

ON A PROBLEM OF ERDŐS AND LOVÁSZ. II:

$$n(r) = O(r)$$

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1. INTRODUCTION

The function $n(r)$ of the title is the least size of a collection of r -sets, any two of which intersect, and such that any set of size $r - 1$ is disjoint from at least one of them. In the language of hypergraphs (see the end of this section for definitions), $n(r)$ is

$$\min\{|\mathcal{H}| : \mathcal{H} \text{ an } r\text{-uniform, intersecting hypergraph with } \tau(\mathcal{H}) = r\}.$$

The problem of estimating $n(r)$ was raised by Erdős and Lovász in [13]. For many years Erdős has listed deciding whether

$$(1) \quad n(r) = O(r)$$

as one of his “three favorite combinatorial problems”. (The other two are the Δ -system problem of Erdős and Rado and the Erdős-Faber-Lovász Conjecture; see, e.g., [12].)

The cover number $\tau(\mathcal{H})$ —the least size of a set (“cover”) meeting all members of \mathcal{H} —is of central interest in the study of hypergraphs (see, e.g., [16]). One may regard the Erdős-Lovász question as an attempt to understand how hard it is to force this number to be large.

Note that some restriction is needed to make the question interesting, since any t pairwise disjoint sets have cover number t . The assumption that \mathcal{H} be intersecting seems the most natural way to avoid this. On the other hand, once this assumption is made, we have $\tau(\mathcal{H}) \leq \min\{|A| : A \in \mathcal{H}\}$, so when the members of \mathcal{H} are of size r , the most one may ask is that $\tau(\mathcal{H}) = r$. Thus we arrive quite naturally at $n(r)$.

The Erdős-Lovász question is also of interest because of its relation to recent developments concerning good packings and edge-covers in large hypergraphs of bounded edge size; this connection is outlined in §5. See also [16, 19] for further discussion of $n(r)$ and related problems.

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The best previous bounds for $n(r)$ are (we write \mathcal{P}_r for any projective plane of order $r - 1$)

$$(2) \quad n(r) \geq 8r/3 - 3 \quad \text{for all } r,$$

$$(3) \quad n(r) \leq 4r^{3/2} \log r \quad \text{if } r \text{ is large enough and there exists a } \mathcal{P}_r,$$

both due to Erdős and Lovász [13], and

$$(4) \quad n(r) < Cr \log r \quad \text{if } r \text{ is large enough and there exists a } \mathcal{P}_r,$$

proved in [17]. The upper bounds (3), (4) are immediate consequences (respectively) of the next two theorems.

Theorem 1.1 [13]. *If \mathcal{H} is a set of $m \geq 4r^{3/2} \log r$ random lines from \mathcal{P}_r , then with probability tending to 1 as $r \rightarrow \infty$, $\tau(\mathcal{H}) = r$.*

Theorem 1.2 [17]. *If \mathcal{H} is a set of $m \geq 22r \log r$ random lines from \mathcal{P}_r , then with probability tending to 1 as $r \rightarrow \infty$, $\tau(\mathcal{H}) = r$.*

Theorem 1.2 (with, of course, C rather than 22) was conjectured in [13]. The correct value of the constant is probably 3; see [17].

Here we prove (1). We will show for some fixed prime power K that for sufficiently large $t \in \mathbb{N}$ and prime power $q \equiv 3 \pmod{4}$ satisfying

$$(5) \quad q < t \leq (1 + K^{-2})q,$$

if

$$(6) \quad r = Kq + t$$

then

$$(7) \quad n(r) \leq 5(K^2 + K)t.$$

Since all sufficiently large r 's are of the form (6), (1) follows.

Thus the constant in our bound on $n(r)$ is about $5K$. No attempt is made here to optimize or even to evaluate K , though it would be interesting to see if it could be brought down to, say, 100, or 1000. It would also be interesting to do something about the still unimproved lower bound (2).

Examples giving (7) are described in the next section, and their correctness is proved in the two sections that follow. The examples are given in dual form, that is, with the roles of vertices and edges interchanged. Thus, for appropriate K , q , t , and $r = Kq + t$, we construct an r -regular hypergraph on $5(K^2 + K)t$ vertices satisfying

$$(8) \quad d(x, y) \geq 1 \quad \text{for all distinct vertices } x, y$$

and having edge cover number r .

In the final section we mention a related problem of J.-C. Meyer, and, intending to put the present work in a more general context, sketch a connection with results of [18].

Terminology. A *hypergraph*, usually denoted \mathcal{H} , is a collection of subsets of a set $V = V(\mathcal{H})$. Members of V are called *vertices*, and members of \mathcal{H} are called *edges*.

The *degree* in \mathcal{H} of a vertex x is, as usual, the number of edges of \mathcal{H} containing x and is denoted $d_{\mathcal{H}}(x)$ or simply $d(x)$. More generally, $d(x, y, \dots)$ denotes the number of edges containing all of the vertices x, y, \dots .

A hypergraph is *k-uniform* (resp. *k-bounded*) if each of its edges has size k (resp. *at most k*) and *r-regular* if each of its vertices has degree r . It is *intersecting* if any two of its edges share a vertex.

A *vertex cover* (clearer would be “cover of edges by vertices”) of \mathcal{H} is a set of vertices meeting every edge of \mathcal{H} , while an *edge cover* is a collection of edges whose union is V . Either of these may be shortened to “cover” if there seems no danger of confusion.

The *vertex* and *edge cover numbers* of \mathcal{H} are the minimum sizes of its vertex and edge covers and are denoted $\tau(\mathcal{H})$ and $\rho(\mathcal{H})$.

A *matching* is a collection of pairwise disjoint edges. It is *perfect* if it is also an (edge) cover.

Finally, the (*point-set*) *dual*, say \mathcal{H}^* , of \mathcal{H} is the hypergraph obtained from \mathcal{H} by reversing the roles of vertices and edges; that is,

$$V(\mathcal{H}^*) = \mathcal{H}, \quad \mathcal{H}^* = \{\{A \in \mathcal{H} : A \ni v\} : v \in V(\mathcal{H})\}.$$

2. EXAMPLE

A. We begin with something akin to expanders (see, e.g., [2]).

Let us say that a bipartite graph G with bipartition $X_1 \cup X_2$ is *expansive* if (for some t) $|X_1| = |X_2| = t$, G is 5-regular, and for $i = 1, 2$ and $X \subseteq X_i$,

$$|N(X)| \geq \begin{cases} 1.1|X| & \text{if } |X| \leq t/2, \\ 3|X| & \text{if } |X| \leq t/10, \end{cases}$$

where $N(X) = \{y : \exists x \in X, y \sim x\}$. (The parameters here are rather arbitrary and by no means optimal.)

It is not hard to see—for instance via either the models of random regular graphs in [1, 4], or the usual random constructions of expanders (e.g., [21])—that such graphs exist for all sufficiently large t . Let us fix one such, called $G(t)$, for each sufficiently large t , and let $\Sigma = \Sigma(t)$ be the dual of $G(t)$. Then Σ is a 5-uniform hypergraph, the disjoint union of two perfect matchings, Σ_1 and Σ_2 , each of size t , such that for $i \in \{1, 2\}$ and Σ' any edge cover of Σ with $|\Sigma_i \setminus \Sigma'| = a$,

$$(9) \quad |\Sigma' \cap \Sigma_{3-i}| \geq \begin{cases} 1.1a & \text{if } a \leq t/2, \\ 3a & \text{if } a \leq t/10. \end{cases}$$

B. Let $C = 1602$. The next ingredients of our construction are a projective plane \mathcal{P} of some fixed order K and for each line l of \mathcal{P} a function

$$\sigma_l : l \rightarrow \{1, 2\}$$

satisfying:

(I) For all $x \in V = V(\mathcal{P})$ and $x \notin m \in \mathcal{P}$,

$$|\{l \ni x : l \sim_\sigma m\}| < K/2 + 2\sqrt{K \log K},$$

where we write $l \sim_\sigma m$ if, with $l \cap m = \{y\}$, we have $\sigma_l(y) = \sigma_m(y)$; and

(II) For all $\tau : V(\mathcal{P}) \rightarrow \{1, 2\}$, if \mathcal{T} is a set of at least $3K/4$ lines l of \mathcal{P} , each satisfying

$$|\{x \in l : \sigma_l(x) \neq \tau(x)\}| \leq C,$$

then there exist $\mathcal{T}_0 \subseteq \mathcal{T}$ with $|\mathcal{T}_0| < 4C$ and a point x of \mathcal{P} such that

$$l \in \mathcal{T} \setminus \mathcal{T}_0 \Rightarrow l \ni x.$$

The existence of such functions for large enough K is given by

Lemma 2.1. *Let C be fixed, \mathcal{P} a projective plane of order K , and for each line l of \mathcal{P} let*

$$\sigma_l : l \rightarrow \{1, 2\}$$

be chosen uniformly at random. Then with probability tending to 1 as $K \rightarrow \infty$, σ satisfies (I) and (II).

This is proved in §3.

We may thus fix K , \mathcal{P} , and functions σ_l as above, with the additional requirement that K be large enough to justify the various inequalities of §4.

Let x_0 be a distinguished point of \mathcal{P} . Set $X = V(\mathcal{P}) \setminus \{x_0\}$, and let $h : X \rightarrow \{l \in \mathcal{P} : x_0 \notin l\}$ be some fixed function satisfying

$$x \in h(x) \quad \forall x, \quad |h^{-1}(l)| \leq 2 \quad \forall l.$$

(That such an h exists is easily seen using the fact that the point-line incidence matrix of \mathcal{P} is nonsingular over the rationals.)

C. Recall that \mathcal{B}^* is a *transversal design* $\text{TD}(K, t)$ with groups G_1, \dots, G_K if G_1, \dots, G_K are pairwise disjoint t -sets and \mathcal{B}^* is a hypergraph on $\bigcup G_i$ satisfying

(a) $|B \cap G_i| = 1$ for all $B \in \mathcal{B}^*$ and $1 \leq i \leq K$, and

(b) for all $0 \leq i < j \leq K$, $x \in G_i$, and $y \in G_j$, there is a unique $B \in \mathcal{B}^*$ with $x, y \in B$.

(Thus \mathcal{B}^* is K -uniform of degree t and has t^2 edges.)

A well-known theorem of Chowla, Erdős, and Straus [8], based on constructions of Bose, Shrikhande, and Parker [7] and Brun's sieve, asserts that the largest size of a collection of mutually orthogonal Latin squares of order t tends

to infinity with t . We need the equivalent (see, e.g., [26] for the connection and the best results to date in this direction)

Theorem 2.2. *For every K' there is a $t(K')$ such that for all $t > t(K')$ there exists a $\text{TD}(K', t)$.*

We suppose henceforth that $t > t(K + 1)$ and the prime power $q \equiv 3 \pmod{4}$ satisfy (5) and fix a $\text{TD}(K, t)$, \mathcal{B}^* , with

$$(10) \quad \rho(\mathcal{B}^*) = t.$$

(This is true, e.g., if \mathcal{B}^* is obtained from a $\text{TD}(K + 1, t)$ by deleting a group, and explains why we took $t > t(K + 1)$ rather than $t(K)$. The condition (10) is not really necessary, but simplifies the proof in §4 at one point.)

D. We need two additional objects, similar to transversal designs (or rather, one additional object which will be used in two ways.)

Let \mathcal{P}_q be a projective plane of order q , \mathcal{O} an oval of \mathcal{P}_q , z_0 a point not on any tangents to \mathcal{O} , and l_1, \dots, l_K exterior lines of \mathcal{O} containing z_0 . (For projective plane definitions, see, e.g., [9].) Let $y_1, \dots, y_{(q+1)/2}$ be points of \mathcal{O} on distinct lines through z_0 .

Let $H_i = l_i \setminus \{z_0\}$ for $1 \leq i \leq K$, and $H_0 = \{y_1, \dots, y_{(q+1)/2}\}$. In addition, let $H'_0 \cup H''_0$ be a partition of H_0 into two sets of size $(q + 1)/4$.

Let \mathcal{B} be the hypergraph with vertices $\bigcup\{H_i : 0 \leq i \leq K\}$ and edges all lines of \mathcal{P}_q not containing z_0 (with the natural containments). Notice that the restriction of \mathcal{B} to $\bigcup\{H_i : 1 \leq i \leq K\}$ is a $\text{TD}(K, q)$ (with groups H_i), whereas for $x \in H_0$ and any other vertex y , there is a (unique) edge of \mathcal{B} containing x and y .

E. For each $x \in X$, let $\Gamma(x)$ be a copy of Σ on a vertex set $V(x)$, with $\Gamma_i(x)$ the corresponding copy of Σ_i ($i = 1, 2$), and

$$V(x) \cap V(y) = \emptyset \quad \text{if } x \neq y.$$

Further, if E_1, \dots, E_t are the edges of $\Gamma_i(x)$, let

$$\Gamma_i^*(x) = \{E_{2j-1} \cup E_{2j} : 1 \leq j \leq t - q\} \cup \{E_j : 2(t - q) + 1 \leq j \leq t\}$$

(so, in particular, $|\Gamma_i^*(x)| = q$). Set $\Gamma = \bigcup \Gamma(x)$, $\Gamma^* = \bigcup \Gamma^*(x)$, both unions ranging over $x \in X$.

Now (9) implies that (for $i = 1, 2$) if $\Gamma' \subseteq \Gamma^*(x) \cup \Gamma(x)$ is a cover with $|\Gamma' \cap (\Gamma_i^*(x) \cup \Gamma_i(x))| = q - a$, then

$$(11)$$

$$|\Gamma' \cap (\Gamma_{3-i}^*(x) \cup \Gamma_{3-i}(x))| \geq \begin{cases} \max\{1.1a - (t - q), .55a\} & \text{if } a \leq t/2, \\ \max\{3a - (t - q), 3a/2\} & \text{if } a \leq t/10. \end{cases}$$

Proof. Since $\bigcup\{E : E \in \Gamma' \cap (\Gamma_i^*(x) \cup \Gamma_i(x))\}$ is the union of at most $q - a + (t - q) = t - a$ edges of $\Gamma_i(x)$, the number of edges of $\Gamma_{3-i}(x)$ needed to cover

the vertices not covered by $\Gamma' \cap \Gamma_i^*(x)$ is, according to (9), at least

$$1.1a \text{ if } a \leq t/2, \quad \text{and} \quad 3a \text{ if } a \leq t/10.$$

But Γ' gives a cover of these vertices by at most

$$\min\{|\Gamma' \cap (\Gamma_{3-i}^*(x) \cup \Gamma_{3-i}(x))| + (t - q), 2|\Gamma' \cap ((\Gamma_{3-i}^*(x) \cup \Gamma_{3-i}(x))|\}$$

edges of $\Gamma_{3-i}(x)$, and this implies (11). \square

We also define for each x and i a hypergraph $\Gamma_i^{**}(x)$, each of whose edges is the union of a few edges of $\Gamma_i^*(x)$. The definition of $\Gamma_i^{**}(x)$ depends on $|h^{-1}(h(x))|$:

(a) If $h^{-1}(h(x)) = \{x\}$, let Π be a partition of the edges of $\Gamma_i^*(x)$ into $(q+1)/2$ blocks, each of size 1 or 2.

(b) If $|h^{-1}(h(x))| = 2$, let Π be a partition of the edges of $\Gamma_i^*(x)$ into $(q+1)/4$ blocks, each of size 4 or less.

In either case, set

$$\Gamma_i^{**}(x) = \left\{ \bigcup_{F \in B} F : B \in \Pi \right\}.$$

(Actually for each x we only need $\Gamma_i^{**}(x)$ when $i = \sigma_{h(x)}(x)$.)

F. For each $l \in \mathcal{P}$ define a hypergraph \mathcal{H}_l on vertex set

$$\bigcup \{V(x) : x \in l \setminus \{x_0\}\}$$

as follows.

(a) If $x_0 \in l$, fix a numbering x_1, \dots, x_K of the points of $l \setminus \{x_0\}$ and bijections

$$\varphi_i^l : G_i \rightarrow \Gamma_{\sigma_l(x_i)}(x_i).$$

(Recall G_i are the groups of the transversal design \mathcal{B}^* .) Let $\varphi^l = \bigcup_{i=1}^K \varphi_i^l$, and set

$$\mathcal{H}_l = \left\{ \bigcup_{p \in B} \varphi^l(p) : B \in \mathcal{B}^* \right\}.$$

For each $A \in \mathcal{H}_l$ set $\sigma_A = \sigma_l$.

(b) For $x_0 \notin l$ the construction is analogous. We use \mathcal{B} in place of \mathcal{B}^* and give special treatment to the points in $h^{-1}(l)$.

If the points of l are x, x_1, \dots, x_K with $h^{-1}(l) = \{x\}$, we fix bijections

$$\varphi_0^l : H_0 \rightarrow \Gamma_{\sigma_l(x)}^{**}(x) \quad \text{and} \quad \varphi_i^l : H_i \rightarrow \Gamma_{\sigma_l(x_i)}^*(x_i)$$

for $1 \leq i \leq K$.

If the points of l are x, y, x_2, \dots, x_K with $h^{-1}(l) = \{x, y\}$, we fix bijections

$$\varphi_0^l : H'_0 \rightarrow \Gamma_{\sigma_l(x)}^{**}(x),$$

$$\varphi_1^l : H''_0 \rightarrow \Gamma_{\sigma_l(y)}^{**}(y),$$

and

$$\varphi_i^l : H_i \rightarrow \Gamma_{\sigma_l(x_i)}^*(x_i)$$

for $2 \leq i \leq K$.

In either case, we let $\varphi^l = \bigcup_{i=0}^K \varphi_i^l$ (so in the second case H_1 is excluded from the domain of φ^l) and

$$\mathcal{H}_l = \left\{ \bigcup_{p \in B} \varphi^l(p) : B \in \mathcal{B} \right\}.$$

Also, in either case, for $A \in \mathcal{H}_l$ we set

$$\sigma_A = \sigma_l|_{\{z : A \cap V(z) \neq \emptyset\}}.$$

G. Finally, set $V(\mathcal{H}) = \bigcup \{V(x) : x \in X\}$ and $\mathcal{H} = \bigcup \{\mathcal{H}_l : l \in \mathcal{P}\}$. It is then straightforward to check that \mathcal{H} is r -regular ($r = Kq + t$) with $5(K^2 + K)t$ vertices and satisfies (8). [For the degrees, note that $v \in V(x)$ has degree t in \mathcal{H}_l if l is the line joining x and x_0 and degree q if l is any other line containing x . For (8): if $v \in V(x)$, $w \in V(y)$ with $x \neq y$, then v, w are in some common edge of \mathcal{H}_l , l the line joining x and y (this uses property (b) of transversal designs); while if $v, w \in V(x)$, then v, w are in some common edge of \mathcal{H}_l , where $l = h(x)$ (see the comment at the end of **D**).]

So the only interesting point is to show $\rho(\mathcal{H}) = r$. Actually, we show a little more ($l(x, y)$ is the line of \mathcal{P} joining x and y):

Theorem 2.3. *If \mathcal{E} is an edge cover of \mathcal{H} of size r , then there exists $x \in X$ such that*

$$|\mathcal{E} \cap \mathcal{H}_l| = \begin{cases} q & \text{if } x \in l \neq l(x, x_0), \\ t & \text{if } l = l(x, x_0). \end{cases}$$

The idea of the construction is roughly as follows. If \mathcal{E} is an edge cover of \mathcal{H} of size r , then for each $x \in X$ the intersections of edges of \mathcal{E} with $V(x)$ give a cover of $\Gamma(x)$. Now (9) says that using edges from both $\Gamma_1(x)$ and $\Gamma_2(x)$ to cover $\Gamma(x)$ is wasteful. But we cannot afford much waste: we must use at least $(K^2 + K)t$ edges of Γ to cover all the $V(x)$'s; on the other hand, any edge of \mathcal{H} is the union of $K + O(1)$ edges of Γ , so that the cover of Γ given by \mathcal{E} has size at most $(K^2 + O(K))t$.

Thus for given $x \in X$, we would like the values $\sigma_A(x)$ for $A \in \mathcal{E}$ meeting $V(x)$ to be mostly the same. But the only way to achieve this—this is the lemma—is to take \mathcal{E} to consist mainly of edges from $\bigcup \{\mathcal{H}_l : l \ni x\}$ for some fixed x .

This gets us to (30). A further analysis in the same spirit then eliminates “mainly” from the above description.

3. LEMMA

Here we prove Lemma 2.1. Condition (I) is a standard calculation, which we omit.

For \mathcal{L} a set of lines of \mathcal{P} and $Z(l)$ a subset of l for each $l \in \mathcal{L}$, we extend our earlier notation by writing $l \sim_{(\sigma, Z)} m$ if either $l \sim m$ or the intersection point of l and m belongs to $Z(l) \cup Z(m)$. We prove condition (II) in the more convenient form

(II') Suppose \mathcal{L} is a subset of \mathcal{P} and for each $l \in \mathcal{L}$, $Z(l)$ is a set of at most C points of l satisfying

$$(12) \quad l \sim_{(\sigma, Z)} m \quad \forall l, m \in \mathcal{L}.$$

Then if $|\mathcal{L}| > 3K/4$, there exist $\mathcal{L}_0 \subseteq \mathcal{L}$ with $|\mathcal{L}_0| < 4C$ and a point x of \mathcal{P} such that

$$l \in \mathcal{L} \setminus \mathcal{L}_0 \Rightarrow l \ni x.$$

Of course it is enough to prove this when $|\mathcal{L}| = 3K/4$ (note all large numbers are integers) and the size of each $Z(l)$ is exactly C . For fixed $\mathcal{L} \subseteq \mathcal{P}$ of this size and C -subsets $Z(l)$, $l \in \mathcal{L}$, as in (II'), let x_1, \dots be the points of \mathcal{P} and set

$$d_i = d_{\mathcal{L}}(x_i), \quad e_i = |\{l \in \mathcal{L} : x_i \in Z(l)\}|.$$

Then

$$(13) \quad \sum_{d_i \geq 2} \binom{d_i}{2} = \binom{|\mathcal{L}|}{2},$$

$$(14) \quad \sum e_i = C|\mathcal{L}|.$$

Consider first the event

$$(15) \quad d_i < K^{1-\delta} \quad \text{for all } i,$$

with (say) $\delta = 1/10$. [Actually (12) is unlikely provided

$$\delta = \omega(\log \log K / \log K);$$

on the other hand (see (19)) we only need that some d_i is at least about $\log^2 K$.]

Then by (13),

$$(16) \quad \sum_{d_i \geq 2} d_i > (9/16)K^{1+\delta},$$

$$(17) \quad \sum_{d_i \geq 2} (d_i - 1) > (9/32)K^{1+\delta}.$$

But in view of (14) this implies (still with \mathcal{F} , Z fixed)

$$(18) \quad \begin{aligned} Pr(()) &= \exp_2[-\sum_{d_i \geq 2} (d_i - e_i - 1)] \\ &< \exp_2[-(9/32)K^{1+\delta} + (3/4)CK]. \end{aligned}$$

On the other hand, the number of possibilities for \mathcal{F} , Z is at most

$$\binom{K^2 + K + 1}{3K/4} \binom{K + 1}{C}^{3K/4}$$

so that the probability of (12) holding for some \mathcal{F} , Z with \mathcal{F} satisfying (15) is $o(1)$.

As a (very weak) consequence we may therefore assume that for some x ,

$$(19) \quad d_{\mathcal{F}}(x) > D^2 \log^2 K,$$

where $D = 3C$. We will show that with probability tending to 1 as $K \rightarrow \infty$, if x satisfies (19) (and \mathcal{F} , Z satisfy (12)), then

$$(20) \quad |\{l \in \mathcal{F} : l \not\ni x\}| < D \log K.$$

Proof. Denote by E the event that there are a point x , lines $l_1, \dots, l_a \not\ni x$, $m_1, \dots, m_b \ni x$, and subsets $Z(l_i) \subseteq l_i$, $Z(m_j) \subseteq m_j$ of size at most C such that no m_j contains the intersection point of two of the l_i 's and

$$(21) \quad l_i \underset{(\sigma, Z)}{\sim} m_j \quad \text{for all } i, j.$$

For given x , l_i 's, m_j 's, and Z , the probability of (21) is at most

$$\exp_2[-ab + C(a + b)].$$

On the other hand, the number of possibilities for x , $\{l_i\}$, $\{m_j\}$, and Z is at most

$$(K^2 + K + 1) \binom{K^2}{a} \binom{K + 1}{b} \binom{K + 1}{C}^{a+b} < \exp[(C + 2)(a + b) \log K].$$

In particular, if we set $a = b = D \log K$, then with probability $1 - o(1)$, E does not occur.

This easily gives (20). For suppose $d_{\mathcal{F}}(x) > D^2 \log^2 K$ and

$$x \notin l_1, \dots, l_a \in \mathcal{F},$$

where, again, $a = D \log K$. There are then at least

$$D^2 \log^2 K - \binom{a}{2} > \frac{1}{2} D^2 \log^2 K > D \log K$$

lines $m \ni x$ containing none of the intersection points of the l_j 's and satisfying

$$m \underset{(\sigma, Z)}{\sim} l_j \quad \text{for all } j.$$

But we have just ruled out such an event, so (20) is proved. \square

Now again let x satisfy (19), and suppose $x \notin l_1, \dots, l_{4C} \in \mathcal{T}$. By (20) we have

$$|\{m : x \in m \in \mathcal{T}\}| > 3K/4 - 3C \log K,$$

which with (I) implies that for $1 \leq i \leq 4C$,

$$\begin{aligned} \left| l_i \cap \left(\bigcup \{Z(m) : x \in m \in \mathcal{T}\} \right) \right| &\geq |\{m : x \in m \in \mathcal{T}, m \not\sim_{\sigma} l_i, m \cap l_i \notin Z(l_i)\}| \\ &> K/4 - 2\sqrt{K \log K} - 3C \log K - C \\ &:= K/4 - F. \end{aligned}$$

There are thus at least $K/4 - F - 4C$ lines m , $x \in m \in \mathcal{T}$, for which $m \cap l_i$ is a point of $Z(m)$ not contained in any other l_j ; whence,

$$3CK/4 \geq \left| \bigcup_{x \in m \in \mathcal{T}} Z(m) \right| \geq (K/4 - F - 4C)4C.$$

But this is false for large K , so the lemma is proved.

4. THEOREM

Let \mathcal{C} be an (edge) cover of \mathcal{H} of size $r = Kq + t$.

It is easy to see, since $\rho(\mathcal{H}_l) \leq t$ for each $l \in \mathcal{P}$ (see (10); actually $\rho(\mathcal{H}_l) \leq q$ if $l \not\ni x_0$), that we may assume

$$(22) \quad |\mathcal{C} \cap \mathcal{H}_l| \leq t \quad \forall l \in \mathcal{P}.$$

For $A \in \mathcal{H}$ and $x \in X$ define

$$\gamma_x(A) = \begin{cases} 0 & \text{if } A \cap V(x) = \emptyset, \\ |\{E \in \Gamma(x) : E \subseteq A\}| & \text{if } A \in \mathcal{H}_{l(x, x_0)}, \\ |\{E \in \Gamma_{\sigma_A(x)}^*(x) : E \subseteq A\}| & \text{otherwise.} \end{cases}$$

(So, for example, $\gamma_x(A) \in \{0, 1\}$ unless $A \in \mathcal{H}_l$ with $h(x) = l$.) Define $\tau : V \rightarrow \{1, 2\}$ by

$$\tau(x) = \begin{cases} 1 & \text{if } \sum \{\gamma_x(A) : A \in \mathcal{C}, \sigma_A(x) = 1\} \geq \sum \{\gamma_x(A) : A \in \mathcal{C}, \sigma_A(x) = 2\} \\ 2 & \text{otherwise.} \end{cases}$$

(Thus, formally, $\tau(x_0) = 1$.)

Again for $A \in \mathcal{H}$ and $x \in X$, set

$$f_x(A) = \begin{cases} \gamma_x(A) & \text{if } \sigma_A(x) = \tau(x), \\ 0 & \text{otherwise,} \end{cases} \quad g_x(A) = \begin{cases} \gamma_x(A) & \text{if } \sigma_A(x) \neq \tau(x), \\ 0 & \text{otherwise,} \end{cases}$$

and $f(A) = \sum_x f_x(A)$, $g(A) = \sum_x g_x(A)$.

Finally, for $l \in \mathcal{P}$, let

$$\beta(l) = |\{x \in l : \sigma_l(x) \neq \tau(x)\}|.$$

Notice (cf. **E** and **F** of §2) that

$$(23) \quad f(A) + g(A) \leq K + 7,$$

and, for $A \in \mathcal{H}_l$,

$$(24) \quad \beta(l) \leq g(A) + |\{x \in l \cap X : A \cap V(x) = \emptyset\}| + \chi(l \ni x_0) \leq g(A) + 2,$$

where χ is the indicator function.

Now (11) implies that for each $x \in X$

$$\sum_{A \in \mathcal{E}} g_x(A) \geq 1.1 \left(q - \sum_{A \in \mathcal{E}} f_x(A) \right) - (t - q),$$

whence,

$$(25) \quad \sum_{A \in \mathcal{E}} g(A) \geq 1.1 \left((K^2 + K)q - \sum_{A \in \mathcal{E}} f(A) \right) - (K^2 + K)(t - q).$$

Combining this with (23) gives

$$\sum_{A \in \mathcal{E}} g(A) \geq 1.1 \left((K^2 + K)q - (K + 7)r + \sum_{A \in \mathcal{E}} g(A) \right) - (K^2 + K)(t - q),$$

or, after a little rearranging,

$$(26) \quad \frac{1}{r} \sum_{A \in \mathcal{E}} g(A) \leq \frac{10}{r} [1.1((K + 7)r - (K^2 + K)q) + (K^2 + K)(t - q)] < 80.$$

Thus, setting

$$\mathcal{E}_1 = \{A \in \mathcal{E} : g(A) \leq 1600\},$$

we have

$$(27) \quad |\mathcal{E}_1| \geq 0.95r.$$

Write $l = l(A)$ if $A \in \mathcal{H}_l$, and set

$$\mathcal{F} = \{l(A) : A \in \mathcal{E}_1\}.$$

Then by (24),

$$(28) \quad \beta(l) \leq 1602 (= C) \quad \forall l \in \mathcal{F},$$

and by (22), (5),

$$(29) \quad |\mathcal{F}| \geq 0.95r/t > 0.95K.$$

Applying (II) to \mathcal{F} (and again using (22)), we conclude that there is some $x \in X \cup \{x_0\}$ for which

$$(30) \quad \left| \mathcal{E} \cap \left(\bigcup_{l \ni x} \mathcal{H}_l \right) \right| \geq |\mathcal{E}_1| - 4Ct > 0.95r - 4Ct.$$

Assume first that $x \neq x_0$. Let l_0, \dots, l_K be the lines of \mathcal{P} containing x , with $l_0 = l(x, x_0)$, and set

$$\begin{aligned} X^* &= X \setminus \{x\} \setminus \bigcup_{i=1}^K h^{-1}(l_i), \\ s_i &= |\mathcal{E} \cap \mathcal{H}_i|, \\ u_i &= |l_i \cap X^*|, \\ \mathcal{E}' &= \mathcal{E} \setminus \bigcup_{i=0}^K \mathcal{H}_i. \end{aligned}$$

For convenience, set

$$q_i = \begin{cases} t & \text{if } i = 0, \\ q & \text{if } 1 \leq i \leq K. \end{cases}$$

The argument we use to finish is analogous to that leading to (26). Notice first of all that for all i ,

$$(31) \quad Kq/19 > |\mathcal{E}'| \geq u_i(q_i - s_i)/8 \geq (K-2)(q_i - s_i)/8.$$

Proof. The first inequality here is weaker than (30), so since u_i is clearly at least $K-2$, we just need to prove the middle inequality.

Let $y \in l_i \cap X^*$. Then $V(y) \cap (\bigcup \{A : A \in \mathcal{E} \cap \mathcal{H}_i\})$ is the union of at most $s_i + t - q_i$ edges of $\Gamma(y)$. Since any $A \cap V(y)$ is the union of at most eight edges of $\Gamma(y)$ (see E of §2), it follows that

$$\left| \mathcal{E} \cap \left(\bigcup \{ \mathcal{H}_l : y \in l \neq l_i \} \right) \right| \geq (t - (s_i + t - q_i))/8 = (q_i - s_i)/8.$$

Since no line of $\mathcal{P} \setminus \{l_i\}$ contains more than one $y \in l_i$, this establishes (31). \square

It follows that for all i ,

$$(32) \quad q_i - s_i < (8/19)Kq/(K-2),$$

and in particular

$$(33) \quad s_i > q_i/2.$$

We now replace τ by $\tau' : X \setminus \{x\} \rightarrow \{1, 2\}$ defined according to the lines through x :

$$\tau'(y) = \sigma_{l(x,y)}(y).$$

Define

$$\begin{aligned} f^*(A) &= \sum \{\gamma_y(A) : y \in X^*, A \cap V(y) \neq \emptyset, \sigma_A(y) = \tau'(y)\}, \\ g^*(A) &= \sum \{\gamma_y(A) : y \in X^*, A \cap V(y) \neq \emptyset, \sigma_A(y) \neq \tau'(y)\}. \end{aligned}$$

In particular, $g^*(A) = 0$ if $A \in \mathcal{E} \setminus \mathcal{E}'$, so by (I),

$$(34) \quad \sum_{A \in \mathcal{E}} g^*(A) = \sum_{A \in \mathcal{E}'} g^*(A) \leq (K/2 + 2\sqrt{K \log K} + O(1))|\mathcal{E}'|.$$

On the other hand (again using (I) for the inequality),

$$f^*(A) \begin{cases} = u_i & \text{if } A \in \mathcal{H}_i, \\ \leq K/2 + 2\sqrt{K \log K} + O(1) & \text{if } A \in \mathcal{H}_l, l \neq i. \end{cases}$$

Recalling that $u_i \geq K - 2$ and noting $|\mathcal{E}'| = \sum_{i=0}^K (q_i - s_i)$, it follows that

$$\begin{aligned} Q &:= \sum_{i=0}^K u_i q_i - \sum_{A \in \mathcal{E}} f^*(A) \\ &\geq \sum_{i=0}^K u_i (q_i - s_i) - (K/2 + 2\sqrt{K \log K} + O(1))|\mathcal{E}'| \\ (35) \quad &\geq (K/2 - 2\sqrt{K \log K} - O(1))|\mathcal{E}'|. \end{aligned}$$

Using (33) with (11) and (9), we have

$$\begin{aligned} \sum_{A \in \mathcal{E}'} g^*(A) &\geq 1.1Q - \sum_{i=0}^K u_i (t - q_i) \\ (36) \quad &\geq 1.1(K/2 - 2\sqrt{K \log K} - O(1))|\mathcal{E}'| - \sum_{i=0}^K u_i (t - q_i). \end{aligned}$$

This together with (34) forces \mathcal{E}' to be small, say

$$|\mathcal{E}'| < 25K(t - q).$$

But then by (32) and (5) we have (overestimating)

$$q_i - s_i < q/10,$$

and by (11),

$$\begin{aligned} \sum_{A \in \mathcal{E}'} g^*(A) &\geq 3Q/2 \\ (37) \quad &\geq 3(K/2 - 2\sqrt{K \log K} - O(1))|\mathcal{E}'|/2. \end{aligned}$$

So again combining with (34) we have

$$(K/4 - 5\sqrt{K \log K} - O(1))|\mathcal{E}'| \leq 0,$$

implying $\mathcal{E}' = \emptyset$.

The proof of Theorem 2.3 is then completed by noting that for any $y \in X \setminus \{x\}$ with $h(y) \neq l(x, y)$, the number of edges of $\mathcal{H}_{l(x, y)}$ required to cover $V(y)$ is t if $l(x, y) = l(x, x_0)$, and q otherwise.

When $x = x_0$, the formal definitions of X^* , s_i , u_i , and \mathcal{E}' are as before—note this now gives $X^* = X$ and $u_i = K$ for all i —and we set $q_i = t$ for all i . The argument then proceeds as before, except that, since the error terms $t - q_i$ are all 0, (36) (with (34)) is already enough to force $\mathcal{E}' = \emptyset$.

Since to cover we must then use at least t edges from \mathcal{H}_l for each $l \ni x_0$, the conclusion in this case is that in fact $x = x_0$ cannot occur.

5. REMARKS

A. A question similar to that of Erdős and Lovász was raised at about the same time by J.-C. Meyer [20], who introduced, for positive integer r , the function

$$m(r) = \min\{|\mathcal{H}| : \mathcal{H} \text{ a maximal intersecting, } r\text{-uniform hypergraph}\}.$$

While Meyer's original conjecture that $m(r) \geq r^2 - r + 1$ was disproved by Füredi [15], who showed

$$(38) \quad m(r) \leq 3r^2/4 \quad \text{if there exists a } \mathcal{P}_{r/2+1},$$

it has not been easy even to guess how $m(r)$ ought to grow. The best results of a fair amount of subsequent effort on the problem are (from [3], [6], and [10] respectively)

$$(39) \quad m(r) \leq r^5 \quad \forall r,$$

$$(40) \quad m(r) < r^2/2 + O(r) \quad \text{if there exists a } \mathcal{P}_r,$$

and

$$(41) \quad m(r) \geq 3r \quad \text{for } r \geq 4.$$

(The lower bound is a slight improvement on $m(r) \geq 8r/3 - 3$, which follows from (2), since trivially $m(r) \geq n(r)$. See also [19] for a proposed construction for $m(r) = o(r^2)$ when there exists a \mathcal{P}_r .)

While the examples for $n(r)$ described above do not seem to give anything for $m(r)$, they seem to me strongly to suggest the truth of

Conjecture 5.1. $m(r) = O(r)$.

B. In view of the examples giving (3) and (4), it is natural to ask whether one can establish (1) with \mathcal{H} 's contained in (the line sets of) projective planes.

(Random \mathcal{H} 's clearly will not do here, since about $3r \log r$ random lines from \mathcal{P}_r are needed to guarantee $d(x) \geq 2 \forall x$, an obvious necessary condition for $\tau = r$; but one might hope to succeed with a more deliberate choice.)

The answer is that such examples do *not* exist, and more generally one cannot prove (1) with \mathcal{H} 's in which all edge intersections are small. Understanding this was a crucial step toward the present work, and a brief sketch of the ideas involved, together with a few as yet unanswered questions, may be of interest. The following results from [18] extend earlier work of Rödl [23], Frankl and Rödl [14], Pippenger (see [24]) and Pippenger and Spencer [22]. (See also [19] for a more extensive discussion of these and related topics.)

For $t : \mathcal{H} \rightarrow \mathbf{R}^+$, define $\bar{t} : 2^V \rightarrow \mathbf{R}^+$ (where $V = V(\mathcal{H})$) by

$$\bar{t}(A) = \sum \{t(B) : B \supseteq A\},$$

and set

$$\alpha_i(t) = \max\{\bar{t}(W) : W \subseteq V, |W| = i\}.$$

Recall that $t : \mathcal{H} \rightarrow \mathbf{R}^+$ is a *fractional cover* if

$$(42) \quad \sum_{A \ni x} t(A) \geq 1 \quad \forall x \in V.$$

We will also say that t is a *fractional tiling* if equality holds in (42).

Theorem 5.2. *Let k be fixed, \mathcal{H} a k -bounded hypergraph, and $t : \mathcal{H} \rightarrow \mathbf{R}^+$ a fractional cover. Then*

$$\rho(\mathcal{H}) \lesssim t(\mathcal{H}) (\alpha_2(t) \rightarrow 0).$$

(We use $t(\mathcal{H})$ for $\sum_{A \in \mathcal{H}} t(A)$ and $f \lesssim g$ for $\limsup(f/g) \leq 1$. A similar result holds for matchings, but we confine ourselves here to covers.)

Corollary 5.3. *Suppose \mathcal{H} is r -regular with at most cr vertices, c fixed, $d(x, y) > 0$ for all $x, y \in V$, and*

$$(43) \quad \max\{d(x, y) : x, y \in V, x \neq y\} = o(r).$$

Then $\rho(\mathcal{H}) < (c/(c+1) + o(1))r$.

In dual form (the form corresponding to $n(r)$) this is the negative result alluded to above:

Corollary 5.4. *Suppose \mathcal{H} is r -uniform, intersecting of size at most cr , c fixed, and satisfies*

$$(44) \quad \max\{|A \cap B| : A, B \in \mathcal{H}, A \neq B\} = o(r).$$

Then $\tau(\mathcal{H}) < (c/(c+1) + o(1))r$.

After a preliminary step that eliminates large edges, the connection between Theorem 5.2 and Corollary 5.3 is provided by the observation that if \mathcal{H} is

r -regular with $n \leq cr$ vertices and satisfies

$$d(x, y) > 0 \quad \forall x, y \in V(\mathcal{H}),$$

then the function $t: \mathcal{H} \rightarrow \mathbf{R}^+$ given by

$$(45) \quad t(A) = |A|/(n+r-1)$$

is a fractional cover of total weight

$$\sum_{A \in \mathcal{H}} t(A) = nr/(n+r-1) \approx nr/(n+r) \leq cr/(c+1).$$

(The condition “ $\alpha_2(t) \rightarrow 0$ ”, of course, derives from (43).)

The following conjecture says that Corollary 5.3 should remain true even if (43) is relaxed to the analogous condition on triples.

Conjecture 5.5. *Suppose \mathcal{H} is r -regular with at most cr vertices, c fixed, $d(x, y) > 0$ for all $x, y \in V$, and*

$$(46) \quad \max\{d(x, y, z) : x, y, z \in V \text{ distinct}\} = o(r).$$

Then $\rho(\mathcal{H}) < (c/(c+1) + o(1))r$.

(I also expect that one cannot replace triples by 4-sets here; there should be examples giving (1) for which (in dual form as usual)

$$\max\{d(W) : W \subseteq V, |W| = 4\} = o(r).$$

While Conjecture 5.5 may seem a minor point, it is of interest in the context of [18]. Roughly, the goal of the work initiated there is to understand, as far as possible, what are the obstructions to good matching and (edge) cover behavior in k -bounded hypergraphs (k fixed) of large degree? Here a working definition of “good cover behavior” is that the conclusion

$$(47) \quad \rho(\mathcal{H}) \lesssim t(\mathcal{H})$$

of Theorem 5.2 holds for appropriate t . (This is similar to asking for $\rho(\mathcal{H}) \sim \rho^*(\mathcal{H})$ but weaker in that one cannot always come close to achieving ρ^* with “appropriate” t .)

Nicest here would be a statement to the effect that the only obstructions are “local”, roughly in the sense that there are integer covers which mimic t on small sets of vertices.

Suppose, for example, that $V(\mathcal{H})$ is partitioned into triples, that each edge of \mathcal{H} meets each triple in either 0 or 2 vertices, and that we drop the requirement that $\bar{t}(\{x, y\})$ be small when x, y are in the same triple. It is then more or less typical (e.g., take \mathcal{H} regular and uniform) for $\rho(\mathcal{H})$ to be about $(4/3)\rho^*(\mathcal{H})$, reflecting the fact that we have $\rho(\Gamma) = (4/3)\rho^*(\Gamma)$ for the underlying graph Γ of pairs for which \bar{t} is allowed to be large.

While the condition “ $\alpha_2(t) \rightarrow 0$ ” precludes such an arrangement, it may be regarded as unnecessarily crude. For example, if $X \cup Y$ is a partition of V ,

then it is probably enough in Theorem 5.2 if we replace $\alpha_2(t)$ by

$$\max\{\bar{t}(\{x, y\}) : x, y \in X \text{ or } x, y \in Y\}.$$

That is, allowing \bar{t} to be large on the edges of a bipartite graph should not create obstructions to good cover behavior.

Surprisingly, it does not seem impossible that one can reach a rather complete understanding of what “local” behavior guarantees (47), provided we at least insist that

$$(48) \quad \alpha_3(t) \rightarrow 0.$$

This would go essentially as follows.

For $t : \mathcal{H} \rightarrow \mathbf{R}^+$ and any $X \subseteq V$, define $t|_X : 2^X \rightarrow \mathbf{R}^+$ by

$$t|_X(A) = \sum \{t(B) : B \cap X = A\} \quad \forall A \subseteq X.$$

Write $\text{MP}(X)$ for the “*matching polytope*” of X :

$$\text{MP}(X) = \text{conv}\{1_M : M \text{ a matching of } 2^X\}.$$

Denote by $b(t)$ the largest b such that for any $X \subseteq V$ with $|X| \leq b$ we have $t|_X \in \text{MP}(X)$. Then one version of what is meant by “nice local behavior” is that $b(t) \rightarrow \infty$. (It is easy to see that this is true if $\alpha_2(t) \rightarrow 0$.)

Conjecture 5.6 [18]. *Let k be fixed, \mathcal{H} a k -bounded hypergraph, and $t : \mathcal{H} \rightarrow \mathbf{R}^+$ a fractional tiling. Then*

$$\rho(\mathcal{H}) \lesssim t(\mathcal{H}) \quad (\alpha_3(t) \rightarrow 0, b(t) \rightarrow \infty).$$

Conjecture 5.5 is in the same vein as Conjecture 5.6, and our interest in the former is mainly motivated by the latter. (It may be that Conjecture 5.6 implies Conjecture 5.5, but I do not quite see this. For the covers given by (45), the condition “ $\alpha_3(t) \rightarrow 0$ ” will, of course, follow from (46) (again, following a preliminary elimination of large edges), and while “ $b(t) \rightarrow \infty$ ” need not hold in general, it does hold in many natural situations and, in particular, in cases that seem to be bad for Conjecture 5.5.)

As mentioned above, Conjecture 5.6 probably leads to an exact understanding of what local conditions guarantee the conclusion “ $\rho(\mathcal{H}) \lesssim t(\mathcal{H})$ ”. A precise statement of this becomes a little messy. Roughly, it involves relaxing “tiling” to “cover” in Conjecture 5.6 and requiring only that *most* of the functions $t|_X$ are *approximately* in $\text{MP}(X)$.

If we try to relax (48) further, allowing \bar{t} to be large on triples, then no such “local” characterization is possible. For example, when \mathcal{H} is itself a graph, Conjecture 5.6 is an easy consequence of either Tutte’s 1-factor theorem [25] or Edmonds’s matching polytope theorem [11]. But if \mathcal{H} is 3-uniform and we relax (48) to the, in this case, vacuous “ $\alpha_4(t) \rightarrow 0$ ” (retaining “ $b(t) \rightarrow 0$ ”), then the resulting conjecture is not true (as shown, for example, by the duals of Bollobás’s examples [5] of cubic graphs of large girth and small “independence

ratio”). Far from being the end of the story, this raises a number of interesting questions, to which we hope to return in a later paper.

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ABSTRACT. We prove a linear upper bound on the function $n(r) = \min\{|\mathcal{H}| : \mathcal{H} \text{ an } r\text{-uniform, intersecting hypergraph with } \tau(\mathcal{H}) = r\}$, thus settling a longstanding problem of Erdős and Lovász.

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