

GLOBAL  $C^\infty$  IRREGULARITY OF THE  
 $\bar{\partial}$ -NEUMANN PROBLEM FOR WORM DOMAINS

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0. INTRODUCTION

Let  $\Omega \subset \mathbf{C}^n$  be a bounded, pseudoconvex domain with  $C^\infty$  boundary, equipped with the standard Hermitian metric inherited from  $\mathbf{C}^n$ . The  $\bar{\partial}$ -Neumann problem for  $(0, 1)$  forms in  $\Omega$  is the boundary value problem

$$\begin{cases} \square u = f & \text{in } \Omega, \\ u \lrcorner \bar{\partial} \rho = 0 & \text{on } \partial\Omega, \\ \bar{\partial} u \lrcorner \bar{\partial} \rho = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\rho$  is a defining function for  $\Omega$ ,  $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ ,  $u, f$  are  $(0, 1)$  forms, and  $\lrcorner$  denotes the interior product of forms. Under the stated hypotheses on  $\Omega$ , this problem is uniquely solvable for every  $f \in L^2(\Omega)$ . The Neumann operator  $N$ , mapping  $f$  to the solution  $u$ , is continuous on  $L^2(\Omega)$ . The Bergman projection  $B$  is the orthogonal projection of  $L^2(\Omega)$  onto the closed subspace of  $L^2$  holomorphic functions on  $\Omega$ , and is related to  $N$  by  $B = I - \bar{\partial}^* N \bar{\partial}$ .

$N$  and  $B$  are  $C^\infty$  pseudolocal if  $\Omega$  is strictly pseudoconvex, or more generally, is of finite type [Ca1]. Both preserve  $C^\infty(\bar{\Omega})$  under certain weaker hypotheses [BS2], [Ca2]. For any pseudoconvex, smoothly bounded  $\Omega$  and any finite exponent  $s$ , there exists a strictly positive weight  $w \in C^\infty(\bar{\Omega})$  such that the Neumann operator and Bergman projection with respect to the Hilbert space  $L^2(\Omega, w(x)dx)$  map the Sobolev space  $H^t(\Omega)$  boundedly to  $H^t(\Omega)$ , for all  $0 \leq t \leq s$  [K1]. It has remained an open question whether  $N$  and  $B$ , defined with respect to the standard metric, preserve  $C^\infty(\bar{\Omega})$  without further hypotheses on  $\Omega$ . An affirmative answer would have significant consequences [BL].

**Theorem.** *There exist pseudoconvex, smoothly bounded domains  $\Omega \Subset \mathbf{C}^2$  for which the Neumann operator on  $(0, 1)$  forms and Bergman projection fail to preserve  $C^\infty(\bar{\Omega})$ .*

Examples are the worm domains, originally introduced by Diederich and Fornæss [DF] for another purpose<sup>1</sup> but long considered likely candidates for  $N$  and  $B$  to fail to be globally regular in  $C^\infty$ .

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<sup>1</sup>Some but not all worm domains have nontrivial Nebenhülle [FS, p. 111], [DF, Theorem 5.1], whereas all worm domains are counterexamples to global regularity.

The proof depends on the observation of Barrett [B] that for each worm domain  $\mathcal{W}$ , for all sufficiently large  $s$ ,  $N$  and  $B$  fail to map  $H^s(\mathcal{W})$  boundedly to  $H^s(\mathcal{W})$ . We establish for each worm domain an *a priori* inequality of the form  $\|Nf\|_{H^s} \leq C_s \|f\|_{H^s}$ , valid for all  $f \in C^\infty(\overline{\mathcal{W}})$  such that  $Nf \in C^\infty(\overline{\mathcal{W}})$ , for a sequence of exponents  $s$  tending to  $\infty$ . If  $N$  were to preserve  $C^\infty(\overline{\mathcal{W}})$ , then since it is a bounded linear operator on  $L^2(\mathcal{W})$  and since  $C^\infty(\overline{\mathcal{W}})$  is dense in  $H^s(\mathcal{W})$ , it would follow that  $N$  maps  $H^s(\mathcal{W})$  boundedly to itself, for a sequence of values of  $s$  tending to  $\infty$ , a contradiction. More accurately, our inequality is valid only for certain subspaces of  $L^2(\mathcal{W})$  preserved by  $\square$ , but this still suffices to contradict the theorem of Barrett.

An analogous counterexample in the real analytic context is already known [Ch1]: there exists a bounded, pseudoconvex domain  $\Omega \subset \mathbf{C}^2$  having real analytic boundary, such that the Szegő projection fails to preserve  $C^\omega(\partial\Omega)$ . That result and its proof are however not closely related to the  $C^\infty$  case.

§§1 through 3 review material on worm domains and the  $\bar{\partial}$ -Neumann problem, and present some routine but tedious reductions. §4 formalizes a class of two-dimensional problems subsuming those to which the reductions lead. The analysis of those problems is contained in §§5 and 6.

## 1. REDUCTION TO THE BOUNDARY

The  $\bar{\partial}$ -Neumann problem is a boundary value problem for an elliptic partial differential equation, and as such is amenable to treatment by the method of reduction to a pseudodifferential equation on the boundary. This reduction has been carried out in detail for domains in  $\mathbf{C}^2$  by Chang, Nagel and Stein [CNS]. We review here certain of their computations and direct consequences thereof.

Assume  $\Omega \subset \mathbf{C}^2$  to be a smoothly bounded domain. The equation  $\square u = f$  on  $\Omega$  for  $(0, 1)$  forms in  $C^\infty(\overline{\Omega})$  is equivalent to an equation  $\square^+ v = g$  on  $\partial\Omega$ , where  $v, g$  are sections of a certain complex line bundle<sup>2</sup>  $\mathcal{B}^{0,1}$ . Let  $\rho$  be a smooth defining function for  $\Omega$  and define  $\bar{\omega}_2 = \bar{\partial}\rho$ , and  $\bar{\omega}_1 = (\partial\rho/\partial z_2)d\bar{z}_1 - (\partial\rho/\partial z_1)d\bar{z}_2$ .

$v$  is related to  $u$  by  $u = Pv + Gf$ , where  $P, G$  are respectively Poisson and Green operators for the elliptic system  $\square u = f$  with Dirichlet boundary conditions. In particular, if  $f \in C^\infty(\overline{\Omega})$ , then  $u \in C^\infty(\overline{\Omega})$  if and only if the same holds for  $v$ . More precisely,  $G$  maps  $H^s(\Omega)$  to  $H^{s+2}(\Omega)$ , while  $P$  maps  $H^s(\partial\Omega)$  to  $H^{s+1/2}(\Omega)$ , for each  $s \geq 0$ . Thus if  $f \in H^s$ , in order to conclude that  $u \in H^s$  it suffices to know that  $v \in H^{s-1/2}$ .

On the other hand,  $g = (\bar{\partial}Gf \lrcorner \bar{\partial}\rho)$ , restricted to  $\partial\Omega$ . If  $f \in H^s$ , then  $\bar{\partial}Gf \in H^{s+1}(\Omega)$ , so its restriction to the boundary belongs to  $H^{s+1/2}(\partial\Omega)$ . Thus in order to show that  $N$  preserves  $H^s(\Omega)$  it suffices to show that if  $\square^+ v \in H^{s+1/2}(\partial\Omega)$ , then  $v \in H^{s-1/2}(\partial\Omega)$ , assuming always that  $s > 1/2$ .

On  $\partial\Omega$  a Cauchy-Riemann operator is the complex vector field  $\bar{L} = (\partial_{\bar{z}_1}\rho)\partial_{\bar{z}_2} - (\partial_{\bar{z}_2}\rho)\partial_{\bar{z}_1}$ . Define  $L$  to be the complex conjugate of  $\bar{L}$ . The characteristic variety<sup>3</sup> of  $\bar{L}$  is a real line bundle  $\Gamma$ . Assuming  $\Omega$  to be pseudoconvex and the set of strictly pseudoconvex points to be dense in  $\partial\Omega$ ,  $\Gamma$  splits smoothly and uniquely as  $\Gamma^+ \cup \Gamma^-$ , where each fiber of  $\Gamma^\pm$  is a single ray, and where  $\Gamma^+$  is distinguished from  $\Gamma^-$  by

<sup>2</sup> $\mathcal{B}^{0,1}$  is defined to be the quotient of the restriction to  $\partial\Omega$  of  $T^{0,1}\mathbf{C}^2$ , modulo the span of  $\bar{\partial}\rho$ . Sections of  $\mathcal{B}^{0,1}$  may be identified with scalar-valued functions times  $\bar{\omega}_1$ , hence with scalar-valued functions.

<sup>3</sup>By the characteristic variety of a pseudodifferential operator we mean the conic subset of the cotangent bundle on which its principal symbol vanishes.

the requirement that the principal symbol of  $[\bar{L}, L]$  is nonpositive on  $\Gamma^+$ , modulo terms spanned by the symbols of the real and imaginary parts of  $\bar{L}$  and a term of order 0. Equivalently, the principal symbol of  $[\bar{L}, \bar{L}^*]$  is nonnegative on  $\Gamma^+$ , modulo the same kinds of error terms.

$\square^+$  is a classical pseudodifferential operator of order +1. Its principal symbol vanishes everywhere on  $\Gamma^+$  but nowhere else. Microlocally in a conic neighborhood of  $\Gamma^+$ ,  $\square^+$  takes the form

$$\square^+ = Q\bar{L}L + F_1\bar{L} + F_2L + F_3,$$

where  $Q$  is an elliptic pseudodifferential operator of order  $-1$ , and each  $F_j$  is a pseudodifferential operator of order less than or equal to  $-1$ . Since  $\square^+$  is elliptic except on  $\Gamma^+$ , for any pseudodifferential operator  $G$  of order zero whose symbol vanishes identically in some neighborhood of  $\Gamma^+$ , one has for all  $u \in C^\infty$  and all  $N < \infty$

$$(1.1) \quad \|Gu\|_{H^{t+1}(\partial\Omega)} \leq C\|\square^+u\|_{H^t(\partial\Omega)} + C_N\|u\|_{H^{-N}(\partial\Omega)}.$$

Let  $A$  be an elliptic pseudodifferential operator of order +1 such that  $A \circ Q$  equals the identity on  $L^2(\partial\Omega)$ , modulo an operator smoothing of infinite order. Composing on the left with  $A$ , the equation  $\square^+v = g$  may be rewritten as  $\mathfrak{L}v = \tilde{g}$  microlocally in a conic neighborhood of  $\Gamma^+$ , where

$$(1.2) \quad \mathfrak{L} = \bar{L}L + B_1\bar{L} + B_2L + B_3,$$

$\|\tilde{g}\|_{H^t} \leq C\|g\|_{H^{t+1}} + C_N\|v\|_{H^{-N}}$  for any finite  $N$ , and each  $B_j$  is an operator of order less than or equal to zero. Therefore, in order to show that the Neumann operator satisfies an *a priori* inequality of the form  $\|Nf\|_{H^s(\Omega)} \leq C\|f\|_{H^s(\Omega)}$  for all  $f \in C^\infty(\bar{\Omega})$  such that  $Nf \in C^\infty(\bar{\Omega})$ , it suffices to establish an *a priori inequality* for all  $v \in C^\infty(\partial\Omega)$  of the form

$$(1.3) \quad \|v\|_{H^t} \leq C\|\mathfrak{L}v\|_{H^t} + C\|v\|_{H^{t'}} + C\|\tilde{Q}v\|_{H^{t+2}},$$

where  $t = s - 1/2$ , for some  $t' < t$  and some pseudodifferential operator  $\tilde{Q}$  of order zero whose symbol vanishes identically in some neighborhood of  $\Gamma^+$ .

## 2. WORM DOMAINS

A worm domain in  $\mathbf{C}^2$  is an open set of the form

$$\mathcal{W} = \{z : |z_1 + e^{i \log |z_2|^2}| < 1 - \phi(\log |z_2|^2)\},$$

where the function  $\phi$  vanishes identically on some interval  $[-r, r]$  of positive length, and is constructed [DF] so as to guarantee that  $\mathcal{W}$  will be pseudoconvex with  $C^\infty$  boundary, and will be strictly pseudoconvex at every boundary point except those on the exceptional annulus  $\mathcal{A} \subset \partial\mathcal{W}$  defined as

$$\mathcal{A} = \{z : z_1 = 0 \text{ and } |\log |z_2|^2| \leq r\}.$$

The circle group acts as a group of automorphisms of  $\mathcal{W}$  by  $z \mapsto R_\theta z = (z_1, e^{i\theta} z_2)$ . It acts on functions by  $R_\theta f(z) = f(R_\theta z)$ , and on  $(0, 1)$  forms by

$$R_\theta(f_1 d\bar{z}_1 + f_2 d\bar{z}_2) = (R_\theta f_1) d\bar{z}_1 + (R_\theta f_2) e^{-i\theta} d\bar{z}_2.$$

The Hilbert space  $L^2_{(0,k)}(\mathcal{W})$  of square integrable  $(0, k)$  forms decomposes as the orthogonal direct sum  $\bigoplus_{j \in \mathbf{Z}} \mathcal{H}_j^k$ , where  $\mathcal{H}_j^k$  is the set of all  $(0, k)$  forms  $f$  satisfying  $R_\theta f \equiv e^{ij\theta} f$ . Here  $\bar{\partial}$  is an unbounded linear operator from  $\mathcal{H}_j^k$  to  $\mathcal{H}_j^{k+1}$ ,  $B$  maps  $\mathcal{H}_j^0$  to itself, and the Neumann operator  $N$  maps  $\mathcal{H}_j^1$  to  $\mathcal{H}_j^1$  boundedly, for each  $j$ .

For each  $k$  and each  $s \geq 0$ , the Sobolev space  $H^s(\mathcal{W})$  likewise decomposes as an orthogonal direct sum of subspaces  $H_j^s$ . It is known that for any smoothly bounded, pseudoconvex domain  $\Omega \subset \mathbf{C}^2$ , for any exponent  $s \geq 0$ , if  $N$  maps  $H^s(\Omega)$  boundedly to itself, then  $B$  also maps  $H^s(\Omega)$  boundedly to itself [BS1], where  $H^s$  denotes in the first instance a space of one forms, and in the second, a space of functions. Because  $N, B$  preserve the summands  $\mathcal{H}_j$ , the same proof shows<sup>4</sup> that for any fixed  $j$ , if  $N$  maps the space  $H_j^s$  of  $(0, 1)$  forms boundedly to itself, then  $B$  maps the space  $H_j^s$  of functions boundedly to itself. Barrett [B] showed that for each worm domain, for all sufficiently large  $s$ , for all  $j$ ,  $B$  fails to map  $H_j^s$  boundedly to itself. Therefore, in order to prove that  $N$ , acting on  $(0, 1)$  forms, fails to preserve  $C^\infty(\bar{\mathcal{W}})$ , it suffices to establish the following result for a single index  $j$ .

**Proposition 1.** *For each worm domain there exists a discrete subset  $S \subset \mathbf{R}^+$  such that for each  $s \notin S$  and each  $j \in \mathbf{Z}$  there exists  $C_{s,j} < \infty$  such that for every  $(0, 1)$  form  $u \in \mathcal{H}_j^1 \cap C^\infty(\bar{\mathcal{W}})$  such that  $Nu \in C^\infty(\bar{\mathcal{W}})$ ,*

$$(2.1) \quad \|Nu\|_{H^s(\mathcal{W})} \leq C_{s,j} \|u\|_{H^s(\mathcal{W})}.$$

The defining function  $\rho = 1 - \phi(\log |z_2|^2) - |z_1 + e^{i \log |z_2|^2}|^2$  for  $\mathcal{W}$  is invariant under  $R_\theta$ , as is the  $(0, 1)$  form  $\bar{\omega}_2$  defined above.  $\bar{\omega}_1$  satisfies  $R_\alpha \bar{\omega}_1 = \exp(-i\alpha) \bar{\omega}_1$  for all  $\alpha$ , but it may also be made invariant by multiplying it by the function  $(z_1, re^{i\theta}) \mapsto e^{i\theta}$ , which is smooth in a neighborhood of  $\bar{\mathcal{W}}$ . We work henceforth with this modified  $\bar{\omega}_1$ .

$\square^+$  commutes with  $R_\theta$  for all  $\theta$ . Indeed,  $\square^+ v = \bar{\partial} P v \lrcorner \bar{\partial} \rho$  [CNS].  $\square$  commutes with  $R_\theta$ , hence so must  $P$ .  $\bar{\partial}$  commutes with  $R_\theta$ , and the Hermitian metric on  $\mathbf{C}^2$  and  $\bar{\partial} \rho$  are likewise  $R_\theta$ -invariant. Thus all ingredients in the above expression for  $\square^+$  are invariant, hence so is  $\square^+$  itself.

Identify square integrable sections of  $\mathcal{B}^{0,1}$  with scalar-valued  $L^2$  functions as above, and decompose  $L^2(\partial\mathcal{W}) = \bigoplus \mathcal{H}_j(\partial\mathcal{W})$ , where  $\mathcal{H}_j$  is the subspace of those functions satisfying  $R_\theta f \equiv e^{ij\theta} f$ . Then  $\square^+$  maps  $\mathcal{H}_j(\partial\mathcal{W}) \cap C^\infty$  to  $\mathcal{H}_j(\partial\mathcal{W})$ . We have seen in §1 that Proposition 1 would be a consequence of the validity of (1.3) for all  $v \in C^\infty(\partial\mathcal{W}) \cap \mathcal{H}_j$ , for  $t = s - 1/2$ .

Fix  $j$  and assume henceforth that  $u$  belongs to  $\mathcal{H}_j^1(\mathcal{W})$  and to  $C^\infty$ . Then the associated boundary function  $v$  belongs to  $\mathcal{H}_j \cap C^\infty(\partial\mathcal{W})$ . Henceforth we work exclusively on the boundary, and simplify notation by writing simply  $\mathcal{H}_j$  rather than  $\mathcal{H}_j(\partial\mathcal{W})$ .

Note that  $\bar{L}, L$  take  $\mathcal{H}_j \cap C^\infty$  to  $\mathcal{H}_{j+1}$  and to  $\mathcal{H}_{j-1}$ , respectively. The operator  $A$  introduced after (1.1) may be constructed to be  $R_\theta$ -invariant, for both  $\square^+$  and  $\bar{L} \circ L$  are invariant while  $\bar{L}, L$  are automorphic of certain degrees, so that averaging the equation  $\square^+ = Q\bar{L}L + F_1\bar{L} + F_2L + F_3$  with respect to  $R_\theta d\theta$  produces an invariant  $Q$  and  $F_3$ , and operators  $F_1, F_2$  automorphic of the appropriate degrees. Thus  $(\bar{L}L + B_1\bar{L} + B_2L + B_3)v \in H^t$  microlocally near  $\Gamma^+$ , where  $B_1, B_2, B_3$  map  $\mathcal{H}_j$  to  $\mathcal{H}_i$  for  $i = j - 1, j + 1, j$  respectively.

<sup>4</sup>This follows from the argument of Boas and Straube [BS1] because all elements of their proof may be chosen to be invariant under the automorphisms  $R_\theta$ .

Since  $\mathcal{W}$  is strictly pseudoconvex at all points not in  $\mathcal{A}$ , it follows as in [K2] that on the complement of any neighborhood of  $\mathcal{A}$ , the  $H^{s+1}$  norm of  $v$  is majorized by  $C\|\square^+v\|_{H^{t+1}(\partial\mathcal{W})} + C\|v\|_{H^{-N}}$ , hence by  $C\|\mathfrak{L}v\|_{H^s} + C\|v\|_{H^{-N}} + C\|\tilde{Q}v\|_{H^{s+2}}$  with  $\tilde{Q}$  as in (1.3). This estimate is one derivative stronger than that which we seek. In particular, it now suffices to control the  $H^s$  norm of  $v$  in an arbitrarily small neighborhood  $U$  of  $\mathcal{A}$ , and to do so microlocally near  $\Gamma^+$ .

Fix a  $C^\infty$  cutoff function  $\varphi$  supported in a small neighborhood of  $\mathcal{A}$  but identically equal to 1 in a smaller neighborhood, and fix an open set  $V$  disjoint from a neighborhood of  $\mathcal{A}$  such that  $\nabla\varphi$  is supported in  $V$ . By Leibniz’s rule and the pseudolocality of pseudodifferential operators, the  $H^s$  norm of  $\mathfrak{L}(\varphi v - v)$  is majorized by  $C\|v\|_{H^{s+1}(V)} + C\|v\|_{H^{-N}}$  for any  $N < \infty$ . Thus by replacing  $v$  with  $\varphi v$  we may reduce matters to the case where  $v$  is supported in an arbitrarily small neighborhood  $W$  of  $\mathcal{A}$ . Therefore it suffices to prove the existence of some  $W$  and an exponent  $s' < s$  such that (1.3) holds (with  $t$  replaced by  $s$ ) for all  $v \in C_0^\infty$  supported in  $W$ .

In a neighborhood of  $\mathcal{A}$  in  $\partial\mathcal{W}$  introduce coordinates  $(x, \theta, t)$ , where

$$z_2 = e^{x+i\theta}, \quad z_1 = e^{i2x}(e^{it}(1 - \phi(2x)) - 1)$$

with  $2|x| < r + \delta$  and  $|t| < \delta$  for some small  $\delta > 0$ . In these coordinates  $\mathcal{A} = \{t = 0, |x| \leq r/2\}$ . Setting

$$\gamma(x, t) = 2[e^{-it} - 1 + \phi(2x) - i\phi'(2x)]/[1 - \phi(2x)],$$

the vector field  $\bar{L} = \partial_x + i\partial_\theta + \gamma\partial_t$  annihilates both  $z_1$  and  $z_2$ . Hence it differs from what was previously denoted as  $\bar{L}$  by multiplication on the left by a nonvanishing factor, which may be verified to have the form  $b(x, t)\exp(i\theta)$ . For  $|x| \leq r/2$ ,  $\phi(x) \equiv 0$  and consequently  $\gamma(x, t) \equiv 2(e^{-it} - 1)$ . Therefore<sup>5</sup>

$$(2.2) \quad \bar{L} = \partial_x + i\partial_\theta + it\alpha(t)\partial_t \quad \text{where } |x| \leq r/2, \text{ with } \operatorname{Re} \alpha(0) \neq 0.$$

The representation  $\mathfrak{L} = \bar{L}L + B_1\bar{L} + B_2L + B_3$  in terms of the new  $\bar{L}$  remains valid with modified coefficients  $B_i$  that now preserve each  $\mathcal{H}_j$ , once the operator formerly denoted by  $\mathfrak{L}$  is multiplied by  $|b|^{-2}$ .

There exists a unique  $C^\infty$  real-valued function  $\mu$ , independent of  $\theta$ , such that  $[\bar{L}, L] = i\mu(x, t)\partial_t$  modulo the span of the real and imaginary parts of  $\bar{L}$ . More precisely, since the coefficients of  $\partial_\theta$  in  $[\bar{L}, L]$  and in  $\operatorname{Re} \bar{L}$  vanish while the coefficient in  $\operatorname{Im} \bar{L}$  is nowhere zero,

$$(2.3) \quad [\bar{L}, L] = i\mu(x, t)\partial_t + i\nu(x, t)\operatorname{Re} \bar{L}$$

for unique real-valued,  $C^\infty$  coefficients  $\mu, \nu$ . The pseudoconvexity of  $\partial\mathcal{W}$  means that  $\mu$  does not change sign. Replacing  $t$  by  $-t$  if necessary, we may assume that  $\mu \geq 0$ .

Fix an integer  $k$ . We identify functions of  $(x, t) \in \mathbf{R}^2$  with elements of  $\mathcal{H}_k$  via the correspondence  $u(x, t) \mapsto u(x, t)e^{ik\theta}$ ;  $\partial_\theta$  then becomes multiplication by  $ik$ . For the remainder of the paper we work in  $\mathbf{R}^2$ . Define

$$\mathcal{D} = \{(x, t) : |x| \leq r/2 \text{ and } t = 0\}.$$

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<sup>5</sup>Boas and Straube [BS2] have shown the  $\bar{\partial}$ -Neumann problem to be globally  $C^\infty$  hypoelliptic whenever there exists a real vector field on the boundary that is transverse to the complex tangent space and has a certain favorable commutation property. If  $\operatorname{Re} \alpha(0)$  were to vanish, then  $\partial_t$  would be such a vector field. Thus nonvanishing of  $\operatorname{Re} \alpha(0)$  is for our purpose an essential feature of worm domains.

By incorporating  $ik$  into the  $B_j$  we may further rewrite  $\mathcal{L}$ , when restricted to  $\mathcal{H}_k$ , as

$$\mathcal{L}_0 = \bar{\ell}\ell + B_1\bar{\ell} + B_2\ell + B_3$$

where  $\bar{\ell}$  is a complex vector field in  $\mathbf{R}^2$  which, for  $|x| \leq r/2$ , takes the form

$$\bar{\ell} = \partial_x + it\alpha(t)\partial_t.$$

Here  $\ell$  denotes the conjugate of  $\bar{\ell}$ , each  $B_j$  is a pseudodifferential operator of order  $\leq 0$  in  $\mathbf{R}^2$ , and  $\text{Re } \alpha(0) \neq 0$ . Note that the commutator of the real and imaginary parts of  $\bar{\ell}$  is not forced to vanish identically, because  $\alpha$  is not real-valued. We have  $[\bar{\ell}, \ell] = i\mu(x, t)\partial_t + i\nu(x, t)\text{Re } \bar{\ell}$  with the same coefficients as in (2.3).

Letting  $(\xi, \tau)$  be Fourier variables dual to  $(x, t)$ , define  $\tilde{\Gamma} = \{(x, t, \xi, \tau) : (x, t) \in \mathcal{D} \text{ and } \xi = 0\}$ .  $\mathcal{L}_0$  is not elliptic at points of  $\mathcal{D}$ , but is elliptic at most points in its complement;  $\tilde{\Gamma}$  is the intersection of the characteristic variety of  $\mathcal{L}_0$  with  $\{(x, t, \xi, \tau) : (x, t) \in \mathcal{D}\}$ . Decompose  $\tilde{\Gamma} = \tilde{\Gamma}^+ \cup \tilde{\Gamma}^-$  where  $\tilde{\Gamma}^+ = \tilde{\Gamma} \cap \{\tau > 0\}$ . Thus the principal symbol of  $i\mu\partial_t$ , namely  $-\mu\tau$ , is nonpositive in a conic neighborhood of  $\tilde{\Gamma}^+$ .

### 3. TWO PSEUDODIFFERENTIAL MANIPULATIONS

By an operator we will always mean a classical pseudodifferential operator, that is, one whose symbol admits a full asymptotic expansion in homogeneous terms of integral degrees.  $\sigma_j(T)$  denotes the  $j$ -th order symbol of  $T$  (in the Kohn-Nirenberg calculus), always with respect to the fixed coordinate system  $(x, t, \xi, \tau)$ . Henceforth we work under the convention that  $A, B, E$  denote operators whose orders are less than or equal to  $0, 0, -1$  respectively, whose meanings are permitted to change freely from one occurrence to the next, even within the same line.  $A$  denotes always an operator having the additional property that  $\sigma_0(A)(x, t, \xi, \tau) \equiv 0$  for all  $(x, t) \in \mathcal{D}$ . Any operator of type  $E$  may be regarded as one of type  $A$ . Two operators are said to agree microlocally in some conic open set if the full symbol of their difference vanishes identically there.

Any operator  $B$  may be written in the form  $B = \beta(x) + E \circ \ell + A$  microlocally in a conic neighborhood of  $\tilde{\Gamma}^+$ , where  $\beta$  denotes both a  $C^\infty$  function and the operator defined by multiplication by that function. Indeed,  $\sigma_0(B)(x, 0, 0, \tau)$  depends only on  $(x, \text{sgn}(\tau))$  and we define  $\beta(x)$  to be this quantity for  $\tau > 0$ . Then where  $\tau > 0$  and  $|x| \leq r/2$ ,  $\sigma_0(B)(x, 0, \xi, \tau)$  is divisible by  $\xi = -i\sigma_1(\ell)(x, 0, \xi, \tau)$ .  $\sigma_{-1}(E)(x, 0, \xi, \tau)$  is then uniquely determined for such  $x, \tau$  by the equation  $\sigma_0(B) = \beta(x) + \sigma_1(\ell) \cdot \sigma_{-1}(E)$ . Define  $E$  to be any operator of order  $-1$  whose principal symbol satisfies this equation when restricted to  $(x, t) \in \mathcal{D}$  and to a conic neighborhood of  $\tilde{\Gamma}^+$ . Then simply define  $A = B - \beta - E \circ \ell$ .

Writing  $B_1 = \beta_1(x) + E_1 \circ \ell + A_1$  and similarly  $B_2 = \beta_2(x) + E_2 \circ \bar{\ell} + A_2$ , and expressing  $E\ell\bar{\ell} = E\bar{\ell}\ell$  plus an operator of order  $\leq 0$ , we obtain  $\mathcal{L}_0 = (I + E)\bar{\ell}\ell + (\beta_1 + A)\bar{\ell} + (\beta_2 + A)\ell + B$ . Composing both sides with a parametrix for  $I + E$  and modifying the definition of  $\mathcal{L}_0$  to include this factor, we have  $\mathcal{L}_0 = \bar{\ell}\ell + (\beta_1 + A)\bar{\ell} + (\beta_2 + A)\ell + B$ . Writing finally  $B = \beta_3(x) + E \circ \ell + A$  results in

$$(3.1) \quad \mathcal{L}_0 = \bar{\ell}\ell + (\beta_1 + A)\bar{\ell} + (\beta_2 + A)\ell + (\beta_3 + A),$$

where the  $\beta_j$  are  $C^\infty$  functions depending only on  $x$ . □

We next reduce the question of *a priori*  $H^s$  inequalities to  $L^2$ , simultaneously for all  $s$ . Fix an operator  $Q$  of order 0 that is elliptic in some conic neighborhood of  $\tilde{\Gamma}^-$ , whose symbol vanishes identically in some conic neighborhood of  $\tilde{\Gamma}^+$ . Fix an exponent  $s > 0$ , for which we seek an *a priori* inequality for all  $v \in C^\infty$  of the form

$$(3.2) \quad \|v\|_{H^s} \leq C\|\mathcal{L}_0 v\|_{H^s} + C\|v\|_{H^{s'}} + C\|Qv\|_{H^{s+2}}$$

for some  $s' < s$ . Having such an inequality for a sequence of exponents  $s$  tending to  $+\infty$  would imply Proposition 1 by the preceding discussion. In particular, the  $H^{s+2}$  norm of  $Qv$  is already under control by virtue of (1.1), while the  $H^0$  norm of  $v$  is harmless because the Neumann operator is bounded on  $L^2$ , and  $\|v\|_{H^{s'}} \leq \varepsilon\|v\|_{H^s} + C_{\varepsilon,N}\|v\|_{H^{-N}}$  for any  $\varepsilon > 0$  and  $N < \infty$ .

Fix a  $C^\infty$ , strictly positive function  $m = m(\xi, \tau)$ , homogeneous of degree 1 for large  $|(\xi, \tau)|$  and identically equal to  $(1 + \tau^2)^{1/2}$  in a conic neighborhood of  $\{\xi = 0\}$ . Define  $\Lambda^s$  to be the Fourier multiplier operator on  $\mathbf{R}^2$  with symbol  $m(\xi, \tau)^s$ .

Substituting  $v = \Lambda^{-s}u$  and  $g = \Lambda^{-s}f$ , estimation of the  $H^s$  norm of  $v$ , modulo a lower order norm, in terms of that of  $g$  is equivalent to estimation of the  $H^0$  norm of  $u$  in terms of that of  $f$ , modulo a negative order norm of  $u$ . The equation  $\mathcal{L}_0 v = g$  becomes  $\Lambda^{-s}\mathcal{L}_0\Lambda^s u = f$ . Write  $\Lambda^{-s}\ell\Lambda^s = \Lambda^{-s}\bar{\ell}\Lambda^s \circ \Lambda^{-s}\ell\Lambda^s$  and similarly for other terms, and note that  $\Lambda^{-s}(\beta_j + A)\Lambda^s = \beta_j + A$  with the same function  $\beta_j$ .

$\Lambda^{\pm s}$  commutes with  $\partial_x$  and with  $\partial_t$ , so  $\Lambda^{-s}(\partial_x + i\alpha t\partial_t)\Lambda^s = \partial_x + \Lambda^{-s}t\Lambda^s \circ \Lambda^{-s}\alpha\Lambda^s\partial_t$ . Applying the Fourier transform gives immediately  $\Lambda^{-s}[t, \Lambda^s]\partial_t = -s + E$ , microlocally in some conic neighborhood of  $\tilde{\Gamma}^+$ , modulo operators smoothing there of infinite order. Since  $\Lambda^{-s}[\alpha, \Lambda^s]\partial_t$  is of order 0,

$$\begin{aligned} \Lambda^{-s}t\Lambda^s \circ \Lambda^{-s}\alpha\Lambda^s\partial_t &= \alpha t\partial_t + \Lambda^{-s}[t, \Lambda^s]\alpha\partial_t + t \circ \Lambda^{-s}[\alpha, \Lambda^s] \circ \partial_t + E \\ &= \alpha \circ (t\partial_t - s) + tB + E, \end{aligned}$$

microlocally near  $\tilde{\Gamma}^+$ . Thus microlocally near  $\tilde{\Gamma}^+$ ,  $\Lambda^{-s}\mathcal{L}_0\Lambda^s = \mathcal{L}_s$  becomes

$$\mathcal{L}_s = \bar{\ell}_s\ell_s + (\beta_1 + A)\bar{\ell}_s + (\beta_2 + A)\ell_s + (\beta_3 + A),$$

where  $\bar{\ell}_s, \ell_s$  are first-order differential operators differing from  $\bar{\ell}, \ell$  respectively by terms of order zero, and taking the forms  $\bar{\ell}_s = \partial_x + i\alpha(t\partial_t - s)$ ,  $\ell_s = \partial_x - i\bar{\alpha}(t\partial_t - s)$  for  $|x| \leq r/2$ , where  $\alpha$  depends only on  $t$  and the  $\beta_i$  only on  $x$ .

To see that  $\bar{\ell}_s$  does take the form claimed for  $|x| \leq r/2$ , express  $\bar{\ell} = \partial_x + i\alpha t\partial_t$  modulo terms  $\gamma(x, t)\partial_x$  and  $\gamma(x, t)\partial_t$  where  $\gamma \equiv 0$  for  $|x| \leq r/2$ . Then  $\Lambda^{-s}[\gamma(x, t)\partial_x, \Lambda^s]$  and  $\Lambda^{-s}[\gamma(x, t)\partial_t, \Lambda^s]$  are operators of the type  $A$ , since they have nonpositive orders and their symbols of order zero vanish identically for  $|x| < r/2$ .  $\square$

#### 4. A TWO-DIMENSIONAL PROBLEM AND PRELIMINARY INEQUALITIES

The remainder of the paper consists of a self-contained analysis of a special class of pseudodifferential equations in a real two-dimensional region. We begin by describing the equations in question and fixing notation, which in some respects differs from that of preceding sections.

Fix an interval  $I = [-r, r] \subset \mathbf{R}$ . Denote by  $(x, t) \in \mathbf{R}^2$  coordinates in a neighborhood  $U$  of  $\mathcal{D} = I \times \{0\}$ . The interval  $\mathcal{D}$  corresponds to the degenerate annulus embedded in the boundary of the worm domain, and will be the focus of attention. The convention concerning the symbols  $A, B, E$  introduced at the outset of §3 remains in force.

Consider a one parameter family of pseudodifferential operators of the form

$$(4.1) \quad \mathcal{L}_s = \bar{L}L + (\beta_1(x) + A)\bar{L} + (\beta_2(x) + A)L + (\beta_3(x) + A),$$

where the  $\beta_j$  are  $C^\infty$  functions. Suppose that  $\bar{L}$ ,  $L$  are first-order differential operators depending on the real parameter  $s$ , and that  $-L$  is the formal adjoint of  $\bar{L}$ , modulo an operator of order zero. Suppose that where  $|x| \leq r$ , they take the special forms<sup>6</sup>

$$\begin{aligned} \bar{L} &= \partial_x + ia(x)(t\partial_t + s) + O(t^2)\partial_t, \\ L &= \partial_x - i\bar{a}(x)(t\partial_t + s) + O(t^2)\partial_t. \end{aligned}$$

Here  $O(t^2)$  denotes multiplication by a smooth function divisible by  $t^2$  on the region  $U$ . Here  $a$  and the coefficients  $\beta_j$  are assumed independent of  $s$ , but  $A$  and the terms  $O(t^2)\partial_t$  are permitted to depend on  $s$ .

Assume that

$$(4.2) \quad \operatorname{Re} a(x) \neq 0 \quad \text{for all } x \in I,$$

and that there exist smooth real-valued coefficients  $\mu, \nu$  such that  $[\bar{L}, L] = i\mu(x, t)\partial_t + i\nu(x, t)\operatorname{Re} \bar{L}$ , satisfying

$$(4.3) \quad \mu \geq 0 \text{ at every point of } U.$$

Because  $\bar{L}^* = -L$  modulo a term of order zero,  $L$  has the same real part as  $\bar{L}$ . A change of variables of the form  $(x, t) \mapsto (x, h(x, t))$ , with  $h(x, 0) \equiv 0$  where  $|x| \leq r$ , therefore reduces matters to the case where the real parts of both  $\bar{L}$  and  $L$  are everywhere parallel to  $\partial_x$ , and  $\bar{L} = \partial_x + i\tilde{a}(x, t)(t\partial_t + s) + O(t^2)\partial_t$  on  $I \times \mathbf{R}$ , with  $\tilde{a}$  real-valued and nonvanishing. Rewrite  $\tilde{a}(x, t) = a(x) + O(t)$ , and incorporate the contribution of  $O(t)$  into the various terms  $O(t^2)\partial_t$  and  $A$ . (4.3) is invariant under diffeomorphism and hence  $\mu$  cannot change sign, so the coefficient of  $t$  in the Taylor expansion of  $\mu(x, t)$  about  $t = 0$  must vanish identically, for  $|x| \leq r$ . This forces  $\partial_x a(x) \equiv 0$  there. Thus

$$(4.4) \quad \bar{L} = \partial_x + ia(t\partial_t + s) + O(t^2)\partial_t$$

for  $|x| \leq r$ , where  $a$  is a nonzero real constant. Moreover,  $\partial_x$  may be expressed in  $U$  as a nonvanishing scalar multiple of  $\bar{L} + L$ , modulo an operator of order zero. From now on we work in these new coordinates.

Define  $\Gamma = \{(x, t, \xi, \tau) : (x, t) \in \mathcal{D} \text{ and } \xi = 0\}$ . Decompose  $\Gamma = \Gamma^+ \cup \Gamma^-$  where  $\Gamma^+ = \{\tau > 0\} \cap \Gamma$ . Then by (4.3), in some conic neighborhood of  $\Gamma^+$  the principal symbol of  $[\bar{L}, L]$  equals a nonpositive symbol, modulo terms in the span of the symbols of  $\bar{L}, L$  and a term of order zero.

The symbol  $\|\cdot\|$ , with no subscript, denotes the norm in  $L^2(U)$ , while  $\|\cdot\|_t$  denotes any fixed norm for the Sobolev space  $H^t$  of functions having  $t$  derivatives in  $L^2$  and supported in  $U$ . The goal of the remainder of the paper is the following *a priori* estimate.

**Proposition 2.** *Let  $\{\mathcal{L}_s\}$  be a family of operators of the form (4.1) satisfying all of the hypotheses introduced above. Then there exists a discrete exceptional set  $S \subset [0, \infty)$  such that for any  $s \notin S$  and any pseudodifferential operator  $Q$  of order 0 whose principal symbol is nonzero in some conic neighborhood of  $\Gamma^-$ , there exist*

<sup>6</sup>No assumption is now made on the vanishing or nonvanishing of the coefficient of  $\partial_t$  in  $\bar{L}$  where  $|x| > r$ , but the strict pseudoconvexity of  $\mathcal{W}$  outside the exceptional annulus was used to reduce Proposition 1 to Proposition 2 below.

$C < \infty$  and a neighborhood  $W$  of  $\mathcal{D}$  such that for every  $C^\infty$  function  $u$  supported in  $W$ ,

$$\|u\| + \|\bar{L}u\| + \|Lu\| \leq C \cdot (\|\mathcal{L}_s u\| + \|u\|_{-1} + \|Qu\|_1).$$

The key conclusions are that there is no loss of derivatives in estimating  $u$  in terms of  $\mathcal{L}_s u$ , and that this holds for a sequence of values of  $s$  tending to  $+\infty$ . The assumption that  $u \in C^\infty$  is essential. All hypotheses of §4 are satisfied by the family of operators  $\mathcal{L}_s$  derived in §2 and §3. Proposition 2 thus implies the validity of (3.2), and hence of Proposition 1, which in turn implies our Theorem.

Our first preliminary estimate is a standard one valid for all  $s \in \mathbf{R}$ .

**Lemma 1.** *For each exponent  $s$  and each  $Q$  there exists  $C < \infty$  such that*

$$\|\partial_x u\| \leq C\|u\| + C\|\mathcal{L}_s u\| + C\|Qu\|_1$$

for every  $u \in C_0^\infty(U)$ .

*Proof.* For  $(x, t) \in \mathcal{D}$ ,  $\sigma_2(\mathcal{L}_s)(x, t, \xi, \tau) = 0$  if and only if  $(x, t, \xi, \tau) \in \Gamma$ . Therefore, the characteristic variety of  $\mathcal{L}_s$  in  $T^*W$  is contained in an arbitrarily small conic neighborhood of  $\Gamma$  as  $\delta \rightarrow 0$ . Consequently, there exists an operator  $\tilde{Q}$  of order zero such that firstly,  $T^*W$  is contained in the union of the two regions where  $\tilde{Q}$  is elliptic and  $\tau > 0$ , and secondly, the symbol of  $\tilde{Q}$  is supported in the union of the two regions where  $\mathcal{L}_s$  is elliptic, and where  $Q$  is elliptic.

Since  $\mathcal{L}_s$  is elliptic outside a small conic neighborhood of  $\Gamma$ , the  $H^2$  norm of  $u$  is majorized away from  $\Gamma$  by  $\|\mathcal{L}_s u\| + \|u\|_{-1}$ , while in a conic neighborhood of  $\Gamma^-$  the  $H^1$  norm of  $u$  is majorized by  $\|Qu\|_1 + \|u\|_{-1}$ .

Write  $\langle f, g \rangle = \int_U f \bar{g} \, dx \, dt$ . By Gårding's inequality and the fact that  $i\mu \cdot i\tau \leq 0$  in the support of the symbol of  $\tilde{Q}$ ,

$$\begin{aligned} -\operatorname{Re}(\langle \bar{L}Lu, u \rangle) &\geq c\|Lu\|^2 + c\|\bar{L}u\|^2 - C\|\bar{L}u\| \cdot \|u\| \\ &\quad - C\|Lu\| \cdot \|u\| - C\|u\|^2 - C\|\tilde{Q}u\|_1^2. \end{aligned}$$

The second condition imposed on  $\tilde{Q}$  ensures that

$$\|\tilde{Q}u\|_1 \leq C\|\mathcal{L}_s u\| + C\|Qu\|_1 + C\|u\|_{-1}.$$

Estimating  $\langle (\mathcal{L}_s - \bar{L}L)u, u \rangle$  by Cauchy-Schwarz thus leads to

$$\|\bar{L}u\| + \|Lu\| \leq C\|\mathcal{L}_s u\| + C\|u\| + C\|Qu\|_1.$$

But  $\partial_x$  may be expressed as a linear combination of  $\bar{L}$  and of  $L$  modulo an operator of order 0. □

**Lemma 2.** *There exists  $C < \infty$  such that for every  $f \in C^1(\mathbf{R})$  and every  $\varepsilon > 0$ ,*

$$\|f\|_{L^2[\varepsilon, 2\varepsilon]} \leq C\|f\|_{L^2[-2\varepsilon, -\varepsilon]} + C\varepsilon\|\partial_x f\|_{L^2(\mathbf{R})}.$$

*Likewise*

$$|f(0) - f(-\varepsilon)| \leq C\varepsilon^{1/2}\|\partial_x f\|_{L^2}.$$

The conclusions are invariant under translation, and the lemma will be invoked in that more general form.

*Proof.* For each  $x \in (\varepsilon, 2\varepsilon)$ ,  $|f(x) - f(x - 3\varepsilon)| \leq \int_{-2\varepsilon}^{2\varepsilon} |\partial_x f(y)| \, dy$  and both conclusions follow from the triangle and Cauchy-Schwarz inequalities. □

To simplify notation define

$$\mathfrak{B} = \|\mathcal{L}_s u\| + \|u\|_{-1} + \|Qu\|_1.$$

Let  $\delta > 0$  be a small constant to be chosen in §6, and assume  $u$  to be supported in

$$W \subset \{(x, t) : |t| < \delta, |x| < r + \delta\}.$$

Applying Lemma 2 to the function  $x \mapsto u(x, t)$  for each  $t$  and applying Lemma 1 gives the following estimate, under the hypotheses of Lemma 1.

**Lemma 3.**

$$\|\partial_x u\| + \|u\| \leq C\|u\|_{L^2(I \times (-\delta, \delta))} + C\mathfrak{B}.$$

5. LIMITING OPERATORS AND MELLIN TRANSFORM

Let  $a$  be a nonvanishing  $C^\infty$ , real-valued function. For  $\zeta \in \mathbf{C}$  define the ordinary differential operator

$$H_\zeta = (\partial_x + i\zeta a(x))(\partial_x - i\zeta a(x)) + \beta_1(x)(\partial_x + i\zeta a(x)) + \beta_2(x)(\partial_x - i\zeta a(x)) + \beta_3(x),$$

acting on functions of  $x \in I$ . Only the case of constant  $a$  will be needed in this paper, but the general case arises in another problem and hence merits discussion.

**Definition 1.**  $\mathfrak{S}$  is defined to be the set of all  $\zeta \in \mathbf{C}$  such that there exists a solution  $g$  of  $H_\zeta g \equiv 0$  on  $I$ , satisfying  $g(-r) = g(r) = 0$ .

For any complex number  $w$  we write  $\langle w \rangle = (1 + |w|^2)^{1/2}$ .

**Lemma 4.**  $\mathfrak{S}$  is a discrete subset of  $\mathbf{C}$ , and for any compact subset  $K$  of  $[0, \infty)$ , the set of all  $\zeta \in \mathfrak{S}$  having real part in  $K$  is finite. For each  $s$  such that  $\mathfrak{S} \cap (s + i\mathbf{R}) = \emptyset$  there exists  $C < \infty$  such that for all  $\zeta \in s + i\mathbf{R}$ , for all  $f, \varphi, \psi \in C^\infty(I)$  satisfying  $H_\zeta f = \varphi + \partial_x \psi$ , one has

$$(5.1) \quad \begin{aligned} & \|f\|_{L^2(I)} + \langle \zeta \rangle^{-1} \|\partial_x f\|_{L^2(I)} \\ & \leq C\langle \zeta \rangle^{-1/2} (|f(-r)| + |f(r)|) + C\langle \zeta \rangle^{-2} \|\varphi\|_{L^2(I)} + C\langle \zeta \rangle^{-1} \|\psi\|_{L^2(I)}. \end{aligned}$$

*Proof.* Throughout this proof, all norms without subscripts denote  $L^2$  norms. The selfadjoint part of  $H_\zeta$ , applied to  $f$ , equals  $(\partial_x - \gamma a(x))(\partial_x + \gamma a(x))f$ , modulo  $O(\langle \gamma \rangle \|f\| + \|\partial_x f\|)$ , in the  $L^2(I)$  norm. Therefore for  $s$  in any fixed compact subset of  $\mathbf{R}$  and any  $\zeta = s + i\gamma \in s + i\mathbf{R}$ , for any  $f$  vanishing at both endpoints of  $I$ ,

$$-\operatorname{Re}\langle H_\zeta f, f \rangle \geq \|\partial_x f\|^2 + \gamma^2 \int_I |f|^2 |a|^2 - O(\langle \gamma \rangle \|f\|^2 + \|f\| \cdot \|\partial_x f\|).$$

The coefficient  $a$  vanishes nowhere, while

$$|\langle H_\zeta f, f \rangle| = |\langle \varphi + \partial_x \psi, f \rangle| \leq \|f\| \cdot \|\varphi\| + \|\partial_x f\| \cdot \|\psi\|.$$

Combining the last two inequalities and invoking the Cauchy-Schwarz inequality and small constant – large constant trick, one obtains

$$(5.2) \quad \gamma^2 \|f\| + |\gamma| \cdot \|\partial_x f\| \leq C\|\varphi\| + C|\gamma| \cdot \|\psi\|$$

for all sufficiently large  $|\gamma|$ , under the additional hypothesis that  $f$  vanishes at both endpoints of  $I$ .

There exists a unique solution  $\phi_\zeta$  of  $H_\zeta \phi_\zeta = 0$  on  $I$ , satisfying  $\phi_\zeta(-r) = 0$ ,  $\partial_x \phi_\zeta(-r) = 1$ . Then  $\phi_\zeta(r)$  is an entire holomorphic function of  $\zeta$ , and  $\zeta \in \mathfrak{S} \Leftrightarrow$

$\phi_\zeta(r) = 0$ . We have seen that  $\zeta \notin \mathfrak{S}$  provided that the imaginary part of  $\zeta$  is sufficiently large, when the real part stays in a bounded set. Thus  $\phi_\zeta(r)$  is nonconstant, so has discrete zeros.<sup>7</sup>

To prove (5.1) let  $\zeta = s + i\gamma$  and  $f \in C^2(I)$  be given, and decompose  $f = g + h$  where  $H_\zeta g \equiv 0$  and  $h$  vanishes at the endpoints of  $I$ . The hypothesis  $\mathfrak{S} \cap (s + i\mathbf{R}) = \emptyset$  means that the Dirichlet nullspace of  $H_\zeta$  is  $\{0\}$ , so by elementary reasoning we conclude that for each  $\gamma$  there exists  $C < \infty$  such that  $\|h\| + \|\partial_x h\| \leq C\|\varphi\| + \|\psi\|$ , since  $H_\zeta h = H_\zeta f = \varphi + \partial_x \psi$ . Moreover, since  $H_\zeta$  depends continuously on  $\zeta$ ,  $C$  may be taken to be independent of  $\zeta$  in any compact subset of  $\mathbf{C} \setminus \mathfrak{S}$ . When  $|\gamma|$  is sufficiently large, on the other hand, (5.2) implies  $\|h\| + \langle \zeta \rangle^{-1} \|\partial_x h\| \leq C \langle \zeta \rangle^{-2} \|\varphi\| + C \langle \zeta \rangle^{-1} \|\psi\|$ . Thus the component  $h$  of  $f$  satisfies (5.1).

We have  $H_\zeta g = 0$ , so that clearly  $\|g\|$  and  $\|\partial_x g\|$  are majorized by  $C|f(r)| + C|f(-r)|$ , uniformly for  $\zeta$  in any compact set disjoint from  $\mathfrak{S}$ . Assuming henceforth that  $|\gamma|$  is large, the equation gives the inequality

$$\|\partial_x^2 g\| \leq C\gamma^2 \|g\| + C|\gamma| \cdot \|\partial_x g\|.$$

Integrating by parts as in the proof of (5.2) yields

$$(5.3) \quad \|\partial_x g\|^2 + \gamma^2 \|g\|^2 \leq C|g(-r)\partial_x g(-r)| + C|g(r)\partial_x g(r)|.$$

To control the right-hand side we use the bound

$$|\partial_x g(-r)| + |\partial_x g(r)| \leq C|\gamma|^{1/2} \|\partial_x g\| + C|\gamma|^{-1/2} \|\partial_x^2 g\|.$$

Indeed, setting  $v = \partial_x g$ , for any  $r' \in [r - |\gamma|^{-1}, r]$

$$|v(r) - v(r')| \leq C \int_{r'}^r |\partial_x v| \leq C|\gamma|^{-1/2} \|\partial_x v\|_{L^2}.$$

Then

$$|v(r)| \leq |\gamma| \int_{r-|\gamma|^{-1}}^r |v(r) - v(r')| dr' + |\gamma| \int_{r-|\gamma|^{-1}}^r |v(r')| dr'$$

and the desired bound follows by Cauchy-Schwarz.

Putting this into (5.3), introducing a parameter  $\lambda \in \mathbf{R}^+$  and applying Cauchy-Schwarz yields

$$\begin{aligned} & \gamma^2 \|g\|^2 + \|\partial_x g\|^2 \\ & \leq C\lambda |g(-r)|^2 + C\lambda |g(r)|^2 + C\lambda^{-1} |\gamma| \cdot \|\partial_x g\|^2 + C\lambda^{-1} |\gamma|^{-1} \|\partial_x^2 g\|^2 \\ & \leq C\lambda |g(-r)|^2 + C\lambda |g(r)|^2 + C\lambda^{-1} |\gamma| \cdot \|\partial_x g\|^2 \\ & \quad + C\lambda^{-1} |\gamma|^{-1} (\gamma^4 \|g\|^2 + \gamma^2 \|\partial_x g\|^2) \\ & \leq C\lambda |g(-r)|^2 + C\lambda |g(r)|^2 + C\lambda^{-1} |\gamma| \cdot \|\partial_x g\|^2 + C\lambda^{-1} |\gamma|^3 \|g\|^2. \end{aligned}$$

Choose  $\lambda$  to be a large constant times  $|\gamma|$ . Then the last two terms on the right-hand side may be absorbed into the left, leaving

$$\gamma^2 \|g\|^2 + \|\partial_x g\|^2 \leq C|\gamma| \cdot |g(-r)|^2 + C|\gamma| \cdot |g(r)|^2.$$

Since  $g = f$  at the endpoints of  $I$ , this is the desired inequality for  $g$ . Adding it to that for  $h$  concludes the proof.  $\square$

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<sup>7</sup>An alternative method of proof would be to combine (5.2) with general results from the perturbation theory of linear operators [Ka], utilizing again the holomorphic dependence of  $H_\zeta$  on  $\zeta$ .

**Definition 2.**

$$S = \{s \in [0, \infty) : \text{there exists } \gamma \in \mathbf{R} \text{ such that } s - \frac{1}{2} + i\gamma \in \mathfrak{S}\}.$$

Lemma 4 guarantees that  $S$  is discrete.

Specialize now to the case where  $a(x) \equiv a$ , the real constant in (4.4). Define

$$\begin{aligned} \mathbb{L}_s = & (\partial_x + ia(t\partial_t + s)) \circ (\partial_x - ia(t\partial_t + s)) \\ & + \beta_1(x)(\partial_x + ia(t\partial_t + s)) + \beta_2(x)(\partial_x - ia(t\partial_t + s)) + \beta_3(x). \end{aligned}$$

Expanding the last term in the expression  $\mathbb{L}_s = \mathcal{L}_s + (\mathbb{L}_s - \mathcal{L}_s)$  gives

$$\mathbb{L}_s u = \Phi + \partial_x \Psi,$$

where

$$(5.4) \quad \begin{aligned} \Phi &= \mathcal{L}_s u + (t\partial_t)^2 Au + t\partial_t Au + Au, \\ \Psi &= t\partial_t Au + Au. \end{aligned}$$

To reach (5.4) we may for instance express  $t\partial_t \circ O(t^2)\partial_t$  as  $(t\partial_t)^2 A + t\partial_t A + A$ , since multiplication by  $t$  is an operator of the type  $A$ . Likewise  $[A, t\partial_t] = t[A, \partial_t] + [A, t]\partial_t$  is an operator of type  $A$ , because  $\sigma_{-1}([A, t]) = c\partial_\tau \sigma_0(A)$  vanishes identically for  $(x, t) \in \mathcal{D}$  since  $\sigma_0(A)$  itself vanishes there.

The partial Mellin transform of  $f$  with respect to the  $t$  variable is defined to be

$$\hat{f}(x, \gamma) = \int_0^\infty f(x, t)t^{-i\gamma} t^{-1} dt,$$

provided that the integral converges. If  $f(x, \cdot) \in C^\infty[0, \infty)$  has bounded support for each  $x$ , then the integral defining  $\hat{f}(x, \gamma)$  converges absolutely whenever  $\gamma$  has strictly positive imaginary part, and  $\hat{f}(x, \gamma)$  extends to a meromorphic function of  $\gamma \in \mathbf{C}$ , whose only possible poles are at  $\gamma = 0, -i, -2i, \dots$ . Clearly

$$(t\partial_t f)^\wedge(x, \gamma) = i\gamma \hat{f}(x, \gamma)$$

for all such  $f$ . Consequently

$$(\mathbb{L}_s u)^\wedge(x, \gamma) = H_{s+i\gamma} \hat{u}(x, \gamma) \quad \text{for all } \gamma \in \mathbf{C} \setminus \{0, -i, -2i, \dots\}.$$

The Mellin inversion and Plancherel formulas read

$$f(x, t) = c \int_{\mathbf{R}} \hat{f}(x, \gamma) t^{i\gamma} d\gamma, \quad \int_0^\infty |f(x, t)|^2 t^{-1} dt = c' \int_{\mathbf{R}} |\hat{f}(x, \gamma)|^2 d\gamma.$$

It follows directly from the definitions that  $(t^{1/2} f)^\wedge(x, \gamma) = \hat{f}(x, \gamma + \frac{i}{2})$  for all  $\gamma \in \mathbf{R}$ . Thus the Plancherel identity may be rewritten as

$$\int_0^\infty |f(x, t)|^2 dt = \int_0^\infty |t^{1/2} f(x, t)|^2 t^{-1} dt = c' \int_{\mathbf{R}} |\hat{f}(x, \gamma + \frac{i}{2})|^2 d\gamma.$$

6. PROOF OF THE MAIN ESTIMATE

We may now estimate  $u$  in terms of  $\mathcal{L}_s u$ . To begin,

$$\iint_{I \times [0, \delta)} |u(x, t)|^2 dx dt = c' \iint_{I \times \mathbf{R}} |\hat{u}(x, \gamma + \frac{i}{2})|^2 d\gamma dx.$$

Assume that  $s \notin S$ , and write  $\zeta = s - \frac{1}{2} + i\gamma$ . With  $\Phi, \Psi$  defined as in (5.4),  $H_\zeta \hat{u}(x, \gamma + \frac{i}{2}) = \hat{\Phi}(x, \gamma + \frac{i}{2}) + \partial_x \hat{\Psi}(x, \gamma + \frac{i}{2})$ . Applying Lemma 4 on  $I$  yields for each  $\gamma \in \mathbf{R}$

$$(6.1) \quad \int_I \hat{u}(x, \gamma + \frac{i}{2}) dx \leq C \int_I |\hat{\Phi}(x, \gamma + \frac{i}{2})|^2 \langle \gamma \rangle^{-4} dx + C \int_I |\hat{\Psi}(x, \gamma + \frac{i}{2})|^2 \langle \gamma \rangle^{-2} dx + C |\hat{u}(-r, \gamma + \frac{i}{2})|^2 + C |\hat{u}(r, \gamma + \frac{i}{2})|^2.$$

**Lemma 5.** *Assume  $W \subset \{(x, t) : |t| < \delta, |x| < r + \delta\}$ . Then there exists  $C < \infty$  such that*

$$\iint_{I \times \mathbf{R}} |\hat{\Phi}(x, \gamma + \frac{i}{2})|^2 \langle \gamma \rangle^{-4} d\gamma dx \leq C\delta^2 \|u\|^2 + C\mathfrak{B}^2$$

and

$$\iint_{I \times \mathbf{R}} |\hat{\Psi}(x, \gamma + \frac{i}{2})|^2 \langle \gamma \rangle^{-2} d\gamma dx \leq C\delta^2 \|\partial_x u\|^2 + C\mathfrak{B}^2.$$

Granting the lemma, we conclude from (6.1) that

$$\|u\|_{L^2(I \times [0, \delta])}^2 \leq C\delta^2 \|u\|^2 + C\delta^2 \|\partial_x u\|^2 + C\mathfrak{B}^2 + C \int_{\mathbf{R}} |u(r, t)|^2 dt + C \int_{\mathbf{R}} |u(-r, t)|^2 dt.$$

But by Lemma 2 and the assumption that  $u(x, t) \equiv 0$  for  $|x| > r + \delta$ , this last term is dominated by  $C\delta \|\partial_x u\|^2$ . Thus

$$\|u\|_{L^2(I \times [0, \delta])}^2 \leq C\delta^2 \|u\|^2 + C\delta \|\partial_x u\|^2 + C\mathfrak{B}^2.$$

All the same reasoning applies on the region  $I \times (-\delta, 0]$ , after the change of variables  $t \mapsto -t$ . Thus

$$\|u\|_{L^2(I \times (-\delta, \delta))} \leq C\delta^{1/2} (\|u\| + \|\partial_x u\|) + C\mathfrak{B}.$$

Combining this with Lemma 3 gives

$$\|u\| + \|\partial_x u\| \leq C\delta^{1/2} (\|u\| + \|\partial_x u\|) + C\mathfrak{B},$$

so choosing  $\delta$  to be sufficiently small gives  $\|u\| \leq C\mathfrak{B}$ , concluding the proof.  $\square$

*Proof of Lemma 5.* The principal term in the double integral of the lemma for  $\Phi$  is of course the contribution of  $\mathcal{L}_s u$ :

$$\begin{aligned} & \iint_{I \times \mathbf{R}} |(\mathcal{L}_s u)^\wedge(x, \gamma + \frac{i}{2})|^2 \langle \gamma \rangle^{-4} d\gamma dx \\ & \leq \iint_{I \times \mathbf{R}} |(\mathcal{L}_s u)^\wedge(x, \gamma + \frac{i}{2})|^2 d\gamma dx \leq C \|\mathcal{L}_s u\|^2, \end{aligned}$$

as desired.

Any operator  $A$  of order  $\leq 0$  satisfying  $\sigma_0(A)(x, t, \xi, \tau) \equiv 0$  for  $(x, t) \in \mathcal{D}$  satisfies

$$\|Au\|^2 \leq C\delta^2\|u\|^2 + C\|u\|_{-1}^2$$

for all  $u$  supported in  $W$ , as  $\delta \rightarrow 0$ . A typical term of  $\Phi$  resulting from  $\mathbb{L}_s u - \mathcal{L}_s u$  is  $(t\partial_t)^2 Au$ . Its contribution to the first double integral in Lemma 5 is

$$\begin{aligned} \iint_{I \times \mathbf{R}} |((t\partial_t)^2 Au)^\wedge(x, \gamma + \frac{i}{2})|^2 \langle \gamma \rangle^{-4} d\gamma dx &\leq C \iint_{I \times \mathbf{R}} |(Au)^\wedge(x, \gamma + \frac{i}{2})|^2 d\gamma dx \\ &= C \iint_{I \times [0, \delta)} |Au(x, t)|^2 dx dt \\ &\leq C\delta^2\|u\|^2 + C\|u\|_{-1}^2. \end{aligned}$$

A typical constituent of  $\Psi$  is the term  $t\partial_t Au$ . Its contribution is dominated by

$$C \iint_{I \times \mathbf{R}} |(t\partial_t Au)^\wedge(x, \gamma + \frac{i}{2})|^2 \langle \gamma \rangle^{-2} d\gamma dx \leq C \iint_{I \times \mathbf{R}} |(Au)^\wedge(x, \gamma + \frac{i}{2})|^2 d\gamma dx$$

and the remainder of the calculation is as above.  $\square$

*Comments.* The author can advance no reason why the method of reduction to the boundary should be essential to this analysis. Working directly on  $\overline{W}$  might well result in a shorter proof. On the other hand, the analysis in §§4–6 applies with minor modification to a broader class of equations unconnected with the  $\bar{\partial}$ -Neumann problem.

Some refinements of this analysis and related observations will appear in [Ch2].

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