FLAT VECTOR BUNDLES, DIRECT IMAGES
AND HIGHER REAL ANALYTIC TORSION

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CONTENTS

The purpose of this paper is to extend the Ray-Singer analytic torsion [RS1] from an invariant of a smooth manifold to an invariant of a smooth parametrized family of manifolds. In addition, we prove a C^{∞}-analog of the Riemann-Roch-Grothendieck theorem for holomorphic submersions. We show that the "higher" analytic torsion enters in a differential form version of this theorem.

Let us first give some of the history of the problem. In the 1930's, Reidemeister and Franz defined a certain invariant of simplicial complexes [Fr, Re]. Let (K, F) be a pair consisting of a finite simplicial complex K and a flat unitary complex vector bundle F on K such that the twisted cohomology $H^*(K; F)$ vanishes. To this data they assigned a real number, now called the Reidemeister torsion. It turns out that the Reidemeister torsion is a topological invariant of K. For a survey article on this invariant, see [M]. The original interest of the Reidemeister torsion was that, unlike other more standard topological invariants, it can distinguish lens spaces which are homotopy-equivalent but not homeomorphic. More generally, the spherical space forms of a given dimension are classified up to isometry by their fundamental groups, along with their Reidemeister torsions.

Ray and Singer asked whether, as for many other real topological invariants, there is an analytic version of the Reidemeister torsion which is defined when M is a closed smooth Riemannian manifold. If F is a flat unitary complex vector bundle on M, they defined a real number in terms of the spectrum of the Laplacian acting on F-valued differential forms on M. They showed that if $H^*(M; F) = 0$ then this number, the analytic torsion, is independent of the choice of Riemannian metric on M, and conjectured that it equals the Reidemeister torsion [RS1]. This conjecture was shown to be true independently by Cheeger [C] and Müller [Mü1]. One can extend the equality between Reidemeister and Ray-Singer torsions to the case of nonunitary F [Mü2, BZ].

The Reidemeister and Ray-Singer torsions have remained somewhat isolated objects, in that it has not been clear how they fit into the more general framework of topology and analysis. On the algebraic topology side, it is well known that

there is a relationship between the Reidemeister torsion and the first algebraic K-group of fields. Soon after the work of Ray and Singer, Wagoner suggested that to extend this relationship, instead of a single manifold one may wish to consider a smooth parametrized family of manifolds. That is, one wishes to define an invariant of a smooth fiber bundle. He conjectured, for reasons coming from concordance theory, that the Reidemeister and Ray-Singer torsions can be extended to invariants of a fiber bundle, and that these invariants are related to the higher algebraic K-theory groups [W].

The program of defining a higher Reidemeister torsion has been recently carried out by Igusa and Klein [I, K]. Their work is based on Waldhausen's algebraic K-theory of spaces and parametrized Morse theory. We summarize their results in Appendix II. However, their constructions are rather involved, and to date the most complicated example for which the higher Reidemeister torsion has been computed is that of a general circle bundle over S^2.

Ray and Singer also considered a holomorphic version of their invariant, defined for Hermitian holomorphic vector bundles on compact complex manifolds [RS2]. Important progress has been made in understanding a families version of this holomorphic analytic torsion. Bismut, Gillet, and Soulé defined a holomorphic torsion form on the base of a Kähler fibration. This holomorphic torsion form enters into a differential form version of the Riemann-Roch-Grothendieck theorem for holomorphic submersions [BGS1, 2, 3]. The degree-0 component of the holomorphic torsion form is given by the holomorphic analytic torsion of the fibers, considered as a function on the base.

In this paper we solve Wagoner's problem of constructing a "higher" analytic torsion in the smooth setting. We define the analytic torsion form of a C^∞-fiber bundle with closed Riemannian fibers. It is a differential form on the base of the fiber bundle whose degree-0 component is given by the analytic torsion of the fibers, considered as a function on the base. We show that there is a C^∞-version of the Riemann-Roch-Grothendieck theorem, which relates the characteristic classes of a flat complex vector bundle, on the total space of a fiber bundle, to those of its "direct image" on the base. The analytic torsion form enters into a differential form version of this theorem. In addition, we show that under appropriate acyclicity conditions, the analytic torsion form gives a smooth topological invariant of the fiber bundle.

Let us state some of our results in detail. We first describe certain characteristic classes of flat bundles. Let B be a smooth manifold, let F be a flat complex vector bundle on B and let h^F be a Hermitian metric on F. With respect to a local covariantly-constant basis of F, h^F is locally a Hermitian matrix-valued function on B. Put

$$(0.1) \qquad \omega(F, h^F) = (h^F)^{-1} dh^F,$$

a globally-defined $\text{End}(F)$-valued 1-form on B.

If k is a positive odd integer, define a k-form on B by

$$(0.2) \qquad c_k(F, h^F) = (2i\pi)^{-(k-1)/2} 2^{-k} \text{Tr}[\omega^k(F, h^F)].$$

Then $c_k(F, h^F)$ is closed and its de Rham cohomology class $c_k(F)$ is indepen-

dent of h^F. The class $c_k(F)$ was previously defined by Kamber and Tondeur [KT, D], and one can think of $c_k(F, h^F)$ as a Chern-Weil-type description of this class.

Now let $Z \to M \xrightarrow{\pi} B$ be a smooth fiber bundle with base B and connected closed fibers $Z_b = \pi^{-1}(b)$. Let F be a flat complex vector bundle on M. Let $H^p(Z; F|_Z)$ denote the flat complex vector bundle on B whose fiber over $b \in B$ is isomorphic to the cohomology group $H^p(Z_b, F|_{Z_b})$. Let TZ be the vertical tangent bundle of the fiber bundle and let $o(TZ)$ be its orientation bundle, a flat real line bundle on M. Let $e(TZ) \in H^{\dim(Z)}(M; o(TZ))$ be the Euler class of TZ.

Theorem 0.1. *For any positive odd integer* k,

$$(0.3) \qquad \sum_{p=0}^{\dim(Z)} (-1)^p c_k(H^p(Z; F|_Z)) = \int_Z e(TZ) \cdot c_k(F) \quad in \ H^k(B; \mathbb{R}).$$

One sees that Theorem 0.1 is an analog of the Riemann-Roch-Grothendieck theorem for holomorphic submersions, in which a holomorphic submersion becomes a smooth fiber bundle, $\bar{\partial}$-flat (i.e., holomorphic) bundles become d-flat bundles, the direct image of F becomes $\sum_{p=0}^{\dim(Z)} (-1)^p H^p(Z; F|_Z)$, the Chern character becomes the c_k classes and the Todd class becomes the Euler class. A corollary of Theorem 0.1 is that if Z is odd-dimensional, then $\sum_{p=0}^{\dim(Z)} (-1)^p c_k(H^p(Z; F|_Z))$ vanishes. In the case $k = 1$, this vanishing can also be seen directly from the existence of a flat unitary metric, the Ray-Singer metric, on the determinant line bundle over B.

We do not know of a purely topological proof of Theorem 0.1. Our proof is analytic in nature, and gives a differential form version of (0.3). Equip the fiber bundle with a horizontal distribution $T^H M$ and a vertical Riemannian metric g^{TZ}, and the flat vector bundle F with a Hermitian metric h^F. The vector bundles $H^p(Z; F|_Z)$ then acquire Hermitian metrics $h^{H^p(Z;F|_Z)}$ from the Hodge isomorphism. We construct a $(k - 1)$-form $\mathcal{T}_{k-1}(T^H M, g^{TZ}, h^F)$ on B such that

Theorem 0.2. *For any positive odd integer* k,

$$d(\mathcal{T}_{k-1}(T^H M, g^{TZ}, h^F)) = \int_Z e(TZ, \nabla^{TZ}) \cdot c_k(F, h^F)$$
$$- \sum_{p=0}^{\dim(Z)} (-1)^p c_k(H^p(Z; F|_Z), h^{H^p(Z;F|_Z)}).$$

The 0-form $\mathcal{T}_0(T^H M, g^{TZ}, h^F)$ is simply the function which to $b \in B$ assigns half of the Ray-Singer analytic torsion of the fiber Z_b over b, computed using the flat vector bundle $F|_{Z_b}$. If Z is odd-dimensional and $k = 1$, then Theorem 0.2 is equivalent to the topological invariance of the Ray-Singer metric

on the determinant line of Z. For $k > 1$, the forms $\mathscr{T}_{k-1}(T^H M, g^{TZ}, h^F)$ can be called "higher" analytic torsion forms on B.

Theorem 0.2 implies that if Z is odd-dimensional and $H^p(Z; F|_Z)$ vanishes for all p, then $\mathscr{T}_{k-1}(T^H M, g^{TZ}, h^F)$ is closed. We show that its de Rham cohomology class $\mathscr{T}_{k-1}(M, F)$ is independent of the choices of $T^H M$, g^{TZ} and h^F. Thus $\mathscr{T}_{k-1}(M, F) \in H^{k-1}(B; \mathbb{R})$ is a smooth invariant of the pair (M, F). In the case $k = 1$, if F is unimodular then $\mathscr{T}_0(M, F)$ is represented by the locally constant function on B which to a point $b \in B$ assigns half of the Reidemeister torsion of the pair $(Z_b, F|_{Z_b})$.

The paper is organized as follows. The first two sections are concerned with finite-dimensional vector bundles, and the last two sections deal with the infinite-dimensional bundles which arise from fibrations. We treat the finite-dimensional case because the formalism is more easily seen in that case, which is free of analytic technicalities, and because the results are of independent interest in the finite-dimensional case.

Section 1 deals with flat superconnections. That is, we have a $\mathbb{Z}_2$-graded vector bundle E on a manifold B and a superconnection A' on E whose square vanishes. Given two flat superconnections A' and A'' and a holomorphic function $f: \mathbb{C} \to \mathbb{C}$, we define an associated closed form on B. If we are given a single flat superconnection A' on E and a Hermitian metric h^E on E, we show how to construct an adjoint flat superconnection A'^* on E, which we then use to define a closed form $f(A', h^E)$ on B. In the special case when A' is an ordinary flat connection, $f(A', h^E)$ essentially reduces to the above-mentioned forms $c_k(E, h^E)$.

In Section 2 we specialize to the case when E is $\mathbb{Z}$-graded and the flat superconnection A' has total degree 1. We can then think of E as a family of cochain complexes parametrized by B, with some additional structure provided by the higher order terms of A'. We introduce a rescaling h_t^E of the metric and examine how the forms $f(A', h_t^E)$ depend on the scaling parameter t. A torsion form $T_f(A', h^E)$ enters into this scaling-dependence. In the special case when B is a point, $T_f(A', h^E)$ is proportionate to the torsion of the cochain complex above B.

Section 3 gives the extension of the results of Sections 1 and 2 to the case of a fiber bundle $Z \to M \xrightarrow{\pi} B$ with connected closed fibers $Z_b = \pi^{-1}(b)$ and a flat (possibly nonunitary) complex vector bundle F on M. Choose a horizontal distribution $T^H M$ on M. The main idea is that the elements of $\Omega(M; F)$, the space of F-valued differential forms on M, can be considered to be forms on B with value in a certain infinite-dimensional vector bundle W, which is such that the fiber of W over $b \in B$ is isomorphic to $\Omega(Z_b; F|_{Z_b})$. The differential d^M on $\Omega(M; F)$ then becomes a flat superconnection A' on W. Upon choosing a vertical Riemannian metric on M and a Hermitian metric on F, we obtain a Hermitian metric on W and an adjoint superconnection A'^* on W. Remarkably, $\frac{1}{2}(A'^* + A')$ is essentially a Levi-Civita superconnection

in the sense of Bismut [B]. This fact allows us to use local heat kernel analysis to extend the formalism of Sections 1 and 2 to our infinite-dimensional setting. We construct the higher analytic torsion form $\mathscr{T}(T^H M, g^{TZ}, h^F)$ and prove Theorems 0.1 and 0.2. We also describe how $\mathscr{T}$ depends on its arguments.

Section 4 deals with the case when the fiber bundle is associated to a principal bundle with compact structure group G. We use equivariant methods to show that the analytic torsion form $\mathscr{T}$ is a linear sum of characteristic classes of the principal bundle, with coefficients that depend on the fiber of the associated bundle and the G-action thereon. We compute $\mathscr{T}$ when M is a circle bundle and F is a complex line bundle, and find that $\mathscr{T}$ is a polynomial in the first Chern class of the circle bundle, with coefficients that are given by polylogarithm functions of the holonomy of $F|_Z$.

Appendix I contains an axiomatic characterization of the torsion form $T_f(A', h^E)$ in the finite-dimensional acyclic case. We also discuss the torsion forms of double complexes. This appendix is a supplement to Section 2. In Appendix II we give a summary of the theory of Reidemeister and higher Reidemeister torsions.

We remark that in view of known relationships between polylogarithms and the algebraic K-theory of fields [L], the appearance of polylogarithms in the computation of $\mathscr{T}$ for circle bundles gives evidence that $\mathscr{T}$ is in fact related to Borel regulators in algebraic K-theory, as predicted in [W]. In the case $B = S^2$, our higher analytic torsion agrees with the higher Reidemeister torsion of Igusa-Klein. This raises the possibility of an extension of the Cheeger-Müller result which would equate the higher Reidemeister torsion with its analytic counterpart. More generally, our work indicates a relationship between analysis on manifolds and algebraic, as opposed to topological, K-theory.

Let us finally remark that one may well ask if it is possible to see the real analytic torsion form of a fiber bundle by endowing B with a Riemannian metric and taking the adiabatic limit of the ordinary analytic torsion $T(M)$ of M, in analogy with what was done by Bismut and Cheeger for the eta form [BC], and by Berthomieu and Bismut for the holomorphic torsion form [BerB]. The answer seems to be negative. In the adiabatic limit, the relevant term in $T(M)$ is $\int_B T(Z_b) \cdot \chi(b)$, where $T(Z_b)$ is the ordinary analytic torsion of the fiber Z_b and $\chi(b)$ is the Gauss-Bonnet-Chern density of B [DM, F]. In effect, the top-dimensionality of $\chi(b)$ blocks out the terms in the analytic torsion form $\mathscr{T}$ of positive degree. This does not mean that there is no higher real analytic torsion, but rather that it cannot be seen by adiabatic limits.

Note. W. Dwyer and B. Williams inform us that they have found a purely topological proof of Theorem 0.1.

I. FLAT SUPERCONNECTIONS, HERMITIAN METRICS
AND THEIR ASSOCIATED CLOSED FORMS

In this section we construct a closed form $f(A', h^E)$ which is associated to a $\mathbb{Z}_2$-graded complex vector bundle E, a flat superconnection A' on E, a metric h^E on E and a holomorphic function $f: \mathbb{C} \to \mathbb{C}$. We show that the de Rham cohomology class of $f(A', h^E)$ is independent of the choice of h^E. We also give the corresponding transgression formula.

The section is organized as follows. In (a) we establish our conventions and briefly recall the superconnection formalism. In (b) we construct a closed form based on a pair of flat superconnections. We introduce the transpose and adjoint of a superconnection in (c) and (d). In (e) we construct the closed form $f(A', h^E)$. In (f) we establish a transgression formula. Finally, in (g) we describe certain characteristic classes of flat vector bundles.

(a) **The superconnection formalism.** For background information on superconnections we refer to [B], [BGV] and [Q1]. Let us just establish some conventions. We use the normalizations of [Sp] for differential forms. Except where otherwise indicated, we will take all vector spaces in this paper to be over $\mathbb{C}$. Let B be a smooth manifold, let $\Lambda(T^*B)$ denote its complexified exterior bundle and let $\Omega(B)$ denote the space of smooth sections of $\Lambda(T^*B)$. The symbol $\widehat{\otimes}$ will denote tensoring over $C^\infty(B)$. Let $E = E_+ \oplus E_-$ be a $\mathbb{Z}_2$-graded finite-dimensional vector bundle on B. We let $C^\infty(B; E)$ denote the smooth sections of E, and $\Omega(B; E)$ denote the smooth sections of $\Lambda(T^*B) \widehat{\otimes} E$.

Let τ be the involution of E defining the $\mathbb{Z}_2$-grading, so that $\tau|_{E_\pm} = \pm I$. Then $\mathrm{End}(E)$ is a $\mathbb{Z}_2$-graded bundle of algebras over B, whose even (resp. odd) elements commute (resp. anticommute) with τ. Given $a \in C^\infty(B; \mathrm{End}(E))$, we define its supertrace $\mathrm{Tr}_s[a] \in C^\infty(B)$ by

$$\mathrm{Tr}_s[a] = \mathrm{Tr}[\tau a].$$

Given $\omega \in \Omega(B)$ and $A \in C^\infty(B; \mathrm{End}(E))$, put

$$(1.1) \qquad \mathrm{Tr}_s[\omega \cdot a] = \omega \, \mathrm{Tr}_s[a].$$

Then Tr_s extends to a linear map from $\Omega(B; \mathrm{End}(E))$ to $\Omega(B)$.

Given $\alpha, \alpha' \in \Omega(B; \mathrm{End}(E))$, we define their supercommutator $[\alpha, \alpha'] \in \Omega(B; \mathrm{End}(E))$ to be

$$[\alpha, \alpha'] = \alpha\alpha' - (-1)^{(\deg \alpha)(\deg \alpha')}\alpha'\alpha.$$

A basic fact is that Tr_s vanishes on supercommutators [Q1].

Let $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$ be a connection on E which preserves the splitting $E = E_+ \oplus E_-$. Let S be an odd element of $\Omega(B; \mathrm{End}(E))$. By definition, $\nabla^E + S$ gives a superconnection A on E. That is, there is a $\mathbb{C}$-linear map

$$A: C^\infty(B; E) \to \Omega(B; E)$$

which is odd with respect to the total $\mathbb{Z}_2$-gradings and satisfies the Leibniz rule. We can extend A to an odd $\mathbb{C}$-linear endomorphism of $\Omega(B; E)$. By definition, the curvature of A is A^2, an even $C^\infty(B)$-linear endomorphism of $\Omega(B; E)$ which is given by multiplication by an even element of $\Omega(B; \mathrm{End}(E))$.

We can expand A as

$$(1.2) \qquad\qquad A = \sum_{j \geq 0} A_j,$$

where A_j is of partial degree j with respect to the $\mathbb{Z}$-grading on $\Lambda(T^*B)$. Note that A_0 is an odd element of $C^\infty(B; \operatorname{End}(E))$ and A_1 is a connection on E.

(b) Flat superconnections.

Definition 1.1. A superconnection A on E is flat if its curvature vanishes, i.e., if $A^2 = 0$.

If A is a flat superconnection written in the form (1.2), then

$$A_0^2 = 0,$$
$$[A_0, A_1] = 0,$$
$$[A_0, A_2] + A_1^2 = 0,$$

$$(1.3) \qquad\qquad \vdots$$

$$[A_0, A_k] + [A_1, A_{k-1}] + \cdots = 0$$

$$\vdots$$

Let A' and A'' be two flat superconnections on E, so that

$$(1.4) \qquad\qquad A'^2 = 0, \qquad A''^2 = 0.$$

Put

$$(1.5) \qquad\qquad A = \tfrac{1}{2}(A'' + A'), \qquad X = \tfrac{1}{2}(A'' - A').$$

Then A is a superconnection on E and X is an odd element of $\Omega(B; \operatorname{End}(E))$.

Proposition 1.2. *The following identities hold:*

$$(1.6) \qquad \begin{array}{c} X^2 = -A^2, \qquad [A, X] = 0, \\ [A', X^2] = 0, \qquad [A'', X^2] = 0, \qquad [A, X^2] = 0. \end{array}$$

Proof. From (1.4) and (1.5), we have that $A^2 = \tfrac{1}{4}[A'', A'] = -X^2$ and $[A, X] = 0$. Then $[A', A^2] = [A'', A^2] = 0$, and (1.6) follows. $\square$

Let $f: \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Put

$$(1.7) \qquad\qquad \alpha = \operatorname{Tr}_s[f(X)] \in \Omega(B).$$

Proposition 1.3. *The form α is closed. Moreover, its even part is*

$$(1.8) \qquad\qquad \alpha^{\text{even}} = (\operatorname{rk}(E_+) - \operatorname{rk}(E_-))f(0).$$

Proof. As $[A, X] = 0$, it follows that

$$(1.9) \qquad\qquad [A, f(X)] = 0.$$

As Tr_s vanishes on supercommutators,

$$(1.10) \qquad\qquad d\alpha = \operatorname{Tr}_s[A, f(X)] = 0.$$

Thus α is closed. Put

$$g(a) = \tfrac{1}{2}(f(a) + f(-a)).$$

Then there is a holomorphic function $h: \mathbb{C} \to \mathbb{C}$ such that $g(a) = h(a^2)$. As X is odd,

(1.11) $$\alpha^{\text{even}} = \text{Tr}_s[h(X^2)].$$

Now

(1.12) $$\frac{\partial}{\partial t} \text{Tr}_s[h(tX^2)] = \text{Tr}_s[X^2 h'(tX^2)] = \frac{1}{2}\text{Tr}_s[X, Xh'(tX^2)] = 0.$$

It follows that

(1.13) $$\text{Tr}_s[h(X^2)] = \text{Tr}_s[h(0)] = (\text{rk}(E_+) - \text{rk}(E_-))h(0).$$

Then (1.8) follows from (1.11) and (1.13). □

Remark 1.4. By Proposition 1.3, if we want to construct interesting closed forms α, then we may restrict ourselves to the case when f is an odd function.

(c) **The transpose of a superconnection.** Let $\overline{E}^*$ be the antidual bundle to E. That is, $\overline{E}^*$ is the bundle of antilinear functionals on E. It inherits a $\mathbb{Z}_2$-grading from that of E. Let

$$\langle \, , \, \rangle: C^\infty(B; \overline{E}^*) \times C^\infty(B; E) \to C^\infty(B)$$

denote the pairing induced from the duality between $\overline{E}^*$ and E; it is linear in the first factor and antilinear in the second factor.

Let $\overline{}^*$ denote the even antilinear map from $\Omega(B; \text{End}(E))$ to $\Omega(B; \text{End}(\overline{E}^*))$ which is defined by the following three relations:

1. For $\alpha, \alpha' \in \Omega(B; \text{End}(E))$,

$$\overline{\alpha\alpha'}^* = \overline{\alpha'}^* \overline{\alpha}^*.$$

2. For $\omega \in \Omega^1(B)$,

$$\overline{\omega}^* = -\overline{\omega}.$$

3. For $a \in C^\infty(B; \text{End}(E))$, we have that $\overline{a}^* \in C^\infty(B; \text{End}(\overline{E}^*))$ is the conjugate transpose of a in the ordinary sense, i.e.,

$$\langle \overline{a}^* s', s \rangle = \langle s', as \rangle$$

for all $s' \in C^\infty(B; \overline{E}^*)$ and $s \in C^\infty(B; E)$.

Given a superconnection A on E, write A as $\nabla^E + S$, where $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$ is a connection on E and S is an odd element of $\Omega(B; \text{End}(E))$. Let $\nabla^{\overline{E}^*}$ be the connection on $\overline{E}^*$ induced from ∇^E.

Definition 1.5. $\overline{A}^*$ is the superconnection on $\overline{E}^*$ given by

(1.14) $$\overline{A}^* = \nabla^{\overline{E}^*} + \overline{S}^*.$$

One can easily check that $\overline{A}^*$ is independent of the decomposition of A as $\nabla^E + S$. If A is flat, then $\overline{A}^*$ is flat.

(d) **The adjoint of a superconnection.** By definition, a Hermitian metric h^E on the $\mathbb{Z}_2$-graded bundle E is a Hermitian metric on E such that E_+ and E_- are orthogonal. The metric h^E induces an even $C^\infty(B)$-linear isomorphism

$$(1.15) \qquad \hat{h}^E : \Omega(B; E) \to \Omega(B; \overline{E}^*).$$

Definition 1.6. The adjoint A^* of a superconnection A is the superconnection on E given by

$$(1.16) \qquad A^* = (\hat{h}^E)^{-1} \overline{A}^* \hat{h}^E.$$

If A is flat, then A^* is flat.

(e) **Associated odd closed forms.** Let $i^{1/2}$ be a square root of i. In what follows, the choice of square root will be irrelevant.

We assume that the $\mathbb{Z}_2$-graded vector bundle E has a flat superconnection A' and a Hermitian metric h^E. We now apply the formalism of Section 1(b), taking A'' to be the adjoint superconnection A'^*. That is, we have

$$(1.17) \qquad A = \tfrac{1}{2}(A'^* + A'), \qquad X = \tfrac{1}{2}(A'^* - A').$$

Let $\varphi \colon \Omega(B) \to \Omega(B)$ be the linear map such that for all homogeneous $\omega \in \Omega(B)$,

$$(1.18) \qquad \varphi\omega = (2i\pi)^{-(\deg \omega)/2}\omega.$$

In what follows, we will say that a holomorphic function $f \colon \mathbb{C} \to \mathbb{C}$ is real if for all $a \in \mathbb{C}$, we have $f(\overline{a}) = \overline{f(a)}$. We will say that a differential form is real if it can be written with real coefficients.

Definition 1.7. Let $f \colon \mathbb{C} \to \mathbb{C}$ be a holomorphic real odd function. Put

$$(1.19) \qquad f(A', h^E) = (2i\pi)^{1/2}\varphi \operatorname{Tr}_s[f(X)] \in \Omega(B).$$

Theorem 1.8. (i) *The form $f(A', h^E)$ is real, odd and closed.*

   (ii) *Let $h^{\overline{E}^*}$ be the induced metric on $\overline{E}^*$. Then $f(A', h^E) = -f(\overline{A'}^*, h^{\overline{E}^*})$.*

   (iii) *If there is an even isomorphism between the triplets (E, A', h^E) and $(\overline{E}^*, \overline{A'}^*, h^{\overline{E}^*})$, then $f(A', h^E) = 0$.*

   (iv) *Suppose that the triplet (E, A', h^E) is the complexification of a real triplet $(E_{\mathbf{R}}, A'_{\mathbf{R}}, h^E)$. Then if $k \equiv 3 \pmod 4$, the degree-k component of $f(A', h^E)$ vanishes.*

Proof. (i) By Proposition 1.3, $f(A', h^E)$ is odd and closed. Now

$$X^* = -X,$$

and so

$$f(X)^* = f(X^*) = f(-X) = -f(X).$$

It is now easy to check that (1.19) implies that $f(A', h^E)$ is real.

   (ii) Using an obvious notation, one can check that

$$X(\overline{A'}^*, h^{\overline{E}^*}) = -\hat{h}^E X(A', h^E)(\hat{h}^E)^{-1},$$

from which the claim follows.

(iii) This follows from statement (ii).

(iv) It is enough to show that $\text{Tr}_s[X^k] = 0$ if $k = 3 \pmod 4$. In general, we have

$$\overline{\text{Tr}_s[X^k]} = (-1)^{k(k+1)/2} \text{Tr}_s[(X^k)^*]$$
$$= (-1)^{k(k+1)/2} \text{Tr}_s[(X^*)^k] = (-1)^{k(k-1)/2} \text{Tr}_s[X^k].$$

In our case, $\text{Tr}_s[X^k]$ is real and the claim follows. □

(f) **A transgression formula.** Let $l \in \mathbb{R}$ parametrize a smooth 1-parameter family of Hermitian metrics h_l^E on E. For each $l \in \mathbb{R}$, form X_l as in (1.17). There is an operator $(h_l^E)^{-1} \frac{\partial h_l^E}{\partial l} \in C^\infty(B; \text{End}(E))$, where we use an obvious notation.

Theorem 1.9. *The form* $\varphi \text{Tr}_s[\frac{1}{2}(h_l^E)^{-1}\frac{\partial h_l^E}{\partial l}f'(X_l)]$ *on* B *is even and real. Moreover,*

$$(1.20) \qquad \frac{\partial}{\partial l} f(A', h_l^E) = d\left(\varphi \text{Tr}_s\left[\frac{1}{2}(h_l^E)^{-1}\frac{\partial h_l^E}{\partial l}f'(X_l)\right]\right).$$

Proof. Let $\mathcal{M}$ be the space of Hermitian metrics on E. Let $\pi: B \times \mathcal{M} \to B$ be projection onto the first factor. Then π^*E is a $\mathbb{Z}_2$-graded vector bundle on $B \times \mathcal{M}$ and π^*A' is a flat superconnection on π^*E. Moreover, π^*E has a canonical metric h^{π^*E} which restricts to h^E on $B \times \{h^E\}$.

Let X^{total} be the odd element of $\Omega(B \times \mathcal{M}; \text{End}(\pi^*E))$ which is associated to the triple $(\pi^*E, \pi^*A', h^{\pi^*E})$ by (1.17). Let $d^{\mathcal{M}}$ (resp. d^B) denote exterior differentiation in the $\mathcal{M}$ (resp. B) direction on $B \times \mathcal{M}$. Put

$$(1.21) \qquad \theta = (h^{\pi^*E})^{-1}(d^{\mathcal{M}}h^{\pi^*E}) \in \Omega^1(B \times \mathcal{M}; \text{End}(\pi^*E)).$$

Let X^{partial} be the map which takes $h^E \in \mathcal{M}$ to $X^{h^E} \in \Omega(B; \text{End}(E))$. We can think of X^{partial} as an element of $C^\infty(\mathcal{M}) \otimes \Omega(B; \text{End}(E))$, which in turn embeds in $\Omega(B \times \mathcal{M}; \text{End}(\pi^*E))$. One can check that

$$(1.22) \qquad X^{\text{total}} = X^{\text{partial}} + \frac{1}{2}\theta.$$

By (1.19),

$$(1.23) \qquad f(\pi^*A', h^{\pi^*E}) = (2i\pi)^{1/2}\varphi \text{Tr}_s[f(X^{\text{total}})].$$

From (1.22) and (1.23),

$$(1.24) \qquad f(\pi^*A', h^{\pi^*E}) = (2i\pi)^{1/2}\varphi \text{Tr}_s[f(X^{\text{partial}} + \frac{1}{2}\theta)].$$

Taking the Taylor expansion of the right-hand side of (1.24) in the variable θ, we obtain

$$(1.25) \qquad \begin{aligned} f(\pi^*A', h^{\pi^*E}) &= (2i\pi)^{1/2}\varphi \text{Tr}_s[f(X^{\text{partial}})] \\ &\quad + (2i\pi)^{1/2}\varphi \text{Tr}_s[\frac{1}{2}\theta f'(X^{\text{partial}})] + \beta, \end{aligned}$$

where β is a form on $B \times \mathcal{M}$ whose partial degree in the Grassmann variables of $\Lambda(T^* \mathcal{M})$ is ≥ 2. By Theorem 1.8, the form $f(\pi^* A', h^{\pi^* E})$ is real, odd and closed. It follows that

$$(1.26) \quad 0 = d^{\mathcal{M}}((2i\pi)^{1/2} \varphi \, \mathrm{Tr}_s[f(X^{\mathrm{partial}})]) + d^B((2i\pi)^{1/2} \varphi \, \mathrm{Tr}_s[\tfrac{1}{2}\theta f'(X^{\mathrm{partial}})]).$$

The smooth 1-parameter family h_l^E of metrics on E is equivalent to a smooth curve $c(l)$ in $\mathcal{M}$. Let $i_{dc/dl}$ denote interior multiplication with the tangent vector of c. Then the part of $i_{dc/dl} f(\pi^* A', h^{\pi^* E})$ which is of partial degree zero in the Grassmann variables of $\Lambda(T^* \mathcal{M})$, namely $\varphi \, \mathrm{Tr}_s[\tfrac{1}{2}(h_l^E)^{-1} \frac{\partial h_l^E}{\partial l} f'(X_l)]$, is an even real form on B. Applying $i_{dc/dl}$ to (1.26), we have

$$
\begin{aligned}
0 &= \frac{\partial}{\partial l} f(A', h_l^E) + i_{dc/dl} d^B \left((2i\pi)^{1/2} \varphi \, \mathrm{Tr}_s \left[\frac{1}{2} \theta f'(X^{\mathrm{partial}}) \right] \right) \\
(1.27) \quad &= \frac{\partial}{\partial l} f(A', h_l^E) - d^B i_{dc/dl} \left((2i\pi)^{1/2} \varphi \, \mathrm{Tr}_s \left[\frac{1}{2} \theta f'(X^{\mathrm{partial}}) \right] \right) \\
&= \frac{\partial}{\partial l} f(A', h_l^E) - d^B \left(\varphi \, \mathrm{Tr}_s \left[\frac{1}{2} (h_l^E)^{-1} \frac{\partial h_l^E}{\partial l} f'(X_l) \right] \right). \quad \square
\end{aligned}
$$

Definition 1.10. Let Q^B be the vector space of real even forms on B. Let $Q^{B,0}$ be the vector space of real exact even forms on B.

Theorem 1.11. *The following identity holds:*

$$(1.28) \quad d \int_0^1 \varphi \, \mathrm{Tr}_s \left[\frac{1}{2} (h_l^E)^{-1} \frac{\partial h_l^E}{\partial l} f'(X_l) \right] dl = f(A', h_1^E) - f(A', h_0^E).$$

In particular, the de Rham cohomology class of $f(A', h^E)$ is independent of h^E. Moreover, the class of the form $\int_0^1 \varphi \, \mathrm{Tr}_s[\tfrac{1}{2}(h_l^E)^{-1} \frac{\partial h_l^E}{\partial l} f'(X_l)] dl$ in $Q^B / Q^{B,0}$ depends only on the metrics h_0^E and h_1^E.

Proof. Equation (1.28) follows from integrating (1.20) with respect to l. We now use the notation of the proof of Theorem 1.9. Let $c: [0, 1] \to \mathcal{M}$ be a path in $\mathcal{M}$. From (1.25), we have

$$\int_c f(\pi^* A', h^{\pi^* E}) = \int_0^1 \varphi \, \mathrm{Tr}_s \left[\frac{1}{2} (h_l^E)^{-1} \frac{\partial h_l^E}{\partial l} f'(X_l) \right] dl.$$

As the form $f(\pi^* A', h^{\pi^* E})$ is closed, the last statement of Theorem 1.11 follows from Stokes' Theorem. $\square$

Definition 1.12. Let $\tilde{f}(A', h_0^E, h_1^E) \in Q^B / Q^{B,0}$ be the class of

$$\int_0^1 \varphi \, \mathrm{Tr}_s \left[\frac{1}{2} (h_l^E)^{-1} \frac{\partial h_l^E}{\partial l} f'(X_l) \right] dl.$$

Equation (1.28) states that

$$(1.29) \quad d\tilde{f}(A', h_0^E, h_1^E) = f(A', h_1^E) - f(A', h_0^E).$$

If A' is an ordinary flat connection, then the class $\tilde{f}(A', h_0^E, h_1^E)$ is the analog for us of the Bott-Chern class [BoC].

In what follows, we will let $f(A')$ denote the de Rham cohomology class of $f(A', h^E)$.

(g) **Chern-type classes of flat vector bundles.** With the notation of Section 1(e), suppose that A' is an ordinary flat connection $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$ on the $\mathbb{Z}_2$-graded vector bundle E. Then

$$(1.30) \qquad\qquad X = \tfrac{1}{2}((\nabla^E)^* - \nabla^E)$$

can be written as $X = \tfrac{1}{2}\omega(E, h^E)$, with

$$(1.31) \qquad\qquad \omega(E, h^E) = (h^E)^{-1}(\nabla^E h^E) \in \Omega^1(B; \mathrm{End}(E)).$$

Equations (0.1) and (1.31) are clearly equivalent. The unitary connection

$$(1.32) \qquad\qquad A = \tfrac{1}{2}((\nabla^E)^* + \nabla^E)$$

on E is given by

$$(1.33) \qquad\qquad A = \nabla^E + \tfrac{1}{2}\omega(E, h^E).$$

For k a positive odd integer, take $f(a) = a^k$. We write

$$(1.34) \qquad c_k(E, h^E) = f(\nabla^E, h^E) = (2i\pi)^{-(k-1)/2} 2^{-k} \, \mathrm{Tr}_s[\omega^k(E, h^E)],$$

a closed k-form on B. Let $c_k(E)$ denote the de Rham cohomology class of $c_k(E, h^E)$; by Theorem 1.11, it is independent of h^E. If the flat vector bundle E admits a covariantly-constant Hermitian metric, then $c_k(E) = 0$.

Proposition 1.13. *If E_1 and E_2 are $\mathbb{Z}_2$-graded flat complex vector bundles on B, then*

$$(1.35) \qquad\qquad c_k(E_1 \oplus E_2) = c_k(E_1) + c_k(E_2)$$

and

$$(1.36) \qquad c_k(E_1 \otimes E_2) = \mathrm{Tr}_s[I_{E_1}] \cdot c_k(E_2) + \mathrm{Tr}_s[I_{E_2}] \cdot c_k(E_1).$$

If E is a $\mathbb{Z}_2$-graded flat complex vector bundle, then

$$(1.37) \qquad\qquad c_k(\overline{E}^*) = -c_k(E).$$

Proof. Equation (1.35) is evident. If h^{E_1} and h^{E_2} are Hermitian metrics on E_1 and E_2 respectively, we have

$$(1.38) \qquad \omega(E_1 \otimes E_2, h^{E_1 \otimes E_2}) = (I_{E_1} \otimes \omega(E_2, h^{E_2})) + (\omega(E_1, h^{E_1}) \otimes I_{E_2}).$$

Then
(1.39)

$$\mathrm{Tr}_s[\omega^k(E_1 \otimes E_2, h^{E_1 \otimes E_2})] = \sum_{p=0}^{k} \binom{k}{p} \mathrm{Tr}_s[\omega^p(E_1, h^{E_1})] \cdot \mathrm{Tr}_s[\omega^{k-p}(E_2, h^{E_2})].$$

By Proposition 1.3, the terms in the sum of (1.39) vanish if p is not equal to 0 or k, from which (1.36) follows. Equation (1.37) follows from Theorem 1.8(ii). $\square$

We wish to give a more intrinsic description of the classes $c_k(E)$. For simplicity, we assume that $E_- = 0$. In the case $k = 1$, $c_1(E) \in H^1(B; \mathbb{R})$ is the cohomology class such that if γ is a smooth closed curve in B and $[\gamma]$ is its homology class, then

$$(1.40) \qquad c_1(E)([\gamma]) = \ln|\det P_\gamma|,$$

where P_γ is the holonomy of E around γ, computed at an arbitrary point of γ.

More generally, put $N = \operatorname{rk}(E)$, $G = GL(N, \mathbb{C})$ and $K = U(N)$. Denote the Lie algebras of G and K by $\gamma = gl(N, \mathbb{C})$ and $\kappa = u(N)$, respectively. The symmetric space G/K is isomorphic to the space of Hermitian metrics on $\mathbb{C}^N$. Suppose that B is connected, let $*$ be a basepoint in B and let $\rho: \pi_1(B, *) \to G$ be the holonomy representation of E at $*$. Letting $\widetilde{B}$ denote the universal cover of B, we can write $E = \widetilde{B} \times_\rho \mathbb{C}^N$. Put $H = \widetilde{B} \times_\rho (G/K)$, a fiber bundle over B with fibers diffeomorphic to G/K. Then Hermitian metrics on E are equivalent to smooth sections of H.

The quotient space γ/κ is isomorphic to the space of Hermitian $N \times N$ matrices, and carries an adjoint representation of K. Define a k-form Φ on γ/κ by sending Hermitian $N \times N$ matrices $M_1, \dots, M_k$ to

$$(1.41) \qquad \Phi(M_1, \dots, M_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{\operatorname{sign}(\sigma)} \operatorname{Tr}[M_{\sigma(1)} \cdots M_{\sigma(k)}].$$

Then Φ is K-invariant and extends to a closed G-invariant k-form on G/K, which we also denote by Φ. Let $\pi_2: \widetilde{B} \times (G/K) \to G/K$ be projection onto the second factor. Then $\pi_2^*(\Phi)$ is a closed form on $\widetilde{B} \times (G/K)$. As Φ is G-invariant, $\pi_2^*(\Phi)$ is $\pi_1(B)$-invariant, and so pulls back from a closed form σ on H. If h^E is a Hermitian metric on E and $s: B \to H$ is the associated section, it follows tautologically that

$$(1.42) \qquad c_k(E, h^E) = (2i\pi)^{-(k-1)/2} 2^{-k} s^*(\sigma).$$

As G/K is contractible, the fiber bundle H is topologically trivial and there is an isomorphism $j: H^*(H; \mathbb{C}) \to H^*(B; \mathbb{C})$. Letting $[\sigma] \in H^k(H; \mathbb{C})$ denote the de Rham cohomology class of σ, we have

$$(1.43) \qquad c_k(E) = (2i\pi)^{-(k-1)/2} 2^{-k} j([\sigma]).$$

Equation (1.43) shows that the classes $c_k(E)$ are the same as those defined by Kamber and Tondeur [KT] and further studied by Dupont [D] (see [D, §4]). Let $GL(N, \mathbb{C})_\delta$ denote $GL(N, \mathbb{C})$ with the discrete topology. The flat bundle E is classified by a homotopy class of maps ν from B to the classifying space $BGL(N, \mathbb{C})_\delta$ and $c_k(E) = \nu^*(c_{k,N})$ for some $c_{k,N} \in H^k(BGL(N, \mathbb{C})_\delta; \mathbb{R})$

[D]. Now the cohomology group $H^*(BGL(N, \mathbb{C})_\delta; \mathbb{R})$ is isomorphic to the (discrete) group cohomology $H^*(GL(N, \mathbb{C}); \mathbb{R})$. We can also consider the continuous group cohomology $H_c^*(GL(N, \mathbb{C}); \mathbb{R})$, meaning the cohomology of the complex of Eilenberg-Mac Lane cochains on $GL(N, \mathbb{C})$ which are continuous in their arguments. By forgetting that the cochains are continuous, there is a map $\mu_N: H_c^*(GL(N, \mathbb{C}); \mathbb{R}) \to H^*(BGL(N, \mathbb{C})_\delta; \mathbb{R})$. It follows from [D] that $c_{k,N} = \mu_N(C_{k,N})$ for some $C_{k,N} \in H_c^k(GL(N, \mathbb{C}); \mathbb{R})$. For example, $C_{1,N}$ is given by the homomorphism $g \to \ln|\det(g)|$ from $GL(N, \mathbb{C})$ to $(\mathbb{R}, +)$.

Furthermore, one can check that with respect to the embedding e_N: $GL(N, \mathbb{C}) \to GL(N+1, \mathbb{C})$, one has $C_{k,N} = (H^k(e_N))(C_{k,N+1})$. The inverse limit $\varprojlim H_c^*(GL(N, \mathbb{C}); \mathbb{R})$ is an exterior algebra with generators in every odd degree [Bo, p. 265]. Thus the classes $c_k(E)$ can be considered to be the indecomposable stable characteristic classes, which can be described by continuous group cochains, of flat complex vector bundles.

A related characteristic class of the flat bundle (E, ∇^E) is its Cheeger-Chern-Simons secondary class, which lies in $H^{\text{odd}}(B; \mathbb{C}/\mathbb{Z})$ [CS]. The imaginary part of the Cheeger-Chern-Simons class lies in $H^{\text{odd}}(B; \mathbb{R})$, and can be explicitly constructed as follows. Let $t \in [0, 1]$ parametrize a smooth 1-parameter family of connections $\{\nabla_t\}_{t \in [0,1]}$ such that ∇_0 is compatible with h^E, and $\nabla_1 = \nabla^E$. Put

$$(1.44) \qquad CCS(\{\nabla_t\}_{t \in [0,1]}) = \int_0^1 dt\, \text{Tr}\left[-\frac{1}{2i\pi} \frac{d\nabla_t}{dt} e^{-\nabla_t^2/2i\pi} \right] \in \Omega(B).$$

Then

$$(1.45) \quad dCCS(\{\nabla_t\}_{t \in [0,1]}) = \text{Tr}[e^{-\nabla_1^2/2i\pi}] - \text{Tr}[e^{-\nabla_0^2/2i\pi}] = \text{rk}(E) - \text{Tr}[e^{-\nabla_0^2/2i\pi}].$$

As ∇_0 is h^E-compatible, $\text{Tr}[e^{-\nabla_0^2/2i\pi}]$ is a real form, and so the form $\text{Im}(CCS(\{\nabla_t\}_{t \in [0,1]}))$ is closed. One can check that the de Rham cohomology class of $\text{Im}(CCS(\{\nabla_t\}_{t \in [0,1]}))$ is independent of h^E and the specific choice of $\{\nabla_t\}_{t \in [0,1]}$, and so defines an invariant $\text{Im}(CCS(E)) \in H^{\text{odd}}(B; \mathbb{R})$ of the flat bundle E. This is the desired construction.

Proposition 1.14. *The Cheeger-Chern-Simons secondary class and the $c_k(E)$ classes are related by*

$$(1.46) \qquad \text{Im}(CCS(E)) = -\frac{1}{2\pi} \sum_{j=0}^{\infty} \frac{2^{2j} j!}{(2j+1)!} c_{2j+1}(E).$$

Proof. Take $\nabla_t = A - tX$. Using the fact that ∇^E is flat, one computes that

$$(1.47) \qquad\qquad\qquad \nabla_t^2 = (t^2 - 1)X^2.$$

Then

$$CCS(\{\nabla_t\}_{t\in[0,1]}) = \int_0^1 dt \operatorname{Tr}\left[\frac{1}{2i\pi}Xe^{-(t^2-1)X^2/2i\pi}\right]$$

(1.48)
$$= -\frac{i}{2\pi}\sum_{j=0}^\infty \frac{1}{j!}\left(\int_0^1 (1-t^2)^j\,dt\right)(2i\pi)^{-j}\operatorname{Tr}[X^{2j+1}]$$

$$= -\frac{i}{2\pi}\sum_{j=0}^\infty \frac{1}{j!}\frac{2^{2j}(j!)^2}{(2j+1)!}c_{2j+1}(E,h^E).$$

The proposition follows from taking the imaginary parts of both sides of (1.48). □

Remark 1.15. Equation (1.46) is equivalent to an equality in the group $H^{\mathrm{odd}}(BGL(N,\mathbb{C})_\delta;\mathbb{R})$ relating the classes $\{c_{k,N}\}$ and the imaginary part of the Chern-Cheeger-Simons secondary class.

II. FLAT SUPERCONNECTIONS OF TOTAL DEGREE 1, HERMITIAN METRICS AND TORSION FORMS

In this section we construct the closed form $f(A',h^E)$ associated to a $\mathbb{Z}$-graded complex vector bundle E which is equipped with a Hermitian metric h^E and a flat superconnection A' of total degree 1. We also introduce a 1-parameter family of metrics $\{h_t^E\}_{t>0}$ on E. We calculate the $t\to+\infty$ limit of $f(A',h_t^E)$. Using the transgression formula of Section 1(f), we construct a form $S_f(A',h^E)$ such that

$$dS_f(A',h^E) = f(A',h^E) - \lim_{t\to+\infty} f(A',h_t^E).$$

In the special case when (E,v) is a flat complex of complex vector bundles equipped with a Hermitian metric h^E, we construct a torsion form $T_f(A',h^E)$ such that

$$dT_f(A',h^E) = \lim_{t\to0} f(A',h_t^E) - \lim_{t\to+\infty} f(A',h_t^E).$$

Finally, we compute the dependence of $S_f(A',h^E)$ and $T_f(A',h^E)$ on the metric h^E.

The section is organized as follows. In (a) we introduce superconnections of total degree 1 on a $\mathbb{Z}$-graded vector bundle E. Given a flat superconnection A' of total degree 1, in (b) we describe the associated cochain map v on E, and the flat vector bundle $H(E,v)$ on B which is the cohomology of (E,v). In (c) we establish a transgression formula for the closed form $f(A',h_t^E)$. In (d) we calculate the $t\to+\infty$ limit of $f(A',h_t^E)$. In (e) we construct the form $S_f(A',h^E)$ and in (f) we construct the torsion form $T_f(A',h^E)$. We discuss a finite-dimensional version of Hodge duality in (g), and give a nontrivial example of the formalism of this section.

In the entire section we use the assumptions and notation of Section 1.

(a) **Superconnections of total degree 1 on a $\mathbb{Z}$-graded vector bundle.** Let B be a smooth manifold. Let $E = \bigoplus_{i=0}^{n} E^i$ be a $\mathbb{Z}$-graded complex vector bundle on B. Put

$$E_+ = \bigoplus_{i \text{ even}} E^i, \qquad E_- = \bigoplus_{i \text{ odd}} E^i.$$

Then $E = E_+ \oplus E_-$ is a $\mathbb{Z}_2$-graded vector bundle, to which we may apply the formalism of Section 1.

Let A' be a superconnection on $E = E_+ \oplus E_-$. As in (1.2), we write A in the form

$$(2.1) \qquad\qquad A' = \sum_{j \geq 0} A'_j,$$

where A'_j is of partial degree j in the Grassmann variables $\Lambda(T^*B)$.

Definition 2.1. We say that A' is of total degree 1 (resp. -1) if
$-A'_1$ is a connection on E which preserves the $\mathbb{Z}$-grading.
–For $j \in \mathbb{N} - \{1\}$, A'_j is an element of $\Omega^j(B\,; \mathrm{Hom}(E^\bullet, E^{\bullet+1-j}))$ (resp. $\Omega^j(B\,; \mathrm{Hom}(E^\bullet, E^{\bullet-1+j})))$.

In what follows we will assume that A' is a flat superconnection of total degree 1. Put

$$(2.2) \qquad\qquad v = A'_0, \qquad \nabla^E = A'_1.$$

Clearly $v \in \Omega^0(B\,; \mathrm{Hom}(E^\bullet, E^{\bullet+1}))$ and ∇^E is a connection on E which preserves the $\mathbb{Z}$-grading.

Proposition 2.2. *We have*

$$(2.3) \qquad v^2 = 0, \qquad [\nabla^E, v] = 0, \qquad (\nabla^E)^2 + [v, A'_2] = 0.$$

Proof. These are just the first three identities in (1.3). $\square$

As $v^2 = 0$, we have a cochain complex of vector bundles

$$(2.4) \qquad\qquad (E, v): 0 \to E^0 \xrightarrow{v} E^1 \xrightarrow{v} \cdots \xrightarrow{v} E^n \to 0.$$

Definition 2.3. For $b \in B$, let $H(E, v)_b = \bigoplus_{i=0}^{n} H^i(E, v)_b$ be the cohomology of the complex $(E, v)_b$.

From Proposition 2.2, v is covariantly constant with respect to the connection ∇^E. It follows that there is a $\mathbb{Z}$-graded complex vector bundle $H(E, v)$ on B whose fiber over $b \in B$ is $H(E, v)_b$. There is also a natural connection on $H(E, v)$ which can be described as follows. Let $\psi\colon \mathrm{Ker}(v) \to H(E, v)$ be the quotient map. For $0 \leq i \leq n$, let s be a smooth section of $H^i(E, v)$. Then there is a smooth section e of $E^i \cap \mathrm{Ker}(v)$ such that $\psi(e) = s$. Given a vector field U on B, we have that $v(\nabla^E_U e) = \nabla^E_U v(e) = 0$, and so $\nabla^E_U e \in \mathrm{Ker}(v)$.

Definition 2.4. Put

$$(2.5) \qquad \nabla_U^{H(E,v)} s = \psi(\nabla_U^E e).$$

Equation (2.5) makes sense, as if $e \in \mathrm{Im}(v)$ then $\nabla_U^E e \in \mathrm{Im}(v)$. Thus $\nabla^{H(E,v)}$ is a connection on $H(e, v)$ which preserves the $\mathbb{Z}$-grading.

Proposition 2.5. *The connection* $\nabla^{H(E,v)}$ *is flat.*

Proof. From (2.3), if s and e are as above, then

$$(2.6) \qquad (\nabla^{H(E,v)})^2 s = \psi((\nabla^E)^2 e) = -\psi([v, A_2']e) = 0. \quad \square$$

(b) Superconnections of total degree 1 and Hermitian metrics. We make the same assumptions as in Section 2(a). By definition, a Hermitian metric on the $\mathbb{Z}$-graded vector bundle $E = \bigoplus_{i=0}^n E^i$ is a Hermitian metric on E such that the E^i's are mutually orthogonal.

Let h^E be a Hermitian metric on E. Let $v^* \in \Omega^0(B; \mathrm{Hom}(E^\bullet, E^{\bullet-1}))$ be the adjoint of v with respect to h^E. Put

$$(2.7) \qquad V = v^* - v.$$

As $V^2 = -(v^*v + v^*v)$, it follows from finite dimensional Hodge theory that for any $b \in B$ there is an isomorphism

$$(2.8) \qquad H(E, v)_b \simeq \mathrm{Ker}(V_b).$$

Thus there is a smooth $\mathbb{Z}$-graded subbundle $\mathrm{Ker}(V)$ of E whose fiber over $b \in B$ is $\mathrm{Ker}(V_b)$. Moreover,

$$(2.9) \qquad H(E, v) \simeq \mathrm{Ker}(V).$$

Being a subbundle of E, $\mathrm{Ker}(V)$ inherits a Hermitian metric from the Hermitian metric h^E on E. Let $h^{H(E,v)}$ denote the Hermitian metric on $H(E, v)$ obtained via the isomorphism (2.9). Let $P^{\mathrm{Ker}(V)}$ be the orthogonal projection of E onto $\mathrm{Ker}(V)$; it clearly preserves the $\mathbb{Z}$-grading. Put

$$(2.10) \qquad \begin{aligned} \omega(E, h^E) &= (h^E)^{-1}(\nabla^E h^E), \\ \omega(H(E, v), h^{H(E,v)}) &= (h^{H(E,v)})^{-1}(\nabla^{H(E,v)} h^{H(E,v)}). \end{aligned}$$

These are 1-forms on B with value in the selfadjoint endomorphisms that preserve the $\mathbb{Z}$-gradings.

Let $(\nabla^E)^*$ (resp. $(\nabla^{H(E,v)})^*$) be the connection on E (resp. $H(E, v)$) which is the adjoint of ∇^E (resp. $\nabla^{H(E,v)}$) with respect to h^E (resp. $h^{H(E,v)}$). These connections still preserve the $\mathbb{Z}$-gradings. Then

$$(2.11) \qquad \begin{aligned} (\nabla^E)^* &= \nabla^E + \omega(E, h^E), \\ (\nabla^{H(E,v)})^* &= \nabla^{H(E,v)} + \omega(H(E, v), h^{H(E,v)}). \end{aligned}$$

Now $P^{\mathrm{Ker}(V)} \nabla^E$ and $P^{\mathrm{Ker}(V)}(\nabla^E)^*$ give connections on $\mathrm{Ker}(V)$. Using the isomorphism (2.9), they can be considered to be connections on $H(E, v)$. Similarly, the 1-form $P^{\mathrm{Ker}(V)} \omega(E, h^E) P^{\mathrm{Ker}(V)}$ can be considered to be an element of $\Omega^1(B; \mathrm{End}(H(E, v)))$.

Proposition 2.6. *The following identities hold:*

$$\nabla^{H(E,v)} = P^{\mathrm{Ker}(V)}\nabla^E,$$

(2.12)
$$(\nabla^{H(E,v)})^* = P^{\mathrm{Ker}(V)}(\nabla^E)^*,$$

$$\omega(H(E,v),h^{H(E,v)}) = P^{\mathrm{Ker}(V)}\omega(E,h^E)P^{\mathrm{Ker}(V)}.$$

Proof. Let e be a smooth section of $\mathrm{Ker}(v)$. As before, for a vector field U on B, $\nabla^E_U e \in \mathrm{Ker}(v)$. By Hodge theory, $P^{\mathrm{Ker}(V)}(\nabla^E_U e) - \nabla^E_U e \in \mathrm{Im}(v)$, and so

(2.13)
$$\psi(\nabla^E_U e) = \psi(P^{\mathrm{Ker}(V)}\nabla^E_U e).$$

Equations (2.5) and (2.13) give the first identity in (2.12).

Let e and e' be smooth sections of E and let U be a vector field on B. Then

(2.14) $$U\langle e,e'\rangle_{h^E} = \langle \nabla^E_U e,e'\rangle_{h^E} + \langle e,\nabla^E_U e'\rangle_{h^E} + \langle (\omega(E,h^E)(U))e,e'\rangle_{h^E}.$$

If in addition e and e' lie in $\mathrm{Ker}(V)$, then (2.14) gives

(2.15)
$$U\langle e,e'\rangle_{h^E} = \langle P^{\mathrm{Ker}(V)}\nabla^E_U e,e'\rangle_{h^E} + \langle e,P^{\mathrm{Ker}(V)}\nabla^E_U e'\rangle_{h^E}$$
$$+ \langle P^{\mathrm{Ker}(V)}(\omega(E,h^E)(U))P^{\mathrm{Ker}(V)}e,e'\rangle_{h^E}.$$

From (2.15) and the first identity in (2.12), we obtain the third identity in (2.12). The second identity in (2.12) now follows from (2.11). □

Let $A'' = A'^*$ be the adjoint of A' with respect to h^E. Then A'' is a flat superconnection of total degree -1. From (2.1),

(2.16)
$$A'' = \sum_{j\geq0} A''_j, \qquad A''_j = A'^*_j.$$

From (2.2),

(2.17)
$$v^* = A''_0, \qquad (\nabla^E)^* = A''_1.$$

(c) The rescaling of the metric. We make the same assumptions and use the same notations as in Section 2(b).

Let $N \in \mathrm{End}(E)$ be the number operator of E, i.e., N acts on E^i by multiplication by i. Extend N to an element of $C^\infty(B;\mathrm{End}(E))$.

Definition 2.7. For $t > 0$, put

(2.18)
$$h^E_t = \bigoplus_{i=0}^n t^i h^{E^i}.$$

Then h^E_t is a metric on E and $h^E = h^E_1$.

Let A''_t be the adjoint of A' with respect to h^E_t. Clearly $A'' = A''_1$. We have

(2.19)
$$A''_t = t^{-N}A''t^N.$$

Using (2.16) and the fact that A'' is of total degree -1, we get

(2.20)
$$A''_t = \sum_{j\geq0} t^{1-j}A''_j.$$

We now use the formalism of Section 1(e). Put

(2.21) $$A_t = \tfrac{1}{2}(A_t'' + A'), \qquad X_t = \tfrac{1}{2}(A_t'' - A').$$

Let $f: \mathbb{C} \to \mathbb{C}$ be a holomorphic real odd function. Following Definition 1.7, for $t > 0$ we put

(2.22) $$f(A', h_t^E) = (2i\pi)^{1/2} \varphi \operatorname{Tr}_s[f(X_t)].$$

Definition 2.8. Put

(2.23) $$f^\wedge(A', h_t^E) = \varphi \operatorname{Tr}_s\left[\frac{N}{2} f'(X_t)\right] \in \Omega(B).$$

Theorem 2.9. *The form* $f^\wedge(A', h_t^E)$ *is real and even. Moreover,*

(2.24) $$\frac{\partial}{\partial t} f(A', h_t^E) = \frac{1}{t} df^\wedge(A', h_t^E).$$

Proof. We have

(2.25) $$(h_t^E)^{-1} \frac{\partial h_t^E}{\partial t} = \frac{N}{t}.$$

The theorem follows from Theorem 1.9 and (2.25). $\square$

Although A' and A_t'' are both flat superconnections on E, they occur in a somewhat asymmetric way. It is possible to make the equations more symmetric by conjugating both A' and A_t'' by $t^{N/2}$.

Definition 2.10. For $t > 0$, let C_t' be the flat superconnection on E of total degree 1 given by

(2.26) $$C_t' = t^{N/2} A' t^{-N/2},$$

and let C_t'' be the flat superconnection on E of total degree -1 given by

(2.27) $$C_t'' = t^{N/2} A_t'' t^{-N/2} = t^{-N/2} A'' t^{N/2}.$$

The superconnections C_t'' and C_t' are adjoint with respect to h^E. Using (2.1), (2.16), (2.26) and (2.27), we get

(2.28) $$C_t' = \sum_{j \geq 0} t^{(1-j)/2} A_j',$$
$$C_t'' = \sum_{j \geq 0} t^{(1-j)/2} A_j''.$$

We again use the formalism of Section 1(e). Put

(2.29) $$C_t = \tfrac{1}{2}(C_t'' + C_t'), \qquad D_t = \tfrac{1}{2}(C_t'' - C_t').$$

From (2.21), (2.26) and (2.27), we have

(2.30) $$C_t = t^{N/2} A_t t^{-N/2}, \qquad D_t = t^{N/2} X_t t^{-N/2}.$$

Theorem 2.11. *For all* $t > 0$,

$$f(A', h_t^E) = f(C_t', h^E),$$
(2.31)
$$f^\wedge(A', h_t^E) = f^\wedge(C_t', h^E).$$

In particular,

(2.32)
$$\frac{\partial}{\partial t} f(C_t', h^E) = \frac{1}{t} df^\wedge(C_t', h^E).$$

Proof. Equation (2.31) follows from (2.30). Then (2.32) is equivalent to (2.24). $\square$

Remark 2.12. There is a simple direct proof of (2.32). By (2.26) and (2.27), we have

(2.33)
$$\frac{\partial}{\partial t} C_t' = -\left[C_t', \frac{N}{2t}\right], \qquad \frac{\partial}{\partial t} C_t'' = \left[C_t'', \frac{N}{2t}\right],$$

and so

(2.34)
$$\frac{\partial}{\partial t} D_t = \left[C_t, \frac{N}{2t}\right].$$

From (1.6),

(2.35)
$$[C_t, D_t] = 0.$$

Using (2.34) and (2.35), we obtain

(2.36)
$$\begin{aligned}
\frac{\partial}{\partial t} f(C_t', h^E) &= \frac{\partial}{\partial t} (2i\pi)^{1/2} \varphi \, \mathrm{Tr}_s[f(D_t)] \\
&= (2i\pi)^{1/2} \varphi \, \mathrm{Tr}_s\left[\left[C_t, \frac{N}{2t}\right] f'(D_t)\right] \\
&= (2i\pi)^{1/2} \varphi \, \mathrm{Tr}_s\left[\left[C_t, \frac{N}{2t} f'(D_t)\right]\right] \\
&= d\left(\varphi \, \mathrm{Tr}_s\left[\frac{N}{2t} f'(D_t)\right]\right) = \frac{1}{t} df^\wedge(C_t', h^E).
\end{aligned}$$

(d) **The** $t \to +\infty$ **limit of** $f(A', h_t^E)$. Suppose that there is a $c > 0$ such that for any $k \in \mathbb{N}$, there exists a $C_k > 0$ such that

(2.37)
$$\sup_{\substack{a \in \mathbb{C} \\ |\mathrm{Re}\, a| \le c}} (1 + |a|)^k |f(a)| \le C_k.$$

As f is holomorphic, the derivatives of f satisfy bounds similar to (2.37). An example of such an f is

$$f(a) = a \exp(a^2).$$

Let $(\omega_t)_{t \in \mathbb{R}_+^* \cup \{+\infty\}}$ be smooth forms on B. We will write

(2.38)
$$\omega_t = \omega_\infty + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right) \quad \text{as } t \to +\infty$$

if for any compact set $K \subset B$ and any $k \in \mathbb{N}$, there is a $C_{k,K} > 0$ such that the sup of the norms of $\omega_t - \omega_\infty$ and its derivatives of order $\leq k$ on K can be bounded above by $\frac{C_{k,K}}{\sqrt{t}}$.

Let $d(H(E, v))$ be the locally constant integer-valued function on B:

$$(2.39) \qquad d(H(E, v)) = \sum_{i=0}^{n} (-1)^i i \operatorname{rk}(H^i(E, v)).$$

Since $\nabla^{H(E,v)}$ is a flat connection on $H(E, v)$ preserving the $\mathbb{Z}$-grading, $\nabla^{H(E,v)}$ is an example of a flat superconnection of total degree 1. Of course,

$$(2.40) \qquad f(\nabla^{H(E,v)}, h^{H(E,v)}) = \sum_{i=0}^{n} (-1)^i f(\nabla^{H^i(E,v)}, h^{H^i(E,v)}).$$

Using (2.11) and (2.40), we have
$$(2.41)$$
$$f(\nabla^{H(E,v)}, h^{H(E,v)}) = \sum_{i=0}^{n} (-1)^i (2i\pi)^{1/2} \varphi \operatorname{Tr}\left[f\left(\frac{\omega}{2}(H^i(E, v), h^{H^i(E,v)}) \right) \right].$$

Theorem 2.13. *As* $t \to +\infty$,

$$(2.42) \qquad \begin{aligned} f(A', h_t^E) &= f(\nabla^{H(E,v)}, h^{H(E,v)}) + \mathscr{O}\left(\frac{1}{\sqrt{t}} \right), \\ f^\wedge(A', h_t^E) &= d(H(E, v)) \frac{f'(0)}{2} + \mathscr{O}\left(\frac{1}{\sqrt{t}} \right). \end{aligned}$$

Proof. Throughout this proof, C will denote a generic positive constant. By (2.2), (2.11), (2.17) and (2.28), we have

$$(2.43) \qquad D_t = \frac{1}{2}\left(\sqrt{t} V + \omega(E, h^E) + \sum_{j \geq 2} t^{(1-j)/2}(A_j'' - A_j') \right).$$

Put
$$F_t = D_t - \frac{\sqrt{t}}{2} V.$$

Then F_t is a sum of forms of degree ≥ 1 in the Grassmann variables of $\Lambda(T^*B)$, and so is nilpotent. Therefore the spectra of D_t and $\frac{\sqrt{t}}{2} V$ are identical.

Let $c > 0$ be such that (2.37) holds. Let $\Delta = \Delta_+ \cup \Delta_-$ be the oriented contour in $\mathbb{C}$:
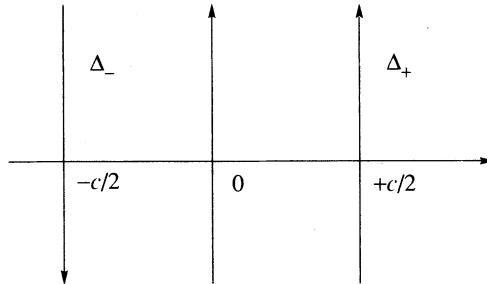
The operator V is skew-adjoint and so its spectrum lies on $i\mathbb{R}$. Then for $t > 0$,

$$(2.44) \qquad f(D_t) = \frac{1}{2\pi i} \int_\Delta \frac{f(\lambda)}{\lambda - D_t} \, d\lambda.$$

For $\lambda \in \Delta$,

$$(2.45) \qquad (\lambda - D_t)^{-1} = \left(1 - \left(\lambda - \frac{\sqrt{t}}{2} V\right)^{-1} F_t\right)^{-1} \left(\lambda - \frac{\sqrt{t}}{2} V\right)^{-1}.$$

By (2.9), $\mathrm{Ker}(V) \simeq H(E, v)$. As V has purely imaginary spectrum, if $K \subset B$ is compact then there is a $C > 0$ such that if $\lambda \in \Delta$ and $t \geq 1$, then

$$(2.46) \qquad \left| \left(\lambda - \frac{\sqrt{t}}{2} V\right)^{-1} - \frac{P^{\mathrm{Ker}(V)}}{\lambda} \right| \leq \frac{C}{\sqrt{t}}(1 + |\lambda|) \quad \text{on } K.$$

Also

$$(2.47) \qquad \begin{aligned} \left(1 - \left(\lambda - \frac{\sqrt{t}}{2} V\right)^{-1} F_t\right)^{-1} &= 1 + \left(\lambda - \frac{\sqrt{t}}{2} V\right)^{-1} F_t \\ &\quad + \left(\left(\lambda - \frac{\sqrt{t}}{2} V\right)^{-1} F_t\right)^2 + \cdots, \end{aligned}$$

and the expansion in (2.47) terminates after a finite number of terms. From (2.43) we know that there is a $C > 0$ such that for $t \geq 1$,

$$(2.48) \qquad \left| F_t - \frac{\omega}{2}(E, h^E) \right| \leq \frac{C}{\sqrt{t}} \quad \text{on } K.$$

From (2.46), (2.47) and (2.48), there exist $C > 0$ and $p \in \mathbb{N}$ such that for $t \geq 1$ and $\lambda \in \Delta$,

$$(2.49) \qquad \begin{aligned} &\left| (1 - (\lambda - \sqrt{t}V)^{-1} F_t)^{-1} - \left(1 - \frac{P^{\mathrm{Ker}(V)}}{\lambda} \frac{\omega}{2}(E, h^E)\right)^{-1} \right| \\ &\leq \frac{C}{\sqrt{t}}(1 + |\lambda|)^p \quad \text{on } K. \end{aligned}$$

Using (2.45), (2.46) and (2.49), we have that for $t \geq 1$ and $\lambda \in \Delta$,

$$(2.50) \qquad \begin{aligned} &\left| (\lambda - D_t)^{-1} - \left(1 - \frac{P^{\mathrm{Ker}(V)}}{\lambda} \frac{\omega}{2}(E, h^E)\right)^{-1} \frac{P^{\mathrm{Ker}(V)}}{\lambda} \right| \\ &\leq \frac{C}{\sqrt{t}}(1 + |\lambda|)^p \quad \text{on } K, \end{aligned}$$

which is equivalent to

$$(2.51) \qquad \begin{aligned} &\left| (\lambda - D_t)^{-1} - P^{\mathrm{Ker}(V)} \left(\lambda - P^{\mathrm{Ker}(V)} \frac{\omega}{2}(E, h^E) P^{\mathrm{Ker}(V)}\right)^{-1} P^{\mathrm{Ker}(V)} \right| \\ &\leq \frac{C}{\sqrt{t}}(1 + |\lambda|)^p \quad \text{on } K. \end{aligned}$$

As $P^{\mathrm{Ker}(V)}\frac{\omega}{2}(E, h^E)P^{\mathrm{Ker}(V)}$ is a 1-form, it is nilpotent, and so its spectrum consists of $\{0\}$. Then

$$(2.52) \quad f\left(P^{\mathrm{Ker}(V)}\frac{\omega}{2}(E, h^E)P^{\mathrm{Ker}(V)}\right) = \frac{1}{2\pi i}\int_\Delta \frac{f(\lambda)}{\lambda - P^{\mathrm{Ker}(V)}\frac{\omega}{2}(E, h^E)P^{\mathrm{Ker}(V)}}\, d\lambda.$$

Using (2.37), (2.44), (2.51) and (2.52), we find that if $t \geq 1$, then

$$(2.53) \quad \left| f(D_t) - P^{\mathrm{Ker}(V)}f\left(P^{\mathrm{Ker}(V)}\frac{\omega}{2}(E, h^E)P^{\mathrm{Ker}(V)}\right)P^{\mathrm{Ker}(V)}\right| \leq \frac{C}{\sqrt{t}} \quad \text{on } K.$$

The first identity in (2.42) now follows from (2.12), (2.31), (2.41) and (2.53).

Still using (2.31) and proceeding as above, we find that as $t \to +\infty$,

$$(2.54) \qquad f^\wedge(A', h_t^E) = f^\wedge(\nabla^{H(E,v)}, h^{H(E,v)}) + \mathscr{O}\left(\frac{1}{\sqrt{t}}\right).$$

Clearly

$$(2.55) \qquad f^\wedge(\nabla^{H(E,v)}, h^{H(E,v)}) = \frac{1}{2}\varphi\sum_{i=0}^n(-1)^i i f'(\nabla^{H^i(E,v)}, h^{H^i(E,v)}).$$

Now f' is an even function. Using Proposition 1.3, we deduce from (2.55) that

$$(2.56) \qquad f^\wedge(\nabla^{H(E,v)}, h^{H(E,v)}) = \sum_{i=0}^n(-1)^i i \,\mathrm{rk}(H^i(E, v))\frac{f'(0)}{2}.$$

The second identity in (2.42) now follows from (2.54)–(2.56). This completes the proof of Theorem 2.13. $\square$

Recall that $f(A')$ (resp. $f(\nabla^{H(E,v)})$) denotes the cohomology class of $f(A', h^E)$ (resp. $f(\nabla^{H(E,v)}, h^{H(E,v)})$).

Theorem 2.14. *As elements of $H^{\mathrm{odd}}(B; \mathbb{R})$,*

$$(2.57) \qquad\qquad f(A') = f(\nabla^{H(E,v)}).$$

Proof. This follows from Theorems 1.11 and 2.13. $\square$

(e) **The form $S_f(A', h^E)$.** We now refine Theorem 2.14 to the level of differential forms.

Definition 2.15. Let $S_f(A', h^E)$ be the real even form on B given by

$$(2.58) \qquad S_f(A', h^E) = -\int_1^{+\infty}\left(f^\wedge(A', h_t^E) - d(H(E, v))\frac{f'(0)}{2}\right)\frac{dt}{t}.$$

Using (2.42), it is clear that the integrand in (2.58) is integrable.

Theorem 2.16. *The following identity holds:*

$$(2.59) \qquad dS_f(A', h^E) = f(A', h^E) - f(\nabla^{H(E,v)}, h^{H(E,v)}).$$

Proof. This follows from Theorems 2.9 and 2.13. $\square$

Let h_0^E and h_1^E be two Hermitian metrics on the $\mathbb{Z}$-graded vector bundle E, and let $h_0^{H(E,v)}$ and $h_1^{H(E,v)}$ be the metrics induced by h_0^E and h_1^E on $H(E,v)$.

We now use the notation of Section 1(f).

Theorem 2.17. *The following identity holds in* $Q^B/Q^{B,0}$:

(2.60)
$$S_f(A', h_1^E) - S_f(A', h_0^E) = \tilde{f}(A', h_0^E, h_1^E) - \tilde{f}(\nabla^{H(E,v)}, h_0^{H(E,v)}, h_1^{H(E,v)}).$$

Proof. Clearly both sides of (2.60) vanish when $h_1^E = h_0^E$. By (1.29) and (2.59), if we apply the operator d to both sides of (2.60) then we get an identity. A simple deformation argument then shows that (2.60) holds.   □

(f) Flat complexes of vector bundles and their torsion forms. In this section we consider the special case when the vector bundle E has not only a flat super-connection, but has a flat ordinary connection. Let

(2.61)
$$(E, v): 0 \to E^0 \xrightarrow{v} E^1 \xrightarrow{v} \cdots \xrightarrow{v} E^n \to 0$$

be a flat complex of complex vector bundles. That is,

(2.62)
$$\nabla^E = \bigoplus_{i=0}^{n} \nabla^{E^i}$$

is a flat connection on $E = \bigoplus_{i=0}^{n} E^i$ and v is a flat chain map, meaning

(2.63)
$$(\nabla^E)^2 = 0, \qquad v^2 = 0, \qquad \nabla_\bullet^E v = 0.$$

Put

(2.64)
$$A' = v + \nabla^E.$$

Then A' is a flat superconnection of total degree 1. With respect to the decomposition (2.1), A' is characterized by the fact that

(2.65)
$$A'_j = 0 \quad \text{for } j \geq 2.$$

We now use the notation of Sections 2(c)–2(e). In particular, for $t > 0$,

(2.66)
$$\begin{aligned} A''_t &= tv^* + (\nabla^E)^*, \\ C'_t &= \sqrt{t}v + \nabla^E, \\ C''_t &= \sqrt{t}v^* + (\nabla^E)^*. \end{aligned}$$

Let $f: \mathbb{C} \to \mathbb{C}$ be a real holomorphic odd function. By (2.66), we see that although $h_t^E = \bigoplus_{i=0}^{n} t^i h^{E^i}$ is not a metric for $t = 0$, A''_t, C'_t, and C''_t still make sense for $t = 0$. Therefore the forms $f(A', h_t^E) = f(C'_t, h^E)$ and $f^\wedge(A, h_t^E) = f^\wedge(C'_t, h^E)$ still make sense for $t = 0$, and depend smoothly on $t \in [0, +\infty)$.

Let $d(E)$ be the locally constant integer-valued function on B given by

(2.67)
$$d(E) = \sum_{i=0}^{n} (-1)^i i \operatorname{rk}(E^i).$$

As in (2.41), we have

$$(2.68) \qquad f(\nabla^E, h^E) = \sum_{i=0}^{n} (-1)^i (2i\pi)^{1/2} \varphi \operatorname{Tr}\left[f\left(\frac{\omega}{2}(E^i, h^{E^i}) \right) \right].$$

Proposition 2.18. *As* $t \to 0$,

$$(2.69) \qquad \begin{aligned} f(A', h_t^E) &= f(\nabla^E, h^E) + \mathscr{O}(t), \\ f^\wedge(A', h_t^E) &= d(E)\frac{f'(0)}{2} + \mathscr{O}(t). \end{aligned}$$

Proof. Clearly

$$(2.70) \qquad \begin{aligned} f(C_0', h^E) &= f(\nabla^E, h^E), \\ f^\wedge(C_0', h^E) &= f^\wedge(\nabla^E, h^E). \end{aligned}$$

Also

$$(2.71) \qquad f^\wedge(\nabla^E, h^E) = \frac{1}{2} \sum_{i=0}^{n} (-1)^i i f'(\nabla^{E^i}, h^{E^i}).$$

As $f'(a)$ is even, Proposition 1.3, (2.70) and (2.71) imply that

$$(2.72) \qquad f'(C_0', h^E) = d(E)\frac{f'(0)}{2}.$$

This completes the proof of Proposition 2.18. □

Theorem 2.19. *As elements of* $H^{\mathrm{odd}}(B; \mathbb{R})$,

$$(2.73) \qquad f(\nabla^E) = f(\nabla^{H(E, v)}).$$

Proof. If $f(a) = a \exp(a^2)$, then Theorem 2.19 follows from Theorems 1.11 and 2.13 and Proposition 2.18. Expanding f in a power series, (2.73) extends to an arbitrary f. □

We now refine Theorem 2.19 to the level of differential forms. Let $f: \mathbb{C} \to \mathbb{C}$ be a real holomorphic odd function such that (2.37) holds.

Definition 2.20. Put

$$(2.74) \qquad \begin{aligned} T_f(A', h^E) = -\int_0^{+\infty} \bigg[& f^\wedge(A', h_t^E) - d(H(E, v))\frac{f'(0)}{2} \\ & -[d(E) - d(H(E, v))]\frac{f'(\frac{i\sqrt{t}}{2})}{2} \bigg] \frac{dt}{t}, \end{aligned}$$

a differential form on B.

Remark 2.21. By Theorem 2.13 and Proposition 2.18, the integrand in (2.74) is integrable. We will call $T_f(A', h^E)$ a torsion form.

Theorem 2.22. *The form* $T_f(A', h^E)$ *is even and real. Moreover,*

$$(2.75) \qquad dT_f(A', h^E) = f(\nabla^E, h^E) - f(\nabla^{H(E,v)}, h^{H(E,v)}).$$

Proof. This follows from Theorems 2.9 and 2.13 and Proposition 2.18. $\square$

Remark 2.23. Let B' be a smooth manifold and let $\alpha: B' \to B$ be a smooth map. Then

$$(2.76) \qquad T_f(\alpha^* A', \alpha^* h^E) = \alpha^* T_f(A', h^E).$$

We now use the same notation as in Theorem 2.17.

Theorem 2.24. *The following identity holds in* $Q^B/Q^{B,0}$:
$$(2.77)$$
$$T_f(A', h_1^E) - T_f(A', h_0^E) = \tilde{f}(\nabla^E, h_0^E, h_1^E) - \tilde{f}(\nabla^{H(E,v)}, h_0^{H(E,v)}, h_1^{H(E,v)}).$$

Proof. Using (2.75), the proof is virtually the same as that of Theorem 2.17. $\square$

We now consider the 0-form component $T_f^{[0]}(A', h^E) \in C^\infty(B)$ of $T_f(A', h^E)$. Given $b \in B$, decompose the degree-0 operator V_b^2 with respect to the $\mathbb{Z}$-grading on E_b as

$$(2.78) \qquad V_b^2 = \bigoplus_{i=0}^n (V_b^2)_i,$$

with $(V_b^2)_i \in \text{End}(E_b^i)$. Let $(V_b^2)_i'$ denote the quotient action of $(V_b^2)_i$ on $E_b^i / \text{Ker}((V_b^2)_i)$.

Theorem 2.25.

$$(2.79) \qquad T_f^{[0]}(A', h^E)(b) = \frac{f'(0)}{2} \sum_{i=0}^n (-1)^i i \ln \det(-(V_b^2)_i').$$

Proof. Let $g: \mathbb{C} \to \mathbb{C}$ be the holomorphic function such that $f'(a) = g(a^2)$. Let $f^{\wedge[0]}(A', h_t^E)(b)$ denote the evaluation of the 0-form component of $f^\wedge(A', h_t^E)$ at b. From Definition 2.8, Theorem 2.11 and (2.66),

$$(2.80) \qquad f^{\wedge[0]}(A', h_t^E)(b) = \frac{1}{2} \sum_{i=0}^n (-1)^i i \, \text{Tr}\left[g\left(\frac{t}{4}(V_b^2)_i\right)\right].$$

Let $\{\lambda_{i,k_i}\}_{k_i=1}^{\text{rk}(E_i)}$ denote the eigenvalues of $-(V_b^2)_i$; they are all nonnegative. Substituting (2.80) into (2.74) gives
$$(2.81)$$
$$T_f^{[0]}(A', h^E)(b) = -\frac{1}{2}\int_0^{+\infty} \sum_{i=0}^n (-1)^i i \sum_{\lambda_{i,k_i}\neq 0} \left[g\left(-\frac{t\lambda_{i,k_i}}{4}\right) - g\left(-\frac{t}{4}\right)\right] \frac{dt}{t}$$

$$= \frac{g(0)}{2} \sum_{i=0}^n (-1)^i i \sum_{\lambda_{i,k_i}\neq 0} \ln(\lambda_{i,k_i}).$$

The theorem follows. $\square$

Theorem 2.25 states that, up to an overall multiplicative constant, the 0-form part of the torsion form $T_f(A', h^E)$ is the function which to a point $b \in B$ assigns the torsion of the chain complex (E_b, v_b) [M, RS1]. Thus $T_f(A', h^E)$ merits being called a "higher" torsion.

Remark 2.26. One can define a torsion form for a general flat superconnection A' of total degree 1 by using zeta-function regularization to control the small-t divergence in the analog of (2.74). The only difference in equation (2.75) is that $f(\nabla^E, h^E)$ would be replaced by the t^0-term in the Laurent expansion of $f(A', h_t^E)$. For simplicity, in this section we have presented the details of the case when A' has only terms of degree 0 and 1 in the Grassmann variables of $\Lambda(T^*B)$, as one does not need zeta-function regularization in this case. In Section 3 we will deal with a superconnection, on an infinite-dimensional bundle, with terms of degree 0, 1, and 2 in the Grassmann variables of $\Lambda(T^*B)$. The degree-2 term of the superconnection, while apparently singular in the small-t limit, will in fact play the role of ensuring that the torsion form will again not need zeta-function regularization.

(g) **Duality.** We make the assumptions of Sections 2(a)–2(e). Recall that $\overline{E}^*$ has the $\mathbb{Z}$-grading given by $\overline{E}^{*i} = \overline{E^{i*}}$.

Theorem 2.27. *Suppose that* $n = \dim(E)$ *is even and there is an isometric isomorphism* $\sigma: E \to \overline{E}^*$ *such that* $\sigma(E^i) = \overline{E}^{*(n-i)}$ *and* $\overline{A'}^* = \sigma A' \sigma^{-1}$. *Then*

$$(2.82) \qquad\qquad f(C_t', h^E) = 0$$

and

$$(2.83) \qquad f^{\wedge}(C_t', h^E) = \frac{n}{4}(\mathrm{rk}(E_+) - \mathrm{rk}(E_-))f'(0).$$

Proof. Equation (2.82) follows from Theorem 1.8(iii). Write $\overline{D}_t^{\,*}$ for the object of (2.29) constructed from $\overline{A'}^*$ and $h^{\overline{E}^*}$. Then as in the proof of Theorem 1.8(ii), we have

$$(2.84) \qquad \mathrm{Tr}_s[Nf'(D_t)] = \mathrm{Tr}_s[Nf'(-\overline{D}_t^{\,*})] = \mathrm{Tr}_s[Nf'(\overline{D}_t^{\,*})].$$

Using the even isomorphism σ, this gives

$$(2.85) \qquad \mathrm{Tr}_s[Nf'(D_t)] = \mathrm{Tr}_s[(n - N)f'(D_t)].$$

Thus

$$(2.86) \qquad \mathrm{Tr}_s[Nf'(D_t)] = \frac{n}{2}\mathrm{Tr}_s[f'(D_t)].$$

The theorem now follows from Proposition 1.3 and Definition 2.8. $\square$

Theorem 2.28. *Assume that* n *is even and* σ *is an isomorphism as in Theorem 2.27. Then with the notation of Definition 2.15,* $S_f(A', h^E) = 0$.

Proof. From Theorem 2.27, $f^{\wedge}(C_t', h^E)$ is independent of t, and so it equals its $t \to +\infty$ limit, which from Theorem 2.13 is $d(H(E, v))\frac{f'(0)}{2}$. Thus the integrand of Definition 2.15 vanishes. $\square$

Theorem 2.29. *Assume that n is even and σ is an isomorphism as in Theorem 2.27. Then with the notation of Definition 2.20, $T_f(A', h^E) = 0$.*

Proof. From Theorem 2.27, $f^\wedge(C_t', h^E)$ is independent of t, and so it equals its $t \to +\infty$ and $t \to 0$ limits, which from Theorem 2.13 and Proposition 2.18 are $d(H(E, v))\frac{f'(0)}{2}$ and $d(E)\frac{f'(0)}{2}$, respectively. Thus the integrand of Definition 2.20 vanishes. $\square$

We now give a useful way to construct isomorphisms σ. Suppose that there is a nondegenerate form $(\cdot, \cdot)$ defined on $\bigoplus_{i=0}^{n}(E^i \otimes E^{n-i})$, linear in the first factor and antilinear in the second factor, such that for all $e_1 \in E^i$ and $e_2 \in E^{n-i}$,

$$(2.87) \qquad (e_1, e_2) = (-1)^{i(n-i)}\overline{(e_2, e_1)}.$$

Extend $(\cdot, \cdot)$ to a $C^\infty(B)$-valued form on $C^\infty(B; E)$, and further to a form on $\Omega(B; \text{End}(E))$ by requiring that for $e_1 \in C^\infty(E^i)$, $e_2 \in C^\infty(E^j)$, $\omega_1 \in \Omega^k(B)$, and $\omega_2 \in \Omega^l(B)$,

$$(2.88) \qquad (\omega_1 \cdot e_1, \omega_2 \cdot e_2) = (-1)^{il}\omega_1 \wedge \overline{\omega_2}(e_1, e_2) \in \Omega^{k+l}(B).$$

Definition 2.30. We say that $(\cdot, \cdot)$ is compatible with A' if for all $e_1 \in C^\infty(B; E^i)$ and $e_2 \in C^\infty(B; E^j)$,

$$(2.89) \qquad d(e_1, e_2) = (A'e_1, e_2) + (-1)^i(e_1, A'e_2).$$

Definition 2.31. Given the A'-compatible form $(\cdot, \cdot)$, let the isomorphism $j: E \to \overline{E}^*$ be given by

$$(2.90) \qquad \langle j(e_1), e_2 \rangle = (e_1, e_2) \quad \text{for all } e_1, e_2 \in E.$$

Define $\sigma: E \to \overline{E}^*$ by

$$(2.91) \qquad \sigma(e) = (-1)^{i(i+1)/2+ni}j(e) \quad \text{for } e \in E^i.$$

Extend σ to an isomorphism $\sigma: \Omega(B; E) \to \Omega(B; \overline{E}^*)$ by

$$(2.92) \qquad \sigma(\omega \cdot e) = (-1)^{nk}\omega \cdot \sigma(e).$$

Theorem 2.32. *We have*

$$(2.93) \qquad \overline{A'}^* = (-1)^n \sigma A' \sigma^{-1}.$$

Proof. As in Section 1(c), split A' as $\nabla^E + S$. Then it suffices for us to show that $\nabla^{\overline{E}^*} = (-1)^n \sigma \nabla^E \sigma^{-1}$ and $\overline{S}^* = (-1)^n \sigma S \sigma^{-1}$.

Let U be a vector field on B. Then for $e, e' \in C^\infty(B; E)$,

$$(2.94) \qquad U(e, e') = U\langle j(e), e' \rangle = \langle \nabla_U^{\overline{E}^*} j(e), e' \rangle + \langle j(e), \nabla_U^E e' \rangle.$$

On the other hand, from (2.89),

$$(2.95) \qquad U(e, e') = (\nabla_U^E e, e') + (e, \nabla_U^E e') = \langle j(\nabla_U^E e), e' \rangle + \langle j(e), \nabla_U^E e' \rangle.$$

Thus $\nabla_U^{\overline{E}^*} j = j \nabla_U^E$. It follows that $\nabla_U^{\overline{E}^*} \sigma = \sigma \nabla_U^E$, and so $\nabla^{\overline{E}^*} \sigma = (-1)^n \sigma \nabla^E$.

Consider the degree-k component of S. We may assume that S is a finite sum of the form $S = \sum \omega \cdot a$, with $a \in C^\infty(B; \mathrm{End}(E^i, E^{i+1-k}))$ and $\omega \in \Omega^k(B)$. Given $e \in C^\infty(E^i)$ and $e' \in C^\infty(E^{n-i})$, equation (2.89) implies that

$$
\begin{aligned}
0 &= \sum[(\omega \cdot ae, e') + (-1)^i(e, \omega \cdot ae')] \\
&= \sum[\omega \cdot (ae, e') + (-1)^{i+ik}\omega \cdot (e, ae')] \\
&= \sum[\omega \cdot \langle j(ae), e'\rangle + (-1)^{i+ik}\omega \cdot \langle j(e), ae'\rangle] \\
&= \sum[\omega \cdot \langle j(ae), e'\rangle + (-1)^{i+ik}\omega \cdot \langle \overline{a}^* j(e), e'\rangle].
\end{aligned}
$$
(2.96)

Thus

(2.97)
$$
0 = \sum[\omega \cdot j(ae) + (-1)^{i+ik}\omega \cdot \overline{a}^* j(e)].
$$

Put

(2.98)
$$
c(i) = (-1)^{i(i+1)/2+ni}.
$$

It is sufficient to check that for all $e \in C^\infty(E^i)$,

(2.99)
$$
\overline{S}^* \sigma(e) = (-1)^n \sigma(Se),
$$

or

(2.100)
$$
(-1)^{k(k+1)/2} \sum \omega \cdot \overline{a}^* \sigma(e) = (-1)^n \sum \sigma(\omega \cdot ae),
$$

or

(2.101)
$$
(-1)^{k(k+1)/2} \sum \omega \cdot \overline{a}^* \sigma(e) = (-1)^{n+nk} \sum \omega \cdot \sigma(ae),
$$

or

(2.102)
$$
(-1)^{k(k+1)/2} c(i) \sum \omega \cdot \overline{a}^* j(e) = (-1)^{n+nk} c(i+1-k) \sum \omega \cdot j(ae).
$$

Equation (2.102) follows from (2.97) and (2.98). $\square$

Definition 2.33. Given $* \in C^\infty(B; \mathrm{End}(E))$, we say that $(E, (\cdot, \cdot), *)$ defines a duality structure on E if

1. $*$ maps $C^\infty(B; E^i)$ isomorphically to $C^\infty(B; E^{n-i})$.
2. For all $e \in C^\infty(B; E^i)$, $*^2(e) = (-1)^{i(n-i)} e$.
3. $\langle e_1, e_2 \rangle_{h^E} \equiv (e_1, *(e_2))$ defines a Hermitian metric on E.

Lemma 2.34. *The map $*$ is an isometry.*

Proof. For $e_1, e_2 \in C^\infty(B; E^i)$, we have

(2.103)
$$
\begin{aligned}
\langle *(e_1), *(e_2)\rangle_{h^E} &= (*(e_1), *^2(e_2)) = (-1)^{i(n-i)}(*(e_1), e_2) = \overline{(e_2, *(e_1))} \\
&= \overline{\langle e_2, e_1\rangle_{h^E}} = \langle e_1, e_2\rangle_{h^E}.
\end{aligned}
$$

The lemma follows. $\square$

We say that $(E, (\cdot, \cdot), *)$ is an A'-compatible duality structure on E if $(\cdot, \cdot)$ is compatible with A' and $(E, (\cdot, \cdot), *)$ is a duality structure on E. We assume that E has such a structure and we give E the corresponding Hermitian metric.

Theorem 2.35. *The isomorphism* σ *of Definition* 2.31 *is an isometry.*

Proof. It is enough to show that j is an isometry. For $e \in C^\infty(B; E)$, we have

$$(2.104)\quad \begin{aligned} \|j(e)\|^2_{h^{\bar{E}^*}} &= \sup_{e' \neq 0} \frac{|\langle j(e), e'\rangle|^2}{\|e'\|^2_{h^E}} = \sup_{e' \neq 0} \frac{|\langle e, e'\rangle|^2}{\|e'\|^2_{h^E}} = \sup_{e' \neq 0} \frac{|\langle e', e\rangle|^2}{\|e'\|^2_{h^E}} \\ &= \sup_{e' \neq 0} \frac{|\langle e', *^{-1}(e)\rangle_{h^E}|^2}{\|e'\|^2_{h^E}} = \|*^{-1}(e)\|^2_{h^E} = \|e\|^2_{h^E}. \end{aligned}$$

The theorem follows. □

Thus if E has an A'-compatible duality structure and n is even, we have an isomorphism as in Theorem 2.27 and we can apply Theorems 2.28 and 2.29.

Define $\nu \colon E \to E$ by

$$(2.105)\qquad\qquad \nu = \sigma^{-1}\hat{h}^E.$$

By (1.16) and (2.93),

$$(2.106)\qquad\qquad A'^* = (-1)^n \nu^{-1} A' \nu.$$

Lemma 2.36. *For* $e \in C^\infty(B; E^i)$,

$$(2.107)\qquad \nu(e) = (-1)^{n(n-1)/2}(-1)^{i(i+1)/2+ni} *(e).$$

Proof. Define $c(i)$ as in (2.98). For all $e' \in C^\infty(B; E^{n-i})$,

$$(2.108)\quad \begin{aligned} \langle \hat{h}^E(e), e'\rangle &= (e, *(e')) = (*(e), *^2(e')) = (-1)^{i(n-i)}(*(e), e') \\ &= (-1)^{i(n-i)}\langle j(*(e)), e'\rangle = (-1)^{i(n-i)} c(n-i)\langle \sigma(*(e)), e'\rangle, \end{aligned}$$

from which the lemma follows. □

Example 2.37. Let $\mathscr{E}$ be a vector bundle over B with positive-definite Hermitian metric $h^{\mathscr{E}}$ and compatible Hermitian connection $\nabla^{\mathscr{E}}$. Take $E = E^0 \oplus E^1$ with $E^0 = E^1 = \mathscr{E}$. Define a superconnection A' by

$$(2.109)\quad A'_0 = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}, \quad A'_1 = \begin{pmatrix} \nabla^{\mathscr{E}} & 0 \\ 0 & \nabla^{\mathscr{E}} \end{pmatrix}, \quad A'_2 = \begin{pmatrix} 0 & -(\nabla^{\mathscr{E}})^2 \\ 0 & 0 \end{pmatrix},$$

the other terms in A' being zero. Then A' is flat. Define a form $(\cdot, \cdot)$ on E by saying that for $e, f \in \mathscr{E}$,

$$(2.110)$$
$$(e_0, f_0) = 0, \quad (e_0, f_1) = -ih^{\mathscr{E}}(e, f), \quad (e_1, f_0) = ih^{\mathscr{E}}(e, f), \quad (e_1, f_1) = 0,$$

where the notation "e_i" indicates that e is considered to be an element of E^i. One can check that $(\cdot, \cdot)$ is compatible with A'. Put

$$(2.111)\qquad\qquad * = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Then $(E, (\cdot, \cdot), *)$ defines a duality structure. The Hermitian metric on E is given by

$$(2.112)\qquad \langle e_0, f_0\rangle_{h^E} = \langle e_1, f_1\rangle_{h^E} = h^{\mathscr{E}}(e, f).$$

One has

$$(2.113) \quad A_0'^* = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \quad A_1'^* = \begin{pmatrix} \nabla^{\mathscr{E}} & 0 \\ 0 & \nabla^{\mathscr{E}} \end{pmatrix}, \quad A_2'^* = \begin{pmatrix} 0 & 0 \\ -(\nabla^{\mathscr{E}})^2 & 0 \end{pmatrix},$$

the other terms in A'^* being zero. Then

$$(2.114) \qquad D_t = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{t}I + \frac{(\nabla^{\mathscr{E}})^2}{\sqrt{t}} \\ -\sqrt{t}I - \frac{(\nabla^{\mathscr{E}})^2}{\sqrt{t}} & 0 \end{pmatrix}.$$

Let $h\colon \mathbb{C} \to \mathbb{C}$ be such that $f'(a) = h(a^2)$. Then

$$
\begin{aligned}
f^\wedge(A', h_t^E) &= f^\wedge(C_t', h^E) = \varphi \operatorname{Tr}_s\left[\frac{N}{2} f'(D_t)\right] = \varphi \operatorname{Tr}_s\left[\frac{N}{2} h(D_t^2)\right] \\
&= -\frac{1}{2}\varphi \operatorname{Tr}\left[h\left(-\frac{1}{4}\left(\sqrt{t}I + \frac{(\nabla^{\mathscr{E}})^2}{\sqrt{t}}\right)^2\right)\right] \\
&= -\frac{1}{2} \operatorname{Tr}\left[f'\left(\frac{1}{2}i\left(\sqrt{t}I + \frac{(\nabla^{\mathscr{E}})^2}{2i\pi\sqrt{t}}\right)\right)\right] \\
&= -\frac{1}{2}\sum_{k=0}^\infty \frac{1}{4^k k!}\left(\frac{i\sqrt{t}}{2}\right)^{-k} f^{(k+1)}\left(\frac{i\sqrt{t}}{2}\right) \operatorname{Tr}\left[\left(-\frac{(\nabla^{\mathscr{E}})^2}{2i\pi}\right)^k\right]
\end{aligned}
$$

(2.115)

and
(2.116)

$$S_f(A', h^E) = \frac{1}{2}\sum_{k=0}^\infty \frac{1}{4^k k!}\left(\int_1^\infty \left(\frac{i\sqrt{t}}{2}\right)^{-k} f^{(k+1)}\left(\frac{i\sqrt{t}}{2}\right)\frac{dt}{t}\right) \operatorname{Tr}\left[\left(-\frac{(\nabla^{\mathscr{E}})^2}{2i\pi}\right)^k\right].$$

We see that $S_f(A', h^E)$ is closed and its de Rham cohomology class is given by a sum of Chern classes of $\mathscr{E}$, with coefficients that depend on f. Example 2.37 is a finite-dimensional model of the circle-bundle results of Section 4.

III. "RIEMANN-ROCH-GROTHENDIECK" FOR FLAT VECTOR BUNDLES AND HIGHER ANALYTIC TORSION FORMS

In this section we extend the finite-dimensional results of Sections 1 and 2 to the case of fiber bundles, using techniques of local index theory. We prove Theorems 0.1 and 0.2 of the Introduction.

The section is organized as follows. Given a smooth fiber bundle $Z \to M \xrightarrow{\pi} B$ with closed fibers, a horizontal distribution $T^H M$ on the fiber bundle, and a flat vector bundle F on M, in (a) we describe a certain infinite-dimensional $\mathbb{Z}$-graded vector bundle W on B. In (b) we show that exterior differentiation on $\Omega(M; F)$ gives a flat superconnection of total degree 1 on W. In (c) we discuss the geometry of fiber bundles with a horizontal distribution and a vertical Riemannian metric. In (d), we compute the adjoint superconnection $(d^M)^*$ and show that $\frac{1}{2}((d^M)^* + d^M)$ is essentially a Levi-Civita superconnection in the

sense of [B]. Following Section 2(c), in section (e) we construct a rescaled superconnection C_t and an operator D_t, and prove a Lichnerowicz-type formula for $C_{4t}^2 - z D_{4t}$. In (f) we describe the flat connection on $H(Z; F|_Z)$.

In (g) we show that for all $t > 0$, the Chern character $\varphi \operatorname{Tr}_s[\exp(-C_t^2)] \in \Omega(B)$ equals $\operatorname{rk}(F)\chi(Z)$. In (h) we take $f(a) = a \exp(a^2)$ and construct $f(C_t', h^W) \in \Omega^{\mathrm{odd}}(B)$. We calculate the $t \to 0$ and $t \to +\infty$ asymptotics of $f(C_h', h^W)$ and prove Theorem 0.1. In (i) we prove a transgression formula for the forms $f(C_t', h^W)$. In (j) we construct an analytic torsion form $\mathcal{T}(T^H M, g^{TZ}, h^F)$ and prove Theorem 0.2. We also compute the dependence of $\mathcal{T}(T^H M, g^{TZ}, h^F)$ on its arguments, and give duality and product theorems. Finally, in (k) we relate the degree-0 component of $\mathcal{T}(T^H M, g^{TZ}, h^F)$ to the Ray-Singer analytic torsion [RS1], and show that the results of (j) imply the anomaly formula of Bismut-Zhang [BZ] concerning Ray-Singer metrics on the determinant line bundle.

In the entire section we use the notation of Sections 1 and 2.

(a) **Smooth fiber bundles.** Let $Z \to M \xrightarrow{\pi} B$ be a smooth fiber bundle with connected closed fibers $Z_b = \pi^{-1}(b)$ of dimension n. Let TZ be the vertical tangent bundle of the fiber bundle and let T^*Z be its dual bundle.

Let $T^H M$ be a horizontal distribution for the fiber bundle, meaning that $T^H M$ is a subbundle of TM such that

$$(3.1) \qquad\qquad TM = T^H M \oplus TZ.$$

Let P^{TZ} denote the projection from TM to TZ. We have

$$(3.2) \qquad\qquad T^H M \simeq \pi^* TB.$$

Then (3.1) and (3.2) give that as bundles of $\mathbb{Z}$-graded algebras over M,

$$(3.3) \qquad\qquad \Lambda(T^*M) \simeq \pi^*(\Lambda(T^*B)) \otimes \Lambda(T^*Z).$$

Let F be a flat complex vector bundle on M and let ∇^F denote its flat connection. Let W be the smooth infinite-dimensional $\mathbb{Z}$-graded vector bundle over B whose fiber over $b \in B$ is $C^\infty(Z_b; (\Lambda(T^*Z) \otimes F)|_{Z_b})$. That is,

$$C^\infty(B; W) \simeq C^\infty(M; \Lambda(T^*Z) \otimes F).$$

Let $\Omega^V(M; F)$ denote the subspace of $\Omega(M; F)$ which is annihilated by interior multiplication with horizontal vectors. Then there is an isomorphism

$$(3.4) \qquad\qquad \Omega^V(M; F) \simeq C^\infty(B; W),$$

where the isomorphism is given by sending an element of $\Omega^V(M; F)$ to its fiberwise restrictions. From (3.3),

$$(3.5) \qquad\qquad \Omega(M; F) \simeq \Omega(B) \,\widehat{\otimes}\, \Omega^V(M; F).$$

Thus we have an isomorphism of $\mathbb{Z}$-graded vector spaces

$$(3.6) \qquad\qquad \Omega(M; F) \simeq \Omega(B; W).$$

(b) A flat superconnection of total degree 1. The exterior differentiation operator d^M, acting on $\Omega(M; F)$, has degree 1 and satisfies $(d^M)^2 = 0$. Furthermore, for all $f \in C^\infty(B)$ and $\omega \in \Omega(M; F)$,

$$(3.7) \qquad d^M((\pi^* f) \cdot \omega) = (\pi^* d^B f) \wedge \omega + (\pi^* f) \cdot d^M(\omega).$$

Thus d^M defines a flat superconnection of total degree 1 on W. We will compute the terms in the decomposition (2.1) of d^M.

Definition 3.1. Let d^Z denote exterior differentiation along fibers. We consider d^Z to be an element of $C^\infty(B; \mathrm{Hom}(W^\bullet, W^{\bullet+1}))$.

If U is a smooth vector field on B, let $U^H \in C^\infty(M; T^H M)$ be its horizontal lift, so that $\pi_* U^H = U$. As the flow generated by U^H sends fibers to fibers diffeomorphically, the Lie differentiation operator L_{U^H} acts on $C^\infty(M; \Lambda(T^*Z) \otimes F)$, and one can easily verify that for $f \in C^\infty(B)$ and $a \in C^\infty(M; \Lambda(T^*Z) \otimes F)$,

$$(3.8) \qquad L_{(fU)^H} a = (\pi^* f) \cdot L_{U^H} a$$

and

$$(3.9) \qquad L_{U^H}((\pi^* f) a) = \pi^*(Uf) \cdot a + (\pi^* f) \cdot L_{U^H} a.$$

Definition 3.2. For $s \in C^\infty(B; W)$ and U a vector field on B, put

$$(3.10) \qquad \nabla_U^W s = L_{U^H} s.$$

From (3.8) and (3.9), ∇^W is a connection on W which preserves the $\mathbb{Z}$-grading.

If U_1 and U_2 are vector fields on B, put

$$(3.11) \qquad T(U_1, U_2) = -P^{TZ}[U_1^H, U_2^H] \in C^\infty(M; TZ).$$

One easily verifies that T gives a TZ-valued horizontal 2-form on M, which one calls the curvature of the fiber bundle.

Definition 3.3. $i_T \in \Omega^2(B; \mathrm{Hom}(W^\bullet, W^{\bullet-1}))$ is the 2-form on B which, to vector fields U and V on B, assigns the operation of interior multiplication by $T(U, V)$ on W.

The next proposition essentially appears in [BGV, Proposition 10.1].

Proposition 3.4. $d^M = d^Z + \nabla^W + i_T$.

Proof. We first show that the superconnection $d^Z + \nabla^W + i_T$ is flat. We have to show:

$$(3.12) \qquad
\begin{aligned}
&1. \quad (d^Z)^2 = 0.\\
&2. \quad d^Z \nabla^W + \nabla^W d^Z = 0.\\
&3. \quad (\nabla^W)^2 + d^Z i_T + i_T d^Z = 0.\\
&4. \quad \nabla^W i_T + i_T \nabla^W = 0.\\
&5. \quad i_T^2 = 0.
\end{aligned}$$

Equation (3.12.1) is trivially true. If U is a vector field on B, let $\{\phi_t\}_{t \in \mathbb{R}}$ denote the flow generated by U^H. As ϕ_t^* commutes with d^Z, it follows that L_{U^H} commutes with d^Z, which implies (3.12.2). If U_1 and U_2 are vector fields on B then

$$(3.13) \quad (\nabla^W)^2(U_1, U_2) = [L_{U_1^H}, L_{U_2^H}] - L_{[U_1, U_2]^H}$$
$$= L_{[U_1^H, U_2^H] - [U_1, U_2]^H} = L_{P^{TZ}[U_1^H, U_2^H]} = -L_{T(U_1, U_2)}.$$

However,

$$(3.14) \quad d^Z \cdot i_{T(U_1, U_2)} + i_{T(U_1, U_2)} \cdot d^Z = L_{T(U_1, U_2)},$$

which gives (3.12.3).

Let ∇^B be a torsion-free connection on TB. If U_1, U_2, and U_3 are vector fields on B then

$$(\nabla^W i_T + i_T \nabla^W)(U_1, U_2, U_3)$$
$$= \nabla_{U_1}^W i_{T(U_2, U_3)} - i_{T(U_2, U_3)} \nabla_{U_1}^W - i_{T(\nabla_{U_1}^B U_2, U_3)}$$
$$- i_{T(U_2, \nabla_{U_1}^B U_3)} + \text{cyclic permutations}$$
$$(3.15) \quad = L_{U_1^H} i_{T(U_2, U_3)} - i_{T(U_2, U_3)} L_{U_1^H} - i_{T(\nabla_{U_1}^B U_2, U_3)}$$
$$- i_{T(U_2, \nabla_{U_1}^B U_3)} + \text{cyclic permutations}$$
$$= i_{[U_1^H, T(U_2, U_3)] - T(\nabla_{U_1}^B U_2, U_3) - T(U_2, \nabla_{U_1}^B U_3)}$$
$$+ \text{cyclic permutations.}$$

On the other hand, by the Jacobi identity,

$$0 = P^{TZ}([U_1^H, [U_2^H, U_3^H]] + \text{cyclic permutations})$$
$$(3.16) \quad = P^{TZ}([U_1^H, -T(U_2, U_3) + [U_2, U_3]^H]) + \text{cyclic permutations}$$
$$= -[U_1^H, T(U_2, U_3)] - T(U_1, [U_2, U_3]) + \text{cyclic permutations.}$$

Comparing (3.15) and (3.16) gives (3.12.4). Equation (3.12.5) is trivially true.

Now as the statement of the proposition is local on M, we may assume without loss of generality that F is the trivial complex line bundle on M. Then letting the superconnection $d^Z + \nabla^W + i_T$ act on $\Omega(B; W) \simeq \Omega(M)$, we have $(d^Z + \nabla^W + i_T)^2 = 0$. It is easy to check that $d^Z + \nabla^W + i_T$ is a derivation of $\Omega(M)$, and that for $f \in C^\infty(M)$,

$$(3.17) \quad (d^Z + \nabla^W + i_T)f = d^M f \in \Omega^1(M).$$

The proposition follows from the standard axiomatic characterization of d^M. $\square$

(c) **Vertical metrics on fiber bundles.** Hereafter, we assume that we have a vertical Riemannian metric on the fiber bundle $Z \to M \xrightarrow{\pi} B$. That is, we have a positive-definite metric g^{TZ} on TZ.

As for notation, we let lower case Greek indices refer to horizontal directions, lower case italic indices refer to vertical directions and upper case italic indices refer to either. We let $\{\tau^J\}$ denote a local basis of 1-forms on M, with dual basis $\{e_J\}$ of tangent vectors. We will always take $\{e_i\}_{i=1}^n$ to be an orthonormal framing of TZ. We will assume that the forms $\{\tau^\alpha\}$ are pulled back from a local basis of 1-forms on B, which we will also denote by $\{\tau^\alpha\}$. Exterior multiplication by a form ϕ will be denoted by $\phi\wedge$ and interior multiplication by a vector v will be denoted by i_v. Using the horizontal distribution and vertical Riemannian metric, we can identify vertical vectors and vertical 1-forms. Exterior multiplication by τ^J will be denoted by E^J and interior multiplication by e_J will be denoted by I^J. We have that $E^j I^k + I^k E^j = \delta^{jk}$. If X is a vertical vector (or 1-form), put

$$(3.18) \qquad c(X) = (X\wedge) - i_X, \qquad \hat{c}(X) = (X\wedge) + i_X.$$

Then for vertical vectors X and Y, we have

$$
\begin{aligned}
& c(X)c(Y) + c(Y)c(X) = -2\langle X, Y\rangle_{g^{TZ}}, \\
(3.19) \qquad & \hat{c}(X)\hat{c}(Y) + \hat{c}(Y)\hat{c}(X) = 2\langle X, Y\rangle_{g^{TZ}}, \\
& c(X)\hat{c}(Y) + \hat{c}(Y)c(X) = 0.
\end{aligned}
$$

Thus c and $\hat{c}$ generate two graded-commuting Clifford algebras. Put $c^i = c(e_i)$ and $\hat{c}^i = \hat{c}(e_i)$.

In calculations we will sometimes assume that B has a Riemannian metric g^{TB} and M has the Riemannian metric $g^{TM} = g^{TZ} \oplus \pi^* g^{TB}$, although all final results will be independent of g^{TB}. Let ∇^{TM} denote the corresponding Levi-Civita connection on M and put $\nabla^{TZ} = P^{TZ}\nabla^{TM}$, a connection on TZ. As shown in [B, Theorem 1.9], ∇^{TZ} is independent of the choice of g^{TB}. The restriction on ∇^{TZ} to a fiber coincides with the Levi-Civita connection of the fiber. We will also denote by ∇^{TZ} the extension to a connection on $\Lambda(T^*Z)$.

We will use the Einstein summation convention freely, and write

$$(3.20) \qquad \omega_{IJK} = \tau^I(\nabla_{e_K}^{TM} e_J).$$

As there is a vertical metric, we may raise and lower vertical indices freely.

The fundamental geometric tensors of the fiber bundle are its curvature (3.11), a TZ-valued horizontal 2-form on M, and the second fundamental form of the fibers, a $(T^H M)^*$-valued vertical symmetric form on M. These tensors are incorporated into a tensor S defined in [B]; the relationship between the notation of [B] and the present notation is given by

$$
\begin{aligned}
(3.21) \qquad & \langle S(e_j)f_\alpha, f_\beta\rangle = \omega_{\beta\alpha j} = -\omega_{\alpha\beta j} = -\omega_{\alpha j\beta} = \omega_{j\alpha\beta} = \omega_{\beta j\alpha} = -\omega_{j\beta\alpha}, \\
& \langle S(e_j)e_k, f_\alpha\rangle = \omega_{\alpha kj} = \omega_{\alpha jk} = -\omega_{j\alpha k} = -\omega_{k\alpha j}.
\end{aligned}
$$

The 2-form T of (3.11) is given in terms of the local framing by

$$(3.22) \qquad T_j = -\omega_{\alpha\beta j}\tau^\alpha \wedge \tau^\beta.$$

Define a horizontal 1-form k on M by

$$(3.23) \qquad k = \omega_{j\alpha j}\tau^\alpha,$$

the mean curvature 1-form to the fibers.

Let $\nabla^{TZ \otimes F}$ be the tensor product of ∇^{TZ} and the flat connection ∇^F.

Proposition 3.5. *As operators on $C^\infty(B; W)$, in terms of a local framing we have*

$$d^Z = E^j \nabla^{TZ \otimes F}_{e_j},$$

(3.24)
$$\nabla^W = E^\alpha (\nabla^{TZ \otimes F}_{e_\alpha} - \omega_{\alpha jk} E^j I^k),$$

$$i_T = -\omega_{\alpha \beta j} E^\alpha E^\beta I^j.$$

Proof. The first and third identities in (3.24) are clear. To prove the second identity, we use the fact that the difference between Lie differentiation and covariant differentiation, acting on $\Omega(M; F)$, is given by

(3.25)
$$L_{e_I} - \nabla^{TM \otimes F}_{e_I} = \omega_{KIJ} E^J I^K.$$

Acting on $\Omega^V(M; F) \simeq C^\infty(B; W)$, we obtain

(3.26)
$$L_{e_\alpha} - \nabla^{TM \otimes F}_{e_\alpha} = \omega_{k\alpha j} E^j I^k + \omega_{k\alpha\beta} E^\beta I^k.$$

Projecting onto TZ then gives

(3.27)
$$\nabla^W_{e_\alpha} - \nabla^{TZ \otimes F}_{e_\alpha} = \omega_{k\alpha j} E^j I^k,$$

from which the second identity in (3.24) follows. □

Remark 3.6. One can rederive Proposition 3.4 using local framings. Suppose that B has a Riemannian metric and that M has the induced Riemannian metric and Levi-Civita connection. Acting on $\Omega^V(M; F) \simeq C^\infty(B; W)$, we have

(3.28)
$$d^M = E^j \nabla^{TM \otimes F}_{e_j} + E^\alpha \nabla^{TM \otimes F}_{e_\alpha}$$
$$= E^j (\nabla^{TZ \otimes F}_{e_j} + \omega_{\alpha kj} E^\alpha I^k) + E^\alpha (\nabla^{TZ \otimes F}_{e_\alpha} + \omega_{\beta k\alpha} E^\beta I^k).$$

Using Proposition 3.5 and the symmetries of (3.21), Proposition 3.4 follows when both sides act on $C^\infty(B; W)$, and hence Proposition 3.4 is an equality of superconnections. Of course, the conclusion of Proposition 3.4 is independent of the choice of the vertical Riemannian metric on the fiber bundle or the Riemannian metric on B.

(d) The adjoint superconnection and the Levi-Civita superconnection. In addition to the assumptions of 2(c), suppose that F is equipped with a Hermitian metric h^F. Let $\nabla^{F,u}$ denote the unitary connection $\frac{1}{2}(\nabla^F + (\nabla^F)^*)$ on F and let $\nabla^{TZ \otimes F, u}$ denote the connection on $\Lambda(T^*Z) \otimes F$ obtained by tensoring ∇^{TZ} and $\nabla^{F,u}$. Let ψ be short for $\omega(F, h^F) \in \Omega^1(M; \text{End}(E))$, defined as in (1.31).

Let $*$ be the fiberwise Hodge duality operator associated to g^{TZ}, which we extend from an operator on $C^\infty(M; \Lambda(T^*Z))$ to an operator on

$C^\infty(M; \Lambda(T^*Z) \otimes F) \simeq C^\infty(B; W)$. Then W acquires a Hermitian metric h^W such that for $s, s' \in C^\infty(B; W)$ and $b \in B$,

$$(3.29) \qquad (\langle s, s' \rangle_{h^W})(b) = \int_{Z_b} \langle s(b) \wedge *s'(b) \rangle_{h^F}.$$

Proposition 3.7. *The adjoint of the superconnection d^M is given by*

$$(3.30) \qquad (d^M)^* = (d^Z)^* + (\nabla^W)^* - T \wedge,$$

where

$$(3.31) \qquad (d^Z)^* = -I^j (\nabla^{TZ \otimes F}_{e_j} + \psi_j)$$

is the fiberwise formal adjoint of d^Z with respect to h^W,

$$(3.32) \qquad (\nabla^W)^* = E^\alpha (\nabla^{TZ \otimes F}_{e_\alpha} - \omega_{\alpha jk} I^j E^k + \psi_\alpha),$$

and $T\wedge$ is exterior multiplication by the T of (3.11).

Proof. Let $\hat{h}^F$ be as in (1.15), let ν be as in (2.107), and let $\mathscr{V}: \Omega(M; F) \to \Omega(M; \overline{F}^*)$ be given by $\mathscr{V} = \nu \hat{\otimes} \hat{h}^F$. Let $d^{M, \overline{F}^*}$ be exterior differentiation on $\Omega(M; \overline{F}^*)$. Following the lines of the proof of (2.106), we have

$$(3.33) \qquad (d^M)^* = (-1)^n \mathscr{V}^{-1} d^{M, \overline{F}^*} \mathscr{V}.$$

One can easily check that

$$(3.34) \qquad \begin{aligned} (-1)^n \mathscr{V}^{-1} E^j \mathscr{V} &= -I^j, \\ (-1)^n \mathscr{V}^{-1} I^j \mathscr{V} &= -E^j, \\ (-1)^n \mathscr{V}^{-1} E^\alpha \mathscr{V} &= E^\alpha, \\ \mathscr{V}^{-1} \nabla^{TZ \otimes \overline{F}^*}_{e_J} \mathscr{V} &= \nabla^{TZ \otimes F}_{e_J} + \psi_J. \end{aligned}$$

The proposition now follows from (3.24) and (3.34). $\square$

Definition 3.8. Put

$$(3.35) \qquad \begin{aligned} D^Z &= d^Z + (d^Z)^*, \\ \nabla^{W, u} &= \tfrac{1}{2}(\nabla^W + (\nabla^W)^*), \\ \omega(W, h^W) &= (\nabla^W)^* - \nabla^W. \end{aligned}$$

Clearly D^Z is a selfadjoint element of $C^\infty(B; \text{End}(W))$, $\nabla^{W, u}$ is a Hermitian connection on W, and $\omega(W, h^W)$ is an element of $\Omega^1(B; \text{End}(W))$. From (3.18) and Propositions 3.5 and 3.7, we obtain

$$(3.36) \qquad D^Z = c^j \nabla^{TZ \otimes F, u}_{e_j} - \tfrac{1}{2} \hat{c}^j \psi_j$$

and

$$\nabla^{W,u} = E^\alpha(\nabla_{e_\alpha}^{TZ\otimes F,u} - \tfrac{1}{2}\omega_{\alpha jk}(E^j I^k + I^k E^j))$$
$$= E^\alpha(\nabla_{e_\alpha}^{TZ\otimes F,u} + \tfrac{1}{2}k_\alpha),$$

(3.37)

$$\omega(W, h^W) = E^\alpha(\omega_{\alpha jk}(-I^j E^k + E^j I^k) + \psi_\alpha)$$
$$= E^\alpha(\omega_{\alpha jk}c^j \hat{c}^k + \psi_\alpha).$$

(Equation (3.36) appears in [BZ, Proposition 4.12], which deals with the case when B is a point.)

In particular, if U is a vector field on B, then

(3.38) $$\nabla_U^{W,u} = \nabla_{U^H}^{TZ\otimes F,u} + \tfrac{1}{2}k(U^H).$$

Following the formalism of Section 1(e), put

(3.39) $$A = \tfrac{1}{2}((d^M)^* + d^M), \qquad X = \tfrac{1}{2}((d^M)^* - d^M).$$

Proposition 3.9. *We have*

(3.40) $$A = \tfrac{1}{2}D^Z + \nabla^{W,u} - \tfrac{1}{2}c(T)$$

and

(3.41) $$X = -\tfrac{1}{2}\hat{c}^j \nabla_{e_j}^{TZ\otimes F,u} + \tfrac{1}{4}c^j \psi_j + \tfrac{1}{2}E^\alpha(\omega_{\alpha jk}c^j \hat{c}^k + \psi_\alpha) - \tfrac{1}{2}\hat{c}(T).$$

Proof. Equation (3.40) follows from Propositions 3.4 and 3.7. From (3.37) and Propositions 3.5 and 3.7, we obtain

(3.42)
$$\tfrac{1}{2}((d^Z)^* - d^Z) = \tfrac{1}{2}(-\hat{c}^j \nabla_{e_j}^{TZ\otimes F} + \tfrac{1}{2}(c - \hat{c})^j \psi_j)$$
$$= \tfrac{1}{2}(-\hat{c}^j \nabla_{e_j}^{TZ\otimes F,u} + \tfrac{1}{2}c^j \psi_j),$$

(3.43) $$\tfrac{1}{2}((\nabla^W)^* - \nabla^W) = \tfrac{1}{2}\omega(W, h^W) = \tfrac{1}{2}E^\alpha(\omega_{\alpha jk}c^j \hat{c}^k + \psi_\alpha),$$

and

(3.44) $$\tfrac{1}{2}(-(T\wedge) - i_T) = -\tfrac{1}{2}\hat{c}(T).$$

Equation (3.41) follows. $\square$

Remark 3.10. For $t > 0$, put

(3.45) $$E_t = \sqrt{t}c^j \nabla_{e_j}^{TZ\otimes F,u} + \nabla^{W,u} - \frac{1}{4\sqrt{t}}c(T),$$

the Levi-Civita superconnection associated to the vertical signature operator coupled to $\nabla^{F,u}$ [B, Section 3]. By (3.40),

(3.46) $$A = E_{1/4} - \tfrac{1}{4}\hat{c}^j \psi_j.$$

Thus A is essentially the same as the Levi-Civita superconnection of parameter $t = \tfrac{1}{4}$; it is precisely the same if h^F is covariantly-constant with respect to ∇^F.

(e) **A Lichnerowicz-type formula.** We now develop the formalism of Section 2(c) in the present infinite-dimensional setting.

For $t > 0$, let h_t^W be the Hermitian metric on W associated to the metrics g^{TZ}/t and h^F on TZ and F, respectively. Let $(d^M)_t^*$ be the adjoint of the superconnection d^M with respect to h_t^W.

Let N be the number operator of W; it acts by multiplication by i on $C^\infty(M; \Lambda^i(T^*Z) \otimes F)$. As in (2.19), we have

$$(3.47) \qquad (d^M)_t^* = t^{-N}(d^M)^* t^N.$$

As in (2.21), put

$$(3.48) \qquad A_t = \tfrac{1}{2}((d^M)_t^* + d^M), \qquad X_t = \tfrac{1}{2}((d^M)_t^* - d^M).$$

Then A_t is a superconnection and X_t is an odd element of $\Omega(B; \operatorname{End}(W))$.

As in Definition 2.10, put

$$(3.49) \qquad \begin{aligned} C_t' &= t^{N/2} d^M t^{-N/2}, \\ C_t'' &= t^{-N/2}(d^M)^* t^{N/2}. \end{aligned}$$

Then C_t'' is the adjoint of C_t' with respect to h^W.

As in (2.29), put

$$(3.50) \qquad C_t = \tfrac{1}{2}(C_t'' + C_t'), \qquad D_t = \tfrac{1}{2}(C_t'' - C_t').$$

Again, C_t is a superconnection and D_t is an odd element of $\Omega(B; \operatorname{End}(W))$. As in Section 2(c), for our purposes one can work equally well with either (A_t, X_t) or (C_t, D_t). To make the comparison easier with the local families index formalism of [B], we will use (C_t, D_t).

As in (1.33), we have

$$(3.51) \qquad \nabla^{F,u} = \nabla^F + \tfrac{1}{2}\omega(F, h^F) = \nabla^F + \tfrac{1}{2}\psi.$$

Let $R^{F,u}$ be the curvature of $\nabla^{F,u}$. From (3.51),

$$(3.52) \qquad R^{F,u} = -\tfrac{1}{4}\psi^2.$$

Let $\nabla^{TZ \otimes F, u}\psi$ be the covariant derivative of ψ. Explicitly,

$$(3.53) \qquad \nabla_{e_j}^{TZ \otimes F, u}\psi_k = \nabla_{e_j}^{TZ}\psi_k + \tfrac{1}{2}[\psi_j, \psi_k].$$

Let R^{TZ} denote the curvature of ∇^{TZ} and put

$$(3.54) \qquad R_{jk}^{TZ} = \langle e_j, R^{TZ} e_k \rangle_{g^{TZ}}.$$

Define $\widehat{R}^{TZ} \in \Omega^2(M; \operatorname{End}(\Lambda(T^*Z)))$ by

$$(3.55) \qquad \widehat{R}^{TZ} = \tfrac{1}{4} R_{jk}^{TZ} \hat{c}^j \hat{c}^k$$

and $\mathscr{R} \in \Omega^2(M; \operatorname{End}(\Lambda(T^*Z) \otimes F))$ by

$$(3.56) \qquad \mathscr{R} = (\widehat{R}^{TZ} \otimes I_F) + (I_{\Lambda(T^*Z)} \otimes R^{F,u}).$$

Let $R \in C^\infty(M)$ be the scalar curvature of the fibers. Let z be an auxiliary odd Grassmann variable which anticommutes with all of the odd Grassmann variables introduced previously.

For $t > 0$, put

(3.57)
$$\mathscr{D}_{e_j} = \nabla^{TZ \otimes F, u}_{e_j} - \frac{1}{2\sqrt{t}} \omega_{\alpha j k} E^\alpha c^k - \frac{1}{4t} \omega_{\alpha \beta j} E^\alpha E^\beta - \frac{1}{2\sqrt{t}} z \hat{c}^j,$$
$$\mathscr{D}^2 = \mathscr{D}_{e_j} \mathscr{D}_{e_j} - \mathscr{D}_{\nabla^{TZ}_{e_j} e_j}.$$

We now prove a crucial Lichnerowicz-type identity.

Theorem 3.11. *For* $t > 0$, *the following identity holds*:

(3.58)
$$\begin{aligned}
C_{4t}^2 - z D_{4t} &= t \left(-\mathscr{D}^2 + \frac{R}{4} \right) + \frac{t}{2} c^i c^j \mathscr{R}(e_i, e_j) \\
&\quad + \sqrt{t} c^i E^\alpha \mathscr{R}(e_i, e_\alpha) + \frac{1}{2} E^\alpha E^\beta \mathscr{R}(e_\alpha, e_\beta) \\
&\quad + t \left(\frac{1}{4} \psi_j^2 + \frac{1}{8} \hat{c}^j \hat{c}^k [\psi_j, \psi_k] - \frac{1}{2} c^j \hat{c}^k (\nabla^{TZ \otimes F, u}_{e_j} \psi_k) \right) \\
&\quad - \frac{\sqrt{t}}{2} E^\alpha \hat{c}^j (\nabla^{TZ \otimes F, u}_{e_\alpha} \psi_j) - \frac{z\sqrt{t}}{2} c^j \psi_j - \frac{z}{2} E^\alpha \psi_\alpha.
\end{aligned}$$

Proof. We first prove the statement when $z = 0$. From (3.46), we have

(3.59)
$$C_{4t} = E_t - \frac{\sqrt{t}}{2} \hat{c}^j \psi_j.$$

Then

(3.60)
$$C_{4t}^2 = E_t^2 + \frac{t}{4} \psi_j^2 + \frac{t}{8} \hat{c}^j \hat{c}^k [\psi_j, \psi_k] - \frac{\sqrt{t}}{2} [E_t, \hat{c}^j \psi_j],$$

where the last term is a graded commutator. From [B, Theorem 3.6],

(3.61)
$$\begin{aligned}
E_t^2 &= t \left(-\mathscr{D}^2 + \frac{R}{4} \right) + \frac{t}{2} c^i c^j \mathscr{R}(e_i, e_j) \\
&\quad + \sqrt{t} c^i E^\alpha \mathscr{R}(e_i, e_\alpha) + \frac{1}{2} E^\alpha E^\beta \mathscr{R}(e_\alpha, e_\beta).
\end{aligned}$$

A simple computation using (3.38) and (3.45) gives

(3.62)
$$[E_t, \hat{c}^j \psi_j] = \sqrt{t} c^j \hat{c}^k (\nabla^{TZ \otimes F, u}_{e_j} \psi_k) + E^\alpha \hat{c}^j (\nabla^{TZ \otimes F, u}_{e_\alpha} \psi_j).$$

Combining (3.60), (3.61) and (3.62) gives the validity of (3.58) when $z = 0$.

As $z^2 = 0$, it remains to show that the coefficients of z in the two sides of (3.58) are the same. That is, we must show that

(3.63)
$$-z D_{4t} = \frac{\sqrt{t}}{2} (\mathscr{D}_j \cdot z \hat{c}^j + z \hat{c}^j \cdot \mathscr{D}_j - z \hat{c}(\nabla^{TZ}_{e_j} e_j)) - \frac{z\sqrt{t}}{2} c^j \psi_j - \frac{z}{2} E^\alpha \psi_\alpha.$$

One computes that

(3.64)
$$\begin{aligned}
&\mathscr{D}_j \cdot z \hat{c}^j + z \hat{c}^j \cdot \mathscr{D}_j - z \hat{c}(\nabla^{TZ}_{e_j} e_j) \\
&= z \left(2 \hat{c}^j \nabla^{TZ \otimes F, u}_{e_j} - \frac{1}{\sqrt{t}} \omega_{\alpha j k} E^\alpha c^j \hat{c}^k - \frac{1}{2t} \omega_{\alpha \beta j} E^\alpha E^\beta \hat{c}^j \right).
\end{aligned}$$

Comparing (3.64) with the expression in (3.41) for $X = D_1$ gives (3.63). $\square$

Remark 3.12. If h^F is covariantly-constant with respect to ∇^F and $z = 0$, then (3.58) is a special case of [B, Theorem 3.5], in which the Dirac-type operator is taken to be the vertical signature operator coupled to $\nabla^{F,u}$. If B is a point and $z = 0$, then (3.58) is equivalent to [BZ, Theorem 4.13].

(f) The flat connection on the cohomology bundle of the fibers.

Definition 3.13. Let $H(Z; F|_Z) = \bigoplus_{i=0}^{\dim(Z)} H^i(Z; F|_Z)$ be the $\mathbb{Z}$-graded vector bundle over B whose fiber over $b \in B$ is the cohomology $H(Z_b; F|_{Z_b})$ of the complex (W_b, d^{Z_b}).

As in Section 2(a), the flat superconnection d^M induces a flat connection $\nabla^{H(Z;F|_Z)}$ on $H(Z; F|_Z)$ which preserves the $\mathbb{Z}$-grading. The connection $\nabla^{H(Z;F|_Z)}$ does not depend on the choice of $T^H M$, and is the canonical flat connection on $H(Z; F|_Z)$.

Put $V = (d^Z)^* - d^Z$, an element of $C^\infty(B; \mathrm{End}(W))$. For each $b \in B$, V_b extends to a densely-defined skew-adjoint operator acting on the L^2-completion of W_b. By Hodge theory, there is an isomorphism

$$(3.65) \qquad H(Z_b; F|_{Z_b}) \simeq \mathrm{Ker}(V_b).$$

Then there is an isomorphism of smooth $\mathbb{Z}$-graded vector bundles on B:

$$(3.66) \qquad H(Z; F|_Z) \simeq \mathrm{Ker}(V).$$

As a subbundle of W, $\mathrm{Ker}(V)$ inherits a Hermitian metric from the Hermitian metric h^W of (3.29). Let $h^{H(Z;F|_Z)}$ denote the Hermitian metric on $H(Z; F|_Z)$ obtained by the isomorphism (3.66).

Let $P^{\mathrm{Ker}(V)}$ be the orthogonal projection of W onto $\mathrm{Ker}(V)$. It is a smooth family of smoothing operators along the fibers. Let $(\nabla^{H(Z;F|_Z)})^*$ be the adjoint of $\nabla^{H(Z;F|_Z)}$ with respect to the Hermitian metric $h^{H(Z;F|_Z)}$. Put

$$(3.67) \qquad \nabla^{H(Z;F|_Z),u} = \tfrac{1}{2}(\nabla^{H(Z;F|_Z)} + (\nabla^{H(Z;F|_Z)})^*),$$

a Hermitian connection on $H(Z; F|_Z)$.

Now $P^{\mathrm{Ker}(V)}\nabla^W$ and $P^{\mathrm{Ker}(V)}(\nabla^W)^*$ give connections on $\mathrm{Ker}(V)$. Using the isomorphism (3.66), they can be considered to be connections on $H(Z; F|_Z)$. Similarly, $P^{\mathrm{Ker}(V)}\omega(W, h^W)P^{\mathrm{Ker}(V)}$ can be considered to be an element of $\Omega^1(B; \mathrm{End}(H(Z; F|_Z)))$.

Proposition 3.14. *The following identities hold:*

$$\nabla^{H(Z;F|_Z)} = P^{\mathrm{Ker}(V)}\nabla^W,$$
$$(3.68) \qquad (\nabla^{H(Z;F|_Z)})^* = P^{\mathrm{Ker}(V)}(\nabla^W)^*,$$
$$\omega(H(Z; F|_Z), h^{H(Z;F|_Z)}) = P^{\mathrm{Ker}(V)}\omega(W, h^W)P^{\mathrm{Ker}(V)}.$$

Proof. The proof is the same as that of Proposition 2.6. $\square$

(g) The Chern character superconnection forms. Let E be a rank-N complex vector bundle on M, let ∇^E be a connection on E and let R^E be its curvature. If P is an ad-invariant polynomial on $gl(N; \mathbb{C})$, put

$$(3.69) \qquad P(E, \nabla^E) = P\left(\frac{-R^E}{2i\pi}\right).$$

Then $P(E, \nabla^E)$ is a closed form, whose de Rham cohomology class will be written as $P(E)$. For $A \in gl(N; \mathbb{C})$, put

$$(3.70) \qquad \mathrm{ch}(A) = \mathrm{Tr}[\exp(A)].$$

The corresponding genus is the Chern character. For N even, let $\mathrm{Pf}: so(N) \to \mathbb{R}$ denote the Pfaffian and put

$$(3.71) \qquad e(TZ, \nabla^{TZ}) = \begin{cases} \mathrm{Pf}\left[\dfrac{R^{TZ}}{2\pi}\right] & \text{if } \dim(Z) \text{ is even,} \\[2mm] 0 & \text{if } \dim(Z) \text{ is odd.} \end{cases}$$

Let $o(TZ)$ be the orientation bundle of TZ, a flat real line bundle on M. Then $e(TZ, \nabla^{TZ})$ is an $o(TZ)$-valued closed form on M which represents the Euler class $e(TZ)$ of TZ, lying in $H^{\dim(Z)}(M; o(TZ))$.

Let $\chi(Z) \in C^\infty(B)$ be the locally constant integer-valued function on B which, to $b \in B$, assigns the Euler characteristic of the fiber Z_b. If $\dim(Z)$ is odd then $\chi(Z) = 0$.

By [B, Theorem 3.4], for any $t > 0$ the form $\varphi \mathrm{Tr}_s[\exp(-C_t^2)] \in \Omega(B)$ is closed and its de Rham cohomology class coincides with the Chern character $\mathrm{ch}(H(Z; F|_Z)) \in H^*(B; \mathbb{R})$. As $H(Z; F|_Z)$ is a flat bundle on B, we have that $\mathrm{ch}(H(Z; F|_Z)) = \mathrm{rk}(F)\chi(Z)$. We now refine this statement to the level of differential forms.

Theorem 3.15. *For any* $t > 0$,

$$(3.72) \qquad \varphi \mathrm{Tr}_s[\exp(-C_t^2)] = \mathrm{rk}(F)\chi(Z).$$

Proof. Without loss of generality, assume that B is connected. We first proceed as in the proof of Proposition 1.3. By (1.6), we have

$$(3.73) \qquad C_t^2 = -D_t^2.$$

Also

$$(3.74) \qquad \begin{aligned} \frac{\partial}{\partial t} \mathrm{Tr}_s[\exp(D_t^2)] &= \mathrm{Tr}_s\left[\left[D_t, \frac{\partial D_t}{\partial t}\right] \exp(D_t^2)\right] \\ &= \mathrm{Tr}_s\left[D_t, \frac{\partial D_t}{\partial t} \exp(D_t^2)\right] = 0. \end{aligned}$$

Suppose temporarily that $\dim(Z)$ is even and TZ is spin. Let $S = S_+ \oplus S_-$ be the corresponding vector bundle of (TZ, g^{TZ}) spinors, a $\mathbb{Z}_2$-graded Hermitian vector bundle on M. Let $\nabla^S = \nabla^{S+} \oplus \nabla^{S-}$ be the Hermitian connection on $S = S_+ \oplus S_-$ induced from ∇^{TZ}. Let $S^* = S_+^* \oplus S_-^*$ be the

dual bundle to S and let $\nabla^{S^*} = \nabla^{S^*_+} \oplus \nabla^{S^*_-}$ be the dual connection on S^*. We have an isomorphism of $\mathbb{Z}_2$-graded vector bundles:

$$\Lambda(T^*Z) \simeq S \otimes S^*.$$

Thus we may apply local index theory techniques to the family of vertical Dirac operators coupled to $\nabla^{S^* \otimes F, u}$. We could work equally well with the methods of [B, Section 4] or [BGV, Chapter 10]. For concreteness, we will follow the latter approach here.

Consider a rescaling in which $\partial_j \to u^{-1/2} \partial_j$, $c^j \to u^{-1/2} E^j - u^{1/2} I^j$, $E^\alpha \to u^{-1/2} E^\alpha$ and $\hat{c}^j \to \hat{c}^j$. One finds from (3.58) that as $u \to 0$, in adapted coordinates the rescaling of $u C_4^2$ approaches

$$(3.75) \qquad\qquad -\left(\partial_j - \frac{1}{4} R_{jk}^{TZ} x^k\right)^2 + \mathscr{R}.$$

Then local index theory techniques give

$$(3.76) \qquad \lim_{t \to 0} \varphi \, \mathrm{Tr}_s[\exp(-C_t^2)] = \int_Z e(TZ, \nabla^{TZ}) \, \mathrm{ch}(F, \nabla^{F, u}).$$

As all of the arguments are local on M, (3.76) is also true if TZ is not spin, or even orientable.

From Proposition 1.3 we have

$$(3.77) \qquad \mathrm{ch}(F, \nabla^{F, u}) = \mathrm{Tr}\left[\exp\left(\frac{-R^{F, u}}{2i\pi}\right)\right] = \mathrm{rk}(F),$$

a locally constant function on M. Then

$$(3.78) \qquad \begin{aligned} \int_Z e(TZ, \nabla^{TZ}) \, \mathrm{ch}(F, \nabla^{F, u}) &= \int_Z e(TZ, \nabla^{TZ}) \, \mathrm{rk}(F) \\ &= \mathrm{rk}(F) \int_Z e(TZ, \nabla^{TZ}) = \mathrm{rk}(F) \chi(Z). \end{aligned}$$

Equation (3.72) now follows from equations (3.73), (3.74), (3.76) and (3.78).

If $\dim(Z)$ is odd, a similar argument can be worked out. In fact, by [BZ, Proposition 4.9], $c(e_1) \cdots c(e_n) \hat{c}(e_1) \cdots \hat{c}(e_n)$ is the only monomial in the $c(e_i)$'s and $\hat{c}(e_i)$'s whose supertrace on $\Lambda(T^*Z) \otimes F$ is nonzero. The implication is that the local index techniques can also be used in the odd-dimensional case. As $t \to 0$, we see that $\hat{R}$ is the only polynomial in the $\hat{c}(e_i)$'s which survives. As it is even in the $\hat{c}(e_i)$'s, it follows that

$$(3.79) \qquad\qquad \lim_{t \to \infty} \varphi \, \mathrm{Tr}_s[\exp(-C_t^2)] = 0,$$

which is consistent with (3.72). $\quad\square$

(h) **A "Riemann-Roch-Grothendieck" theorem for flat vector bundles.** We continue the extension of the formalism of Section 2 to the infinite-dimensional setting. Put

$$(3.80) \qquad\qquad f(a) = a \exp(a^2),$$

a holomorphic real odd function. The inequalities (2.37) hold for f. We have

$$(3.81) \qquad\qquad f'(a) = (1 + 2a^2)\exp(a^2).$$

By Definition 1.7,

$$f(\nabla^F, h^F) = (2i\pi)^{1/2}\varphi \operatorname{Tr}_s\left[f\left(\frac{\omega}{2}(F, h^F)\right)\right] \in \Omega(M),$$

$$(3.82) \qquad f(\nabla^{H(Z\,;\,F|_Z)}, h^{H(Z\,;\,F|_Z)})$$

$$= (2i\pi)^{1/2}\varphi \operatorname{Tr}_s\left[f\left(\frac{\omega}{2}(H(Z\,;\,F|_Z), h^{H(Z\,;\,F|_Z)})\right)\right] \in \Omega(B).$$

We obtain from (3.41) and (3.50) that for any $t > 0$, the operator $-D_t^2$ is a fiberwise-elliptic differential operator. Then $f(D_t^2)$ is a fiberwise trace class operator. Following the notation of Sections 1 and 2, put

$$(3.83) \qquad\qquad f(C_t', h^W) = (2i\pi)^{1/2}\varphi \operatorname{Tr}_s[f(D_t)] \in \Omega(B).$$

Theorem 3.16. *For any $t > 0$, the form $f(C_t', h^W)$ is real, odd and closed. Its de Rham cohomology class is independent of t, $T^H M$, g^{TZ} and h^F. As $t \to 0$,*

$$(3.84)$$

$$f(C_t', h^W) = \begin{cases} \int_Z e(TZ, \nabla^{TZ})f(\nabla^F, h^F) + \mathcal{O}(t) & \text{if } \dim(Z) \text{ is even}, \\ \mathcal{O}(\sqrt{t}) & \text{if } \dim(Z) \text{ is odd}. \end{cases}$$

As $t \to +\infty$,

$$(3.85) \qquad f(C_t', h^W) = f(\nabla^{H(Z\,;\,F|_Z)}, h^{H(Z\,;\,F|_Z)}) + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right).$$

Proof. Proceeding as in the proof of Proposition 1.3 and Theorem 1.8, it follows that $f(C_t', h^W)$ is real, odd and closed. Proceeding as in the proof of Theorem 2.11, it follows that its de Rham cohomology class is independent of t. Proceeding as in the proof of Theorem 1.11, it follows that the de Rham cohomology class is independent of $T^H M$, g^{TZ} and h^F.

Let z be an odd Grassmann variable. Given $\alpha \in \Omega(B) \otimes \mathbb{C}[z]$, we can write α in the form

$$(3.86) \qquad\qquad \alpha = \alpha_0 + z\alpha_1$$

with $\alpha_0, \alpha_1 \in \Omega(B)$. Put

$$(3.87) \qquad\qquad \alpha^z = \alpha_1.$$

As $C_t^2 = -D_t^2$, we have

$$(3.88) \qquad\qquad \operatorname{Tr}_s[f(D_t)] = \operatorname{Tr}_s[\exp(-C_t^2 + zD_t)]^z.$$

For $t > 0$, let $\psi_t \in \operatorname{End}(\Omega(B) \otimes \mathbb{C}[z])$ be such that if $\alpha \in \Omega(B) \otimes \mathbb{C}[z]$ has total degree k, then $\psi_t\alpha = t^{-k/2}\alpha$. Then

$$(3.89) \qquad \operatorname{Tr}_s[\exp(-C_t^2 + zD_t)] = \psi_t \operatorname{Tr}_s[\exp(t(-C_1^2 + zD_1))].$$

By standard results on heat kernels, we know that $\operatorname{Tr}_s[\exp(t(-C_1^2 + zD_1))]$ has an asymptotic expansion in t as $t \to 0$, which only contains integral powers of

t if $\dim(Z)$ is even, and only contains half-integral powers of t if $\dim(Z)$ is odd. From (3.88) and (3.89), we conclude that $f(C'_t, h^W)$ has an asymptotic expansion in t of the type just described.

To calculate the $t \to 0$ limit of $\text{Tr}_s[\exp(-C_{4t}^2 + zD_{4t})]$, we use the Lichnerowicz formula of Theorem 3.11. First, assume that $\dim(Z)$ is even and TZ is spin. We proceed as in the proof of Theorem 3.15. Consider a rescaling in which $\partial_j \to u^{-1/2}\partial_j$, $c^j \to u^{-1/2}E^j - u^{1/2}I^j$, $E^\alpha \to u^{-1/2}E^\alpha$, $\hat{c}^j \to \hat{c}^j$ and $z \to u^{-1/2}z$. One finds from (3.58) that as $u \to 0$, in adapted coordinates the rescaling of $u(C_4^2 - zD_4)$ approaches

$$(3.90) \qquad -(\partial_j - \tfrac{1}{4}R_{jk}^{TZ}x^k - \tfrac{1}{2}z\hat{c}^j)^2 + \mathscr{R} - \tfrac{1}{2}z\psi.$$

As $z\hat{c}^j$ commutes with the other terms in $\partial_j - \tfrac{1}{4}R_{jk}^{TZ}x^k - \tfrac{1}{2}z\hat{c}^j$, we have

(3.91)
$$e^{-t(\partial_j - \tfrac{1}{4}R_{jk}^{TZ}x^k - \tfrac{1}{2}z\hat{c}^j)^2}(0,0)$$
$$= e^{-t(\partial_j - \tfrac{1}{4}R_{jk}^{TZ}x^k)^2}(0,0) + (tz\hat{c}^j(\partial_j - \tfrac{1}{4}R_{jk}^{TZ}x^k)e^{-t(\partial_j - \tfrac{1}{4}R_{jk}^{TZ}x^k)^2})(0,0)$$
$$= e^{-t(\partial_j - \tfrac{1}{4}R_{jk}^{TZ}x^k)^2}(0,0).$$

Then local index theory techniques give

(3.92)
$$\lim_{t\to 0}(2i\pi)^{1/2}\varphi\,\text{Tr}_s[\exp(-C_t^2 + zD_t)]^z$$
$$= \int_Z e(TZ, \nabla^{TZ})\sqrt{2i\pi}\varphi\,\text{Tr}_s\left[\exp\left(-R^{F,u} + \frac{z}{2}\omega(F, h^F)\right)\right]^z.$$

Putting together (3.52), (3.82), (3.88) and (3.92), we find

$$(3.93) \qquad \lim_{t\to 0}f(C'_t, h^W) = \int_Z e(TZ, \nabla^{TZ})f(\nabla^F, h^F).$$

As all of the arguments are local on M, (3.93) is true in full generality when $\dim(Z)$ is even. The first identity in (3.84) follows.

When $\dim(Z)$ is odd, the same arguments as in the proof of Theorem 3.15 give that

$$(3.94) \qquad \lim_{t\to 0}f(C'_t, h^W) = 0.$$

The second identity in (3.84) follows.

To establish (3.85), we proceed as in the proof of Theorem 2.13. From Propositions 3.4 and 3.7 and Definition 3.8, we have
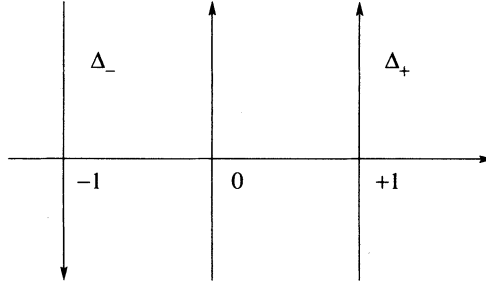
$$(3.95) \qquad D_t = \frac{\sqrt{t}}{2}((d^Z)^* - d^Z) + \frac{1}{2}\omega(W, h^W) - \frac{\hat{c}(T)}{2\sqrt{t}}.$$

Put

(3.96)
$$V = (d^Z)^* - d^Z,$$
$$F_t = \frac{1}{2}\omega(W, h^W) - \frac{\hat{c}(T)}{2\sqrt{t}}.$$

As in the proof of Theorem 2.13, we see that the spectra of D_t and $\frac{\sqrt{t}}{2}V$ coincide.

Let $\Delta = \Delta_+ \cup \Delta_-$ be the contour in $\mathbb{C}$:

Then

$$(3.97) \qquad f(D_t) = \frac{1}{2\pi i} \int_\Delta \frac{f(\lambda)}{\lambda - D_t} \, d\lambda.$$

To study the asymptotics of (3.96) as $t \to +\infty$, we may proceed formally as in the proof of Theorem 2.13. A similar problem was considered in [BGV, Theorem 9.23] and [BerB, Section 5]. We find that as $t \to +\infty$,

$$(3.98) \qquad \begin{aligned} &\sqrt{2i\pi}\, \varphi \, \mathrm{Tr_s}[(D_t)] \\ &= \sqrt{2i\pi}\, \varphi \, \mathrm{Tr_s}\left[P^{\mathrm{Ker}(V)} f\left(\frac{\omega}{2}(W, h^W)\right) P^{\mathrm{Ker}(V)} \right] + \mathscr{O}\left(\frac{1}{\sqrt{t}}\right). \end{aligned}$$

Equation (3.85) now follows from Proposition 3.14 and (3.98). □

For k a positive odd integer, recall the definition of $c_k(F) \in H^k(M; \mathbb{R})$ from Section 1(g). Similarly, define $c_k(H(Z; F|_Z)) \in H^k(B; \mathbb{R})$.

Theorem 3.17. *For any positive odd integer k,*

$$(3.99) \qquad c_k(H(Z; F|_Z)) = \int_Z e(TZ) \cdot c_k(F) \quad \text{in } H^k(B; \mathbb{R}).$$

In particular, if $\dim(Z)$ is odd or if F admits a Hermitian metric which is covariantly-constant with respect to ∇^F then

$$(3.100) \qquad c_k(H(Z; F|_Z)) = 0.$$

Proof. From Theorem 3.16, we have that as cohomology classes,

$$(3.101) \qquad f(\nabla^{H(Z;F|_Z)}, h^{\nabla^{H(Z;F|_Z)}}) = \int_Z e(TZ, \nabla^{TZ}) f(\nabla^F, h^F).$$

We have

$$(3.102) \qquad \begin{aligned} f(\nabla^F, h^F) &= \sum_{i=1}^{+\infty} \frac{c_{2i+1}(F, h^F)}{i!}, \\ f(\nabla^{H(Z;F|_Z)}, h^{H(Z;F|_Z)}) &= \sum_{i=1}^{+\infty} \frac{c_{2i+1}(H(Z; F|_Z), h^{H(Z;F|_Z)})}{i!}. \end{aligned}$$

Equation (3.99) follows from matching terms of equal degree on the two sides of (3.101).

If $\dim(Z)$ is odd then $e(TZ) = 0$. If h^F is covariantly-constant with respect to ∇^F then $c_k(F) = 0$. In either case, (3.100) follows from (3.99). $\square$

Remark 3.18. As mentioned in the Introduction, Theorem 3.17 is a C^∞-analog of the Riemann-Roch-Grothendieck theorem for holomorphic submersions.

Remark 3.19. The composition of multiplication by the vertical Euler class and integration over the fiber is the Becker-Gottlieb transfer [BG] in rational cohomology.

(i) A transgression formula. Recall that N is the number operator of W. In keeping with the notation of Section 2(c), for $t > 0$ put

$$(3.103) \qquad f^\wedge(C_t', h^W) = \varphi \operatorname{Tr}_s \left[\frac{N}{2} f'(D_t) \right].$$

From (3.81), we have

$$(3.104) \qquad f^\wedge(C_t', h^W) = \varphi \operatorname{Tr}_s \left[\frac{N}{2} (1 + 2D_t^2) \exp(D_t^2) \right].$$

Theorem 3.20. *For any* $t > 0$, *the form* $f^\wedge(C_t', h^W)$ *is real and even. Moreover,*

$$(3.105) \qquad \frac{\partial}{\partial t} f(C_t', h^W) = \frac{1}{t} df^\wedge(C_t', h^W).$$

Proof. The proof is formally the same as that of Theorem 2.9. See also Remark 2.12 for a simple direct proof. $\square$

Put

$$(3.106) \qquad \chi'(Z; F) = \sum_{i=0}^{\dim(Z)} (-1)^i i \operatorname{rk}(H^i(Z; F|_Z)),$$

an integer-valued locally constant function on B.

Theorem 3.21. *As* $t \to 0$,

$$(3.107) \quad f^\wedge(C_t', h^W) = \begin{cases} \frac{1}{4} \dim(Z) \operatorname{rk}(F) \chi(Z) + \mathcal{O}(t) & \text{if } \dim(Z) \text{ is even,} \\ \mathcal{O}(\sqrt{t}) & \text{if } \dim(Z) \text{ is odd.} \end{cases}$$

As $t \to +\infty$,

$$(3.108) \qquad f^\wedge(C_t', h^W) = \frac{\chi'(Z; F)}{2} + \mathcal{O}\left(\frac{1}{\sqrt{t}} \right).$$

Proof. Put $\widehat{M} = M \times \mathbb{R}_+^*$ and $\widehat{B} = B \times \mathbb{R}_+^*$. Define $\hat{\pi} : \widehat{M} \to \widehat{B}$ by $\hat{\pi}(x, s) = (\pi(x), s)$. Let ρ be the projection $\widehat{M} \to M$ and let ρ' be the projection $\widehat{M} \to \mathbb{R}_+^*$.

Let $\widehat{Z}$ be the fiber of $\hat{\pi}$. Then $T\widehat{Z} = \rho^* TZ$. Let $g^{T\widehat{Z}}$ be the metric on $T\widehat{Z}$ which restricts to $\rho^* g^{TZ}/s$ on $M \times \{s\}$. Put

$$(3.109) \qquad T^H \widehat{M} = \rho^* T^H M \oplus \rho'^* T\mathbb{R}_+^*.$$

One can show that $\nabla^{T\widehat{Z}} = \rho^*\nabla^{TZ} + ds(\frac{\partial}{\partial s} - \frac{1}{2s})$ and $R^{T\widehat{Z}} = \rho^*R^{TZ}$. In particular, $R^{T\widehat{Z}}(\frac{\partial}{\partial s}, \cdot) = 0$. Clearly $(\rho^*F, \rho^*\nabla^F)$ is a flat vector bundle on $\widehat{M}$.

Using the product structure on $\widehat{M}$, we can write

(3.110)
$$d^{\widehat{M}} = d^M + ds\,\partial_s,$$
$$(d^{\widehat{M}})^* = s^{-(N-\dim(Z)/2)}(d^M + ds\,\partial_s)s^{N-\dim(Z)/2}.$$

(The $\dim(Z)$-term arises from the dependence of the volume form on s .) Then

(3.111)
$$(d^{\widehat{M}})^* = (d^M)_s^* + ds\left(\partial_s + \frac{1}{s}\left(N - \frac{\dim(Z)}{2}\right)\right).$$

Following (3.39), put

(3.112)
$$\widehat{X} = \frac{1}{2}((d^{\widehat{M}})^* - d^{\widehat{M}}) = X_s + \frac{ds}{2s}\left(N - \frac{\dim(Z)}{2}\right).$$

Defining $\widehat{D}_t$ as in (3.50), we have

(3.113)
$$\widehat{D}_t = s^{-N/2}D_{st}s^{N/2} + \frac{ds}{2s}\left(N - \frac{\dim(Z)}{2}\right).$$

We deduce that

(3.114)
$$f(\widehat{C}_t', h^{\widehat{W}}) = f(C_{st}, h^W) + \frac{ds}{s}f^\wedge(C_{st}, h^W)$$
$$- \frac{\dim(Z)}{4s}ds\,\varphi\,\mathrm{Tr}_s[f'(D_{st})].$$

As in the proof of Theorem 3.15, the fact that $f'(a)$ is an even function implies that $\varphi\,\mathrm{Tr}_s[f'(D_t)]$ is independent of t , and the method of proof of (3.85) shows that it equals $\mathrm{rk}(F)\chi(Z)f'(0)$. Thus

(3.115)
$$f(\widehat{C}_t', h^{\widehat{W}}) = f(C_{st}, h^W) + \frac{ds}{s}f^\wedge(C_{st}, h^W)$$
$$- ds\frac{\dim(Z)}{4s}\mathrm{rk}(F)\chi(Z)f'(0).$$

Equation (3.84) gives the $t \to 0$ asymptotics of the left-hand side of (3.115). In particular, using the fact that $R^{T\widehat{Z}}(\frac{\partial}{\partial s}, \cdot) = 0$ we see that the $t \to 0$ limit of the left-hand side of (3.115) has no ds term. As $f'(0) = 1$, equation (3.107) follows from (3.84) and (3.115).

Let $\sigma: \widehat{B} \to B$ be projection onto the first factor. Letting N now be the number operator of $H(Z; F|_Z)$, we have

$$H(\widehat{Z}; \rho^*F|_{\widehat{Z}}) = \sigma^*H(Z; F|_Z),$$

(3.116)
$$\omega(H(\widehat{Z}; \rho^*F|_{\widehat{Z}}), h^{H(\widehat{Z}; \rho^*F|_{\widehat{Z}})})$$
$$= \omega(H(Z; F|_Z), h^{H(Z;F|_Z)}) + \frac{ds}{s}\left(N - \frac{\dim(Z)}{2}\right).$$

Thus
(3.117)
$$f(\nabla^{H(\hat{Z};\rho^*F|_{\hat{Z}})}, h^{H(\hat{Z};\rho^*F|_{\hat{Z}})})$$

$$= f(\nabla^{H(Z;F|_Z)}, h^{H(Z;F|_Z)}) + \frac{ds}{2s}\left(\chi'(Z;F) - \frac{\dim(Z)}{2}\mathrm{rk}(F)\chi(Z)\right)f'(0).$$

Equation (3.85) gives the $t \to +\infty$ asymptotics of the left-hand side of (3.115). In particular, the $t \to +\infty$ limit equals the right-hand side of (3.117). Comparing the ds-terms, equation (3.108) follows. □

(j) **Higher analytic torsion forms.** We now construct analytic torsion forms along the lines of Definition 2.20.

Definition 3.22. The analytic torsion form $\mathcal{T}(T^H M, g^{TZ}, h^F) \in \Omega(B)$ is given by
(3.118)
$$\mathcal{T}(T^H M, g^{TZ}, h^F)$$

$$= -\int_0^{+\infty}\left[f^\wedge(C_t', h^W) - \frac{\chi'(Z;F)}{2}f'(0)\right.$$

$$\left. - \left(\frac{\dim(Z)\,\mathrm{rk}(F)\chi(Z)}{4} - \frac{\chi'(Z;F)}{2}\right)f'\left(\frac{i\sqrt{t}}{2}\right)\right]\frac{dt}{t}.$$

It follows from Theorem 3.21 that the integrand of (3.118) is integrable on $[0, \infty]$.

Theorem 3.23. *The form* $\mathcal{T}(T^H M, g^{TZ}, h^F)$ *is even and real. Moreover,*
$$d\mathcal{T}(T^H M, g^{TZ}, h^F)$$
(3.119)
$$= \int_Z e(TZ, \nabla^{TZ})f(\nabla^F, h^F) - f(\nabla^{H(Z;F|_Z)}, h^{H(Z;F|_Z)}).$$

Proof. This follows from Theorems 3.16 and 3.20. □

We now describe how $\mathcal{T}(T^H M, g^{TZ}, h^F)$ depends on its arguments. Let $(T^H M, g^{TZ}, h^F)$ and $(T'^H M, g'^{TZ}, h'^F)$ be two triples. We will mark the objects associated to the second triple with a $'$.

Let $\hat{e}(TZ, \nabla^{TZ}, \nabla'^{TZ}) \in Q^M/Q^{M,0}$ be the secondary class associated to the Euler class. Its representatives are forms of degree $\dim(Z) - 1$ such that
(3.120) $$d\hat{e}(TZ, \nabla^{TZ}, \nabla'^{TZ}) = e(TZ, \nabla'^{TZ}) - e(TZ, \nabla^{TZ}).$$

If $\dim(Z)$ is odd, we take $\hat{e}(TZ, \nabla^{TZ}, \nabla'^{TZ})$ to be zero.

Theorem 3.24. *The following identity holds in* $Q^B/Q^{B,0}$:
$$\mathcal{T}(T'^H M, g'^{TZ}, h'^F) - \mathcal{T}(T^H M, g^{TZ}, h^F)$$

$$= \int_Z \hat{e}(TZ, \nabla^{TZ}, \nabla'^{TZ})f(\nabla^F, h^F)$$
(3.121)
$$+ \int_Z e(TZ, \nabla'^{TZ})\tilde{f}(\nabla^F, h^F, h'^F)$$

$$- \tilde{f}(\nabla^{H(Z;F|_Z)}, h^{H(Z;F|_Z)}, h'^{H(Z;F|_Z)}).$$

Proof. First, a horizontal distribution on M is simply a splitting of the exact sequence

$$0 \to TZ \to TM \to \pi^* TB \to 0.$$

As the space of splitting maps is affine, it follows that any pair of horizontal distributions can be connected by a smooth path of horizontal distributions. Let $s \in [0, 1]$ parametrize a smooth path $\{T_s^H M\}_{s \in [0,1]}$ such that $T_0^H M = T^H M$ and $T_1^H M = T'^H M$. Similarly, let g_s^{TZ} and h_s^F be metrics on TZ and F, depending smoothly on $s \in [0, 1]$, which coincide with g^{TZ} and h^F at $s = 0$, and with g'^{TZ} and h'^F at $s = 1$. Let $\tilde{\pi} \colon M \times [0, 1] \to B \times [0, 1]$ be the obvious projection, with fiber $\tilde{Z}$. Let $\tilde{F}$ be the lift of F to $M \times [0, 1]$.

Now $T^H(M \times [0, 1])_{(\cdot, s)} = T_s^H M \times \mathbb{R}$ defines a horizontal subbundle $T^H(M \times [0, 1])$ of $T(M \times [0, 1])$, and $T\tilde{Z}$ and $\tilde{F}$ are naturally equipped with metrics $g^{T\tilde{Z}}$ and $h^{\tilde{F}}$. By Theorem 3.23,

$$(3.122) \qquad \begin{aligned} &d\mathscr{F}(T^H(M \times [0, 1]), g^{T\tilde{Z}}, h^{\tilde{F}}) \\ &= \int_{\tilde{Z}} e(T\tilde{Z}, \nabla^{T\tilde{Z}}) f(\nabla^{\tilde{F}}, h^{\tilde{F}}) - f(\nabla^{H(\tilde{Z}; \tilde{F}|_{\tilde{Z}})}, h^{H(\tilde{Z}; \tilde{F}|_{\tilde{Z}})}). \end{aligned}$$

Let $\tilde{\sigma} \colon B \times [0, 1] \to B$ be projection onto the first factor. Then there is an equality of pairs

$$(3.123) \qquad (H(\tilde{Z}; \tilde{F}|_{\tilde{Z}}), \nabla^{H(\tilde{Z}; \tilde{F}|_{\tilde{Z}})}) = \tilde{\sigma}^*(H(Z; F|_Z), \nabla^{H(Z; F|_Z)}).$$

The restriction of $\mathscr{F}(T^H(M \times [0, 1]), g^{T\tilde{Z}}, h^{\tilde{F}})$ to $B \times \{0\}$ (resp. $B \times \{1\}$) coincides with $\mathscr{F}(T^H M, g^{TZ}, h^F)$ (resp. $\mathscr{F}(T'^H M, g'^{TZ}, h'^F)$). Comparing the ds-terms of the two sides of equation (3.122) and integrating with respect to s yields equation (3.121). $\square$

Corollary 3.25. *If* $\dim(Z)$ *is odd or if* h^F *is covariantly-constant with respect to* ∇^F *then the class of* $\mathscr{F}(T^H M, g^{TZ}, h^F)$ *in* $Q^B / Q^{B,0}$ *is independent of* $T^H M$.

If $\dim(Z)$ *is odd and* $H(Z; F|_Z) = 0$ *then* $\mathscr{F}(T^H M, g^{TZ}, h^F)$ *is a closed form whose de Rham cohomology class is independent of* $T^H M$, g^{TZ}, *and* h^F.

Proof. This follows from Theorems 3.23 and 3.24. $\square$

Theorem 3.26. *If* h^F *is covariantly-constant with respect to* ∇^F, $\dim(Z)$ *is even and* TZ *is orientable then* $\mathscr{F}(T^H M, g^{TZ}, h^F) = 0$.

Proof. Define a form $(\cdot, \cdot)$ on W by

$$(3.124) \qquad (s_1, s_2)(b) = \int_{Z_b} \langle s_1(b) \wedge \overline{s_2(b)} \rangle_{h^F}$$

for all $s_1, s_2 \in C^\infty(B; W) \simeq C^\infty(M; \Lambda(T^* Z) \otimes F)$. Let $*$ be the fiberwise Hodge duality operator associated to g^{TZ}. Then one can check that $(W, (\cdot, \cdot), *)$ is a d^M-compatible duality structure in the sense of Definition 2.33. The proof is now formally the same as that of Theorem 2.29. $\square$

Remark 3.27. More generally, if $\dim(Z)$ is even, h^F is arbitrary, and $h^{\overline{F}^*}$ is the induced Hermitian metric on the antidual bundle $\overline{F}^*$ then

$$(3.125) \qquad \mathcal{T}(T^H M, g^{TZ}, h^F) = -\mathcal{T}(T^H M, g^{TZ}, h^{\overline{F}^*}).$$

Proposition 3.28. *For $i \in \{1, 2\}$, let $Z_i \to M_i \overset{\pi_i}{\to} B$ be a fiber bundle over B with connected closed fibers, a horizonal distribution $T^H M_i$ and a vertical Riemannian metric g^{TZ_i}. Let F_i be a flat complex vector bundle on M_i, with Hermitian metric h^{F_i}. Put $Z = Z_1 \times Z_2$. Let $Z \to M \overset{\pi}{\to} B$ be the product fiber bundle, with the product horizontal distribution $T^H M$ and the product vertical Riemannian metric g^{TZ}. If $p_i : M \to M_i$ denotes the projection map, put $F = p_1^*(F_1) \otimes p_2^*(F_2)$, with the Hermitian metric $h^F = p_1^*(h^{F_1}) \otimes p_2^*(h^{F_2})$. Then*

$$(3.126) \qquad \begin{aligned} \mathcal{T}(T^H M, g^{TZ}, h^F) &= \mathrm{rk}(F_1)\chi(Z_1)\mathcal{T}(T^H M_2, g^{TZ_2}, h^{F_2}) \\ &\quad + \mathrm{rk}(F_2)\chi(Z_2)\mathcal{T}(T^H M_1, g^{TZ_1}, h^{F_1}). \end{aligned}$$

Proof. We have

$$(3.127) \qquad \begin{aligned} \Omega(M ; F) &= \Omega(M_1 ; F_1) \otimes_{\Omega(B)} \Omega(M_2 ; F_2), \\ N &= (N_1 \otimes I) + (I \otimes N_2), \\ D_t^2 &= (D_{1,t}^2 \otimes I) + (I \otimes D_{2,t}^2). \end{aligned}$$

As $(1 + 2(a + b))e^{a+b} = (1 + 2a)e^a \cdot (1 + 2b)e^b - 4ae^a \cdot be^b$, we obtain

$$(3.128) \qquad \begin{aligned} \mathrm{Tr}_s\left[\frac{N}{2}f'(D_t)\right] &= \mathrm{Tr}_s[f'(D_{1,t})]\,\mathrm{Tr}_s\left[\frac{N_2}{2}f'(D_{2,t})\right] \\ &\quad + \mathrm{Tr}_s\left[\frac{N_1}{2}f'(D_{1,t})\right]\mathrm{Tr}_s[f'(D_{2,t})] \\ &\quad - 4\,\mathrm{Tr}_s[f(D_{1,t}^2)]\,\mathrm{Tr}_s\left[\frac{N_2}{2}f(D_{2,t}^2)\right] \\ &\quad - 4\,\mathrm{Tr}_s\left[\frac{N_1}{2}f(D_{1,t}^2)\right]\mathrm{Tr}_s[f(D_{2,t}^2)]. \end{aligned}$$

The method of proof of Theorem 3.15 gives that for $i \in \{1, 2\}$,

$$(3.129) \qquad \begin{aligned} \mathrm{Tr}_s[f(D_{i,t})] &= 0, \\ \mathrm{Tr}_s[f'(D_{i,t})] &= \mathrm{rk}(F_i)\chi(Z_i). \end{aligned}$$

The theorem now follows from Definition 3.22. □

(k) Relationship to the Ray-Singer torsion and the anomaly formula of Bismut-Zhang. We now consider the 0-form component $\mathcal{T}^{[0]}(T^H M, g^{TZ}, h^F) \in C^\infty(B)$ of the form $\mathcal{T}(T^H M, g^{TZ}, h^F)$. Recall that $V = (d^Z)^* - d^Z$. Let $(V^2)'$ denote the quotient action of V^2 on $W/\mathrm{Ker}(V)$. Clearly $-(V^2)'$ is a positive operator. For $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > \frac{\dim(Z)}{2}$, put

$$(3.130) \qquad \theta(s) = -\mathrm{Tr}_s[N(-(V^2)')^{-s}].$$

Then $\theta(s)$ extends to a meromorphic function of $s \in \mathbb{C}$ which is holomorphic near $s = 0$ [Se]. By definition, the Ray-Singer analytic torsion of the de Rham complex (W, d^Z) is $\frac{\partial \theta}{\partial s}(0)$ [RS1].

Theorem 3.29. *We have*

$$(3.131) \qquad \mathscr{T}^{[0]}(T^H M, g^{TZ}, h^F) = \frac{1}{2}\frac{\partial \theta}{\partial s}(0).$$

Proof. From [BZ, Theorem 7.10], we have that as $t \to 0$,

$$(3.132)$$

$$\mathrm{Tr}_s[N \exp(tV^2)] = \begin{cases} \dfrac{\dim(Z)\,\mathrm{rk}(F)\chi(Z)}{2} + \mathscr{O}(t) & \text{if } \dim(Z) \text{ is even}, \\[2mm] \dfrac{c}{\sqrt{t}} + \mathscr{O}(\sqrt{t}) & \text{if } \dim(Z) \text{ is odd}, \end{cases}$$

where we will not need the exact value of the constant c. Put $g(a) = (1+2a)e^a$. We abbreviate $\frac{\dim(Z)\,\mathrm{rk}(F)\chi(Z)}{2} - \chi'(Z; F)$ by c'. Then

$$(3.133) \qquad \begin{aligned} \theta(s) &= -\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{Tr}_s[Ne^{t(V^2)'}]\,dt \\[2mm] &= -\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}(\mathrm{Tr}_s[Ne^{t(V^2)}] - \chi'(Z; F))\,dt = -c' + sA(s), \end{aligned}$$

where the function $A(s)$ is holomorphic around $s = 0$. It follows that

$$(3.134) \qquad \frac{1}{2}\frac{\partial \theta}{\partial s}(0) = \frac{1}{2}A(0).$$

On the other hand, from Definition 3.22,

$$(3.135)$$

$$\begin{aligned} &\mathscr{T}^{[0]}(T^H M, g^{TZ}, h^F) \\[2mm] &= -\int_0^{+\infty}\left(\mathrm{Tr}_s\left[\frac{N}{2}g\left(\frac{tV^2}{4}\right)\right] - \frac{\chi'(Z; F)}{2}g(0) - \frac{c'}{2}g\left(-\frac{t}{4}\right)\right)\frac{dt}{t} \\[2mm] &= -\int_0^{+\infty}\left(\mathrm{Tr}_s\left[\frac{N}{2}g(tV^2)\right] - \frac{\chi'(Z; F)}{2}g(0) - \frac{c'}{2}g(-t)\right)\frac{dt}{t} \\[2mm] &= -\int_0^{+\infty}\left(\mathrm{Tr}_s\left[\frac{N}{2}g(t(V^2)')\right] - \frac{c'}{2}g(-t)\right)\frac{dt}{t} \\[2mm] &= -\int_0^{+\infty}\left(\mathrm{Tr}_s\left[\frac{N}{2}e^{t(V^2)'}\right] + t\frac{d}{dt}\mathrm{Tr}_s[Ne^{t(V^2)'}] - \frac{c'}{2}g(-t)\right)\frac{dt}{t} \\[2mm] &= -\frac{d}{ds}\bigg|_{s=0}\frac{1}{\Gamma(s)}\int_0^{+\infty}t^{s-1}\left(\mathrm{Tr}_s\left[\frac{N}{2}e^{t(V^2)'}\right] + t\frac{d}{dt}\mathrm{Tr}_s[Ne^{t(V^2)'}] - \frac{c'}{2}g(-t)\right)dt \\[2mm] &= \left(\frac{1}{2}\frac{d}{ds}\bigg|_{s=0}(1 - 2s)\theta(s)\right) - c' \\[2mm] &= \frac{1}{2}A(0). \end{aligned}$$

The theorem follows. $\quad\square$

As in [BZ, Section 2a)], if λ is a complex line, let λ^{-1} denote the dual line. If E is a complex vector space, put

$$\det E = \Lambda^{\max}(E).$$

Put

$$(3.136) \qquad \det H(Z; F|_Z) = \bigotimes_{i=0}^{\dim(Z)} (\det H^i(Z; F|_Z))^{(-1)^i},$$

a complex line bundle on B.

Identifying $H(Z; F|_Z)$ with $\mathrm{Ker}(V)$, the line bundle $\det H(Z; F|_Z)$ inherits a metric $|\ |_{\det H(Z; F|_Z)}$ which is induced from the restriction of the L^2-metric (3.29) to $\mathrm{Ker}(V)$.

Definition 3.30 [BZ]. The Ray-Singer metric $\|\ \|^{RS}_{\det H(Z; F|_Z)}$ on $\det H(Z; F|_Z)$ is given by

$$(3.137) \qquad \|\ \|^{RS}_{\det H(Z; F|_Z)} = |\ |_{\det H(Z; F|_Z)} \exp\left\{\frac{1}{2}\frac{\partial\theta}{\partial s}(0)\right\}.$$

The Ray-Singer metric is constructed in analogy to the Quillen metric [Q2, BGS3] for holomorphic vector bundles on complex manifolds. One can check that the degree-1 part of (3.119) is equivalent to a statement about $\det H(Z; F|_Z)$, namely
(3.138)

$$\{f(\det H(Z; F|_Z)), \|\ \|^{RS}_{\det H(Z; F|_Z)})\}^{[1]} = \left\{\int_Z e(TZ, \nabla^{TZ})f(\nabla^F, h^F)\right\}^{[1]}.$$

Similarly, if (g^{TZ}, h^F) and (g'^{TZ}, h'^F) are pairs of metrics on TZ and F then the degree-0 part of (3.121) is equivalent to

$$(3.139) \qquad \begin{aligned} &\log\left(\frac{\|\ \|'^{RS}_{\det H(Z; F|_Z)}}{\|\ \|^{RS}_{\det H(Z; F|_Z)}}\right) \\ &= \left\{\int_Z \tilde{f}(\nabla^F, h^F, h'^F)e(TZ, \nabla^{TZ}) \right. \\ &\qquad \left. - \int_Z f(\nabla^F, h'^F)\hat{e}(TZ, \nabla^{TZ}, \nabla'^{TZ})\right\}^{[0]}, \end{aligned}$$

which in turn is equivalent to the anomaly formula of [BZ, Theorem 4.7]. In fact, by considering a fiber bundle over $\mathbb{R}$ with fiber Z, one sees that (3.138) and (3.139) are equivalent.

IV. COMPACT STRUCTURE GROUPS

In this section we use equivariant methods to compute the analytic torsion form of a fiber bundle with compact structure group.

The section is organized as follows. Given a fiber bundle $M = P \times_G Z$, with G compact, in (a) we give a way to construct flat complex vector bundles on M by equivariant means. In (b) we give the equivariant versions of the operators of Section 3. In (c) we compute the analytic torsion form of $P \times_G Z$ and in (d) we give explicit formulas when M is a circle bundle and F is a flat complex line bundle.

(a) **Equivariant flat vector bundles.** Basic information on fiber bundles with compact structure group can be found in [Be, Chapter 9]. With the assumptions of Section 3, suppose that the holonomy group of the bundle $Z \to M \xrightarrow{\pi} B$ is a compact Lie group G. Then there is a principal G-bundle P and an action of G on Z such that $M = P \times_G Z = (P \times Z)/\sim$, where the equivalence relation $\sim$ is given by $(p, z) \sim (p \cdot \gamma^{-1}, \gamma \cdot z)$ for all $p \in P$, $z \in Z$, and $\gamma \in G$. Given a connection Θ on P, we obtain a horizontal distribution $T^H M$ on M. We can assume that Z has a G-invariant Riemannian metric g^Z. This gives a vertical Riemannian metric g^{TZ} on M. The fibers of M are then totally geodesic, and so the only relevant geometric tensor of the fiber bundle is the curvature tensor.

We now give an equivariant construction of flat complex vector bundles on M. That is, we wish to extend flat vector bundles on Z equivariantly to flat vector bundles on M. It is clear that this cannot be done in complete generality, as not every flat vector bundle on Z extends topologically to a flat vector bundle on M. (For example, if $Z = S^1$, $B = S^2$ and $M = S^3$ then only a trivial flat vector bundle on Z will extend to a flat vector bundle on M.) Hence our construction will produce a certain class of flat vector bundles F_Z on Z, namely those of Lemma 4.3, which do extend equivariantly to flat vector bundles F on M.

Let Γ be the fundamental group of Z and let $\pi_Z : \widetilde{Z} \to Z$ be the projection map from the universal cover of Z to Z. Define a group G' by

$$(4.1) \qquad G' = \{(\gamma, \phi) \in G \times \text{Diff}(\widetilde{Z}) : \pi_Z \circ \phi = \gamma \cdot \pi_Z\}.$$

There are homomorphisms $\alpha : G' \to G$ and $\beta : G' \to \text{Diff}(\widetilde{Z})$ given by $\alpha((\gamma, \phi)) = \gamma$ and $\beta((\gamma, \phi)) = \phi$, and by definition $\pi_Z \circ \beta(\gamma') = \alpha(\gamma') \cdot \pi_Z$ for all $\gamma' \in G'$. Define an equivalence relation $\approx$ on $P \times \widetilde{Z}$ by

$$(4.2) \qquad (p, \tilde{z}) \approx (p \cdot (\alpha(\gamma'))^{-1}, \beta(\gamma') \cdot \tilde{z})$$

for all $p \in P$, $\tilde{z} \in \widetilde{Z}$, and $\gamma' \in G'$. Put $P \times_{G'} \widetilde{Z} = (P \times \widetilde{Z})/\approx$.

Lemma 4.1. *There is a diffeomorphism between* $P \times_{G'} \widetilde{Z}$ *and* M.

Proof. Define $\mathscr{E} : P \times \widetilde{Z} \to P \times Z$ by $\mathscr{E}(p, \tilde{z}) = (p, \pi_Z(\tilde{z}))$. Then

$$(4.3) \qquad \begin{aligned} \mathscr{E}(p \cdot (\alpha(\gamma'))^{-1}, \beta(\gamma') \cdot \tilde{z}) &= (p \cdot (\alpha(\gamma'))^{-1}, \pi_Z(\beta(\gamma') \cdot \tilde{z})) \\ &= (p \cdot (\alpha(\gamma'))^{-1}, \alpha(\gamma') \cdot \pi_Z(\tilde{z})) \sim \mathscr{E}(p, \tilde{z}). \end{aligned}$$

Thus $\mathscr{E}$ descends to a map $\mathscr{D} : P \times_{G'} \widetilde{Z} \to M$. Clearly $\mathscr{D}$ is onto. To show that $\mathscr{D}$ is one-to-one, it is enough to show that if $\mathscr{E}(p, \tilde{z}) \sim \mathscr{E}(p', \tilde{z}')$ then

$(p, \tilde{z}) \approx (p', \tilde{z}')$. So suppose that $\mathcal{E}(p, \tilde{z}) \sim \mathcal{E}(p', \tilde{z}')$. Then there exists a $\gamma \in G$ such that $(p \cdot \gamma^{-1}, \gamma \cdot \pi(\tilde{z})) = (p', \pi(\tilde{z}'))$. By the lifting theorem, there exists a $\phi \in \mathrm{Diff}(\tilde{Z})$ such that $\pi_Z \circ \phi = \gamma \cdot \pi_Z$ and $\phi(\tilde{z}) = \tilde{z}'$. Putting $\gamma' = (\gamma, \phi)$, it follows that $(p, \tilde{z}) \approx (p', \tilde{z}')$. Thus $\mathscr{D}$ is a bijection. One can check that it is a diffeomorphism. $\square$

There is an exact sequence

$$(4.4) \qquad\qquad 1 \to \Gamma \to G' \to G \to 1$$

and a corresponding homotopy exact sequence

$$(4.5) \qquad\qquad \cdots \to \pi_1(G) \to \Gamma \to \pi_0(G') \to \pi_0(G) \to 0.$$

Let $\rho: \pi_0(G') \to GL(N, \mathbb{C})$ be a representation of $\pi_0(G')$ and let

$$(4.6) \qquad\qquad \rho_{G'}: G' \to \pi_0(G') \xrightarrow{\rho} GL(N, \mathbb{C})$$

be the corresponding representation of G'.

Definition 4.2. The vector bundle F on $M = P \times_{G'} \tilde{Z}$ is given by $F = (P \times \tilde{Z} \times \mathbb{C}^N)/G'$, where $\gamma' \in G'$ acts on $(p, \tilde{z}, v) \in (P \times \tilde{Z} \times \mathbb{C}^N)$ by

$$(4.7) \qquad \gamma' \cdot (p, \tilde{z}, v) = (p \cdot (\alpha(\gamma'))^{-1}, \beta(\gamma') \cdot \tilde{z}, \rho_{G'}(\gamma') \cdot v).$$

Lemma 4.3. *The restriction F_Z of F to a fiber Z is isomorphic to the flat bundle on Z given by the representation $\Gamma \to \pi_0(G') \xrightarrow{\rho} GL(N, C)$.*

Proof. The projection $\mathscr{P}: F \to B$ is the map from $(P \times \tilde{Z} \times \mathbb{C}^N)/G'$ to P/G induced by $(p, \tilde{z}, v) \to p$. Then for $p \in P$, the preimage $\mathscr{P}^{-1}(pG)$ of $pG \in B$ is isomorphic to $(pG \times \tilde{Z} \times \mathbb{C}^N)/G' \simeq (\tilde{Z} \times \mathbb{C}^N)/\Gamma$. $\square$

Definition 4.4. With the notation of Definition 4.2, we say that G acts on (Z, F_Z), and write $F = P \times_G F_Z$ and $(M, F) = P \times_G (Z, F_Z)$.

Let d^{Z, F_Z} be the exterior derivative on $\Omega(Z; F_Z)$. Let g be the Lie algebra of G. Given $x \in g$, let X be the corresponding vector field on Z, let $c(X) = (X\wedge) - i_X$ and $\hat{c}(X) = (X\wedge) + i_X$ be the Clifford multiplications on $\Omega(Z; F_Z)$, and let L_X be Lie differentiation in the X-direction on $\Omega(Z; F_Z)$.

Proposition 4.5. *The vector bundle F is flat.*

Proof. For any $x \in g$, its moment $\mu^{F_Z}(x) \in C^\infty(Z; \mathrm{End}(F_Z))$, relative to the flat connection on F_Z, is given by $\mu^{F_Z}(x) = L_X - d^{Z, F_Z} i_X - i_X d^{Z, F_Z}$ [BGV, Definition 7.5]. As the representation $\Gamma \to \pi_0(G') \xrightarrow{\rho} GL(N, C)$ factors through $\pi_0(G')$, it follows that the moment vanishes. As the curvature of F_Z also vanishes, the proposition follows from [BGV, Lemma 7.37.2]. $\square$

(b) **Equivariant computational methods.** We use the notation and results of [BGV, Chapter 7 and Section 10.7]. Let $\{x_a\}$ be a basis of the Lie algebra

g. Let $\Theta = \Theta^a x_a$ and $\Omega = \Omega^a x_a$ be the connection and curvature forms on P, respectively. The space $(\Omega(P) \otimes \Omega(Z\,;\,F_Z))_{\text{basic}}$ of basic forms is given by
(4.8)
$$(\Omega(P) \otimes \Omega(Z\,;\,F_Z))_{\text{basic}} = \{\tau \in \Omega(P) \otimes \Omega(Z\,;\,F_Z):$$
$$\tau \text{ is } G\text{-invariant and for all } x \in g, \; i_X \tau = 0\},$$

where the interior multiplication acts only on the first factor in $\Omega(P) \otimes \Omega(Z\,;\,F_Z)$.

There is an identification between $\Omega(B\,;\,W) \simeq \Omega(M\,;\,F)$ and $(\Omega(P) \otimes \Omega(Z\,;\,F_Z))_{\text{basic}}$. The operators d^Z, ∇^W, and i_T of Section 3(b), when acting on $\Omega(B\,;\,W)$, arise as the restrictions (to the basic subspace) of operators on $\Omega(P) \otimes \Omega(Z\,;\,F_Z)$ given by

$$(d^Z)'(\tau_1 \otimes \tau_2) = (-1)^{\deg \tau_1} \tau_1 \otimes d^{Z,F_Z} \tau_2,$$

(4.9) $\qquad (\nabla^W)'(\tau_1 \otimes \tau_2) = d^P \tau_1 \otimes \tau_2 + (\Theta^a \wedge \tau_1) \otimes L_{X_a} \tau_2,$

$$(i_T)'(\tau_1 \otimes \tau_2) = -(\Omega^a \wedge \tau_1) \otimes i_{X_a} \tau_2.$$

Suppose that the vector bundle $\widetilde{F}_Z = \widetilde{Z} \times \mathbb{C}^N$ has a G'-invariant Hermitian metric $h^{\widetilde{F}_Z}$. This passes to a Hermitian metric h^{F_Z} on F_Z, and a Hermitian metric h^F on F. Define $\omega(F_Z,\,h^{F_Z}) \in \Omega^1(Z, \text{End}(F_Z))$ as in (1.31).

Lemma 4.6. *The form* $\omega(F,\,h^F) \in \Omega^1(M\,;\,\text{End}(F))$ *is the pushdown to* M *of the basic form* $1 \otimes \omega(F_Z,\,h^{F_Z}) \in \Omega^0(P) \otimes \Omega^1(Z,\,F_Z)$.

Proof. Define $\hat{h}^{F_Z}$ as in (1.15). Then $\hat{h}^F$ is induced from the operator $I \otimes \hat{h}^{F_Z}$ on $\Omega(P) \otimes \Omega(Z,\,F_Z)$. As h^{F_Z} is G-invariant, $\hat{h}^{F_Z}$ commutes with L_X. Now as

(4.10) $\qquad\qquad \omega(F,\,h^F) = (\hat{h}^F)^{-1} d^M \hat{h}^F - d^M,$

the lemma follows from Proposition 3.4 and (4.9). $\quad\square$

Let $\mathbb{C}[g]$ denote the space of complex-valued polynomial functions on the Lie algebra g. Recall that the space of G-equivariant differential forms on Z is defined to be

(4.11) $\qquad\qquad \Omega_G(Z) = (\mathbb{C}[g] \otimes \Omega(Z))^G,$

the G-invariant elements of $\mathbb{C}[g] \otimes \Omega(Z)$. If $\mathscr{B}$ is a G-vector bundle on Z then the space of G-equivariant $\mathscr{B}$-valued differential forms on Z is defined to be

(4.12) $\qquad\qquad \Omega_G(Z\,;\,\mathscr{B}) = (\mathbb{C}[g] \otimes \Omega(Z\,;\,\mathscr{B}))^G.$

Writing an element $\sigma \in \Omega_G(Z)$ as $\sigma(x)$, $x \in g$, the equivariant Chern-Weil homomorphism $\Phi: \Omega_G(Z) \to (\Omega(P) \otimes \Omega(Z))_{\text{basic}}$ essentially consists of replacing x by the curvature form Ω.

Proposition 4.7. *As forms on* B,

(4.13) $\qquad \displaystyle\int_Z e(TZ, \nabla^{TZ}) f(\nabla^F,\,h^F) = f(\nabla^{H(Z;F|_Z)},\,h^{H(Z;F|_Z)}) = 0.$

Proof. The result of the equivariant computation must be a polynomial in the curvature form Ω. However, any such polynomial is an even form, whereas $\int_Z e(TZ, \nabla^{TZ}) f(\nabla^F, h^F)$ and $f(\nabla^{H(Z;F|_Z)}, h^{H(Z;F|_Z)})$ are odd forms. $\square$

Let $(d^{Z,F_Z})^*$ be the formal adjoint to d^{Z,F_Z}. Then the operators $(d^Z)^*$, $(\nabla^W)^*$, and $T\wedge$ of Proposition 3.7, acting on $\Omega(B;W)$, are the restrictions of operators on $\Omega(P) \otimes \Omega(Z, F_Z)$ given by

$$(4.14) \qquad \begin{aligned} (d^Z)^{*'}(\tau_1 \otimes \tau_2) &= (-1)^{\deg \tau_1} \tau_1 \otimes (d^{Z,F_Z})^* \tau_2, \\ (\nabla^W)^{*'}(\tau_1 \otimes \tau_2) &= (\nabla^W)'(\tau_1 \otimes \tau_2), \\ (T\wedge)'(\tau_1 \otimes \tau_2) &= -(\Omega^a \wedge \tau_1) \otimes (X_a \wedge \tau_2). \end{aligned}$$

Thus the operators C_t and D_t of (3.50) are the restrictions of operators $\widehat{C}_t$ and $\widehat{D}_t$ on $\Omega(P) \otimes \Omega(Z, F_Z)$ which, when acting on $\Omega(Z, F_Z)$, can be written as

$$(4.15) \qquad \begin{aligned} \widehat{C}_t &= \frac{\sqrt{t}}{2}[(d^{Z,F_Z})^* + d^{Z,F_Z}] + (\nabla^W)' + \frac{1}{2\sqrt{t}} c(\Omega), \\ \widehat{D}_t &= \frac{\sqrt{t}}{2}[(d^{Z,F_Z})^* - d^{Z,F_Z}] + \frac{1}{2\sqrt{t}} \hat{c}(\Omega). \end{aligned}$$

(c) Analytic torsion form of a fiber bundle with compact structure group. We will compute the supertraces of operators on $C^\infty(B;W)$ by equivariant means. We first discuss the formalism in the finite-dimensional case. Let G be a compact Lie group and let P be a principal G-bundle. Let $\mathscr{N}$ be a finite-dimensional $\mathbb{Z}_2$-graded G-module. Let W be the vector bundle $P \times_G \mathscr{N}$. Let Q be an element of $\Omega_G(\text{pt.}; \text{End}(\mathscr{N})) = (\mathbb{C}[g] \otimes \text{End}(\mathscr{N}))^G$. Let us write Q as $Q(x)$, with $x \in g$. There is a corresponding element Q' of $\Omega(B; \text{End}(W))$ which is represented by the basic form $Q(\Omega) \in \Omega(P) \otimes \text{End}(\mathscr{N})$.

We can compute $\text{Tr}_s[Q'] \in \Omega(B)$ as follows. There is a supertrace

$$(4.16) \qquad \text{Tr}_s : \Omega_G(\text{pt.}; \text{End}(\mathscr{N})) \to \Omega_G(\text{pt.}),$$

where $\Omega_G(\text{pt.}) = \mathbb{C}[g]^G$. Composing with the Chern-Weil homomorphism $\Phi : \Omega_G(\text{pt.}) \to \Omega(B)$, one has

$$(4.17) \qquad \text{Tr}_s[Q'] = \Phi(\text{Tr}_s[Q]).$$

In the infinite-dimensional case of interest to us, $\mathscr{N} = \Omega(Z; F_Z)$. Our supertraces will now be formal power series in x, as opposed to polynomials in x. For $x \in g$, put

$$(4.18) \qquad \widehat{D}_t(x) = \frac{\sqrt{t}}{2}[(d^{Z,F_Z})^* - d^{Z,F_Z}] + \frac{1}{2\sqrt{t}} \hat{c}\left(\frac{X}{2i\pi}\right).$$

Proposition 4.8. *For all $t > 0$, the form $f(C'_t, h^W) \in \Omega(B)$ vanishes.*

Proof. The same argument as in the proof of Proposition 4.7 applies. $\square$

Proposition 4.9. *For all* $t > 0$, *the form* $f^\wedge(C_t', h^W) \in \Omega(B)$ *is a linear combination of characteristic forms of the principal* G-*bundle* P, *with coefficients that only depend on* (Z, F_Z, g^Z, h^{F_Z}) *and the* G-*action thereon.*

Proof. Define $Q \in \Omega_G(\mathrm{pt.}; \mathrm{End}(\mathcal{N}))$ by

$$(4.19) \qquad Q(x) = \frac{N}{2} f'(\widehat{D}_t(x)).$$

Then $f^\wedge(C_t', h^W) = \mathrm{Tr}_s Q'$. The above discussion of how to compute supertraces equivariantly extends to the infinite-dimensional setting. The computation of the supertrace $\mathrm{Tr}_s[Q] \in \Omega_G(\mathrm{pt.})$ only involves (Z, F_Z, g^Z, h^{F_Z}) and the G-action thereon. As $f^\wedge(C_t', h^W) = \Phi(\mathrm{Tr}_s[Q])$ lies in the image of the Chern-Weil homomorphism, the proposition follows. $\square$

If $(M, F) = P \times_G (Z, F_Z)$ has the horizontal distribution $T^H M$ induced from the connection Θ on P, the vertical Riemannian metric g^{TZ} induced from the Riemannian metric g^Z on Z, and the Hermitian metric h^F induced from the Hermitian metric h^{F_Z} on F_Z, we will denote the analytic torsion form $\mathscr{T}(T^H M, g^{TZ}, h^F) \in \Omega(B)$ by $\mathscr{T}(P \times_G (Z, F_Z))$.

Corollary 4.10. *The analytic torsion form* $\mathscr{T}(P \times_G (Z, F_Z)) \in \Omega(B)$ *is a linear combination of characteristic forms of the principal* G-*bundle* P, *with coefficients that only depend on* (Z, F_Z, g^Z, h^{F_Z}) *and the* G-*action thereon.*

Proof. This follows from Definition 3.22 and Proposition 4.8. $\square$

Corollary 4.11. *Let* $Z \to M \xrightarrow{\pi} B$ *be a fiber bundle with compact structure group* G. *Let* F *be a flat bundle on* M *which is constructed as in Definition 4.2. Suppose that* $\dim(Z)$ *is odd and* $H(Z; F_Z) = 0$. *Let* $(T^H M, g^{TZ}, h^F)$ *be a triplet consisting of a horizontal distribution on* M, *vertical Riemannian metric on* M *and Hermitian metric on* F. *Then the de Rham cohomology class of* $\mathscr{T}(T^H M, g^{TZ}, h^F)$ *is a linear combination of characteristic classes of the principal* G-*bundle* P, *with coefficients that only depend on the topological type of* (Z, F_Z) *and the* G-*action thereon.*

Proof. This follows from Corollaries 3.25 and 4.10. $\square$

(d) **Circle bundles.** Let P be a principal $U(1)$-bundle with connection Θ and curvature Ω. Take Z to be a circle of length 2π, upon which $U(1)$ acts by an r-fold covering, $|r| > 1$. (The case $r = 0$ can be easily handled separately.) Put $M = P \times_{U(1)} S^1$.

Lemma 4.12. *Defining* G' *as in* (4.1), *we have* $G' = (\mathbb{Z}/r\mathbb{Z}) \times \mathbb{R}$.

Proof. Letting $\mathbb{R}$ act on $\widetilde{Z} = \mathbb{R}$ by translation, we have

$$(4.20) \qquad G' = \{(w, t) \in U(1) \times \mathbb{R}: w^r = e^{it}\}.$$

There is an isomorphism $i: G' \to (\mathbb{Z}/r\mathbb{Z}) \times \mathbb{R}$ given by $i(w, t) = (we^{-it/r}, t)$. $\square$

The sequences (4.4) and (4.5) become

$$(4.21) \qquad 1 \to \mathbb{Z} \to (\mathbb{Z}/r\mathbb{Z}) \times \mathbb{R} \to U(1) \to 1$$

and

(4.22) $\cdots \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/r\mathbb{Z} \to 0 \to 0.$

Let $\zeta \neq 1$ be an rth root of unity and let $\rho: \mathbb{Z}/r\mathbb{Z} \to \mathbb{C}^*$ be the corresponding representation of $\mathbb{Z}/r\mathbb{Z}$. Then F_{S^1} is the flat complex line bundle on the circle with holonomy group generated by ζ. As $\zeta \neq 1$, we have that $H(S^1; F_{S^1}) = 0$. Give F_{S^1} the Hermitian metric induced from the standard Hermitian metric on $\widetilde{Z} \times \mathbb{C}$.

Proposition 4.13. *The analytic torsion form* $\mathcal{T}(P \times_{U(1)} (S^1, F_{S^1})) \in \Omega(B)$ *is the pushdown to* B *of the basic form on* P *given by*

$$(4.23) \qquad \mathcal{T} = \frac{1}{2} \sum_{j=0}^{\infty} \frac{(2j+1)!}{2^{2j}(j!)^2} \left(\sum_{m \neq 0} \frac{1}{m^j |m|} \zeta^m \right) \left(\frac{r\Omega}{4\pi^2} \right)^j.$$

Proof. Write $\zeta = e^{2i\pi b}$ with $rb \in \mathbb{Z}$, $b \notin \mathbb{Z}$. Defining $\widehat{D}_t(x)$ as in (4.18), we first compute the eigenvalues of $\widehat{D}_t^2(x)$, acting on $\Omega^1(S^1, F_{S^1})$. Let $l(s)\,ds \in \Omega^1(\widetilde{Z}) \otimes \mathbb{C}$ be a Γ-invariant 1-form on $\mathbb{R}$, i.e., $l(s+2\pi) = e^{2i\pi b}l(s)$. Using the fact that $iy \in u(1)$ is represented by the vector field $ry\partial_s$ on $\mathbb{R}$, we obtain

$$(4.24) \qquad \widehat{D}_t^2(iy)(l(s)\,ds) = \left(\left(-\frac{\sqrt{t}}{2}\partial_s + \frac{1}{2\sqrt{t}}\frac{ry}{2i\pi} \right)^2 l(s) \right) ds.$$

Then a basis of eigenvectors for $\widehat{D}_t^2(iy)$ is given by $\{e^{i(n+b)s}ds\}_{n=-\infty}^{\infty}$, with eigenvalues $\{-\frac{t}{4}(n+b+\frac{riy}{2i\pi t})^2\}_{n=-\infty}^{\infty}$. Putting $g(a) = (1+2a)e^a$, we have

$$(4.25) \qquad f^\wedge(C_t', h^W) = -\frac{1}{2}\Phi\left(\sum_{n=-\infty}^{\infty} g\left(-\frac{t}{4}\left(n+b+\frac{riy}{2i\pi t} \right)^2 \right) \right),$$

where Φ is the Chern-Weil homomorphism which replaces iy by Ω.

By the Poisson summation formula,

$$(4.26) \qquad \sum_{n=-\infty}^{\infty} e^{-\frac{t}{4}(n+b)^2} = \sqrt{\frac{4\pi}{t}} \sum_{m=-\infty}^{\infty} e^{-4\pi^2 m^2/t} e^{2i\pi mb}.$$

Acting on both sides of (4.26) with $1 + 2t\frac{d}{dt}$ gives

$$(4.27) \qquad \sum_{n=-\infty}^{\infty} g\left(-\frac{t}{4}(n+b)^2 \right) = 16\pi^{5/2}t^{-3/2} \sum_{m \neq 0} m^2 e^{-4\pi^2 m^2/t} e^{2i\pi mb}.$$

Formally replacing the b of (4.27) by $b + \frac{riy}{2i\pi t}$ and using the fact that $\chi(S^1) = \chi'(S^1; F_{S^1}) = 0$, equation (3.118) becomes

$$\mathcal{T} = \frac{1}{2}\Phi\left(\int_0^\infty 16\pi^{5/2}t^{-3/2}\sum_{m\neq0}m^2 e^{-(4\pi^2m^2-mriy)/t}e^{2i\pi mb}\frac{dt}{t}\right)$$

$$= \frac{1}{2}\Phi\left(\int_0^\infty 16\pi^{5/2}u^{3/2}\sum_{m\neq0}m^2 e^{-(4\pi^2m^2-mriy)u}e^{2i\pi mb}\frac{du}{u}\right)$$

$$= \frac{1}{2}\Phi\left(16\pi^{5/2}\Gamma\left(\frac{3}{2}\right)\sum_{m\neq0}m^2(4\pi^2m^2-mriy)^{-3/2}e^{2i\pi mb}\right)$$

(4.28)

$$= \frac{1}{2}\Phi\left(\sum_{m\neq0}\frac{1}{|m|}\left(1-\frac{riy}{4\pi^2m}\right)^{-3/2}e^{2i\pi mb}\right)$$

$$= \frac{1}{2}\Phi\left(\sum_{j=0}^\infty\sum_{m\neq0}\frac{1}{|m|}\frac{(2j+1)!}{2^{2j}(j!)^2}\left(\frac{riy}{4\pi^2m}\right)^j\zeta^m\right)$$

$$= \frac{1}{2}\sum_{j=0}^\infty\frac{(2j+1)!}{2^{2j}(j!)^2}\left(\sum_{m\neq0}\frac{1}{m^j|m|}\zeta^m\right)\left(\frac{r\Omega}{4\pi^2}\right)^j. \quad\square$$

Let $c_1(M) \in H^2(B;\mathbb{Z})$ be the first Chern class of a circle bundle M. Letting $\beta: H^2(B;\mathbb{Z}) \to H^2(B;\mathbb{R})$ be the change-of-coefficients map, $\beta(c_1(M))$ is represented by the basic form $-\frac{r\Omega}{2i\pi}$ on P. For $\zeta \in \mathbb{C}$ such that $|\zeta| \leq 1$ and $\zeta \neq 1$, let $\mathrm{Li}_j(\zeta)$ be the polylogarithm function

(4.29)
$$\mathrm{Li}_j(\zeta) = \sum_{m=1}^\infty\frac{\zeta^m}{m^j}.$$

Corollary 4.14. *Let* M' *be an oriented circle bundle over a connected base* B. *Let* r *be an integer greater than one and let* $\rho: \mathbb{Z}/r\mathbb{Z} \to \mathrm{Aut}(\mathbb{C})$ *be a nontrivial representation which is generated by an* r*th root of unity* ζ. *Give* M' *the structure of a principal* $U(1)$*-bundle and let* M *be the quotient of* M' *by the action of the* $\mathbb{Z}/r\mathbb{Z}$ *subgroup of* $U(1)$. *Let* F *be the flat complex line bundle* $M' \times_\rho \mathbb{C}$ *over* M. *Let* (T^HM, g^{TS^1}, h^F) *be a triplet consisting of a horizontal distribution on* M, *a vertical Riemannian metric on* TS^1 *and a Hermitian metric on* F. *Then* $\mathcal{T}(T^HM, g^{TS^1}, h^F)$ *is a closed form on* B *and its de Rham cohomology class* $[\mathcal{T}(T^HM, g^{TS^1}, h^F)] \in H^*(B;\mathbb{R})$ *is given by*

$$[\mathcal{T}(T^HM, g^{TS^1}, h^F)]$$

(4.30)
$$= \sum_{j\text{ even}}(-1)^{j/2}(2\pi)^{-j}\frac{(2j+1)!}{2^{2j}(j!)^2}\mathrm{Re}(\mathrm{Li}_{j+1}(\zeta))(\beta(c_1(M)))^j$$

$$+ \sum_{j\text{ odd}}(-1)^{(j-1)/2}(2\pi)^{-j}\frac{(2j+1)!}{2^{2j}(j!)^2}\mathrm{Im}(\mathrm{Li}_{j+1}(\zeta))(\beta(c_1(M)))^j.$$

Proof. This follows from Corollary 3.25 and Proposition 4.13. $\square$

APPENDIX I

(a) **An axiomatic characterization of the torsion forms in the acyclic case.** We make the same assumptions as in Section 2(f). In addition, we assume that the complex (E, v) is acyclic, i.e.,

$$(A1.1) \qquad\qquad\qquad H(E, v) = 0.$$

We will say that the flat Hermitian complex (E, A', h^E) splits if there exist flat Hermitian vector bundles $(F^i, \nabla^{F^i}, h^{F^i})$ such that (E, v) is the complex

$$(A1.2) \quad 0 \to F^0 \to F^0 \oplus F^1 \to F^1 \oplus F^2 \to \cdots \to F^{n-2} \oplus F^{n-1} \to F^{n-1} \to 0,$$

and moreover, for $1 \le i \le n-1$, $E^i = F^{i-1} \oplus F^i$ is equipped with the metric $h^{E^i} = h^{F^{i-1}} \oplus h^{F^i}$.

Theorem A1.1. (a) *The following identity holds*:

$$(A1.3) \qquad\qquad\qquad dT_f(A', h^E) = f(\nabla^E, h^E).$$

   (b) *If B' is a smooth manifold and $\alpha\colon B' \to B$ is a smooth map, then*

$$(A1.4) \qquad\qquad T_f(\alpha^* A', \alpha^* h^E) = \alpha^* T_f(A', h^E).$$

   (c) *If (E, A', h^E) splits, then $T_f(A', h^E) = 0$.*

   (d) *$T_f(A', h^E)$ depends smoothly on A' and h^E.*

Proof. Equation (A1.3) follows from Theorem 2.22, and equation (A1.4) follows from Remark 2.23. If E splits then we have

$$(A1.5) \quad [\omega(E, h^E), v] = 0, \qquad [\omega(E, h^E), v^*] = 0, \qquad vv^* + v^*v = 1.$$

From (2.21), we have

$$(A1.6) \qquad\qquad\qquad X_t = \tfrac{1}{2}(\omega(E, h^E) + tv^* - v).$$

From (A1.5) and (A1.6) we obtain

$$(A1.7) \qquad\qquad\qquad X_t^2 = \tfrac{1}{4}(\omega^2(E, h^E) - t).$$

As $f'(a)$ is an even function, there is a holomorphic function $g(a)$ such that $f'(a) = g(a^2)$. Then

$$(A1.8)
\begin{aligned}
f^\wedge(A', h_t^E) &= \varphi \operatorname{Tr}_s\left[\frac{N}{2} g(X_t^2)\right] \\
&= \sum_{i=0}^n (-1)^i i\varphi \operatorname{Tr}\left[\frac{1}{2} g\left(\frac{1}{4}\omega^2(E^i, h^{E^i}) - \frac{t}{4}\right)\right].
\end{aligned}$$

Recalling the definition of $d(E)$ from (2.67), Proposition 1.3 and (A1.8) imply

$$(A1.9) \qquad f^\wedge(A', h_t^E) = d(E)\frac{g(-\frac{t}{4})}{2} = d(E)\frac{f'(\frac{i\sqrt{t}}{2})}{2}.$$

Part (c) of the theorem now follows from putting (A1.9) into Definition 2.20.

Finally, $T_f(A', h^E)$ depends smoothly on A' and h^E by construction, giving part (d) of the theorem. □

Now we imitate [BGS1, Theorem 1.29], where an axiomatic characterization of Bott-Chern classes was given.

Theorem A1.2. *Given a manifold B, let $T'_f(A', h^E)$ be a real even form on B verifying the conditions* (a)–(d) *of Theorem* A1.1. *Then*

$$(A1.10) \qquad T'_f(A', h^E) = T_f(A', h^E) \quad in \ Q^B/Q^{B,0}.$$

Proof. Suppose first that $n = 2$. In this case, the complex (E, v) can be written in the form

$$0 \to L \underset{v}{\to} M \underset{v}{\to} M/L \to 0,$$

with v being the obvious injection or projection map. Let ∇^L, ∇^M, and $\nabla^{M/L}$ be the flat connections on L, M and M/L, respectively.

As smooth vector bundles, we have an isomorphism

$$(A1.11) \qquad\qquad M = L \oplus M/L,$$

and L is a flat subbundle of M. Let $^0\nabla^M$ be the sum connection $^0\nabla^M = \nabla^L \oplus \nabla^{M/L}$. Put

$$(A1.12) \qquad \alpha = \nabla^M - {}^0\nabla^M \in \Omega^1(B; \mathrm{Hom}(M/L, L)).$$

In matrix form, we can write

$$(A1.13) \qquad \nabla^M = \begin{pmatrix} \nabla^L & \alpha \\ 0 & \nabla^{M/L} \end{pmatrix}, \qquad \nabla^E = \begin{pmatrix} \nabla^L & 0 & 0 & 0 \\ 0 & \nabla^L & \alpha & 0 \\ 0 & 0 & \nabla^{M/L} & 0 \\ 0 & 0 & 0 & \nabla^{M/L} \end{pmatrix}.$$

The flatness of ∇^M implies

$$(A1.14) \qquad\qquad \nabla^L\alpha + \alpha\nabla^{M/L} = 0,$$

i.e., α is covariantly-constant.

The idea of the proof will be to assume first that M has a direct sum Hermitian metric, and then effectively deform α linearly to zero, keeping track of how T'_f changes during the deformation process. When α vanishes, we are in the split situation. Using the hypotheses on T'_f, we will derive an explicit formula for T'_f, which will prove the theorem. One must only be careful to check that all of the equations make sense as one approaches the split situation.

More precisely, let h^L and $h^{M/L}$ be Hermitian metrics on L and M/L, respectively. In what follows, the parameter s will always lie in $(0, 1]$. Let h_s^M be the Hermitian metric on M given by

$$(A1.15) \qquad\qquad h_s^M = h^L \oplus \frac{h^{M/L}}{s}.$$

Let $h_s^E = \bigoplus_{i=0}^2 h_s^{E^i}$ be the Hermitian metric on E which coincides with h^L on $E^0 = L$, with h_s^M on $E^1 = M$ and with $\frac{h^{M/L}}{s}$ on $E^2 = M/L$.

By assumption, T_f' satisfies conditions (a) and (b) of Theorem A1.1, and proceeding as in the proof of Theorem 2.17, we find

$$(A1.16) \qquad T_f'(A', h_1^E) - T_f'(A', h_s^E) = \tilde{f}(\nabla^E, h_s^E, h_1^E) \quad \text{in } Q^B/Q^{B,0}.$$

Let $p \in \text{End}(M)$ be such that $p = 0$ on L and $p = I_{M/L}$ on M/L. Let $q \in \text{End}(E)$ be such that $q = 0$ on $E^0 = L$, $q = p$ on $E^1 = M$, and $q = I_{M/L}$ on $E^2 = M/L$. In matrix form,

$$(A1.17) \qquad p = \begin{pmatrix} 0 & 0 \\ 0 & I_{M/L} \end{pmatrix}, \qquad q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{M/L} & 0 \\ 0 & 0 & 0 & I_{M/L} \end{pmatrix}.$$

Let ∇^{L*} (resp. $\nabla^{M/L*}$) be the adjoint of ∇^L (resp. $\nabla^{M/L}$) with respect to h^L (resp. $h^{M/L}$). Then ${}^0\nabla^{M*} = \nabla^{L*} \oplus \nabla^{M/L*}$ is the adjoint of ${}^0\nabla^M$ with respect to h_s^M for any $s \in (0, 1]$. Let ∇_s^{M*} (resp. ∇_s^{E*}) be the adjoint of ∇^M (resp. ∇^E) with respect to h_s^M (resp. h_s^E). Let α^* be the adjoint of α with respect to h_1^M. Then one can check that

$$(A1.18) \qquad \begin{aligned} \nabla_s^{M*} &= {}^0\nabla^{M*} + s\alpha^*, \\ \nabla_s^{E*} &= \nabla^{L*} \oplus \nabla_s^{M*} \oplus \nabla^{M/L*}. \end{aligned}$$

Put

$$(A1.19) \qquad \begin{aligned} X^L &= \tfrac{1}{2}(\nabla^{L*} - \nabla^L), \\ X^{M/L} &= \tfrac{1}{2}(\nabla^{M/L*} - \nabla^{M/L}), \\ X' &= \tfrac{1}{2}({}^0\nabla^{M*} - {}^0\nabla^M), \\ X_s &= \tfrac{1}{2}(\nabla_s^{E*} - \nabla^E). \end{aligned}$$

Using (A1.12) and (A1.18), we have

$$(A1.20) \qquad X' = \begin{pmatrix} X^L & 0 \\ 0 & X^{M/L} \end{pmatrix}, \qquad X_s = \begin{pmatrix} X^L & 0 & 0 & 0 \\ 0 & X^L & -\frac{\alpha}{2} & 0 \\ 0 & \frac{s\alpha^*}{2} & X^{M/L} & 0 \\ 0 & 0 & 0 & X^{M/L} \end{pmatrix}.$$

By Definition 1.12, we have

$$(A1.21) \qquad \tilde{f}(\nabla^E, h_s^E, h_1^E) = -\int_s^1 \varphi \, \text{Tr}_s \left[\frac{q}{2} f'(X_u) \right] \frac{du}{u} \quad \text{in } Q^B/Q^{B,0}.$$

From Proposition 1.3, (A1.17), and (A1.20), we obtain

$$(A1.22) \quad \text{Tr}_s \left[\frac{q}{2} f'(X_s) \right] = -\text{Tr} \left[\frac{p}{2} f' \left(X' + \frac{1}{2}(s\alpha^* - \alpha) \right) \right] + \text{rk}(M/L) \frac{f'(0)}{2}.$$

We have the identity

$$(A1.23) \qquad s^{-p/2}\left(X' + \frac{1}{2}(s\alpha^* - \alpha)\right)s^{p/2} = X' + \frac{\sqrt{s}}{2}(\alpha^* - \alpha).$$

Using (A1.23), equation (A1.22) becomes

$$(A1.24) \quad \mathrm{Tr}_s\left[\frac{q}{2}f'(X_s)\right] = -\mathrm{Tr}\left[\frac{p}{2}f'\left(X' + \frac{\sqrt{s}}{2}(\alpha^* - \alpha)\right)\right] + \mathrm{rk}(M/L)\frac{f'(0)}{2}.$$

Then as $s \to 0$,

$$(A1.25) \qquad \mathrm{Tr}_s\left[\frac{q}{2}f'(X_s)\right] = \mathscr{O}(s).$$

Thus as $s \to 0$,

$$(A1.26) \qquad -\int_s^1 \varphi\,\mathrm{Tr}_s\left[\frac{q}{2}f'(X_u)\right]\frac{du}{u} \to -\int_0^1 \varphi\,\mathrm{Tr}_s\left[\frac{q}{2}f'(X_u)\right]\frac{du}{u}.$$

One computes that

$$(A1.27) \qquad s^{-q/2}A's^{q/2} = v + (\nabla^L \oplus (^0\nabla^M + \sqrt{s}\alpha) \oplus \nabla^{M/L}).$$

Put

$$(A1.28) \qquad A'_0 = v + (\nabla^L \oplus {}^0\nabla^M \oplus \nabla^{M/L}).$$

From (A1.27), we see that as $s \to 0$,

$$(A1.29) \qquad s^{-q/2}A's^{q/2} = A'_0 + \mathscr{O}(\sqrt{s}).$$

Tautologically,

$$(A1.30) \qquad T'_f(A', h_s^E) = T'_f(s^{-q/2}A's^{q/2}, h^E).$$

As T'_f satisfies condition (d) of Theorem A1.1, equations (A1.29) and (A1.30) give that as $s \to 0$,

$$(A1.31) \qquad T'_f(A', h_s^E) \to T'_f(A'_0, h^E).$$

Clearly (E, A'_0, h^E) splits, and so by condition (c) of Theorem A1.1,

$$(A1.32) \qquad T'_f(A'_0, h^E) = 0.$$

Taking $s \to 0$ in (A1.16) and using (A1.21), (A1.26), (A1.31) and (A1.32), we obtain

$$(A1.33) \qquad T'_f(A', h_1^E) = -\int_0^1 \varphi\,\mathrm{Tr}_s\left[\frac{q}{2}f'(X_u)\right]\frac{du}{u} \quad \text{in } Q^B/Q^{B,0}.$$

The class of $T'_f(A', h_1^E)$ in $Q^B/Q^{B,0}$ is uniquely determined by (A1.33). Equation (A1.10) follows when $h^E = h_1^E$. If h^E is an arbitrary metric on the $\mathbb{Z}$-graded vector bundle E, then by using conditions (a), (b) and (c) and proceeding as in the proof of Theorem 2.17, we get the validity of (A1.10) in full generality.

We have now proved the theorem in the case $n = 2$. The proof for arbitrary n follows along similar lines, and is left to the reader. $\square$

Put

$$\det E^i = \Lambda^{\max}(E^i), \qquad 0 \le i \le n,$$

(A1.34)

$$\det E = \bigotimes_{i=0}^{n}(\det(E^i))^{(-1)^i}.$$

As (E, v) is acyclic, $\det E$ has a canonical nonzero section $T(v)$ [BGS1, Section 1a)]. To construct $T(v)$, take $s_0 \in \det E^0$, $s_0 \ne 0$. Take $s_1 \in \Lambda^{\dim E^1 - \dim E^0} E^1$ such that $v s_0 \wedge s_1 \ne 0$ in $\Lambda^{\max} E^1$, $s_2 \in \Lambda^{\dim E^2 - \dim E^1 + \dim E^0} E^2$ such that $v s_1 \wedge s_2 \ne 0$ in $\Lambda^{\max} E^2$, etc. Put

(A1.35) $$T(v) = s_0 \otimes (v s_0 \wedge s_1)^{-1} \otimes (v s_1 \wedge s_2) \otimes \cdots \otimes (v s_{n-1} \wedge s_n)^{(-1)^n}.$$

Then $T(v)$ is independent of the choices of the s_i's. Moreover, $T(v)$ is a flat section of $\det E$.

Let h^E be a Hermitian metric on the $\mathbb{Z}$-graded vector bundle E. Let $\| \ \|_{\det E}^{h^E}$ be the induced metric on $\det E$. Let $(T_f(A', h^E))^{[0]}$ be the degree-0 component of $T_f(A', h^E)$.

Theorem A1.3. *The following identity holds:*

(A1.36) $$(T_f(A', h^E))^{[0]} = f'(0) \operatorname{Log}(\|T(v)\|_{\det E}^{h^E}).$$

Proof. From (A1.3), we have

(A1.37) $$d(T_f(A', h^E)^{[0]}) = \frac{f'(0)}{2} \operatorname{Tr}_s[\omega(E, h^E)].$$

As $T(v)$ is a flat section of $\det E$,

(A1.38) $$d(\operatorname{Log}(\|T(v)\|_{\det E}^{h^E})) = \tfrac{1}{2} \operatorname{Tr}_s[\omega(E, h^E)].$$

Moreover, one easily verifies that if (E, A', h^E) splits then

(A1.39) $$\operatorname{Log}(\|T(v)\|_{\det E}^{h^E}) = 0.$$

The theorem now follows from Theorem A1.2 and (A1.37)–(A1.39). $\square$

(b) **Analytic torsion forms of double complexes.** We make the same assumptions as in Section 2(f).

Let $(E_i^j, v, v')_{0 \le i \le n, 0 \le j \le n'}$ be a flat double complex of complex vector bundles on B. Then v maps $E_\bullet^\bullet$ into $E_\bullet^{\bullet+1}$ and v' maps $E_\bullet^\bullet$ into $E_{\bullet+1}^\bullet$. Moreover $v + v'$ is also a chain map, i.e., $(v + v')^2 = 0$.

Put $E = \bigoplus_{0 \le i \le n, 0 \le j \le n'} E_i^j$. Let $\nabla^E = \bigoplus_{0 \le i \le n, 0 \le j \le n'} \nabla^{E_i^j}$ be the flat connection on E. Put

(A1.40)
$$\begin{aligned}
A'_\bullet &= \nabla^E + v, \\
A'^\bullet &= \nabla^E + v', \\
A'^\bullet_\bullet &= \nabla^E + v + v'.
\end{aligned}$$

Then $A'_\bullet$ (resp. $A'^\bullet$) induces corresponding flat superconnections A'_i (resp. A'^j) on $E_i = \bigoplus_{0 \le j \le n'} E_i^j$ (resp. $E^j = \bigoplus_{0 \le i \le n} E_i^j$). Let $h^E = \bigoplus_{0 \le i \le n, 0 \le j \le n'} h^{E_i^j}$ be a Hermitian metric on E.

We assume that the $(E_i^\bullet, v)$ complexes and the $(E_\bullet^j, v')$ complexes are acyclic. Then $(E_\bullet^\bullet, v + v')$ is also acyclic. Put

(A1.41) $$f(E_\bullet^\bullet, h^E) = \sum_{\substack{0 \le i \le n \\ 0 \le j \le n'}} (-1)^{i+j} f(\nabla^{E_i^j}, h^{E_i^j}).$$

Then there is a form $T_f(A'^\bullet_\bullet, h^E)$ associated to the flat Hermitian complex $(E, v + v', h^E)$ which satisfies

(A1.42) $$dT_f(A'^\bullet_\bullet, h^E) = f(\nabla^E, h^E).$$

For $0 \le i \le n$ (resp. $0 \le j \le n'$), put $h^{E_i} = \bigoplus_{0 \le j \le n'} h^{E_i^j}$ (resp. $h^{E^j} = \bigoplus_{0 \le i \le n} h^{E_i^j}$). As before, we can construct the forms $T_f(A'_i, h^{E_i})$, $T_f(A'^j, h^{E^j})$.

Now we imitate [BGS1, Theorem 1.20].

Theorem A1.4. *The following identities hold:*

(A1.43)
$$T_f(A'^\bullet_\bullet, h^E) = \sum_{i=0}^n (-1)^i T_f(A'_i, h^{E_i}) \quad in \ Q^B/Q^{B,0},$$

$$T_f(A'^\bullet_\bullet, h^E) = \sum_{j=0}^{n'} (-1)^j T_f(A'^j, h^{E^j}) \quad in \ Q^B/Q^{B,0}.$$

Proof. We could use Theorem A1.1 to prove the theorem. Instead, we will give a direct proof. Let $N_H \in \mathrm{End}(E)$ act on E_i^j by multiplication by j. For $s > 0$, set

$$h_s^E = \bigoplus_{\substack{0 \le i \le n \\ 0 \le j \le n'}} s^j h^{E_i^j}.$$

From Definition 1.12 and Theorem 2.24, we have

(A1.44) $$\frac{\partial}{\partial s} T_f(A'^\bullet_\bullet, h_s^E) = \varphi \, \mathrm{Tr_s}\left[\frac{N_H}{2s} f'\left(\frac{\omega}{2}(E, h_s^E)\right)\right] \quad in \ Q^B/Q^{B,0}.$$

Then from Proposition 1.3, we have

(A1.45) $$\varphi \, \mathrm{Tr_s}\left[\frac{N_H}{2s} f'\left(\frac{\omega}{2}(E, h_s^E)\right)\right] = \mathrm{Tr_s}\left[\frac{N_H}{2s}\right] f'(0).$$

As the complexes $(E_\bullet^j, v')$ are acyclic, we have

(A1.46) $$\mathrm{Tr_s}[N_H] = 0.$$

From (A1.44) and (A1.46), we deduce that for $s > 0$,

(A1.47) $$T_f(A'^\bullet_\bullet, h_s^E) = T_f(A'^\bullet_\bullet, h^E) \quad in \ Q^B/Q^{B,0}.$$

Let $A_{\bullet,s}^{\prime\prime\bullet}$ be the adjoint of $A_{\bullet}^{\prime\bullet}$, with respect to h_s^E. Put

$$(A1.48) \qquad C_s' = s^{N_H/2} A_{\bullet}^{\prime\bullet} s^{-N_H/2}, \qquad C_s'' = s^{-N_H/2} A_{\bullet,s}^{\prime\prime\bullet} s^{N_H/2}.$$

Let v^* and $v^{\prime*}$ be the adjoints of v and v' with respect to h^E. Then

$$(A1.49) \qquad C_s' = \nabla^E + \sqrt{s}v + v', \qquad C_s'' = \nabla^E + \sqrt{s}v^* + v^{\prime*}.$$

For any $s > 0$, we have

$$(A1.50) \qquad T_f(A_{\bullet}^{\prime\bullet}, h_s^E) = T_f(C_{\bullet,s}^{\prime\bullet}, h^E).$$

Also, C_s' is well defined for $s = 0$. Using the fact that the $(E_{\bullet}^j, v')$ complexes are acyclic, we find that as $s \to 0$,

$$(A1.51) \qquad T_f(C_{\bullet,s}^{\prime\bullet}, h^E) \to T_f(C_{\bullet,0}^{\prime\bullet}, h^E).$$

Put

$$(A1.52) \qquad D = \tfrac{1}{2}(C_0'' - C_0').$$

Let D^j be the restriction of D to E^j. Then

$$(A1.53) \qquad \varphi \operatorname{Tr}_s\left[\frac{N_H}{2}f'(D)\right] = \sum_{j=0}^{n'}(-1)^j\frac{j}{2}\varphi \operatorname{Tr}_s[f'(D^j)].$$

Using Proposition 1.3, (A1.53), and the acyclicity of $(E_{\bullet}^j, v')$, we obtain

$$(A1.54) \qquad \varphi \operatorname{Tr}_s\left[\frac{N_H}{2}f'(D)\right] = 0.$$

From (A1.54), we deduce that

$$(A1.55) \qquad T_f(C_0', h^E) = \sum_{j=0}^{n'}(-1)^j T_f(A^{\prime j}, h^{E^j}).$$

The second equality in (A1.43) follows from (A1.47), (A1.50), (A1.51) and (A1.55). The first equality follows by exchanging the roles of v and v'. $\quad\square$

APPENDIX II. REIDEMEISTER TORSION AND HIGHER REIDEMEISTER TORSION

In this appendix we review some facts about the Reidemeister torsion of Reidemeister and Franz and the higher Reidemeister torsion of Igusa and Klein. The results of this appendix are independent of the rest of the paper.

The Reidemeister torsion is a classical invariant of non-simply-connected manifolds (see [M] for a survey of Whitehead and Reidemeister torsions). Let Z be a compact manifold with fundamental group π. Let $\mathbb{F}$ be a field. Let $K_1(\mathbb{F})$ be the first algebraic K-group of $\mathbb{F}$; it is isomorphic to $\mathbb{F}^* = \mathbb{F} - \{0\}$. Let $\rho: \pi \to GL(N, \mathbb{F})$ be a representation. Suppose that $H^*(Z; E_\rho)$, the cohomology of Z defined with the local system E_ρ induced from ρ, vanishes.

By means of a cellular decomposition of Z, one can define the Reidemeister torsion $T(Z, \rho)$, an element of

$$(A2.1) \qquad \mathrm{Wh}_1^\rho(\mathbb{F}) = \mathrm{Cokernel}\left(\pm 1 \times \frac{\pi}{[\pi, \pi]} \xrightarrow{\alpha} K_1(\mathbb{F})\right),$$

where $\alpha(\pm 1 \times [g]) = \pm\det(\rho(g))$. One can show that $T(Z, \rho)$ is a simple-homotopy invariant of Z, and hence is, in particular, a homeomorphism invariant.

Suppose that $\mathbb{F}$ has a "conjugation" automorphism of order two, and define the unitary group $U(N, \mathbb{F})$ accordingly. Suppose that ρ takes its value in $U(N, \mathbb{F})$. If $\sigma: \mathbb{F} \to \mathbb{C}$ is a conjugation-preserving homomorphism then the map $\delta: \mathbb{F}^*(\simeq K_1(\mathbb{F})) \to \mathbb{R}$ given by $\delta(f) = \ln|\sigma(f)|$ extends to a map $\bar{\delta}: \mathrm{Wh}_1^\rho(\mathbb{F}) \to \mathbb{R}$. Then $\bar{\delta}(T(Z, \rho)) \in \mathbb{R}$ is a topological invariant of Z. An analytic version of this real invariant, the analytic torsion, was proposed by Ray and Singer [RS1]. The equality between the analytic torsion and $\bar{\delta}(T(Z, \rho))$ was proven independently by Cheeger [C] and Müller [Mü1]. This equality was extended to representations with unitary determinant in [Mü2]. See [BZ] for the relationship between analytic and combinatorial torsions in the case of general representations.

The Whitehead and Reidemeister torsions are intimately related to the first algebraic K-groups of $\mathbb{Z}\pi$ and $\mathbb{F}$, respectively. J. Wagoner conjectured that they can be extended to invariants related to higher algebraic K-groups [W]. We will only discuss the extension of the Reidemeister torsion.

Let us first recall Quillen's definition of higher algebraic K-groups [Q3], when applied to $\mathbb{F}$. Let $GL(\mathbb{F})$ denote the direct limit of the groups $GL(N, \mathbb{F})$ with respect to inclusion. Let $BGL(\mathbb{F})_\delta$ be the corresponding classifying space (where the δ subscript indicates the discrete topology on $GL(\mathbb{F})$) and let $BGL(\mathbb{F})_\delta^+$ be the result of applying Quillen's plus construction to $BGL(\mathbb{F})_\delta$. Put

$$(A2.2) \qquad K(\mathbb{F}) = \mathbb{Z} \times BGL(\mathbb{F})_\delta^+.$$

Then by definition, $K_i(\mathbb{F}) = \pi_i(K(\mathbb{F}))$.

There is a homomorphism $\bar{\sigma}: K_{k+1}(\mathbb{F}) \to K_{k+1}(\mathbb{C})$ induced by σ. Let z be a primitive element in degree $(k+1)$ of $H_c^*(GL(\mathbb{C}); \mathbb{R})$, the cohomology of the complex of Eilenberg-Mac Lane group cochains which are continuous in their entries. By forgetting the continuity of its defining group cocycle, z maps to an element $\bar{z}$ of the group cohomology $H^{k+1}(GL(\mathbb{C}); \mathbb{R})$. On the other hand, the Hurewicz homomorphism gives a map

$$(A2.3) \qquad \begin{aligned} K_{k+1}(\mathbb{C}) &= \pi_{k+1}(BGL(\mathbb{C})_\delta^+) \to H_{k+1}(BGL(\mathbb{C})_\delta^+; \mathbb{R}) \\ &\simeq H_{k+1}(BGL(\mathbb{C})_\delta; \mathbb{R}) \simeq H_{k+1}(GL(\mathbb{C}); \mathbb{R}). \end{aligned}$$

If k is even then by pairing the image of the Hurewicz homomorphism with $\bar{z}$, we obtain a nonzero map from $K_{k+1}(\mathbb{C})$ to $\mathbb{R}$, the Borel regulator [Bo]. Let $\Delta: K_{k+1}(\mathbb{F}) \to \mathbb{R}$ denote the composition of $\bar{\sigma}$ and the Borel regulator.

Now suppose that one has a fiber bundle $Z \to M \to S^k$ with compact connected fiber Z. Let π' be the fundamental group of M and suppose that

one has a representation $\rho': \pi' \to U(N, \mathbb{F})$ such that the cohomology groups of the fibers, defined using the restriction of ρ', vanish. To such a bundle and representation, Wagoner proposed to assign a higher Reidemeister torsion $\mathscr{T}(M, \rho')$, an element of a certain quotient $K'_{k+1}(\mathbb{F})$ of $K_{k+1}(\mathbb{F})$. If k is even, then Δ descends to a map $\overline{\Delta}: K'_{k+1}(\mathbb{F}) \to \mathbb{R}$, and one would obtain a real invariant $\overline{\Delta}(\mathscr{T}(M, \rho'))$ of the fiber bundle. The standard Reidemeister would be recovered as the special case $k = 0$, with S^0 being a point, $\Delta = \delta$ and $\overline{\Delta} = \overline{\delta}$.

One application envisaged in [W] was the detection of homotopy groups of diffeomorphism groups. Let (Z, ρ) be acyclic as above and let $\mathrm{Diff}(Z, \rho)$ denote the group of diffeomorphisms of Z which fix a basepoint and induce the identity on $\pi / \mathrm{Ker}(\rho)$. Given a basepoint-preserving map $\beta: S^{k-1} \to \mathrm{Diff}(Z, \rho)$, use the clutching construction to form a fiber bundle $Z \to M \to S^k$ and extend ρ to a representation ρ' of π'. Then one would obtain an invariant of the class of β in $\pi_{k-1}(\mathrm{Diff}(Z, \rho))$ by taking the higher Reidemeister torsion of the fiber bundle.

The construction of the higher Reidemeister torsion of [I, K] uses a "Whitehead space". Let $M_n(\pi')$ be the group of $n \times n$ matrices with entries consisting of zeros and of elements of π', with one element of π' in each row and each column. The direct limit under inclusion is denoted $M(\pi')$. A Barratt-Priddy-type theorem states

(A2.4) $$\mathbb{Z} \times BM(\pi')^+ \simeq Q_+(B\pi'),$$

where $Q_+(B\pi') = \Omega^\infty \Sigma^\infty (B\pi'_+)$ is the application of the stable homotopy functor to the disjoint union of $B\pi'$ and a basepoint [Lo].

The representation ρ' gives a homomorphism from $M_n(\pi')$ to $GL(nN, \mathbb{F})$, and hence a map

(A2.5) $$\hat{\rho}: \mathbb{Z} \times BM(\pi')^+ \to K(\mathbb{F}).$$

The homotopy fiber of $\hat{\rho}$ has a delooping, which is called the Whitehead space $\mathrm{Wh}^{\rho'}(\mathbb{F})$. Thus there is a homotopy fibration

(A2.6) $$\Omega \mathrm{Wh}^{\rho'}(\mathbb{F}) \to Q_+(B\pi') \to K(\mathbb{F}).$$

By definition, the higher Whitehead group is $\mathrm{Wh}_i^{\rho'}(\mathbb{F}) = \pi_i(\mathrm{Wh}^{\rho'}(\mathbb{F}))$. One sees from the homotopy exact sequence of (A2.6) that the definition of $\mathrm{Wh}_1^{\rho'}(\mathbb{F})$ coincides with that of (A2.1).

Given a fiber bundle $Z \to M \xrightarrow{\pi} B$ with compact connected fiber Z and a representation $\rho': \pi' \to U(N, \mathbb{F})$ of the fundamental group of M which is acyclic on the fibers, Igusa and Klein use parametrized Morse theory to construct a homotopy class $\tau \in [B, \Omega \mathrm{Wh}^{\rho'}(\mathbb{F})]$ of maps from B to $\Omega \mathrm{Wh}^{\rho'}(\mathbb{F})$. In the special case when B is a point, $\tau \in \pi_0(\Omega \mathrm{Wh}^{\rho'}(\mathbb{F})) \simeq \mathrm{Wh}_1^{\rho'}(\mathbb{F})$ coincides with the Reidemeister torsion $T(Z, \rho')$

The Becker-Gottlieb transfer of the fiber bundle is a homotopy class of maps from B to $Q_+(M)$ [BG]. The classifying map $M \to B\pi'$ induces a map

$Q_+(M) \to Q_+(B\pi')$. It is shown that the compositions $B \to Q_+(M) \to Q_+(B\pi')$ and $B \xrightarrow{\tau} \Omega\text{Wh}^{\rho'}(\mathbb{F}) \to Q_+(B\pi')$ coincide up to homotopy. Thus if the transfer of the fiber bundle is trivial then τ can be lifted to a homotopy class $\hat{\tau} \in [SB, K(\mathbb{F})]$ of maps from the suspension of B to $K(\mathbb{F})$. The Borel regulator, thought of as an element of $H^{k+1}(K(\mathbb{F}); \mathbb{R})$, pulls back under $\hat{\tau}$ to give an element of $H^k(B; \mathbb{R})$ which is independent of the choice of lift of τ. This is the higher real Reidemeister torsion. In the case $k = 0$, one recovers the real Reidemeister torsion $\bar{\delta}(T(Z, \rho))$.

If the fiber Z of the bundle is closed and odd-dimensional then the transfer of the fiber bundle is rationally trivial. So in this case one can again define the higher real Reidemeister torsion. The reader can compare this with the second result of our Corollary 3.25, stating that under the same conditions, the analytic torsion form gives a well-defined de Rham cohomology class of B. (In fact, Corollary 3.25 does not require unitariness of the representation.) Igusa and Klein computed the pairing between the higher Reidemeister torsion and the fundamental homology class of B when $B = S^2$, $Z = S^1$, M is the lens space $L(r, 1)$, $\mathbb{F} = \mathbb{Q}(\zeta)$, the cyclotomic number field generated by a primitive rth root ζ of unity, and $\rho': \mathbb{Z}/r\mathbb{Z} \to \mathbb{F}$ is given by $\rho'(n) = \zeta^n$. The result was a numerical constant times $r\,\text{Im}(\text{Li}_2(\zeta))$. The reader can compare this with (4.30).

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ABSTRACT. We prove a Riemann-Roch-Grothendieck-type theorem concerning the direct image of a flat vector bundle under a submersion of smooth manifolds. We refine this theorem to the level of differential forms. We construct associated secondary invariants, the analytic torsion forms, which coincide in degree 0 with the Ray-Singer real analytic torsion.

RÉSUMÉ. On démontre un analogue du théorème de Riemann-Roch-Grothen-dieck pour l'image directe d'un fibré plat par une submersion. On raffine ce théorème au niveau des formes différentielles. On construit des invariants sec-ondaires, les formes de torsion analytique, qui coïncident, en degré 0, avec la torsion de Ray-Singer.

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