on the choice of the volume θ on E. We indicate this by $\omega(\theta)$. If we replace θ by $\lambda\theta$, $\lambda \in \mathbf{R}$, then

$$\omega(\lambda\theta) = \prod_{q=0}^n \lambda^{(-1)^q r_q} \omega(\theta) = \lambda^{\chi(K)} \omega(\theta)$$

where $\chi(K)$ is the Euler characteristic of K. Thus, if K is a triangulation of a closed oriented manifold of odd dimension, then ω is independent of the choice of a volume on E. If $\chi(K) \neq 0$, we fix a volume on E and set $\omega = \omega(\theta)$.

Then we define the R-torsion $\tau_K(\rho)$ by

$$\tau_K(\rho) = \left[\tau(C_\bullet(K; E), \omega) \right] \in \left(\det H_*(K; E) \right)^{-1}/\{\pm 1\}$$

and this definition does not depend on the choices that we made to define the volume element ω for $C_\bullet(K; E)$. Let $\mu \in \det H_*(K; E)$ be a volume . Then we have a natural isomorphism

$$\left(\det H_*(K; E) \right)^{-1}/\{\pm 1\} \cong \mathbf{R}^+,$$

and $\tau_K(\rho)$ can be identified with a positive real number which we denote by $\tau_K(\rho; \mu)$. An important fact is that $\tau_K(\rho; \mu)$ is invariant under subdivision. The proof is similar to the corresponding proof for orthogonal representations in [Mi1]. Thus $\tau_K(\rho)$ is a combinatorial invariant.

Let M be a compact smooth manifold, and let $\rho : \pi_1(M) \to \mathrm{GL}(E)$ be a unimodular representation. The manifold M has a distinguished class of triangulations, the so-called smooth triangulations. Any two of these have a common subdivision (cf. [Mu]). Thus, for any choice of a volume $\mu \in \det H_*(M)$, we get an invariant $\tau_M(\rho; \mu) \in \mathbf{R}^+$.

Definition 1.11. The positive real number $\tau_M(\rho; \mu)$ is called the *Reidemeister torsion* (or *R-torsion*) of the manifold M with respect to ρ and μ.

Recall that a representation $\rho : \pi_1(M) \to \mathrm{GL}(E)$ is called *acyclic* if $H^*(M; E_\rho) = 0$.

If ρ is acyclic, the R-torsion $\tau_M(\rho)$ is a positive real number which is an invariant of the manifold M and the representation ρ.

As above, the R-torsion $\tau_M(\rho; \mu)$ has several important properties. Here we recall two of them which we need in §3 to prove the equality of analytic torsion and R-torsion.

Let M be a closed orientable manifold of dimension n, and let $\rho : \pi_1(M) \to \mathrm{GL}(E)$ be a unimodular representation with associated flat bundle E_ρ. Let ρ^* be the contragredient representation. Then ρ^* is also unimodular and the associated flat bundle is the dual vector bundle E_ρ^* of E_ρ. For each q, the Poincaré duality isomorphism induces an isomorphism

$$\lambda_q : \det H_q(M; E^*) \overset{\sim}{\longrightarrow} \left(\det H_{n-q}(M; E)\right)^{-1}.$$

The following result is a slight extension of Milnor's duality theorem [Mi2].

Proposition 1.12. *Assume that, for each q, we have chosen volumes $\mu_q \in \det H_q(M; E)$ and $\mu_q^* \in \det H_q(M; E^*)$ satisfying $\lambda_q(\mu_q^*) = (\mu_{n-q})^{-1}$. Then we have*

$$\tau_M(\rho; \mu)\, \tau_M(\rho^*; \mu^*)^{(-1)^n} = 1.$$

Let M_1, M_2 be two compact smooth manifolds, and let $\rho_i : \pi_1(M_i) \to \mathrm{GL}(E_i)$, $i = 1, 2$, be unimodular representations with associated flat bundles E_{ρ_1}, E_{ρ_2}. Note that $\pi_1(M_1 \times M_2) = \pi_1(M_1) \times \pi_1(M_2)$ and the representation $\rho_1 \otimes \rho_2 : \pi_1(M_1 \times M_2) \to \mathrm{GL}(E_1 \otimes E_2)$ is unimodular. Denote by $p_i : M_1 \times M_2 \to M_i$, $i = 1, 2$, the canonical projection. Then the flat bundle over $M_1 \times M_2$ defined by $\rho_1 \otimes \rho_2$ equals $p_1^* E_{\rho_1} \otimes p_2^* E_{\rho_2}$. Let $\mu_i \in \det H_*(M_i; E_i)$, $i = 1, 2$, be volumes. By the Kunneth formula we obtain an isomorphism

$$\det H_*(M_1 \times M_2; E_1 \otimes E_2)$$
$$= \det H_*(M_1; E_1)^{\chi(M_2; E_2)} \otimes \det H_*(M_2; E_2)^{\chi(M_1; E_1)}.$$

Let $\mu_1 \otimes \mu_2$ be the element of $\det H_*(M_1 \times M_2; E_1 \otimes E_2)$ which corresponds to $\mu_1^{\chi(M_2; E_2)} \otimes \mu_2^{\chi(M_1; E_1)}$ under this isomorphism. As a consequence of Proposition 1.6 we get

Proposition 1.13. *With the notation above, we have the equality*

$$\tau_{M_1 \times M_2}(\rho_1 \otimes \rho_2; \mu_1 \otimes \mu_2) = \tau_{M_1}(\rho_1; \mu_1)^{\chi(M_2; E_2)} \cdot \tau_{M_2}(\rho_2; \mu_2)^{\chi(M_1; E_1)}.$$

The volumes μ that we are considering arise in the following way. We choose metrics on M and on the flat bundle E_ρ defined by ρ. If $\partial M \neq \varnothing$ we also impose boundary conditions for the Laplacian on forms. Then the space of harmonic forms has an inner product and, via the De Rham isomorphism combined with Poincaré duality, we get inner products in $H_*(M; E)$ or $H_*(M, \partial M; E)$, depending on the boundary conditions. These inner products define volumes. In particular, for this choice of volumes the condition of Proposition 1.12 is satisfied.

Remark. We may also work with complex coefficients and define the R-torsion in this case. As above, a representation $\rho : \pi_1(M) \to \mathrm{GL}(E)$ on a complex finite-dimensional vector space E is called unimodular if $|\det \rho(\gamma)| = 1$ for all $\gamma \in \pi_1(M)$. For a unimodular representation ρ the R-torsion $\tau_M(\rho; \mu)$ is defined in the same way as in the real case.

Let $i : \mathrm{GL}(N, \mathbf{C}) \to \mathrm{GL}(2N, \mathbf{R})$ be the standard embedding which sends $g = A + \sqrt{-1}B$, $A, B \in \mathrm{Mat}(N, \mathbf{R})$ to $\left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right) \in \mathrm{GL}(2N, \mathbf{R})$. If $|\det(g)| = 1$ then $|\det(i(g))| = 1$. Hence, if $\rho : \pi_1(M) \to \mathrm{GL}(N, \mathbf{C})$ is unimodular, then $i \circ \rho : \pi_1(M) \to \mathrm{GL}(2N, \mathbf{R})$ is also unimodular and we are back to our previous framework.

2. ANALYTIC TORSION FOR UNIMODULAR REPRESENTATIONS

In this section we define the analytic torsion and prove some of its properties. Let (M, g) be a compact Riemannian manifold of dimension n. For simplicity we assume that M is orientable. Let $\rho : \pi_1(M) \to \mathrm{GL}(E)$ be a representation of the fundamental group of M on a real vector space of dimension N. The representation ρ defines a flat vector bundle E_ρ over M. We choose a metric h on E_ρ. It induces a $\mathbf{R}$-linear isomorphism $\# : E_\rho \xrightarrow{\sim} E_\rho^*$ where E_ρ^* is the dual vector bundle. Let $\Lambda^p(E)$ be the space of C^∞ p-forms on M with values in E_ρ, i.e., the space of smooth sections of the vector bundle $\Lambda^p(T^*M) \otimes E_\rho$. Then $\#$ extends to an isomorphism

$$\# : \Lambda^p(E) \xrightarrow{\sim} \Lambda^p(E^*)$$

for each p. Furthermore, the Riemannian metric on M defines a linear mapping

$$* : \Lambda^p(E) \longrightarrow \Lambda^{n-p}(E)$$

for each p (cf. [MM, §2]) which satisfies $** = (-1)^{p(n-p)}$ on $\Lambda^p(E)$, and it is easy to see that $*$ and $\#$ commute. The usual exterior product of differential forms combined with the evaluation map $\mathrm{tr} : E_\rho \otimes E_\rho^* \to \mathbf{R}$ induces the following exterior product for vector-valued forms

$$\wedge : \Lambda^p(E) \otimes \Lambda^q(E^*) \longrightarrow \Lambda^{p+q}(M)$$

where $\Lambda^*(M)$ is the space of smooth differential forms on M (cf. [MM, §2]). Then an inner product on $\Lambda^p(E)$ is defined by

$$\langle \omega, \omega' \rangle = \int_M \omega \wedge * \circ \# \omega'.$$

Let $L^2\Lambda^p(E)$ denote the completion of $\Lambda^p(E)$ with respect to the norm defined by this inner product. Since E_ρ is flat, we have the De Rham complex

$$\Lambda^0(E) \xrightarrow{d_0} \Lambda^1(E) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Lambda^n(E) .$$

The formal adjoint of d_p on $\Lambda^p(E)$ is given by

$$\delta_p = (-1)^{np+n+1} * \circ \#^{-1} d_p \# \circ * .$$

Remark. We may, of course, combine $*$ and $\#$ into a single operator $\tilde{*}$. The advantage of this notation is that the dependence on the Riemannian metric on M and on the Euclidean metric on E_ρ is separated by $*$ and $\#$.

Now we define the Laplacian on p-forms as usual by

$$\Delta_p = \delta_p d_p + d_{p-1}\delta_{p-1}.$$

Note that Δ_p does not depend on the choice of an orientation on M, and therefore we can define Δ_p also if M is not orientable. One has to use the formalism of densities (see [C, §3] for details).

Now assume that $\partial M = \varnothing$. Then the Laplacian is a symmetric, positive semidefinite, elliptic operator with pure point spectrum

$$0 \le \lambda_0 \le \lambda_1 \le \cdots \to \infty .$$

Let $\mathscr{H}^p(E)$ denote the kernel of Δ_p. This is the space of E_ρ-valued harmonic p-forms on M, and the De Rham map induces an isomorphism

$$(2.1) \qquad\qquad \mathscr{H}^p(E) \xrightarrow{\sim} H^p(M;E)$$

where $H^p(M;E)$ is the cohomology of M with coefficients in the flat bundle E_ρ. Let P_q denote the orthogonal projection of $L^2\Lambda^q(E)$ onto the subspace $\mathscr{H}^q(E)$. The zeta function associated to the Laplacian Δ_q on $\Lambda^q(E)$ is defined by

$$\zeta_q(s;\rho) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}(e^{-t\Delta_q} - P_q)\, dt = \sum_{\lambda_j > 0} \lambda_j^{-s}, \qquad \operatorname{Re}(s) > n/2.$$

It is proved in [Se] that $\zeta_q(s;\rho)$ extends to a meromorphic function of $s \in \mathbf{C}$ which is holomorphic at $s = 0$. As in the case of an orthogonal representation, we define the *analytic torsion* $T_M(\rho;h)$ by

$$(2.2) \qquad T_M(\rho;h) = \exp\left(\frac{1}{2}\sum_{q=0}^n (-1)^q q \frac{d}{ds}\zeta_q(s;\rho)\big|_{s=0}\right).$$

Note that $T_M(\rho;h)$ also depends on the Riemannian metric g, but we do not indicate this dependence explicitly.

First we establish a formula for the variation of the analytic torsion with respect to the Riemannian and Euclidean structure. To obtain such a formula we may proceed essentially the same way as in [RS, Theorems 2.1 and 7.3]. Consider one-parameter families h_u, g_v of metrics on E_ρ and M, respectively. Let $\Delta_q(u,v)$ be the Laplacian with respect to (h_u, g_v). Set

$$(2.3) \qquad\qquad \alpha = \#^{-1}\dot{\#} \quad\text{and}\quad \beta = *^{-1}\dot{*},$$

where

$$\# = d\#/du \quad \text{and} \quad \dot{*} = d * /dv.$$

We regard α and β as differential operators of order zero. Then it follows from Lemma 1.7.7 in [G] that, for $t \to 0$, there exist asymptotic expansions of the form

$$(2.4) \qquad \text{Tr}\big(\alpha e^{-t\Delta_q(u,v)}\big) \sim \sum_{k=0}^{\infty} c_{q,k}(u,v)\, t^{-n/2+k},$$

$$(2.5) \qquad \text{Tr}\big(\beta e^{-t\Delta_q(u,v)}\big) \sim \sum_{k=0}^{\infty} d_{q,k}(u,v)\, t^{-n/2+k}.$$

Theorem 2.6. *Let M be a closed oriented manifold of dimension n, and let ρ be a representation of $\pi_1(M)$ on E. Let h_u and g_v be one-parameter families of metrics on E_ρ and M, respectively. Let $P_q(u, v)$ denote the harmonic projection with respect to (h_u, g_v). Furthermore, let $c_q(u, v)$ and $d_q(u, v)$ denote the constant terms of the asymptotic expansions (2.4) and (2.5), respectively. Then*

$$\frac{\partial}{\partial u}\log T_M(\rho\,;u,v) = \frac{1}{2}\sum_{q=0}^{n}(-1)^q\,\text{Tr}\big(\alpha_u P_q(u,v)\big) - \frac{1}{2}\sum_{q=0}^{n}(-1)^q\,c_q(u,v),$$

$$\frac{\partial}{\partial v}\log T_M(\rho\,;u,v) = \frac{1}{2}\sum_{q=0}^{n}(-1)^q\,\text{Tr}\big(\beta_v P_q(u,v)\big) - \frac{1}{2}\sum_{q=0}^{n}(-1)^q\,d_q(u,v).$$

If n is odd, the local terms c_q and d_q vanish and, by (2.1), we get

Corollary 2.7. *Assume that $\dim M$ is odd and $\rho : \pi_1(M) \to \text{GL}(E)$ is acyclic. Then $T_M(\rho\,;h)$ has the same value for any choice of a Riemannian metric on M and a metric h on E_ρ.*

Thus, for an odd-dimensional manifold M and an acyclic representation ρ, $T_M(\rho\,;h)$ is an invariant $T_M(\rho)$ of the manifold M and of the representation ρ. This justifies the following

Definition 2.8. Let M be an odd-dimensional closed manifold and ρ an acyclic representation of $\pi_1(M)$. Then the analytic torsion $T_M(\rho)$ of ρ is the common value of $T_M(\rho\,;h)$ for any choice of a metric h on E_ρ.

Thus for closed manifolds of odd dimension our definition of analytic torsion works well. The situation is different if the dimension of M is even. For orthogonal (or unitary) representations the individual local terms do not vanish in general, but their alternating sum does. This can be seen as follows. Assume that E is equipped with an inner product and $\rho : \pi_1(M) \to \text{GL}(E)$ preserves the inner product, that is, ρ is orthogonal. Then $*$ commutes with Δ. Hence

$$e^{-t\Delta_{n-q}}*^{-1} = *^{-1}e^{-t\Delta_q}.$$

Multiplying by $\dot{*}$ and taking the trace gives

$$\text{Tr}\big(*^{-1}\dot{*}e^{-t\Delta_{n-q}}\big) = \text{Tr}\big(e^{-t\Delta_{n-q}}*^{-1}\dot{*}\big) = \text{Tr}\big(*^{-1}e^{-t\Delta_q}\dot{*}\big) = \text{Tr}\big(\dot{*}\,*^{-1}e^{-t\Delta_q}\big).$$

Now observe that $\bar{*}*^{-1} = -*^{-1}\bar{*}$. Then

$$\mathrm{Tr}\big(*^{-1}\bar{*}e^{-t\Delta_{n-q}}\big) = -\mathrm{Tr}\big(*^{-1}\bar{*}e^{-t\Delta_q}\big),$$

which implies $d_{q,n/2} = -d_{n-q,n/2}$. Therefore, $\sum_{q=0}^{n}(-1)^q d_{q,n/2} = 0$.

This is false, however, if the metric on E_ρ is not compatible with the flat connection. As an example consider a closed surface Σ of genus $g \geq 2$ and choose on Σ the metric of constant curvature -1. Let E_ρ be any flat orthogonal bundle over Σ. We denote by $\langle \cdot, \cdot \rangle$ the canonical metric on E_ρ which is compatible with the flat connection. Let $f \in C^\infty(\Sigma)$ and $u \in \mathbf{R}$. Then we define a new metric h_u by

$$h_u(v, w) = e^{uf(z)}\langle v, w\rangle_z \quad \text{for} \quad v, w \in E_{\rho,z}, \ z \in \Sigma.$$

If we use the canonical metric to identify E_ρ and E_ρ^*, then $\#_u : E_\rho \to E_\rho$ is given by

$$\#_u v = e^{uf(z)}v, \qquad v \in E_{\rho,z}, \ z \in \Sigma.$$

Hence $\alpha_u = \#^{-1}\dot{\#} = f \cdot \mathrm{Id}$. Let $K_{q,u}(z, z', t)$ be the heat kernel on E_ρ-valued q-forms of Σ where E_ρ is equipped with h_u. Then

$$\mathrm{tr}\big(\alpha_u(z)K_{q,u}(z, z, t)\big) = f(z)\,\mathrm{tr}\big(K_{q,u}(z, z, t)\big).$$

Let m be the rank of E_ρ and $K_q(z, z', t)$ the heat kernel for the Laplacian on q-forms. Then, for $u = 0$, $\mathrm{tr}\, K_{q,0}(z, z, t) = m\,\mathrm{tr}\, K_q(z, z, t)$. Furthermore, the first three coefficients of the asymptotic expansion of $\mathrm{tr}\, K_q(z, z, t)$ are well known (cf. [G, p. 330]). Since, by assumption, the curvature of Σ is $\equiv -1$, it follows that

$$c_0(0) - c_1(0) + c_2(0) = -\frac{m}{2\pi}\int_\Sigma f(z)\,dz.$$

This shows that in the even-dimensional case the local contribution to the variational formula for $T_M(\rho; h)$ is nonzero in general. (See [B-Z] for further results about this problem.)

The analytic torsion for orthogonal representations satisfies a number of functorial properties (cf. [RS, §2]) which reflect known properties of the R-torsion. They continue to hold, with some modifications, for arbitrary finite-dimensional representations.

Let E_ρ be a flat bundle over M defined by the representation $\rho: \pi_1(M) \to \mathrm{GL}(E)$. We denote by ρ^* the contragredient representation. The associated flat bundle is the dual bundle E_ρ^* of E_ρ. Any metric h on E_ρ induces a metric on E_ρ^* which we denote by h^*.

Proposition 2.9. *Suppose that M is a closed orientable manifold of dimension n. Let ρ be a finite-dimensional real representation of $\pi_1(M)$ and E_ρ the associated flat vector bundle. For any choice of a metric h on E_ρ, we have*

$$T_M(\rho; h)\, T_M(\rho^*; h^*)^{(-1)^n} = 1.$$

Proof. We may essentially proceed as in the proof of Theorem 2.3 in [RS]. Let Δ_ρ (resp. Δ_{ρ^*}) denote the Laplacian on E_ρ-valued (resp. E_ρ^*-valued) differential forms of M. The zeta functions $\zeta_q(s\,;\rho)$ and $\zeta_q(s\,;\rho^*)$ of the operators $\Delta_{\rho,q}$ and $\Delta_{\rho^*,q}$, respectively, are defined as above. By (2.2), it suffices to show that

$$(2.10) \qquad \sum_{q=0}^{n}(-1)^q q\left(\zeta_q(s\,;\rho)+(-1)^n\zeta_q(s\,;\rho^*)\right)\equiv 0.$$

Let $\lambda > 0$ be an eigenvalue of $\Delta_{\rho,q}$, and let $\mathscr{E}_q(\lambda\,;\rho)\subset\Lambda^q(E)$ be the corresponding eigenspace. We introduce the following subspaces

$$\mathscr{E}_q'(\lambda\,;\rho)=\{\phi\in\mathscr{E}_q(\lambda\,;\rho)\,|\,d\phi=0\}\quad\text{and}\quad\mathscr{E}_q''(\lambda\,;\rho)=\{\phi\in\mathscr{E}_q(\lambda\,;\rho)\,|\,\delta\phi=0\}.$$

As in [RS, pp. 154–155], it follows that

$$\mathscr{E}_q(\lambda\,;\rho)=\mathscr{E}_q'(\lambda\,;\rho)\oplus\mathscr{E}_q''(\lambda\,;\rho)$$

and $\lambda^{-1/2}d$ defines an isometry of $\mathscr{E}_q''(\lambda\,;\rho)$ onto $\mathscr{E}_{q+1}'(\lambda\,;\rho)$, with inverse $\lambda^{-1/2}\delta$. Let $m_q'(\lambda\,;\rho)$ and $m_q''(\lambda\,;\rho)$ be the dimension of the spaces $\mathscr{E}_q'(\lambda\,;\rho)$ and $\mathscr{E}_q''(\lambda\,;\rho)$, respectively. Then the multiplicity of λ is

$$m_q(\lambda\,;\rho)=m_q'(\lambda\,;\rho)+m_q''(\lambda\,;\rho)=m_q'(\lambda\,;\rho)+m_{q+1}'(\lambda\,;\rho),$$

and we obtain
(2.11)

$$\sum_{q=0}^{n}(-1)^q q\,\zeta_q(s\,;\rho)=\sum_{q=1}^{n}(-1)^q\sum_{\lambda>0}m_q'(\lambda\,;\rho)\lambda^{-s}=-\sum_{q=0}^{n-1}(-1)^q\sum_{\lambda>0}m_q''(\lambda\,;\rho)\lambda^{-s}$$

and similarly for ρ^*. Let d,δ denote the differential and the codifferential with respect to E_ρ and d',δ' the corresponding operators with respect to $E_{\rho^*}=E_\rho^*$. Then $*\circ\#:\Lambda^q(E)\to\Lambda^{n-q}(E^*)$ satisfies $*\circ\#d\delta=\delta'd'*\circ\#$. Hence, $*\circ\#$ defines an isometry of $\mathscr{E}_q'(\lambda\,;\rho)$ onto $\mathscr{E}_{n-q}''(\lambda\,;\rho^*)$ showing that $m_q'(\lambda\,;\rho)=m_{n-q}''(\lambda\,;\rho^*)$. Combined with (2.11), we obtain (2.10).   Q.E.D.

Since $\left(H_*(M\,;E)\right)^*\cong H_*(M\,;E^*)$, the representation ρ is acyclic iff the contragredient representation $\rho^*:\pi_1(M)\to\mathrm{GL}(E^*)$ is acyclic and we get

Corollary 2.12. *Let M be a closed orientable manifold of even dimension, and let $\rho:\pi_1(M)\to\mathrm{GL}(E)$ be an acylic representation. Then*

$$T_M(\rho)\,T_M(\rho^*)=1.$$

Assume that E is equipped with an inner product so that $E\cong E^*$. Then ρ is orthogonal iff $\rho^*=\rho$. Hence, for orthogonal representations, Corollary 2.12 implies $T_M(\rho)=1$, which agrees with Theorem 2.3 in [RS].

Next consider two closed oriented Riemannian manifolds M_i, $i=1,2$, and let E_{ρ_i} be a flat bundle over M_i defined by a representation $\rho_i:\pi_1(M_i)\to\mathrm{GL}(E_i)$. We choose a metric h_i on E_{ρ_i}. Furthermore, let

$$p_i:M_1\times M_2\to M_i,\qquad i=1,2,$$

be the canonical projection. The flat bundle $p_1^*(E_{\rho_1}) \otimes p_2^*(E_{\rho_2})$ over $M_1 \times M_2$ is associated to the representation $\rho_1 \otimes \rho_2$, and we denote by $h_1 \times h_2$ the product metric on this bundle. We also assume that $M_1 \times M_2$ is equipped with the product metric.

Proposition 2.13. *With the notation above we have the equality*

$$\log T_{M_1 \times M_2}(\rho_1 \otimes \rho_2; h_1 \times h_2)$$
$$= \chi(M_2; E_2) \log T_{M_1}(\rho_1; h_1) + \chi(M_1; E_1) \log T_{M_2}(\rho_2; h_2),$$

where $\chi(M_i; E_i)$ denotes the Euler characteristic.

This is proved by an easy generalization of the proof of Theorem 2.5 of [RS].

Now assume that the representation $\rho : \pi_1(M) \to \mathrm{GL}(E)$ is unimodular. Then the Reidemeister torsion $\tau_M(\rho)$ is defined to be an element of the one-dimensional vector space $\left(\det H_*(M; E)\right)^{-1}$. In order to get a real number we have to choose a volume $\mu \in \det H_*(M; E)$. In the present context we choose μ as follows. Let h be a metric on E_ρ. Then the inner product on $\Lambda^*(E)$ induces an inner product on the space of E_ρ-valued harmonic forms $\mathscr{H}^*(E)$ and we use the De Rham isomorphism (2.1) combined with Poincaré duality to introduce an inner product in $H_*(M; E)$. The inner product in $H_q(M; E)$ defines a volume μ_q which, up to sign, is uniquely determined by the metric h on E_ρ.

Definition 2.14. Let $\mu_h = \bigotimes_{q=0}^{n} (\mu_q)^{(-1)^q} \in \det H_*(M; E)$ be the volume defined by the metric h on E. Then we set

$$\tau_M(\rho; h) = \tau_M(\rho; \mu_h)$$

where the latter is the R-torsion of M with respect to ρ and μ_h (cf. §1).

Again the R-torsion depends not only on h, but also on g, unless ρ is acyclic. The computation of the variational formula for the R-torsion is similar to the orthogonal case and is an easy extension of the proof of Theorem 7.6 in [RS]. The result is

Theorem 2.15. *Let M be a closed oriented manifold and $\rho : \pi_1(M) \to \mathrm{GL}(E)$ a unimodular representation. Let h_u and g_v be one-parameter families of metrics on E_ρ and M, respectively. Denote by $\tau_M(\rho; u, v)$ the R-torsion and by $P_q(u, v)$ the harmonic projection with respect to (h_u, g_v). Then*

$$\frac{\partial}{\partial u} \log \tau_M(\rho; u, v) = \frac{1}{2} \sum_{q=0}^{n} (-1)^q \, \mathrm{Tr}(\alpha_u P_q(u, v)),$$

$$\frac{\partial}{\partial v} \log \tau_M(\rho; u, v) = \frac{1}{2} \sum_{q=0}^{n} (-1)^q \, \mathrm{Tr}(\beta_v P_q(u, v)),$$

where α_u and β_v are defined by (2.3).

If we combine Theorems 2.6 and 2.15, we get

Corollary 2.16. *Let M be a closed oriented manifold, and let $\rho : \pi_1(M) \to$ GL(E) be a unimodular representation. If* $\dim M$ *is odd, then*

$$\log T_M(\rho\,;h) - \log \tau_M(\rho\,;h)$$

has the same value for any choice of a metric h on E_ρ and a Riemannian metric on M. If $\dim M$ *is even, then the variation of* $\log T_M(\rho\,;h) - \log \tau_M(\rho\,;h)$ *can be computed as follows. Let h_u and g_v be one-parameter families of metrics on E_ρ and M, respectively. Let the notation be the same as in Theorem 2.12. Then*

$$\frac{\partial}{\partial u}\left(\log T_M(\rho\,;u,v) - \log \tau_M(\rho\,;u,v)\right) = \frac{1}{2}\sum_{q=0}^{n}(-1)^{q+1}c_q(u,v),$$

$$\frac{\partial}{\partial v}\left(\log T_M(\rho\,;u,v) - \log \tau_M(\rho\,;u,v)\right) = \frac{1}{2}\sum_{q=0}^{n}(-1)^{q+1}d_q(u,v).$$

The example discussed above shows that for even-dimensional manifolds the variation of $\log T_M(\rho\,;h) - \log \tau_M(\rho\,;h)$ may not be zero.

As in the case of orthogonal representations Corollary 2.16 is the key result to prove the equality of $T_M(\rho\,;h)$ and $\tau_M(\rho\,;h)$ for closed manifolds of odd dimension. For this purpose it is essential to have the proper generalization of Corollary 2.16 to the case $\partial M \neq \varnothing$. To define the analytic torsion in this case we have to introduce boundary conditions for Δ_q. Let $i : \partial M \to M$ be the inclusion, and let $i^* : \Lambda^*(E) \to \Lambda^*(E|\partial M)$ be the induced map on E_ρ-valued differential forms. A differential form $\omega \in \Lambda^*(E)$ is said to satisfy *absolute* boundary conditions if

$$i^*(*\omega) = 0 \quad \text{and} \quad i^*(*d\omega) = 0,$$

and if ω satisfies absolute boundary conditions then $*\omega$ is said to satisfy *relative* boundary conditions.

Let $\text{Dom}(\Delta_a)$ (resp. $\text{Dom}(\Delta_r)$) be the subspace of $\Lambda^*(E)$ consisting of all those forms which satisfy absolute (resp. relative) boundary conditions on ∂M. If ω, θ satisfy either boundary conditions, then

$$\langle d\omega, \theta \rangle = \langle \omega, \delta\theta \rangle$$

and the restriction of the Laplacian to the corresponding domains defines symmetric positive semidefinite, operators

$$\Delta_a : \text{Dom}(\Delta_a) \longrightarrow \Lambda^*(E) \quad \text{and} \quad \Delta_r : \text{Dom}(\Delta_r) \longrightarrow \Lambda^*(E)$$

The corresponding selfadjoint extensions on $L^2\Lambda^*(E)$ have pure point spectrum $0 \leq \lambda_{a,0} \leq \lambda_{a,1} \leq \cdots$ and $0 \leq \lambda_{r,0} \leq \lambda_{r,1} \leq \cdots$, respectively. Let $\mathscr{H}_a^*(E)$ and $\mathscr{H}_r^*(E)$ denote the spaces of harmonic forms for absolute and relative boundary conditions, respectively. Then the De Rham map induces isomorphisms

$$(2.17) \qquad \mathscr{H}_a^*(E) \xrightarrow{\sim} H^*(M\,;E) \quad \text{and} \quad \mathscr{H}_r^*(E) \xrightarrow{\sim} H^*(M,\partial M\,;E).$$

The analytic torsion $T_M^a(\rho\,;h)$ (resp. $T_M^r(\rho\,;h)$) for absolute (resp. relative) boundary conditions is defined by formula (2.2) with Δ_q replaced by $\Delta_{a,q}$ (resp. $\Delta_{r,q}$).

In order to compute the variation of the analytic torsion we may essentially proceed as in the case of a closed manifold. There are, however, additional complications due to the nonempty boundary.

First we have to study the variation of the trace of the heat operator. We fix absolute boundary conditions. The case of relative boundary conditions follows similarly. Let h_u be a one-parameter family of metrics on E_ρ, and let $\Delta_{a,q}(u)$ denote the Laplacian and $P_{a,q}(u)$ the harmonic projection with respect to h_u and the choice of absolute boundary conditions. Let $K_q(t; u)$ denote the kernel of $e^{-t\Delta_{a,q}(u)}$, and let $K_q = K'_q + K''_q + K'''_q$ denote the Hodge decomposition of K_q into its exact, coexact, and harmonic components. To compute the variation of $\mathrm{Tr}\big(e^{-t\Delta_{a,q}(u)}\big)$ we simply follow the proof of Theorem 3.10 in [C] and we note that Duhamel's principle (3.9) which is used in the course of the proof remains valid in our case. As above, let $\alpha = \#^{-1}\dot\#$. Then

$$
(2.18) \quad \frac{\partial}{\partial u}\mathrm{Tr}\big(e^{-t\Delta_{a,q}(u)}\big) = t\frac{\partial}{\partial t}\Big\{\mathrm{Tr}\big(\alpha K'_{q+1}(t; u)\big) - \mathrm{Tr}\big(\alpha K''_q(t; u)\big)
$$
$$
+ \mathrm{Tr}\big(\alpha K'_q(t; u)\big) - \mathrm{Tr}\big(\alpha K''_{q-1}(t; u)\big)\Big\}.
$$

Applying (2.18), it follows that

$$
\frac{\partial}{\partial u}\sum_{q=0}^{n}(-1)^q q\,\mathrm{Tr}\big(e^{-t\Delta_{a,q}(u)} - P_{a,q}(u)\big)
$$

$$
= t\frac{\partial}{\partial t}\sum_{q=0}^{n}(-1)^q\,\mathrm{Tr}\big(\alpha\big(e^{-t\Delta_{a,q}(u)} - P_{a,q}(u)\big)\big).
$$

The asymptotic expansion of $\mathrm{Tr}\big(\alpha e^{-t\Delta_{a,q}(u)}\big)$ now contains additional terms coming from the boundary

$$
(2.19) \quad \mathrm{Tr}\big(\alpha e^{-t\Delta_{a,q}(u)}\big) \sim \sum_{k=0}^{\infty} c^a_{q,k}(u)\,t^{-n/2+k} + \sum_{k=0}^{\infty} b^a_{q,k}(u)\,t^{-n/2+k/2}.
$$

Now we can proceed in exactly the same way as in the closed case, and we get

$$
(2.20) \quad \frac{\partial}{\partial u}\log T^a_M(\rho; h_u) = \frac{1}{2}\sum_{q=0}^{n}(-1)^q\big\{\mathrm{Tr}\big(\alpha P_{a,q}(u)\big) - b^a_{q,n}(u)\big\}
$$

if $\dim M$ is odd.

A similar relation holds for the variation with respect to the Riemannian metric. Then α has to be replaced by $\beta = *_v^{-1}\dot*_v$ and the coefficients $b^a_{q,n}(u)$ by $\tilde b^a_{q,n}(v)$ occurring in the corresponding asymptotic expansion of $\mathrm{Tr}\big(\beta e^{-t\Delta_{a,q}(u)}\big)$.

To define the R-torsion $\tau^a_M(\rho; h)$ we employ the De Rham isomorphism to introduce a volume element for $H^*(M; E)$. The computation of the variation of $\log\tau^a_M(\rho; h_u)$ remains the same

$$
(2.21) \quad \frac{\partial}{\partial u}\log\tau^a_M(\rho; h_u) = \frac{1}{2}\sum_{q=0}^{n}(-1)^q\,\mathrm{Tr}\big(\alpha P_{a,q}(u)\big).
$$

Putting (2.20) and (2.21) together we obtain

Theorem 2.22. *Let M be a closed manifold of odd dimension and E_ρ a flat vector bundle over M. Impose absolute boundary conditions. Let h_u and g_v be one-parameter families of metrics on E_ρ and M, respectively. Denote by $T_M^a(\rho; u, v)$ the analytic torsion and by $\tau_M^a(\rho; u, v)$ the R-torsion with respect to (h_u, g_v). Then*

$$\frac{\partial}{\partial u}\left(\log T_M^a(\rho; u, v) - \log \tau_M^a(\rho; u, v)\right) = \frac{1}{2}\sum_{q=0}^{n}(-1)^{q+1}\, b_{q,n}^a(u, v),$$

$$\frac{\partial}{\partial v}\left(\log T_M^a(\rho; u, v) - \log \tau_M^a(\rho; u, v)\right) = \frac{1}{2}\sum_{q=0}^{n}(-1)^{q+1}\, \tilde{b}_{q,n}^a(u, v)$$

where the coefficients $b_{q,n}^a$ and $\tilde{b}_{q,n}^a$ are determined by the asymptotic expansion (2.19) and its counterpart for the variation with respect to g_v. A similar result holds for relative boundary conditions.

An immediate consequence of Theorem 2.22 is

Corollary 2.23. *Let the assumption and notation be the same as in Theorem 2.22. Then for any values (u_2, v_2) and (u_1, v_1) of the parameters the difference*

$$\left(\log T_M^a(\rho; u_2, v_2) - \log \tau_M^a(\rho; u_2, v_2)\right) - \left(\log T_M^a(\rho; u_1, v_1) - \log \tau_M^a(\rho; u_1, v_1)\right)$$

depends only on the germs of h_{u_2}, h_{u_1}, g_{v_2}, g_{v_1} restricted to ∂M and is completely independent of the geometry and topology of $\mathrm{int}(M)$. The same holds for relative boundary conditions.

Proof. Set
$$f(u, v) = \log T_M^a(\rho; u, v) - \log \tau_M^a(\rho; u, v).$$
Then, by Theorem 2.22,

$$f(u_2, v_2) - f(u_1, v_1)$$
$$= \int_{u_1}^{u_2}\frac{\partial}{\partial u}f(u, v_2)\,du + \int_{v_1}^{v_2}\frac{\partial}{\partial v}f(u_1, v)\,dv$$
$$= \frac{1}{2}\sum_{q=0}^{n}(-1)^{q+1}\left\{\int_{u_1}^{u_2}b_{q,n}^a(u, v_2)\,du + \int_{v_1}^{v_2}\tilde{b}_{q,n}^a(u_1, v)\,dv\right\}.$$

By the construction of the asymptotic expansion (2.19) and its analogue for $\mathrm{Tr}\left(\beta e^{-t\Delta_{a,q}(v)}\right)$ the coefficients $b_{q,n}^a$ and $\tilde{b}_{q,n}^a$ depend only on the germs of h_{u_2}, h_{u_1}, g_{v_2}, g_{v_1} restricted to ∂M.   Q.E.D.

3. THE EQUALITY OF ANALYTIC TORSION AND R-TORSION

Let M be a closed oriented Riemannian manifold of odd dimension, and let $\rho : \pi_1(M) \to \mathrm{GL}(E)$ be a unimodular representation with associated flat bundle E_ρ. Let h be an Euclidean metric on E_ρ. To establish the equality of the analytic torsion $T_M(\rho; h)$ and the R-torsion $\tau_M(\rho; h)$ we shall follow

Cheeger's proof for the case of orthogonal representations [C]. As explained in the introduction, this proof is well suited for this purpose because the relevant analysis is done locally near a given handle and this can easily be adapted to the case of unimodular representations.

We recall the basic steps in Cheeger's proof and indicate how they have to be modified (if at all). By Corollary 2.16, $\log T_M(\rho ; h) - \log \tau_M(\rho ; h)$ is independent of the choice of h and of the metric on M. Set

$$(3.1) \qquad e_M(\rho) = \log T_M(\rho ; h) - \log \tau_M(\rho ; h).$$

Furthermore, both $T_M(\rho ; h)$ and $\tau_M(\rho ; h)$ are independent of the choice of orientation.

Let M_0 and M_1 be two closed smooth manifolds of odd dimension n, and assume that M_1 is obtained from M_0 by surgery on some embedded k-sphere $S^k \subset M_0$ (see below for the precise definition of this statement). Let E_0 be a flat bundle over M_0 defined by a unimodular representation ρ_0 of $\pi_1(M_0)$, and assume that E_0 extends over the trace of the surgery to a flat bundle E_1 over M_1 defined by a unimodular representation ρ_1 of $\pi_1(M_1)$. By an argument similar to the one used in the introduction of [C], it suffices to show that $e_{M_0}(\rho_0) = 0$ implies $e_{M_1}(\rho_1) = 0$ for all such pairs (M_0, ρ_0), (M_1, ρ_1). A slight complication arises because this can be proved directly only for $0 < k < n - 1$.

Now recall that surgery on an embedded k-sphere $S^k \subset M_0$ means the following:

(a) The tubular neighborhood $N(S^k)$ is a product

$$N(S^k) = S^k \times D^{n-k}.$$

(b) There is an embedded $(n - k - 1)$-sphere $S^{n-k-1} \subset M_1$ whose tubular neighborhood is also a product

$$N(S^{n-k-1}) = S^{n-k-1} \times D^{k+1}.$$

(c) There is a manifold $\widetilde{M}$ with boundary $\partial \widetilde{M} = S^k \times S^{n-k-1}$ such that

$$M_0 = N(S^k) \underset{\partial \widetilde{M}}{\cup} \widetilde{M} \quad \text{and} \quad M_1 = N(S^{n-k-1}) \underset{\partial \widetilde{M}}{\cup} \widetilde{M}$$

where the union means that the common boundaries of $\widetilde{M}$, $N(S^k)$, and $N(S^{n-k-1})$ are identified with the obvious identification and the manifolds are given the standard differentiable structures.

The fact that E_0 extends over the trace of the surgery to give E_1 is equivalent to the existence of a flat unimodular bundle $\widetilde{E}$ over $\widetilde{M}$ which extends as a flat unimodular bundle over $M_0 = N(S^k) \cup \widetilde{M}$ and $M_1 = N(S^{n-k-1}) \cup \widetilde{M}$. Recall that a flat bundle over a contractible space is trivial as a flat bundle. Since $S^k \times \{p\} \subset \partial \widetilde{M}$ bounds $D^{k+1} \times \{p\}$, it follows that $E_0|S^k$, $S^k \subset M_0$, and therefore $E_0|N(S^k)$ is trivial as a flat bundle. Similarly, $E_1|N(S^{n-k-1})$ is trivial as a flat bundle. This is the important fact which allows us to extend Cheeger's analysis.

We shall employ the following notation. If X is a Riemannian manifold and $Y \subset X$ a submanifold then we shall denote by $N_u(Y)$ the tubular neighborhood of Y in X consisting of all normal vectors to Y of length $\leq u$.

Now we introduce metrics on M_0 and M_1 such that the tubular neighborhoods $N_1(S^k)$ and $N_1(S^{n-k-1})$ are isometrically the product of the unit spheres S^k, S^{n-k-1} and the unit balls D^{n-k}, D^{k+1} equipped with the standard flat metrics. Furthermore, since $E_0|N_1(S^k)$ and $E_1|N_1(S^{n-k-1})$ are trivial, we may fix trivializations

$$(3.2) \quad E_0|N_1(S^k) \cong N_1(S^k) \times \mathbf{R}^N \quad \text{and} \quad E_1|N_1(S^{n-k-1}) \cong N_1(S^{n-k-1}) \times \mathbf{R}^N$$

and choose metrics h_0, h_1 on E_0, E_1 in such a way that they coincide on $E_0|N_1(S^k)$ and $E_1|N_1(S^{n-k-1})$ with the product metrics given by the trivializations (3.2). Set

$$M_u = \begin{cases} M_0 - N_u(S^k) & \text{if } 0 < u \leq 1/3, \\ M_1 - N_{1-u}(S^{n-k-1}) & \text{if } 2/3 \leq u < 1; \end{cases}$$

$$E_u = \begin{cases} E_0|M_u & \text{if } 0 < u \leq 1/3, \\ E_1|M_u & \text{if } 2/3 \leq u < 1. \end{cases}$$

We may think of M_u (resp. E_u) as being $\widetilde{M}$ (resp. $\widetilde{E}$) equipped with the metric g_u (resp. h_u). For $1/3 \leq u \leq 2/3$ let g_u (resp. h_u) be any smooth family of metrics connecting these two families but subject to the condition that, near $\partial \widetilde{M} = S^k \times S^{n-k-1}$, g_u (resp. h_u) is fixed independent of M_0, M_1. As above, we denote by M_u (resp. E_u), $0 < u < 1$, the manifold $\widetilde{M}$ (resp. the flat bundle $\widetilde{E}$) equipped with g_u (resp. h_u).

The structure of the proof of the equality of the two torsions is now the same as in [C]. Let $\rho : \pi_1(\widetilde{M}) \to \mathrm{GL}(E)$ be the unimodular representation that defines $\widetilde{E}$ and set

$$e_u = \begin{cases} \log T_M(\rho \, ; h_u) - \log \tau_M(\rho \, ; h_u) & \text{if } 0 < u < 1, \\ \log T_{M_i}(\rho_i \, ; h_i) - \log \tau_{M_i}(\rho_i \, ; h_i) & \text{if } u = i \in \{0, 1\}. \end{cases}$$

Then

$$e_1 - e_0 = (e_1 - e_{1-u}) + (e_{1-u} - e_u) + (e_u - e_0) = A_u + B_u + C_u.$$

Let M_0', M_1', E_0', E_1' be a similar set of data where M_1' is again obtained from M_0' by surgery on a k-sphere S^k (with the same k). Then

$$e_1' - e_0' = A_u' + B_u' + C_u'$$

with the obvious notation. By Corollary 2.23, we have $B_u = B_u'$ and the core of the proof is

Theorem 3.3. *Let M_0, M_1, E_0, E_1 and M_0', M_1', E_0', E_1' be quatruples as above. Then, if $0 < k \leq n - 1$, $\lim_{u \to 0}(A_u - A_u') = 0$ and, if $0 \leq k < n - 1$, $\lim_{u \to 0}(C_u - C_u') = 0$.*

By symmetry, it suffices to consider C_u. To prove the Theorem for C_u, we first consider

$$(3.4) \quad \left(\log T_{M_u}(\rho; h_u) - \log T_{M_0}(\rho_0; h_0)\right) - \left(\log T_{M_u'}(\rho'; h_u') - \log T_{M_0'}(\rho_0'; h_0')\right)$$

and investigate its behaviour as $u \to 0$. Let $\Delta_q(u)$ be the Laplacian on E_u-valued q-forms of M_u, $1 > u > 0$, satisfying absolute boundary conditions. Let $\zeta_q(s; u)$ be the zeta function associated to $\Delta_q(u)$. Then

$$\log T_{M_u}(\rho; h_u) = \frac{1}{2} \sum_{q=0}^{n} (-1)^q q \frac{\partial}{\partial s} \zeta_q(s; u)\big|_{s=0}$$

and we have to study the behaviour of $\frac{\partial}{\partial s} \zeta_q(s; u)\big|_{s=0}$ as $u \to 0$.

Recall that there exists an asymptotic expansion

$$(3.5) \quad \mathrm{Tr}(e^{-t\Delta_q(u)}) = \sum_{j=0}^{m} a_{q,j}(u) t^{-n/2+j} + \sum_{j=0}^{2m} b_{q,j}(u) t^{-n/2+j/2} + O(t^{-n/2+m+1/2})$$

as $t \to 0$. The coefficients $a_{q,j}(u)$ are locally computable in the sense that they are obtained by integrating local densities which in any coordinate system depend in a universal fashion on the coefficients $g_{i,j}(x)$ and $h_{i,j}(x)$ of the metrics and a finite number of their partial derivatives. In the same sense, the coefficients $b_{q,j}(u)$ are locally computable in terms of the extrinsic geometry of the boundary, e.g., its induced metric and second fundamental form (cf. [G] for details). Set

$$c_{q,j}(u) = \begin{cases} b_{q,j}(u) & \text{if } j = 2l+1, \\ a_{q,j/2}(u) + b_{q,j}(u) & \text{if } j = 2l. \end{cases}$$

Furthermore, let $m > n/2$ and denote by $\mu_u(t)$ the difference of $\mathrm{Tr}(e^{-t\Delta_q(u)})$ and its asymptotic expansion up to order m. Denote by $P_q(u)$ the harmonic projection. Then, using the definition of $\zeta_q(s; u)$ in terms of the heat kernel, we get

$$(3.6) \quad \begin{aligned} \frac{\partial}{\partial s} \zeta_q(s; u)\big|_{s=0} &= \int_1^\infty t^{-1} \left(\mathrm{Tr}(e^{-t\Delta_q(u)}) - P_q(u)\right) dt + \int_0^1 t^{-1} \mu_u(t)\, dt \\ &\quad + \sum_{j \neq n} \frac{c_{q,j}(u)}{-n/2+j/2} + \left(c_{q,n}(u) - \dim H^q(M_u; E_u)\right)\gamma \end{aligned}$$

where $\gamma = -\Gamma'(1)$ is Euler's constant. Now we have to investigate the behaviour of the individual terms as $u \to 0$. Let Δ_q be the Laplacian on E_0-valued q-forms on M_0. Then we have a corresponding asymptotic expansion as $t \to 0$ of the form

$$\mathrm{Tr}(e^{-t\Delta_q}) \sim \sum_{j=0}^{m} a_{q,j} t^{-n/2+j} + O(t^{-n/2+m+1}).$$

The nature of the coefficients $a_{q,j}(u)$ and $a_{q,j}$ implies immediately that

$$\lim_{u \to 0} a_{q,j}(u) = a_{q,j}, \qquad q = 0, \ldots, m.$$

Furthermore, by assumption, the boundary terms occurring in (3.5) are the same for both M_u and M'_u. Hence the contribution to (3.4) made by the coefficients of the asymptotic expansion (3.5) and its analogues for M'_u, M_0, and M'_0 will cancel out in the limit.

Next consider $\dim H^q(M_u \, ; E_u)$. Recall that $M_u \subset M_0$ and $E_u = E_0 | M_u$. Hence there is the long exact cohomology sequence

(3.7)
$$\cdots \to H^q(M_0, M_u \, ; E_0) \xrightarrow{j} H^q(M_0 \, ; E_0)$$
$$\xrightarrow{l} H^q(M_u \, ; E_u) \xrightarrow{\delta} H^{q+1}(M_0, M_u \, ; E_0) \to \cdots$$

and, by excision,

$$H^q(M_0, M_u \, ; E_0) \cong H^q(N(S^k), \partial N(S^k) \, ; E_0 | N(S^k)).$$

Using (3.2), we obtain

(3.8)
$$H^q(M_0, M_u \, ; E_0) \cong H^q(N(S^k), \partial N(S^k)) \otimes \mathbf{R}^N.$$

Since M'_1 is obtained from M'_0 by surgery on an embedded k-sphere $S^k \subset M_0$ with the same k, (3.8) also holds for (M'_0, M'_u). Let $k_q = \dim(\ker l_q)$. Then, by the exactness of (3.7), we get
(3.9)
$$\dim H^q(M_u \, ; E_u) = \dim H^q(M_0 \, ; E_0) - k_q + \dim H^{q+1}(M_0, M_u \, ; E_0) - k_{q+1}.$$

Let k'_q have the same meaning with respect to (M'_0, M'_u). Then the contribution to (3.4) of the last term in (3.6) and its analogues for M'_u, M_0, M'_0 is

(3.10)
$$\gamma \sum_{q=0}^{n} (-1)^q q \big((k_q + k_{q+1}) - (k'_q + k'_{q+1}) \big).$$

This term will be further discussed below.

The essential part is to study the behaviour, as $u \to 0$, of the first integral on the right-hand side of (3.6). For this purpose we need two estimates describing the behaviour of the heat kernel as $u \to 0$. Let $K_u(x, y, t)$ and $K_0(x, y, t)$ denote the kernel of $e^{-t\Delta_q(u)}$ and $e^{-t\Delta_q}$, respectively.

Theorem 3.11. *Let $k < n - 1$. Given $T, m, u_0 > 0$, there exists $C_m(T, u_0)$ such that, for $t \le T$ and $x, y \in M_{u_0} \subset M_u$,*

$$\|K_0(x, y, t) - K_u(x, y, t)\| \le C_m(T, u_0) \, t^m \begin{cases} u^{n-k-2} & \text{if } n - k > 2, \\ \dfrac{1}{|\log u|}, & \text{if } n - k = 2. \end{cases}$$

This theorem states that away from $S^k \subset M_0$, the heat kernel of (M_u, E_u) converges uniformly to that of (M_0, E_0) as $u \to 0$.

The next result describes the behaviour of the heat kernel near S^k. For $u < u_0$, let A_{u, u_0}^{n-k} denote the annulus obtained by removing the $(n - k)$-ball D_u^{n-k} of radius u from the $(n - k)$-ball $D_{u_0}^{n-k}$ of radius u_0. Let $K_{u, 1}(t)$ be

the heat kernel on q-forms of $S^k \times A^{n-k}_{u,1}$ with coefficients in the trivial flat $\mathbf{R}^N$-bundle satisfying absolute boundary conditions. Introduce polar coordinates on $A^{n-k}_{u,1}$ and write $z \in S^k \times A^{n-k}_{u,1}$ as $z = (x, r)$ where $x \in S^k \times S^{n-k-1}$ and $r \in (u, 1)$. Fix $u_0 < 1$.

Theorem 3.12. *Let $k < n - 1$. Given T, m, there exists a constant $C_m(T)$ such that*

(1) *For $r_1, r_2 < u_0$, $x_1, x_2 \in S^k \times S^{n-k-1}$,*

$$\|K_{u,1}((x_1, r_1), (x_2, r_2), t) - K_u((x_1, r_1), (x_2, r_2), t)\|$$

$$\leq C_m(T)t^m \begin{cases} (1 + u^{n-k-2}r_1^{-(n-k-1)})(1 + u^{n-k-2}r_2^{-(n-k-1)}) \\ \qquad\qquad\qquad\qquad\qquad\qquad if\ n - k > 2, \\[2mm] \left(1 + \dfrac{r_1^{-1}}{|\log u|}\right)\left(1 + \dfrac{r_2^{-1}}{|\log u|}\right), \quad if\ n - k = 2. \end{cases}$$

(2)

$$\left| \int_{S^k \times A^{n-k}_{u,u_0}} \left\{ \mathrm{tr}\big(K_{u,1}((x, r), (x, r), t)\big) - \mathrm{tr}\big(K_u((x, r), (x, r), t)\big) \right\} \right|$$

$$\leq C_m(T)t^m \begin{cases} (u_0^{n-k-1} + u^{n-k-3})(u_0 - u), & if\ n - k > 2, \\[2mm] u_0(u_0 - u) + \dfrac{\log(u_0/u)}{\log^2 u}, & if\ n - k = 2. \end{cases}$$

The proof of Theorem 3.11 and Theorem 3.12, which, for orthogonal flat bundles, is given in §7 of [C], depends on two kinds of results. First of all, some standard estimates for the heat kernel are used (cf. [C, §5]). These estimates are derived from Duhamel's principle together with the Sobolev inequality. Both Duhamel's principle and the Sobolev inequality do not require any special assumption on the bundle E_ρ, and therefore the estimates obtained by Cheeger in §5 of [C] are also valid in our setting. The other part that is important for the proof of Theorems 3.11 and 3.12 is the local analysis near S^k. One has to construct a parametrix for $\Delta_q(u)$ satisfying absolute boundary conditions which allows us to study its behaviour as $u \to 0$. The parametrix is obtained by patching an interior parametrix and a parametrix near the boundary. The interior parametrix is obtained by general constructions which again do not require special assumptions. To construct the parametrix near the boundary it suffices to construct the Green's operator for Δ on $\mathbf{R}^N$-valued differential forms on $S^k \times A^{n-k}_{u,1}$ satisfying absolute boundary conditions. By our assumption, the metrics on $S^k \times A^{n-k}_{u,1}$ and

$$E_u|(S^k \times A^{n-k}_{u,1}) \cong S^k \times A^{n-k}_{u,1} \times \mathbf{R}^N$$

are the standard product metrics. Therefore, all the local analysis done by Cheeger in [C, §§6, 7] can be applied to our case without any change. This suffices to prove Theorems 3.11 and 3.12. We just follow the proof of these theorems in [C, §7] line by line.

In order to estimate $\mathrm{Tr}(e^{-t\Delta_q(u)})$ for large t one has to study the small eigenvalues, that is, those eigenvalues of $\Delta_q(u)$ which converge to zero as $u \to 0$. Set

$$r(q) = \dim H^{q-(n-k-1)}(S^k).$$

The behaviour of the small eigenvalues is described by

Proposition 3.13. *There exists* $\lambda > 0$ *with the following property. For all* $0 < \varepsilon < \lambda$, *there exists* $\delta > 0$ *such that, for* $u < \delta$, $\Delta_q(u)$ *has exactly*

$$\dim H^q(M_0; E_0) + N \cdot r(q)$$

eigenvalues $< \lambda$ *and all of them are* $< \varepsilon$.

The proof is exactly the same as the proof of Proposition 7.19 in [C].

Let $0 < \lambda_1^q(u) \le \lambda_2^q(u) \le \cdots \le \lambda_{s(q)}^q(u) < \lambda$ be the nonzero eigenvalues described by Proposition 3.13 . By Proposition 3.13, we have

$$\dim H^q(M_u; E_u) + s(q) = \dim H^q(M_0; E_0) + \dim H^{q+1}(M_0, M_u; E_0).$$

Comparing this with (3.9), we get

(3.14) $$k_q + k_{q+1} = s(q).$$

Now let $\Delta_q'(u)$ denote the restriction of $\Delta_q(u)$ to the orthogonal complement in $L^2 \Lambda^q(E_u)$ of the sum of the eigenspaces corresponding to the small eigenvalues $< \lambda$ described by Proposition 3.13. Note that $\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\lambda t} \, dt$. Taking the derivative at $s = 0$ gives

$$\int_1^\infty t^{-1} e^{-\lambda t} \, dt = -\log \lambda - \int_0^1 t^{-1}(e^{-\lambda t} - 1) \, dt - \gamma$$

where $\gamma = -\Gamma'(1)$. Then the first integral on the right-hand side of (3.6) equals

$$-\sum_{j=1}^{s(q)} \log \lambda_j^q(u) + \int_1^\infty t^{-1} \mathrm{Tr}(e^{-t\Delta_q'(u)}) \, dt - \int_0^1 t^{-1}\Big(\sum_{j=1}^{s(q)} e^{-\lambda_j^q(u)t} - s(q)\Big) \, dt - s(q)\gamma.$$

By (3.14), the contribution of the last term to (3.4) cancels (3.10). Furthermore, by Proposition 3.13, the second integral tends to zero as $u \to 0$, and for any given $\varepsilon > 0$ there exists T' such that, for $T > T'$, $u < \delta$,

$$\int_T^\infty t^{-1} \mathrm{Tr}(e^{-t\Delta_q'(u)}) \, dt < \varepsilon.$$

Fix u_0, $T > 0$. Then
(3.15)

$$\int_1^T t^{-1} \mathrm{Tr}(e^{-t\Delta_q'(u)}) \, dt = \int_1^T t^{-1} \int_{M_{u_0}} \mathrm{tr} K_u^q(x, x, t) \, dx \, dt$$

$$+ \int_1^T t^{-1} \int_{S^k \times A_{u,u_0}^{n-k}} \mathrm{tr} K_u^q(x, x, t) \, dx \, dt$$

$$- \sum_{j=1}^{s(q)} \int_1^T t^{-1} e^{-\lambda_j^q(u)t} \, dt - \log T \dim H^q(M_u; E_u).$$

By Proposition 3.13, the last two terms tend to

$$-\log T \left(s(q) + \dim H^q(M_u \, ; E_u) \right)$$

as $u \to 0$, and, by Proposition 3.13, this equals

$$-\log T \left(\dim H^q(M_0 \, ; E_0) + N \cdot r(q) \right).$$

The same holds if we replace M_u by M_u' so that the combined contribution to (3.3) of the last two terms in (3.15) and the analogous terms with respect to M_u' will cancel out in the limit $u \to 0$.

Finally, one uses Theorems 3.11 and 3.12 to tread the remaining integrals. Let $\underline{K}_{u,1}(t)$ denote the kernel obtained by projecting the heat kernel $K_{u,1}(t)$ defined above onto the orthogonal complement of the harmonic forms. Then the final result can be stated as

(3.16)

$$\lim_{u_0 \to 0} \lim_{u \to 0} \left| \int_1^\infty t^{-1} \operatorname{Tr}\!\left(e^{-t\Delta_q(u)} - P^q(u)\right) dt - \int_1^\infty t^{-1} \operatorname{Tr}(e^{-t\Delta_q} - P^q)\, dt \right.$$

$$\left. - \int_1^\infty t^{-1} \int_{S^k \times A_{u,u_0}^{n-k}} \operatorname{Tr}(\underline{K}_{u,1}(x,x,t))\, dx\, dt + \sum_{j=1}^{s(q)} \log(\lambda_j^q(u)) \right| = 0.$$

Next we have to investigate the integrals $\int_0^1 t^{-1} \mu_u(t)\, dt$ and $\int_0^1 t^{-1} \mu_0(t)\, dt$. Recall that $\mu_u(x,t)$, $\mu_0(x,t)$, and $\mu_{u,1}(x,t)$ are the differences between $\operatorname{tr} K_u^q(x,x,t)$, $\operatorname{tr} K_0^q(x,x,t)$, and $\operatorname{tr} K_{u,1}^q(x,x,t)$, respectively, and their asymptotic expansion up to order $m > n/2$. Then the argument which led to (3.16) can also be used to prove

(3.17)

$$\lim_{u_0 \to 0} \lim_{u \to 0} \left| \int_0^1 t^{-1} \mu_u(t)\, dt - \int_0^1 t^{-1} \int_{M_0} \mu_0(x,t)\, dx\, dt \right.$$

$$\left. - \int_0^1 t^{-1} \int_{S^k \times A_{u,u_0}^{n-k}} \mu_{u,1}(x,t)\, dx\, dt \right| = 0.$$

Now consider M_0', M_u', E_0', E_u'. Let $0 < \lambda_1'^q(u) \le \cdots \le \lambda_{s(q)}'^q(u) < \lambda$ be the nonzero eigenvalues described by Proposition 3.13 with respect to M_u', E_u'. Subtracting the corresponding versions of (3.16) and (3.17) for M_0', M_u', E_0', E_u' from (3.16) and (3.17), respectively, and using (3.6) we can summarize our results by

(3.18)

$$\lim_{u \to 0} \left\{ \left(\log T_{M_u}(\rho \, ; h_u) - \log T_{M_0}(\rho_0 \, ; h_0) \right) \right.$$

$$- \left(\log T_{M'}(\rho' \, ; h_u') - \log T_{M_0'}(\rho_0' \, ; h_0') \right)$$

$$\left. + \sum_{q=0}^n (-1)^q q \sum_{j=1}^{s(q)} \log(\lambda_j^q(u)/\lambda_j'^q(u)) \right\} = 0.$$

It remains to investigate the corresponding expression for the R-torsion. First we consider

$$\log \tau_{M_u}(\rho \, ; h_u) - \log \tau_{M_0}(\rho_0 \, ; h_0).$$

Let K_0 be a smooth triangulation of M_0 and $K' \subset K_0$ a subcomplex which induces a triangulation of the submanifold $\widetilde{M} \subset M_0$. Recall that M_u denotes the manifold $\widetilde{M}$ equipped with the metric g_u. Furthermore, $\widetilde{E} = E_0|\widetilde{M}$ and E_u is the flat bundle $\widetilde{E}$ equipped with the metric h_u. Then we get a short exact sequence

$$(3.19) \qquad 0 \to C_\bullet(K'; \widetilde{E}) \longrightarrow C_\bullet(K_0; E_0) \longrightarrow C_\bullet(K_0, K'; E_0) \to 0$$

of chain complexes. Each of these chain complexes has a distinguished volume determined by preferred bases and the metrics h_0, g_0, h_u, g_u determine volumes $\mu_0 \in \det H_*(M_0; E_0)$, $\mu_u \in \det H_*(M_u; E_u)$. Since $M_0 - \widetilde{M} \cong N(S^k)$, $E_0|N(S^k) \cong N(S^k) \times \mathbf{R}^N$, and the metrics are the standard product metrics, the torsion of the relative chain complex equals $\tau_{N(S^k)}(\mathbf{1})^N$ where $\mathbf{1}$ stands for the trivial one-dimensional representation. Moreover, it is easy to verify that the torsion of each complex

$$0 \to C_q(K'; \widetilde{E}) \longrightarrow C_q(K_0; E_0) \longrightarrow C_q(K_0, K'; E_0) \to 0$$

equals 1. By Theorem 3.2 in [Mil], it follows that

$$\log \tau_{M_0}(\rho_0; h_0) = \log \tau_{M_u}(\rho, h_u) + N \cdot \log \tau_{N(S^k)}(\mathbf{1}) + \log \tau(\mathscr{H}_\bullet; \mu_u)$$

where $\mathscr{H}_\bullet$ denotes the long exact homology sequence of (3.19) regarded as an acyclic chain complex of length $3n$ equipped with the volumes μ_u, μ_0. Subtracting the corresponding equation for M_0', M_u', E_0', E_u', we get

$$(3.20) \quad \begin{aligned} \lim_{u \to 0} \Big\{ &\big(\log \tau_{M_u}(\rho; h_u) - \log \tau_{M_0}(\rho_0; h_0)\big) \\ &- \big(\log \tau_{M_u'}(\rho'; h_u') - \log \tau_{M_0'}(\rho_0'; h_0')\big) \\ &+ \big(\log \tau(\mathscr{H}_\bullet; \mu_u) - \log \tau(\mathscr{H}_\bullet'; \mu_u')\big) \Big\} = 0. \end{aligned}$$

Now one has to study the behaviour of $\log \tau(\mathscr{H}_\bullet; \mu_u)$ and $\log \tau(\mathscr{H}_\bullet'; \mu_u')$ as $u \to 0$. This is completely analogous to [C] and we leave it to the reader (cf. [C, p. 316]). As result we obtain that, in the limit $u \to 0$, $\log \tau(\mathscr{H}_\bullet; \mu_u) - \log \tau(\mathscr{H}_\bullet'; \mu_u')$ cancels the contribution of the small eigenvalues in (3.18). This completes the proof of Theorem 3.3.

Now we can proceed exactly in the same way as in [C]. Write $d_k = e_1 - e_0$ where $e_i = \log T_{M_i}(\rho_i; h_i) - \log \tau_{M_i}(\rho_i; h_i)$ and M_1 is obtained from M_0 by surgery on an embedded k-sphere.

Proposition 3.21. *We have $d_k = 0$ for $1 \leq k \leq n - 2$. Moreover, d_0 is well defined.*

We just follow the proofs of Proposition 8.20 and 8.21 of [C] line by line to obtain the proof of Proposition 3.21.

We can now prove Theorem 1 of the introduction. To prove this theorem we proceed in exactly the same way as on p. 318 of [C], using Propositions 1.12, 1.13, 2.9, 2.13.

Remark. Everything that has been said in the previous sections works also for finite-dimensional complex representations of $\pi_1(M)$.

4. SOME EXAMPLES

In this section we shall discuss two examples where nonorthogonal or nonunitary representations of the fundamental group arise naturally and the torsion is of relevance in this context

4.1. Locally symmetric manifolds.
Let G be a connected semisimple Lie group with finite center. We also assume that G has no compact factors. Let K be a maximal compact subgroup of G. The Lie algebras of G and K will be denoted by $\mathfrak{g}$ and $\mathfrak{k}$, respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{k}$, and let θ be the Cartan involution of $(\mathfrak{g}, \mathfrak{k})$. The quotient space $X = G/K$ is then a symmetric Riemannian manifold and G is the identity component of the group of orientation preserving isometries of X. We shall denote by $x_0 \in X$ the coset eK of the identity $e \in G$.

Let $\Gamma \subset G$ be a discrete, torsionfree, co-compact subgroup of G. Then $M = \Gamma \backslash X$ is a compact locally symmetric manifold covered by X with $\pi_1(M) = \Gamma$.

Let $\rho : G \to \mathrm{GL}(E)$ be a representation of G on a finite-dimensional complex vector space E. If we restrict ρ to Γ, we obtain a representation $\rho_\Gamma : \Gamma \to \mathrm{GL}(E)$ with associated flat vector bundle E_ρ over $\Gamma \backslash X$. Since E is a Γ-module, the group cohomology $H^*(\Gamma; E)$ is defined and we have the equality

$$H^*(\Gamma \backslash X; E_\rho) = H^*(\Gamma; E).$$

There is a different way to describe the bundle E_ρ. Let $\rho_K : K \to \mathrm{GL}(E)$ be the restriction of ρ to K, and consider the fibration

$$(4.1) \qquad\qquad \Gamma \backslash G \longrightarrow \Gamma \backslash X$$

which is principal with structure group K. Then ρ_K defines the induced bundle $\widetilde{E}_\rho$ over $\Gamma \backslash X$ whose global C^∞-sections are the C^∞-functions $f : \Gamma \backslash G \longrightarrow E$ which satisfy

$$f(gk) = \rho(k)^{-1}(f(g)) \quad \text{for all } g \in G, \, k \in K.$$

Lemma 4.2. *The bundles E_ρ and $\widetilde{E}_\rho$ are naturally isomorphic. If $\tilde{f} : \Gamma \backslash G \to E$ is a section of $\widetilde{E}_\rho$, set $f(x) = f(gx_0) = \rho(g)\tilde{f}(g)$ where $x = gx_0$. Then $f : X \to E$ defines a section of E_ρ and $\tilde{f} \mapsto f$ establishes an isomorphism of the corresponding spaces of C^∞-sections.*

For the proof see Proposition 3.1 in [MM].

Now observe that the flat bundles E_ρ defined by a representation $\rho : G \to \mathrm{GL}(E)$ fit into the setting of the previous sections. Namely, we have

Lemma 4.3. *Any finite-dimensional representation $\rho : G \to \mathrm{GL}(E)$ satisfies $\det \rho(g) = 1$ for all $g \in G$.*

Proof. Let $N = \dim_{\mathbb{C}} E$. Then G acts on $\Lambda^N E$ via the character $\chi = \det \circ \rho$. Since G is semisimple, we have $D\mathfrak{g} = \mathfrak{g}$ and, therefore, $d\chi = 0$. But G is connected, which implies $\chi \equiv 1$. Q.E.D.

Thus each of the flat bundles E_ρ constructed above is unimodular. Next we equip E_ρ with a canonical metric. Following Matsushima and Murakami in [MM], we call a hermitian inner product $\langle u, v \rangle$ on E *admissible* if

(a) $\langle \rho(Y)u, v \rangle = -\langle u, \rho(Y)v \rangle$ for all $Y \in \mathfrak{k}$, $u, v \in E$;

(b) $\langle \rho(Y)u, v \rangle = \langle u, \rho(Y)v \rangle$ for all $Y \in \mathfrak{p}$, $u, v \in E$.

The existence of an admissible inner product on any G-module E is proved in [MM, Lemma 3.1]. Condition (a) means that $\langle \cdot, \cdot \rangle$ is invariant under $\rho(K)$, and therefore it defines a hermitian metric on E_ρ. This is the *canonical metric* on E_ρ defined by the admissible inner product on E.

This choice of an inner product on E_ρ allows us to use harmonic analysis to study the Laplacian Δ on E_ρ-valued differential forms on $\Gamma \backslash X$. The tangent space to X at x_0 can be identified with $\mathfrak{p}$ and the tangent bundle of $\Gamma \backslash X$ is the bundle induced from (4.1) by the adjoint representation $\mathrm{ad}_\mathfrak{p} : K \to \mathrm{GL}(\mathfrak{p})$. Therefore, we have a natural identification

$$\Lambda^q(\Gamma \backslash X, E) = \{ \varphi : \Gamma \backslash G \to \Lambda^q \mathfrak{p}^* \otimes E \mid \varphi \text{ is } C^\infty \text{ and }$$

$$\varphi(gk) = \left(\Lambda^q \mathrm{ad}_\mathfrak{p}^*(k^{-1}) \otimes \rho(k^{-1}) \right)(\varphi(g)), \ g \in G, \ k \in K \}.$$

Concerning the cohomology $H^*(\Gamma; E)$ there exist plenty of results, in particular, vanishing theorems (cf. [BW, Chapter VII]) telling us which representations are acyclic. We mention one of them. Let $\mathfrak{h}^+ \subset \mathfrak{k}$ be a maximal abelian subalgebra of $\mathfrak{k}$. Let $\mathfrak{h}$ be the centralizer of $\mathfrak{h}^+$ in $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let Φ be a root system of $(\mathfrak{g}_\mathbf{C}, \mathfrak{h}_\mathbf{C})$, and let Φ_k be a root system of $(\mathfrak{k}_\mathbf{C}, \mathfrak{h}_\mathbf{C}^+)$. Fix a set Φ_k^+ of positive roots for Φ_k, and let Φ^+ be a system of positive roots for Φ compatible with Φ_k^+ (see [BW, p. 65]). This means, in particular, that $\theta\alpha \in \Phi^+$ whenever $\alpha \in \Phi^+$. As usually set

$$2\underline{\rho} = \sum_{\alpha \in \Phi^+} \alpha.$$

Theorem 4.4 (Borel-Wallach). *Let E be an irreducible finite-dimensional G-module with highest weight $\Lambda - \underline{\rho}$. If $\theta\Lambda \neq \Lambda$ then $H^*(\Gamma; E) = 0$.*

For the proof see Theorem 6.7 of Chapter VII in [BW].

Let $\mathfrak{h}_\mathbf{C}^- = \{ H \in \mathfrak{h}_\mathbf{C} \mid \theta H = -H \}$. Then $\mathfrak{h}_\mathbf{C}^* = \left(\mathfrak{h}_\mathbf{C}^+ \right)^* \oplus \left(\mathfrak{h}_\mathbf{C}^- \right)^*$. Moreover, assume that $\mathrm{rk}\, G > \mathrm{rk}\, K$. Then $\dim \mathfrak{h}_\mathbf{C}^- \geq 1$ and the highest weight of a generic representation satisfies $\theta\Lambda \neq \Lambda$. Note that the condition $\mathrm{rk}\, G > \mathrm{rk}\, K$ is satisfied whenever $\dim G/K$ is odd, which is the case we are mainly interested in. Thus, for compact locally symmetric manifolds $\Gamma \backslash X$ of odd dimension, Theorem 4.4 produces a large class of acyclic unimodular representations of Γ.

As an example we shall discuss three-dimensional hyperbolic manifolds. In this case we have $G = \mathrm{SL}(2, \mathbf{C})$ and $K = \mathrm{SU}(2)$, and $H^3 = \mathrm{SL}(2, \mathbf{C})/\mathrm{SU}(2)$ is the three-dimensional hyperbolic space. Let $\Gamma \subset \mathrm{SL}(2, \mathbf{C})$ be as above, that is, a discrete, torsionfree, co-compact subgroup. Since $\mathrm{SL}(2, \mathbf{C})$ is simple, Γ is an irreducible discrete subgroup.

To describe the irreducible finite-dimensional representations of $\mathrm{SL}(2, \mathbf{C})$, we have to consider $\mathfrak{sl}(2, \mathbf{C})$ as Lie algebra over $\mathbf{R}$ which we denote by

$\mathfrak{sl}(2,\mathbf{C})^{\mathbf{R}}$. It is well known that

$$\mathfrak{sl}(2,\mathbf{C})^{\mathbf{R}} \cong \mathfrak{so}(3,1) \quad \text{and} \quad \mathfrak{su}(2) \cong \mathfrak{so}(3,\mathbf{R})$$

where $\mathfrak{so}(3,1)$ is the Lie algebra of $\mathrm{SO}(3,1) \subset \mathrm{GL}(4,\mathbf{R})$. Furthermore, there exists an isomorphism

$$(4.5) \qquad \varphi : \mathfrak{so}(3,1) \underset{\mathbf{R}}{\otimes} \mathbf{C} \cong \mathfrak{so}(4,\mathbf{C}) \cong \mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C})$$

such that

$$\varphi(\mathfrak{so}(3,1)) = \{(X,\overline{X}) \mid X \in \mathfrak{sl}(2,\mathbf{C})\}.$$

Next recall that the irreducible finite-dimensional representations of the complex Lie algebra $\mathfrak{sl}(2,\mathbf{C})$ are parametrized by $p \in \mathbf{N}$. Given $p \in \mathbf{N}$, let W_p be the corresponding irreducible $\mathfrak{sl}(2,\mathbf{C})$-module. Then $W_1 = \mathbf{C}^2$ and $\rho_1 : \mathfrak{sl}(2,\mathbf{C}) \to \mathfrak{gl}(\mathbf{C}^2)$ is the standard representation. Furthermore, for each $p \in \mathbf{N}$, W_p is the pth symmetric power of the module $W_1 = \mathbf{C}^2$.

Let E be a finite-dimensional complex vector space. Then the complex conjugate $\overline{E}$ of E is defined to be the dual vector space of the vector space of all antilinear forms on E. Every $x \in E$ determines a unique element $\overline{x} \in \overline{E}$. In particular, we can consider the complex conjugate $\overline{W}_q$ of the $\mathfrak{sl}(2,\mathbf{C})$-module W_q. Then $\mathfrak{sl}(2,\mathbf{C})$ acts on $\overline{W}_q$ by

$$\overline{\rho}_q(X)\overline{w} = \overline{\rho_q(X)w}, \qquad X \in \mathfrak{sl}(2,\mathbf{C}),\ \overline{w} \in \overline{W}_q,$$

and we get an antiholomorphic representation

$$\overline{\rho}_q : \mathfrak{sl}(2,\mathbf{C}) \longrightarrow \mathfrak{gl}(\overline{W}_q).$$

Given $(p,q) \in \mathbf{N}^2$, set $\tilde{\rho}_{p,q} = \rho_p \otimes \overline{\rho}_q$. Then

$$\tilde{\rho}_{p,q} : \mathfrak{sl}(2,\mathbf{C}) \longrightarrow \mathfrak{gl}(W_p \otimes \overline{W}_q)$$

is an irreducible representation of $\mathfrak{sl}(2,\mathbf{C})^{\mathbf{R}}$. Since $\mathrm{SL}(2,\mathbf{C})$ is simply connected, $\tilde{\rho}_{p,q}$ can be lifted to $\mathrm{SL}(2,\mathbf{C})$. Hence, for each $(p,q) \in \mathbf{N}^2$, we get an irreducible representation

$$\rho_{p,q} : \mathrm{SL}(2,\mathbf{C}) \longrightarrow \mathrm{GL}(W_p \otimes \overline{W}_q);$$

any irreducible representation of the real Lie group $\mathrm{SL}(2,\mathbf{C})$ is obtained in this way. This is a special case of a more general result (cf. [Wa, Theorem 3.1.1.1]). With respect to this parametrization, the Cartan involution acts by $\theta(p,q) = (q,p)$. Employing Theorem 4.4, we obtain

Lemma 4.6. *Let Γ be a discrete, torsionfree, co-compact subgroup of $\mathrm{SL}(2,\mathbf{C})$. For $(p,q) \in \mathbf{N}^2$, let $E_{p,q}$ be the irreducible $\mathrm{SL}(2,\mathbf{C})$-module described above. Then*

$$H^*(\Gamma\backslash H^3, E_{p,q}) = 0 \quad \text{if } p \neq q.$$

We note that the complexification of the standard representation of $\mathfrak{so}(3,1)$ on $\mathbf{R}^4$ is equivalent to the $\mathfrak{sl}(2,\mathbf{C})$-module $E_{1,1}$. Thus we can restate Lemma

4.6 as follows. If a given irreducible $\mathfrak{sl}(2, \mathbf{C})$-module E_Λ with heighest weight Λ has nontrivial cohomology, then Λ is a multiple of the heighest weight of the standard representation of $\mathfrak{so}(3, 1)$. This agrees with Theorem 1 in [R].

For all these acyclic representations ρ, the analytic torsion $T_M(\rho)$ and the R-torsion $\tau_M(\rho)$ are defined and independent of any choice of metrics. This is of interest in connection with the results of Moscovici and Stanton [MS].

4.2. Chern-Simons gauge theory.
Chern-Simons gauge theory is a three-dimensional gauge field theory with pure Chern-Simons action. The connection with analytic torsion was first noticed by Schwarz [S] in the case of an abelian gauge group. For nonabelian gauge groups the link to analytic torsion was established by Witten [W1] via the perturbative expansion of the corresponding functional integral. For a compact gauge group, the torsion invariants occurring in the perturbative expansion are associated to unitary representations of the fundamental group of the underlying 3-manifold. However, if we allow noncompact semisimple gauge groups, then one has to deal with the analytic torsion attached to unimodular representations.

Now we shall describe this in more detail. Let M be a compact oriented three-dimensional manifold without boundary and G a Lie group. We start with the case where G is compact, and for simplicity we take G to be $SU(N)$. Consider the space $\mathscr{A}$ of all G-connections on the trivial G-bundle over M. In fact, every principal G-bundle over M is trivial. The space $\mathscr{A}$ may be identified with the space $\Lambda^1(M, \mathfrak{g})$ of differential 1-forms on M with values in the Lie algebra g of G. For a given connection $A \in \mathscr{A}$, the Chern-Simons action is defined to be

$$(4.7) \qquad I(A) = \frac{1}{4\pi} \int_M \mathrm{Tr}(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A)$$

where Tr is the trace of $\mathfrak{su}(N)$ in the standard representation. This is a real-valued nonlinear functional on $\mathscr{A}$. The gauge group $\mathscr{G} = \mathrm{Map}(M, G)$ acts on $\mathscr{A}$ by the usual prescription $A^g = g^{-1}Ag + g^{-1}dg$, $g \in G$, $A \in \mathscr{A}$. The Chern-Simons functional I is not invariant under the action of $\mathscr{G}$, but it satisfies $I(A^g) = I(A) + 2\pi m$ for some $m \in \mathbf{Z}$ depending on $g \in \mathscr{G}$. Let $k \in \mathbf{N}$. Then $e^{ikI(A)}$ is a $\mathscr{G}$-invariant function on $\mathscr{A}$ and Witten's invariant of M is defined as the path integral

$$(4.8) \qquad Z_M(k) = \int e^{ikI(A)} \mathscr{D}A$$

where the integration is over all gauge equivalence classes of connections. This, however, has to be considered as a formal expression because no measure $\mathscr{D}A$ has yet been constructed. Part of this theory can be made rigorous, and Witten gave an explicit recipe for computing $Z_M(k)$. Moreover, using the theory of quantum groups, Reshetikhin and Turaev [RT] introduced invariants $\widetilde{Z}_M(q)$ of a 3-manifold M depending on a root of unity $q = e^{2\pi i/r}$, and they suggest that $\widetilde{Z}_M(e^{2\pi i/(k+2)})$ coincides with $Z_M(k)$ after normalization.

Instead of the full path integral (4.8) one can also study the perturbative formulation, i.e., one perturbs about the critical points $A^{(0)}$ of the action (4.7).

In the present context this method is again not based on solid ground, but it gives very interesting results. The Euler-Lagrange equation for (4.7) is

$$dA + A \wedge A = 0,$$

which shows that the critical points of (4.7) are precisely the connections with vanishing curvature, that is, the flat connections on the bundle $P = M \times G$. A flat connection A is determined up to gauge equivalence by a representation

$$(4.9) \qquad\qquad \alpha : \pi_1(M) \to G$$

up to conjugacy. Hence the space of gauge equivalence classes of flat connections on P can be identified with $\mathrm{Hom}(\pi_1(M), G)/G$ where G acts on $\mathrm{Hom}(\pi_1(M), G)$ by conjugation. Assume that the topology of M is such that there exists only a finite number of gauge equivalence classes of flat connections on P, say $A_1, \ldots, A_m$ and let $\alpha_1, \ldots, \alpha_m$ denote the corresponding homomorphisms (4.9).

Let A be a flat connection, and for simplicity assume that A is irreducible. To work out the leading perturbative approximation to the contribution of the critical point A to the path integral $Z_M(k)$, gauge fixing is needed. This is achieved by picking a Riemannian metric g on M. Let $\alpha : \pi_1(M) \to G$ be the holonomy representation of A and $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ the adjoint representation. Then $\mathrm{Ad} \circ \alpha$ is a representation

$$(4.10) \qquad\qquad \rho_\alpha : \pi_1(M) \longrightarrow \mathrm{GL}(\mathfrak{g}),$$

and we denote the associated flat bundle by $\mathfrak{g}_\alpha$. Since $\mathfrak{g}$ is compact, the Killing form is negative definite on $\mathfrak{g}$. Therefore, the negative of the Killing form defines a G-invariant inner product on $\mathfrak{g}$, and with respect to this choice of an inner product the representation (4.10) is orthogonal. Finally, we note that the De Rham complex $\Lambda^*(M; \mathfrak{g}_\alpha)$ is equivalent to the complex of $\mathfrak{g}$-valued differential forms

$$0 \to \Lambda^0(M; \mathfrak{g}) \xrightarrow{d_A} \Lambda^1(M; \mathfrak{g}) \xrightarrow{d_A} \Lambda^2(M; \mathfrak{g}) \xrightarrow{d_A} \Lambda^3(M; \mathfrak{g}) \to 0$$

where $\Lambda^q(M; \mathfrak{g}) = \Lambda^q(M) \otimes \mathfrak{g}$ and d_A is the covariant derivative with respect to the connection A. Since A is flat, we have $d_A^2 = 0$. By assumption, A is irreducible and isolated modulo gauge equivalence. Therefore, we have $H^0(M; \mathfrak{g}_\alpha) = H^1(M; \mathfrak{g}_\alpha) = 0$ and Poincaré duality implies $H^*(M; \mathfrak{g}_\alpha) = 0$, so that ρ_α is acyclic.

To describe the final result we need some more notation. Let $c_2(G)$ be the value of the Casimir operator of G in the adjoint representation, normalized so that $c_2(\mathrm{SU}(N)) = 2N$. Furthermore, let $I(g)$ be the Chern-Simons invariant of the Levi-Civita connection of g with respect to a given trivialization of the tangent bundle of M, and let $\eta(g)$ be the η-invariant of the metric g.

Assume now that the representatives $A_1, \ldots, A_m$ of the gauge equivalence classes of flat connections are all irreducible. Then Witten's formula for the stationary phase approximation (or one loop approximation) of the path integral (4.8) is

$$(4.11) \qquad Z_M(k) \sim \frac{1}{\#Z(G)} e^{i\pi d \left(\frac{1}{2}\eta(g) + \frac{1}{24\pi} I(g) \right)} \sum_{j=1}^m e^{i(k + c_2(G)/2) I(A_j)} \sqrt{T_M(\rho_{\alpha_j})}$$

where $Z(G)$ is the center of G, d is the dimension of G, and $T_M(\rho_{\alpha_j})$ is the analytic torsion of the flat connection ρ_{α_j}. In fact, formula (2.23) in [W1] has to be slightly corrected (cf. [FG, (1.32)]).

Freed and Gompf [FG] have done explicit computations in a number of cases supporting the believe that (4.11) gives the correct asymptotic behaviour of Witten's invariant.

Since each ρ_{α_j} is acyclic, $T_M(\rho_{\alpha_j})$ is independent of the choice of the metric g on M and, by [C, Mü], it coincides with the R-torsion $\tau_M(\rho_{\alpha_j})$. Furthermore, by the Atiyah-Patodi-Singer theorem, $\frac{1}{2}\eta(g)+\frac{1}{24\pi}I(g)$ is also independent of the metric. It depends only on the trivialization of the tangent bundle. As we know, the R-torsion $\tau_M(\rho_{\alpha_j})$ can be computed from a triangulation K of M in a pure combinatorial way. This suggests that one may be able to develop a rigorous treatment of the path integral (4.8) on the combinatorial level and derive the asymptotic behaviour (4.11) in this way.

There exist also conjectures how (4.11) has to be modified if we give up the assumption that the gauge equivalence classes of flat connections are isolated and irreducible (cf., [FG, (1.36)]).

So far we considered the case of a compact gauge group. Witten has also started to investigate Chern-Simons theory with noncompact gauge group [W3]. This is of great interest, for example, in the application to 3-dimensional gravity. More precisely, $(2 + 1)$-dimensional gravity is related to Chern-Simons gauge theory with gauge group SL(2, **C**), ISO(2, 1), or SL(2, **R**)×SL(2, **R**) depending on whether the cosmological constant is positive, zero, or negative [W2]. For a general noncompact Lie group G, the quantization of Chern-Simons gauge theory w ith gauge group G is not yet understood. For SL(2, **R**), however, the quantization is well understood [K]. One can also study the perturbative expansion of the corresponding path integral [B-NW].

The perturbative treatment of Chern-Simons gauge theory with noncompact gauge group requires again gauge fixing. Since the Killing form is indefinite, there exists no obvious gauge fixing as in the compact case and different approaches are possible [B-NW]. For a semisimple Lie group G, the most natural gauge fixing seems to be the unitary gauge fixing described in [B-NW, §4]. Let A be a flat connection on the trivial G-bundle over M with holonomy representation $\alpha : \pi_1(M) \to G$. As above, let $\mathfrak{g}_\alpha$ be the flat bundle defined by $\rho_\alpha = \text{Ad} \circ \alpha$. Then the unitary gauge fixing amounts to the choice of a Riemannian metric g on M and a Hermitian metric h on $\mathfrak{g}_\alpha$. We observe that $\rho_\alpha : \pi_1(M) \to \text{GL}(\mathfrak{g})$ is unimodular. In fact, since $\mathfrak{g}$ is semisimple, the Killing form is nondegenerate. Hence, for each $g \in G$, $\text{Ad}(g)$ preserves a nondegenerate symmetric bilinear form on $\mathfrak{g}$ which implies that $|\det \text{Ad}(g)| = 1$. This is precisely the setting of §2.

Under the same assumption as above, one gets a formula for the one loop approximation of the path integral which is similar to (4.11). The analytic torsion $T_M(\rho_{\alpha_j})$ is now defined by (2.2). For the discussion of the phase factor see §4 of [B-NW]. By assumption, each representation ρ_{α_j} is acyclic, and, therefore, by Corollary 2.7, $T_M(\rho_{\alpha_j})$ is independent of the choice of the metric on M and

$\mathfrak{g}_\alpha$. Moreover, by Theorem 1, $T_M(\rho_{\alpha_j})$ equals the R-torsion $\tau_M(\rho_{\alpha_j})$ which has again a pure combinatorial describtion. This suggests that Chern-Simons gauge theory with a noncompact but semisimple gauge group should also be accessible to a combinatorial treatment.

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MAX-PLANCK-INSTITUT FÜR MATHEMATIK, GOTTFRIED-CLAREN-STR. 26, D-5300 BONN 3, GERMANY

E-mail address: mueller@mpim-bonn.mpg.de