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GENERALIZATIONS OF MÜNTZ'S THEOREM VIA A REMEZ-TYPE INEQUALITY FOR MÜNTZ SPACES

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Dedicated to the memory of Paul Erdős

1. Introduction

Müntz's beautiful, classical theorem characterizes sequences $\Lambda := (\lambda_i)_{i=0}^\infty$ with

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \tag{1.1}$$

for which the Müntz space $M(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is dense in $C[0,1]$. Here, and in what follows, $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ denotes the collection of finite linear combinations of the functions $x^{\lambda_0}, x^{\lambda_1}, \dots$ with real coefficients, and $C(A)$ is the space of all real-valued continuous functions on $A \subset [0,\infty)$ equipped with the uniform norm. If $A := [a,b]$ is a finite closed interval, then the notation $C[a,b] := C([a,b])$ will be used. Throughout this paper $\Lambda := (\lambda_i)_{i=0}^\infty$ denotes a sequence satisfying (1.1). Müntz's Theorem ([11], [18], [27], [30]) states the following.

Theorem. *$M(\Lambda)$ is dense in $C[0,1]$ if and only if $\sum_{i=1}^\infty 1/\lambda_i = \infty$.*

The original Müntz Theorem proved by Müntz [18] in 1914, by Szász [27] in 1916, and anticipated by Bernstein [3] was only for sequences of exponents tending to infinity. The point 0 is special in the study of Müntz spaces. Even replacing $[0,1]$ by an interval $[a,b] \subset [0,\infty)$ in Müntz's Theorem is a non-trivial issue. This is, in large measure, due to Clarkson and Erdős [12] and Schwartz [24] whose works include the result that if $\sum_{i=1}^\infty 1/\lambda_i < \infty$, then every function belonging to the uniform closure of $M(\Lambda)$ on $[a,b]$ can be extended analytically throughout the region

$$\{z \in \mathbb{C} \setminus (-\infty,0] : |z| < b\}.$$

There are many generalizations and variations of Müntz's Theorem ([1], [4], [5], [6], [7], [8], [9], [17], [19], [24], [26], [28], [29]). There are also still many open problems. The proper generalizations to many variables are still open.

In Section 6 of this paper we show that the interval $[0,1]$ in Müntz's Theorem can be replaced by an arbitrary compact set $A \subset [0,\infty)$ of positive Lebesgue measure.

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That is, if $A \subset [0,\infty)$ is a compact set of positive Lebesgue measure, then $M(\Lambda)$ is dense in $C(A)$ if and only if $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$.

If A contains an interval, then this follows from the already mentioned results of Clarkson, Erdős, and Schwartz. However, their results and methods cannot handle the case that, for example, $A \subset [0,1]$ is a Cantor-type set of positive measure. Note also that the scaling $x \to \alpha x$ reduces the case that $A \subset [0,\infty)$ is compact to the case that $A \subset [0,1]$ is closed, so working on $[0,1]$ is perfectly general.

In the case that $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, analyticity properties of the functions belonging to the uniform closure of $M(\Lambda)$ on A are also established.

An analogue of the above result is also proved in $L_w^q(A)$, where w is a non-negative integrable weight function on A with $\int_A w > 0$, and $q \in (0,\infty)$.

Speculations about the above extensions of Müntz's Theorem are probably as old as Müntz's Theorem itself.

Somorjai [26] and Bak and Newman [2], [20] proved that

$$R(\Lambda) := \{p/q : p, q \in M(\Lambda)\}$$

is always dense in $C[0,1]$. (Though (1.1) is assumed throughout this paper, the above result holds for an arbitrary $\Lambda := (\lambda_i)_{i=0}^{\infty}$ containing infinitely many distinct real numbers.) This surprising result says that while the set $M(\Lambda)$ of Müntz polynomials may be far from dense, the set $R(\Lambda)$ of Müntz rationals is always dense in $C[0,1]$, no matter what the underlying sequence Λ. In light of this result, Newman, in 1978 [20, p. 50], raises "the very sane, if very prosaic question": Are the functions

$$\prod_{j=1}^{k} \left(\sum_{i=0}^{n_j} a_{i,j} x^{i^2} \right), \qquad a_{i,j} \in \mathbb{R}, \quad n_j \in \mathbb{N},$$

dense in $C[0,1]$ for some fixed $k \geq 2$? In other words does the "extra multiplication" have the same power that the "extra division" has in the Bak-Newman-Somorjai result? Newman speculated that it did not.

Denote the set of the above products by H_k. Since every natural number is the sum of four squares, H_4 contains all the monomials x^n, $n = 0, 1, 2, \dots$. However, H_k is not a linear space, so Müntz's Theorem itself cannot be applied to resolve the denseness or non-denseness of H_4 in $C[0,1]$.

Section 7 of this paper deals with products of Müntz spaces and, in particular, answers the above question of Newman in the negative. For

$$\Lambda_j := (\lambda_{i,j})_{i=0}^{\infty}, \qquad 0 = \lambda_{0,j} < \lambda_{1,j} < \lambda_{2,j} < \cdots, \qquad j = 1, 2, \dots, \tag{1.2}$$

we define the sets

$$M(\Lambda_1, \Lambda_2, \dots, \Lambda_k) := \left\{ p = \prod_{j=1}^{k} p_j : p_j \in M(\Lambda_j) \right\}.$$

Bounded Remez-, Bernstein-, and Nikolskii-type inequalities are established for $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$ in the case that

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_{i,j}} < \infty, \qquad j = 1, 2, \dots, k. \tag{1.3}$$

Any of these obviously implies that if (1.2) and (1.3) hold and $A \subset [0,\infty)$ is a compact set of positive Lebesgue measure, then $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$ is not dense in

$C(A)$. In particular, H_4 is not dense in $C[0,1]$, which answers Newman's problem negatively. In addition, under the assumptions (1.2) and (1.3), our methods give an "almost characterization" of the uniform closure of $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$ on A in terms of analyticity properties. This will likely be discussed in a later publication of the authors.

The results of Sections 6 and 7 can be proved fairly simply, once one has established the bounded Remez-type inequality of Section 5 for non-dense Müntz spaces $M(\Lambda)$. This is the central result of the paper, and is a result we believe should be a basic tool for dealing with problems about Müntz spaces, in addition to those discussed in Sections 6 and 7.

Let $\mathcal{P}_n$ denote the set of all algebraic polynomials of degree at most n with real coefficients. For a fixed $s \in (0,1)$ let

$$\mathcal{P}_n(s) := \{p \in \mathcal{P}_n : m(\{x \in [0,1] : |p(x)| \le 1\}) \ge s\},$$

where $m(\cdot)$ denotes the linear Lebesgue measure. The classical Remez inequality concerns the problem of bounding the uniform norm of a polynomial $p \in \mathcal{P}_n$ on $[0,1]$ given that its modulus is bounded by 1 on a subset of $[0,1]$ of Lebesgue measure at least s. That is, how large can $\|p\|_{[0,1]}$ (the uniform norm of p on $[0,1]$) be if $p \in \mathcal{P}_n(s)$? The answer is given in terms of the Chebyshev polynomials. The extremal polynomials for the above problem are the Chebyshev polynomials $\pm T_n(x) := \pm \cos(n \arccos h(x))$, where h is a linear function which scales $[0,s]$ or $[1-s,1]$ onto $[-1,1]$.

For various proofs, extensions, and applications, see [13], [14], [15], [22], [23].

We generalize the Remez inequality in the following way. Let

$$M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}.$$

That is, $M_n(\Lambda)$ is the collection of Müntz polynomials

$$p(x) := \sum_{i=0}^{n} a_i x^{\lambda_i}, \qquad a_i \in \mathbb{R}.$$

We seek to find

$$\max\left\{\frac{|p(0)|}{\|p\|_A} : 0 \ne p \in M_n(\Lambda), \quad A \subset [0,1], \ m(A) \ge s\right\} \tag{1}$$

and

$$\max\left\{\frac{|p(1)|}{\|p\|_A} : 0 \ne p \in M_n(\Lambda), \quad A \subset [0,1], \ m(A) \ge s\right\}. \tag{2}$$

These two problems are no longer equivalent as they are in the polynomial case (since $x \to 1-x$ does not preserve membership in $M_n(\Lambda)$) and they have different answers. However, these two problems can be handled in essentially the same way. In Section 5 we concentrate on problem (1). Lemma 5.4 shows that an extremal function for problem (1) is the (generalized) Chebyshev polynomial

$$T_n := T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; [1-s,1]\}$$

for $M_n(\Lambda)$ on $[1-s,1]$ defined in Section 2. This reduces problem (1) to the interval case, $A = [1-s,1]$. The interval case can then be handled by a bounded Chebyshev-type inequality

$$\|p\|_{[0,1]} \le c\|p\|_{[1-s,1]}, \qquad p \in M(\Lambda) = \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\},$$

established in Section 3 for every $\Lambda := (\lambda_i)_{i=0}^\infty$ satisfying $\sum_{i=1}^\infty 1/\lambda_i < \infty$, where the constant c depends only on Λ and s (and not on the "length" of p). This we first prove under the gap condition

$$\inf\{\lambda_{i+1} - \lambda_i : i \in \mathbb{N}\} > 0\,.$$

However, some comparison lemmas of Section 4 will allow us to drop this condition. This leads to the central result of the paper, a bounded Remez-type inequality for non-dense Müntz spaces (see Theorem 5.1) which states the following. For every $\Lambda := (\lambda_i)_{i=0}^\infty$ with $\sum_{i=1}^\infty 1/\lambda_i < \infty$, and for every $s > 0$, there exists a constant c depending only on Λ and s (and not on ϱ, A, or the "length" of p) so that

$$\|p\|_{[0,\varrho]} \le c\|p\|_A$$

for every $p \in M(\Lambda) = \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ and for every set $A \subset [\varrho, 1]$ of Lebesgue measure at least s.

One might note that the existence of such a bounded Remez-type inequality for a Müntz space $M(\Lambda)$ is equivalent to the non-denseness of $M(\Lambda)$ in $C[0,1]$. Indeed, if $M(\Lambda)$ is not dense in $C[0,1]$, then by Müntz's Theorem, $\sum_{i=1}^\infty 1/\lambda_i < \infty$, and hence Theorem 5.1 implies that the above Remez-type inequality holds for $M(\Lambda)$. On the other hand, if $M(\Lambda)$ is dense in $C[0,1]$, then

$$\sup_{p \in M(\Lambda)} \frac{|p(0)|}{\|p\|_{[1-s,1]}} = \infty.$$

This follows from

$$\sup_{p \in C[0,1]} \frac{|p(0)|}{\|p\|_{[1-s,1]}} = \infty$$

by approximation.

In [8] the above result is established for lacunary Müntz spaces, that is, for $M(\Lambda)$ with

$$\inf\{\lambda_{i+1}/\lambda_i : i \in \mathbb{N}\} > 1\,.$$

Yet another remarkable consequence of the bounded Remez-type inequality of Theorem 5.1 is that the pointwise and locally uniform convergence of a sequence $(p_i)_{i=1}^\infty \subset M(\Lambda)$ on $[0,b)$ are equivalent whenever $\sum_{i=1}^\infty 1/\lambda_i < \infty$. See Theorem 6.3. In fact, one can characterize the non-dense Müntz spaces within the Müntz spaces $M(\Lambda)$ as exactly those in which locally uniform and pointwise convergence on $[0,b)$ are equivalent.

2. NOTATION

The notations

$$\|p\|_A := \sup_{x \in A} |p(x)|\,,$$

$$\|p\|_{L^q_w(A)} := \left(\int_A |p|^q w\right)^{1/q}\,,$$

and

$$\|p\|_{L^q(A)} := \left(\int_A |p|^q\right)^{1/q}$$

are used throughout this paper for measurable functions p defined on a measurable set $A \subset [0,\infty)$, for non-negative measurable weight functions w defined on A, and for $q \in (0,\infty)$. The space of all real-valued continuous functions on a set $A \subset [0,\infty)$ equipped with the uniform norm is denoted by $C(A)$.

The space $L^q_w(A)$ is defined as the collection of equivalence classes of real-valued measurable functions for which $\|f\|_{L^q_w(A))} < \infty$. The equivalence classes are defined by the equivalence relation $f \sim g$ if $fw = gw$ almost everywhere on A. When $A := [a,b]$ is a finite closed interval, we use the notation $L^q_w[a,b] := L^q_w(A)$. When $w := 1$, we use the notation $L^q[a,b] := L^q_w[a,b]$. Again, it is always our understanding that the space $L^q_w(A)$ is equipped with the $L^q_w(A)$ norm.

Throughout this paper $\Lambda := (\lambda_i)_{i=0}^\infty$ denotes a sequence of real numbers satisfying

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots .$$

The non-negative-valued functions x^{λ_i} are well-defined on $[0,\infty)$. The system

$$(x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n})$$

is called a (finite) Müntz system. The linear space

$$M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$$

over $\mathbb{R}$ is called a (finite) Müntz space. That is, the Müntz space $M_n(\Lambda)$ is the collection of Müntz polynomials

$$p(x) = \sum_{i=0}^{n} a_i x^{\lambda_i}\,, \qquad a_i \in \mathbb{R}\,.$$

The set

$$M(\Lambda) := \bigcup_{n=0}^{\infty} M_n(\Lambda) = \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

is called the (infinite) Müntz space associated with Λ.

One of the most basic properties of a Müntz space $M_n(\Lambda)$ is the fact that it is a Chebyshev space on every $A \subset [0,\infty)$ containing at least $n+1$ points. That is, $M(\Lambda) \subset C(A)$ and every $p \in M_n(\Lambda)$ having at least $n+1$ (distinct) zeros in A is identically 0. In fact, Müntz spaces are the "canonical" examples for Chebyshev spaces and the following properties of Müntz spaces $M_n(\Lambda)$ are well known (see, for example, [9], [11], [21]).

Theorem 2.1 (Unique Interpolation Property). *For every*

$$0 \le x_0 < x_1 < \cdots < x_n \quad \text{and} \quad y_0, y_1, \dots, y_n \in \mathbb{R}\,,$$

there exists a unique $p \in M_n(\Lambda)$ so that

$$p(x_j) = y_j\,, \qquad j = 0, 1, \dots, n\,.$$

Theorem 2.2 (Existence of Chebyshev Polynomials). *Let A be a compact subset of $[0,\infty)$ containing at least $n+1$ points. Then there exists a unique (extended) Chebyshev polynomial*

$$T_n := T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; A\}$$

for $M_n(\Lambda)$ *on* A *defined by*

$$T_n(x) = c\left(x^{\lambda_n} - \sum_{i=0}^{n-1} a_i x^{\lambda_i}\right),$$

where the numbers $a_0, a_1, \dots, a_{n-1} \in \mathbb{R}$ *are chosen to minimize*

$$\left\| x^{\lambda_n} - \sum_{i=0}^{n-1} a_i x^{\lambda_i} \right\|_A ,$$

and where $c \in \mathbb{R}$ *is a normalization constant chosen so that*

$$\|T_n\|_A = 1$$

and the sign of c *is determined by*

$$T_n(\max A) > 0 .$$

Theorem 2.3 (Alternation Characterization). *The Chebyshev polynomial*

$$T_n := T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; A\} \in M_n(\Lambda)$$

is uniquely characterized by the existence of an alternation set

$$\{x_0 < x_1 < \cdots < x_n\} \subset A$$

for which

$$T_n(x_j) = (-1)^{n-j} = (-1)^{n-j}\|T_n\|_A , \qquad j = 0, 1, \dots, n .$$

3. BOUNDED CHEBYSHEV AND BERNSTEIN TYPE INEQUALITIES FOR $M(\Lambda)$

The main results of this section are the following two theorems.

Theorem 3.1. *Suppose* $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. *Let* $s \in (0,1)$. *Then there exists a constant* c *depending only on* $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ *and* s *(and not on the "length" of* p*) so that*

$$\|p\|_{[0,1]} \le c\,\|p\|_{[1-s,1]}$$

for every $p \in M(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$.

Theorem 3.2. *Suppose* $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$ *and* $\lambda_1 \ge 1$. *Let* $\varepsilon \in (0,1)$. *Then there exists a constant* c *depending only on* $\Lambda := (\lambda_i)_{i=0}^{\infty}$ *and* ε *(and not on the "length" of* p*) so that*

$$\|p'\|_{[0,1-\varepsilon]} \le c\,\|p\|_{[0,1]}$$

for every $p \in M(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$.

To prove the above two theorems we need three lemmas. Lemmas 3.3 and 3.4 establish the conclusion of Theorems 3.2 and 3.1, respectively, under the gap condition

$$\inf\{\lambda_{i+1} - \lambda_i : i \in \mathbb{N}\} > 0 , \tag{3.1}$$

which is then dropped with the aid of Lemma 4.5.

Lemma 3.3. *Suppose* $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, $\lambda_1 \geq 1$, *and the gap condition* (3.1) *holds. Let* $\varepsilon \in (0,1)$. *Then there exists a constant* c *depending only on* $\Lambda := (\lambda_i)_{i=0}^{\infty}$ *and* ε (*and not on the "length" of* p) *so that*

$$\|p'\|_{[0,1-\varepsilon]} \leq c\,\|p\|_{[0,1]}$$

for every $p \in M(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$.

Proof. Clarkson and Erdős [12] observed that under the conditions of the lemma, there exists a constant $c_1(\varepsilon)$ depending only on $\Lambda = (\lambda_i)_{i=0}^{\infty}$ and $\varepsilon \in (0,1)$ (and not on i and n) so that

$$\begin{aligned} |a_i| &\leq c_1(\varepsilon)(1+\varepsilon)^{\lambda_i}\,\|p\|_{L^2[0,1]} \\ &\leq c_1(\varepsilon)(1+\varepsilon)^{\lambda_i}\,\|p\|_{[0,1]} \end{aligned}$$

for every $p \in M(\Lambda)$ of the form

$$p(x) = \sum_{i=0}^{n} a_i x^{\lambda_i}\,, \qquad a_i \in \mathbb{R}\,.$$

Therefore, if $y \in [0, 1-\varepsilon]$ and $\|p\|_{[0,1]} \leq 1$, then

$$\begin{aligned} |p'(y)| &= \left|\sum_{i=1}^{n} \lambda_i a_i y^{\lambda_i - 1}\right| \leq \sum_{i=1}^{n} |a_i|\lambda_i |y|^{\lambda_i - 1} \\ &\leq \sum_{i=1}^{\infty} c_1(\varepsilon)\lambda_i (1+\varepsilon)^{\lambda_i}(1-\varepsilon)^{\lambda_i - 1} \\ &= \frac{c_1(\varepsilon)}{1-\varepsilon}\sum_{i=1}^{\infty} \lambda_i (1-\varepsilon^2)^{\lambda_i} \\ &\leq \frac{c_1(\varepsilon)}{1-\varepsilon}\sum_{i=1}^{k} \lambda_i (1-\varepsilon^2)^{\lambda_i} + \sum_{i=k+1}^{\infty} i^{-2} =: c\,, \end{aligned}$$

where $k \in \mathbb{N}$ is chosen so that

$$(1-\varepsilon^2)^{\lambda_i} \leq \frac{1}{\lambda_i i^2}\,, \qquad i = k+1, k+2, \dots\,.$$

This proves the lemma. □

Lemma 3.4. *Suppose* $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$ *and the gap condition* (3.1) *holds. Let* $s \in (0,1)$. *Then there exists a constant* c *depending only on* $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ *and* s (*and not on the "length" of* p) *so that*

$$\|p\|_{[0,1]} \leq c\,\|p\|_{[1-s,1]}$$

for every $p \in M(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$.

Proof. Using the scaling $x \to x^{1/\lambda_1}$, without loss of generality we may assume that $\lambda_1 = 1$. Suppose there exist

$$p_i \in M(\Lambda)\,, \qquad i = 1, 2, \dots\,,$$

so that

$$A_i := \|p_i\|_{[0,1]} \to \infty\,,$$

while

$$\|p_i\|_{[1-s,1]} = 1\,, \qquad i = 1, 2, \dots\,.$$

Let $q_i := p_i/A_i$. Note that $\|q_i\|_{[0,1]} = 1$ and

$$\lim_{i\to\infty} \|q_i\|_{[1-s,1]} = 0\,. \tag{3.2}$$

By Lemma 3.3, there exists a constant c depending only on $\Lambda = (\lambda_i)_{i=0}^{\infty}$ and ε so that

$$\|q_i'\|_{[0,1-\varepsilon]} \le c\,\|q_i\|_{[0,1]} = c$$

for every $\varepsilon \in (0,1)$. Hence $(q_i)_{i=1}^{\infty}$ is a sequence of uniformly bounded and equicontinuous functions on every closed subinterval of $[0,1)$. So, by the Arzela-Ascoli Theorem, we may extract a uniformly convergent subsequence on $[0, 1-s/2]$. By a theorem of Clarkson and Erdős [12], this subsequence converges uniformly to a function F analytic on $(0, 1-s/2)$. Combining this with (3.2) and the Unicity Theorem, we can deduce that F is identically zero. This is a contradiction since $\|q_i\|_{[0,1]} = 1$ and

$$\|q_i\|_{[0,1-s]} = \|q_i\|_{[0,1]}$$

for every sufficiently large i. The lemma is now proved. □

Proof of Theorem 3.1. Observe that $\lim\limits_{i\to\infty} \lambda_i/i = \infty$. Therefore, there is an $m \in \mathbb{N}$ so that $\lambda_i > 2i$ whenever $i > m$. Let $\Gamma := (\gamma_i)_{i=1}^{\infty}$ be defined by

$$\gamma_i := \begin{cases} \min\{\lambda_i, i\} & \text{if } i = 0, 1, \dots, m\,, \\ \lambda_i/2 + i & \text{if } i = m+1, m+2, \dots\,. \end{cases}$$

Then $0 = \gamma_0 < \gamma_1 < \gamma_2 < \cdots, \sum_{i=1}^{\infty} 1/\gamma_i < \infty$,

$$\gamma_i \le \lambda_i\,, \qquad i = 0, 1, 2, \dots\,,$$

and

$$\inf\{\gamma_{i+1} - \gamma_i : i \in \mathbb{N}\} > 0\,.$$

Now Lemma 3.4 and the first part of Lemma 4.5 will yield the theorem. □

Proof of Theorem 3.2. Without loss of generality we may assume that $\lambda_1 = 1$. Let $\Gamma := (\gamma_i)_{i=0}^{\infty}$ be defined as in the proof of Theorem 3.1. Now Lemma 3.3, Theorem 3.1, and the second part of Lemma 4.5 yield the theorem. □

4. COMPARISON LEMMAS

One of the basic properties of a Müntz system

$$(x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n})\,, \qquad 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n\,,$$

is that it is a Descartes system on every interval $[a,b] \subset (0,\infty)$; see [21]. The following comparison lemma, due to Pinkus and Smith [25], is valid for every Descartes system.

Lemma 4.1. *Suppose* $(f_0, f_1, \dots, f_n)$ *is a Descartes system on* $[a, b]$. *Suppose*

$$p = f_\alpha + \sum_{i=1}^{k} a_i f_{\lambda_i}\,, \qquad a_i \in \mathbb{R}\,,$$

$$q = f_\alpha + \sum_{i=1}^{k} b_i f_{\gamma_i}, \qquad b_i \in \mathbb{R}\,,$$

where $0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_k \le n,\ \ 0 \le \gamma_1 < \gamma_2 < \cdots < \gamma_k \le n,$

$$0 \le \gamma_i \le \lambda_i < \alpha\,, \qquad i = 1, 2, \dots, m\,,$$

and

$$\alpha < \lambda_i \le \gamma_i \le n\,, \qquad i = m+1, m+2, \dots, k\,,$$

with strict inequality for at least one index $i = 1, 2, \dots, k$. *Then*

$$p(x_i) = q(x_i) = 0\,, \quad i = 1, 2, \dots, k\,,$$

with distinct $x_i \in [a, b]$ *implies*

$$|p(x)| \le |q(x)|$$

for every $x \in [a, b]$ *with strict inequality for every*

$$x \in [a, b] \setminus \{x_1, x_2, \dots, x_k\}\,.$$

To formulate the next lemmas we introduce the following notation. Let

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n\,, \quad 0 = \gamma_0 < \gamma_1 < \cdots < \gamma_n\,,$$

and

$$\gamma_i \le \lambda_i\,, \qquad i = 1, 2, \dots, n\,.$$

Let

$$M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$$

and

$$M_n(\Gamma) := \operatorname{span}\{x^{\gamma_0}, x^{\gamma_1}, \dots, x^{\gamma_n}\}\,.$$

Let $s \in (0, 1)$ be fixed. Let

$$T_{n,\lambda} := T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; [1-s, 1]\}$$

and

$$T_{n,\gamma} := T_n\{\gamma_0, \gamma_1, \dots, \gamma_n; [1-s, 1]\}$$

denote the Chebyshev polynomials on $[1-s, 1]$ for $M_n(\Lambda)$ and $M_n(\Gamma)$, respectively (see Theorem 2.2).

Lemma 4.2. *Let* $y \in [0, 1-s)$ *be fixed. Then both*

$$\max_{0 \ne p \in M_n(\Lambda)} \frac{|p(y)|}{\|p\|_{[1-s,1]}}$$

and

$$\max_{0 \ne p \in M_n(\Lambda)} \frac{|p'(y)|}{\|p\|_{[1-s,1]}}$$

are attained by $p = T_{n,\lambda}$. *In the second case we assume* $\lambda_1 \ge 1$ *if* $y = 0$.

Proof. A simple compactness argument shows that the maxima in the lemma are attained by some $p^* \in M_n(\Lambda)$ and $q^* \in M_n(\Lambda)$, which can be identified as $T_{n,\lambda}$ by a standard variational method. See for example [16, p. 295], [23, p. 101] where arguments of this variety are given. $\square$

Lemma 4.3. *We have*

$$|T_{n,\lambda}(0)| \le |T_{n,\gamma}(0)| \,.$$

Further, if $\lambda_1 = \gamma_1 = 1$, *then*

$$|T'_{n,\lambda}(0)| \le |T'_{n,\gamma}(0)| \,.$$

Proof. Let $p \in M_n(\Gamma)$ interpolate $T_{n,\lambda}$ at the n zeros of $T_{n,\lambda}$ in $[1-s,1]$ and at 0. It follows from Lemma 4.1 that

$$|p(x)| \le |T_{n,\lambda}(x)| \,, \qquad x \in [0,1] \,.$$

In particular,

$$\|p\|_{[1-s,1]} \le \|T_{n,\lambda}\|_{[1-s,1]} = 1 \,,$$

which, together with $p(0) = T_{n,\lambda}(0)$ and Lemma 4.2, gives

$$|T_{n,\lambda}(0)| = |p(0)| \le \frac{|p(0)|}{\|p\|_{[1-s,1]}} \le \frac{|T_{n,\gamma}(0)|}{\|T_{n,\gamma}\|_{[1-s,1]}} = |T_{n,\gamma}(0)| \,.$$

This proves the first part of the lemma.

The second part of the lemma can be proved in essentially the same way. Let $0 \ne p \in M_n(\Gamma)$ interpolate $T_{n,\lambda}$ at the n zeros of $T_{n,\lambda}$ in $[1-s,1]$. Note that $p'(0) \ne 0$, otherwise

$$p \in \operatorname{span}\{x^{\gamma_j} : j = 0, 1, \dots, n \,, \;\; j \ne 1\}$$

cannot have n zeros in $[1-s,1]$. Similarly $T'_{n,\lambda}(0) \ne 0$. Normalize p so that

$$p'(0) = T'_{n,\lambda}(0) \,.$$

It follows from Lemma 4.1 that

$$|p(x)| \le |T_{n,\lambda}(x)| \,, \qquad x \in [0,1] \,.$$

In particular

$$\|p\|_{[1-s,1]} \le \|T_{n,\lambda}\|_{[1-s,1]} = 1 \,,$$

which, together with $p'(0) = T'_{n,\lambda}(0)$ and Lemma 4.2, yields

$$|T'_{n,\lambda}(0)| = |p'(0)| \le \frac{|p'(0)|}{\|p\|_{[1-s,1]}} \le \frac{|T'_{n,\gamma}(0)|}{\|T_{n,\gamma}\|_{[1-s,1]}} = |T'_{n,\gamma}(0)| \,.$$

This proves the second part of the lemma. $\square$

Lemma 4.4. *The functions* $|T_{n,\lambda}|$ *and* $|T_{n,\gamma}|$ *are decreasing on* $[0, 1-s]$. *Further, if* $\lambda_1 = \gamma_1 = 1$, *then* $|T'_{n,\lambda}|$ *and* $|T'_{n,\gamma}|$ *are also decreasing on* $[0, 1-s]$.

Proof. Suppose $|T_{n,\lambda}|$ is not monotone decreasing on $[0, 1-s]$. Then

$$T'_{n,\lambda} \in \operatorname{span}\{x^{\lambda_1 - 1}, x^{\lambda_2 - 1}, \dots, x^{\lambda_n - 1}\}$$

must have at least n zeros in $(0,1)$, which is impossible. This proves the first statement of the lemma.

Suppose now $\lambda_1 = 1$ and $|T'_{n,\lambda}|$ is not monotone decreasing on $[0, 1-s]$. Then

$$T''_{n,\lambda} \in \operatorname{span}\{x^{\lambda_2 - 2}, x^{\lambda_3 - 2}, \dots, x^{\lambda_n - 2}\}$$

must have at least $n-1$ zeros in $(0,1)$, which is impossible. This proves the second statement of the lemma. □

The main result of this section is the following lemma. It plays a crucial role in the proof of Theorems 3.1 and 3.2 of the previous section.

Lemma 4.5. *We have*

$$\max_{0 \neq p \in M_n(\Lambda)} \frac{\|p\|_{[0,1]}}{\|p\|_{[1-s,1]}} \leq \max_{0 \neq p \in M_n(\Gamma)} \frac{\|p\|_{[0,1]}}{\|p\|_{[1-s,1]}} .$$

Further, if $\lambda_1 = \gamma_1 = 1$, *then*

$$\max_{0 \neq p \in M_n(\Lambda)} \frac{\|p'\|_{[0,1-s]}}{\|p\|_{[1-s,1]}} \leq \max_{0 \neq p \in M_n(\Gamma)} \frac{\|p'\|_{[0,1-s]}}{\|p\|_{[1-s,1]}} .$$

Proof. Combining Lemmas 4.2, 4.3, and 4.4, we obtain for every $y \in [0, 1-s)$ that

$$\begin{aligned}\max_{0 \neq p \in M_n(\Lambda)} \frac{|p(y)|}{\|p\|_{[1-s,1]}} &= \frac{|T_{n,\lambda}(y)|}{\|T_{n,\lambda}\|_{[1-s,1]}} = |T_{n,\lambda}(y)| \leq |T_{n,\lambda}(0)| \leq |T_{n,\gamma}(0)| \\ &= \frac{|T_{n,\gamma}(0)|}{\|T_{n,\gamma}\|_{[1-s,1]}} \leq \max_{0 \neq p \in M_n(\Gamma)} \frac{\|p\|_{[0,1-s]}}{\|p\|_{[1-s,1]}} ,\end{aligned}$$

which implies the first inequality of the lemma.

Similarly, combining Lemmas 4.2, 4.3, and 4.4, we obtain for every $y \in [0, 1-s)$ that

$$\begin{aligned}\max_{0 \neq p \in M_n(\Lambda)} \frac{|p'(y)|}{\|p\|_{[1-s,1]}} &= \frac{|T'_{n,\lambda}(y)|}{\|T_{n,\lambda}\|_{[1-s,1]}} = |T'_{n,\lambda}(y)| \leq |T'_{n,\lambda}(0)| \leq |T'_{n,\gamma}(0)| \\ &= \frac{|T'_{n,\gamma}(0)|}{\|T_{n,\gamma}\|_{[1-s,1]}} \leq \max_{0 \neq p \in M_n(\Gamma)} \frac{\|p'\|_{[0,1-s]}}{\|p\|_{[1-s,1]}} ,\end{aligned}$$

which implies the second inequality of the lemma. □

5. BOUNDED REMEZ-TYPE INEQUALITY FOR NON-DENSE MÜNTZ SPACES

The central result of this paper is the following.

Theorem 5.1. *Suppose* $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. *Let* $s > 0$. *Then there exists a constant* c *depending only on* $\Lambda := (\lambda_i)_{i=0}^{\infty}$ *and* s *(and not on* ϱ, A, *or the "length" of* p*) so that*

$$\|p\|_{[0,\varrho]} \leq c\, \|p\|_A$$

for every $p \in M(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ *and for every set* $A \subset [\varrho, 1]$ *of Lebesgue measure at least* s.

The proof of Theorem 5.1 is based on interpolation. By the Unique Interpolation Property of Chebyshev spaces, associated with

$$0 \leq x_0 < x_1 < \cdots < x_n ,$$

we can uniquely define

$$\ell_k := \ell_k\{x_0, x_1, \dots, x_n\} \in M_n(\Lambda) , \qquad k = 0, 1, \dots, n ,$$

so that

$$\ell_k\{x_0, x_1, \dots, x_n\}(x_j) = \delta_{j,k} := \begin{cases} 1 & \text{if } j = k\,, \\ 0 & \text{if } j \neq k\,. \end{cases}$$

Lemma 5.2. *Let*

$$0 < x_0 < x_1 < \cdots < x_n \qquad \textit{and} \qquad 0 < \tilde{x}_0 < \tilde{x}_1 < \cdots < \tilde{x}_n\,.$$

Suppose $0 \leq k \leq n$ *and*

$$\begin{array}{ll} x_j \leq \tilde{x}_j & \textit{for } j = 0, 1, \dots, k-1, \\ x_j = \tilde{x}_j & \textit{for } j = k, \\ x_j \geq \tilde{x}_j & \textit{for } j = k+1, k+2, \dots, n\,. \end{array}$$

For the sake of brevity let

$$\ell_k := \ell_k\{x_0, x_1, \dots, x_n\}$$

and

$$\tilde{\ell}_k := \ell_k\{\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_n\}\,.$$

Then

$$|\ell_k(0)| \leq |\tilde{\ell}_k(0)|\,.$$

Proof. It is sufficient to prove the lemma in the case that there is an index m so that $0 \leq m \leq n$, $m \neq k$, and

$$\begin{array}{ll} x_j = \tilde{x}_j & \text{for } j = 0, 1, \dots, n\,, \quad j \neq m\,, \\ x_m < \tilde{x}_m & \text{if } m < k\,, \\ x_m > \tilde{x}_m & \text{if } m > k. \end{array}$$

The general case of the lemma then follows from repeated applications of the above special cases. Note that in the above special cases

$$\ell_k - \tilde{\ell}_k \in M_n(\Lambda)$$

has a zero at each of the points

$$x_0, x_1, \dots, x_{m-1}, x_{m+1}, x_{m+2}, \dots, x_n\,,$$

hence it changes sign at each of these points, and does not have any other zero in $[0, \infty)$. It is also obvious that

$$\operatorname{sign}(\ell_k(x)) = \operatorname{sign}(\tilde{\ell}_k(x))\,, \qquad x \in [0, x_0]\,.$$

This, together with the previous observation and the inequality $x_0 \leq \tilde{x}_0$, yields that

$$|\ell_k(0)| \leq |\tilde{\ell}_k(0)|\,,$$

otherwise $\ell_k - \tilde{\ell}_k$ would have a zero in (x_{m-1}, x_{m+1}), where $x_{-1} := 0$ and $x_{n+1} := \infty$, which is impossible. □

By a simple scaling we can extend Lemma 5.2 as follows. We use the notation introduced in Lemma 5.2.

Lemma 5.3. *Let*

$$0 < x_0 < x_1 < \cdots < x_n \qquad \textit{and} \qquad 0 < \tilde{x}_0 < \tilde{x}_1 < \cdots < \tilde{x}_n\,.$$

Suppose $0 \leq k \leq n$*,* $\alpha \geq 0$*, and*

$$\begin{aligned} x_j &\le \tilde{x}_j - \alpha \quad \textit{for } j = 0, 1, \dots, k-1, \\ x_j &= \tilde{x}_j - \alpha \quad \textit{for } j = k, \\ x_j &\ge \tilde{x}_j - \alpha \quad \textit{for } j = k+1, k+2, \dots, n. \end{aligned}$$

Then

$$|\ell_k(0)| \le |\tilde{\ell}_k(0)| .$$

Proof. If $\alpha = 0$, then Lemma 5.2 yields the lemma. So we may suppose that $\alpha > 0$. Let

$$\beta := \frac{x_k}{\tilde{x}_k} = \frac{\tilde{x}_k - \alpha}{\tilde{x}_k} ,$$

$$x_j^* := \beta \tilde{x}_j , \qquad j = 0, 1, \dots, n ,$$

and

$$\ell_k^* := \ell_k\{x_0^*, x_1^*, \dots, x_n^*\} , \qquad k = 0, 1, \dots, n .$$

Obviously

$$\tilde{\ell}_k(\beta x) = \ell_k^*(x) , \qquad x \in [0, \infty) ,$$

and

$$\begin{aligned} x_j &\le x_j^* \quad \text{for } j = 0, 1, \dots, k-1, \\ x_j &= x_j^* \quad \text{for } j = k, \\ x_j &\ge x_j \quad \text{for } j = k+1, k+2, \dots, n. \end{aligned}$$

Hence Lemma 5.2 implies that

$$|\ell_k(0)| \le |\ell_k^*(0)| = |\tilde{\ell}_k(0)| ,$$

which finishes the proof. □

The next two lemmas are interesting for their own right. They show that the appropriately placed Chebyshev polynomial is always extremal for the Remez-type inequality we are considering.

Lemma 5.4. *Let $A \subset [0, 1]$ be a closed set of Lebesgue measure at least $s \in (0, 1)$. Then*

$$|p(0)| \le |T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; [1-s, 1]\}(0)| \cdot \|p\|_A$$

for every $p \in M_n(\Lambda)$.

Proof. If $0 \in A$, then the statement is trivial. So assume that $0 \notin A$. Let

$$\tilde{x}_0 < \tilde{x}_1 < \cdots < \tilde{x}_n$$

denote the extreme points of

$$T_n := T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; [1-s, 1]\}$$

in $[1-s, 1]$, that is,

$$T_n(\tilde{x}_j) = (-1)^{n-j} , \qquad j = 0, 1, \dots, n .$$

Let $x_j \in A$, $j = 0, 1, \dots, n$, be defined by

$$m([x_j, 1] \cap A) = m([\tilde{x}_j, \tilde{x}_n]) = \tilde{x}_n - \tilde{x}_j .$$

Since A is a closed subset of $[0,1]$ with $m(A) \geq s$, such points $x_j \in A$ exist. Let $p \in M_n(\Lambda)$. Then, using Lemma 5.3, we obtain

$$\begin{aligned}
|p(0)| &= \left| \sum_{k=0}^{n} p(x_k)\ell_k(0) \right| \\
&\leq \left(\sum_{k=0}^{n} |\ell_k(0)| \right) \|p\|_A \\
&\leq \left(\sum_{k=0}^{n} |\tilde{\ell}_k(0)| \right) \|p\|_A \\
&= \left| \sum_{k=0}^{n} (-1)^{n-k} \tilde{\ell}_k(0) \right| \|p\|_A \\
&= \left| \sum_{k=0}^{n} T_n(\tilde{x}_k)\tilde{\ell}_k(0) \right| \|p\|_A \\
&= |T_n(0)| \cdot \|p\|_A
\end{aligned}$$

and the lemma follows. In the rest of the proof we justify each line above.

Note that $p \in M_n(\Lambda)$ and $\sum_{k=0}^{n} p(x_k)\ell_k \in M_n(\Lambda)$ agree at $x_0, x_1, \dots, x_n$. Since $M_n(\Lambda)$ is a Chebyshev space of dimension $n+1$ on $[0,1]$, we can deduce that $p = \sum_{k=0}^{n} p(x_k)\ell_k$, and the first line above follows by substituting 0. The second line follows by the triangle inequality. Note that $x_k \in A$, so $|p(x_k)| \leq \|p\|_A$ for each k. To see the third line we need the inequalities $|\ell_k(0)| \leq |\tilde{\ell}_k(0)|$ for each k. These follow from Lemma 5.3. The assumptions of Lemma 5.3 are satisfied since the construction obviously implies the inequalities $0 < x_j \leq \tilde{x}_j$ and $\tilde{x}_i - \tilde{x}_j \leq x_i - x_j$ for every $0 \leq j < i \leq n$. The fourth line follows from the observation that $\tilde{\ell}_k(0) = -\tilde{\ell}_{k+1}(0)$ for each $k = 0, 1, \dots, n-1$. This can be deduced from the fact that $\tilde{\ell}_k$ changes sign exactly at

$$\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{k-1}, \tilde{x}_{k+1}, \tilde{x}_{k+2}, \dots, \tilde{x}_n,$$

while $\tilde{\ell}_k(\tilde{x}_k) = 1$. The fifth line uses the fact that

$$T_n(\tilde{x}_k) = (-1)^{n-k}, \qquad k = 0, 1, \dots, n.$$

Finally, the last line follows by observing that $T_n \in M_n(\Lambda)$ and $\sum_{k=0}^{n} T_n(\tilde{x}_k)\tilde{\ell}_k \in M_n(\Lambda)$ agree at $\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_n$. Since $M_n(\Lambda)$ is a Chebyshev space of dimension $n+1$ on $[0,1]$, we can deduce that $T_n = \sum_{k=0}^{n} T_n(\tilde{x}_k)\tilde{\ell}_k$, and on substituting 0, we obtain the last line. □

Lemma 5.5. *Let A be a closed subset of* $[0,1]$ *with Lebesgue measure at least* $s \in (0,1)$. *Then*

$$|p(y)| \leq |T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; [1-s, 1]\}(0)| \cdot \|p\|_A$$

for every $p \in M_n(\Lambda)$ *and* $y \in [0, \inf A)$.

Proof. Let $y \in [0, \inf A)$ be fixed. Simple compactness and perturbation arguments show that

$$\max_{0 \neq p \in M_n(\Lambda)} \frac{|p(y)|}{\|p\|_A}$$

is attained by

$$T_{n,A} := T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; A\}\,.$$

Note that $\lambda_0 = 0$ implies that $T_{n,A}$ is decreasing on $[0, \inf A]$, otherwise

$$T'_{n,A} \in \operatorname{span}\{x^{\lambda_1 - 1}, x^{\lambda_2 - 1}, \dots, x^{\lambda_n - 1}\}$$

must have at least $n+1$ zeros in $(0,1]$, which is impossible. Hence, by Lemma 5.4,

$$\frac{|p(y)|}{\|p\|_A} \le \frac{|T_{n,A}(y)|}{\|T_{n,A}\|_A} = |T_{n,A}(y)| \le |T_{n,A}(0)| \le |T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; [1-s, 1]\}(0)|$$

for every $0 \ne p \in M_n(\Lambda)$. This finishes the proof. □

Proof of Theorem 5.1. Without loss of generality we may assume that A is closed. Let

$$T_n := T_n\{\lambda_0, \lambda_1, \dots, \lambda_n; [1-s, 1]\}.$$

By Theorem 3.1, there exists a constant c depending only on $\Lambda := (\lambda_i)_{i=0}^\infty$ and s (and not on the "length" of p) so that

$$\|T_n\|_{[0,1]} \le c\|T_n\|_{[1-s,1]} = c$$

for every $p \in M(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$. By combining this with Lemma 5.5, there exists a constant c depending only on $\Lambda := (\lambda_i)_{i=0}^\infty$ and s (and not on ϱ, A, or the "length" of p) so that

$$\|p\|_{[0,\varrho]} \le c\|p\|_A$$

for every $p \in M(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ and for every set $A \subset [\varrho, 1]$ of Lebesgue measure at least s. □

The next theorem establishes an L^q version of Theorem 5.1.

Theorem 5.6. *Suppose* $\sum_{i=1}^\infty 1/\lambda_i < \infty$. *Let* $s > 0$ *and* $q \in (0, \infty)$. *Then there exists a constant* c *depending only on* $\Lambda := (\lambda_i)_{i=0}^\infty$, s, *and* q (*and not on* ϱ, A, *or the "length" of* p) *so that*

$$\|p\|_{[0,\varrho]} \le c\,\|p\|_{L^q(A)}$$

for every $p \in M(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ *and for every set* $A \subset [\varrho, 1]$ *of Lebesgue measure at least* s.

Proof. Note that $m(A) \ge s$ implies that

$$m\left(\left\{x \in A : |p(x)| \ge \left(\frac{2}{s}\right)^{1/q} \|p\|_{L^q(A)}\right\}\right) \le \frac{s}{2}\,,$$

hence

$$m\left(\left\{x \in A : |p(x)| < \left(\frac{2}{s}\right)^{1/q} \|p\|_{L^q(A)}\right\}\right) \ge \frac{s}{2}\,.$$

The theorem now follows from Theorem 5.1. □

6. MÜNTZ'S THEOREM ON COMPACT SETS OF POSITIVE MEASURE

The results of this section are straightforward consequences of the Remez-type inequality of Theorem 5.1.

Theorem 6.1. *Suppose* $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$ *and* $A \subset [0, \infty)$ *is a set of positive Lebesgue measure. Then* $M(\Lambda)$ *is not dense in* $C(A)$*.*

Moreover, if the gap condition

$$\inf\{\lambda_{i+1} - \lambda_i : i \in \mathbb{N}\} > 0 \tag{6.1}$$

holds, then every function $f \in C(A)$ *from the uniform closure of* $M(\Lambda)$ *on* A *is of the form*

$$f(x) = \sum_{i=0}^{\infty} a_i x^{\lambda_i}\,, \qquad x \in A \cap [0, r_A)\,,$$

where

$$r_A := \sup\{x \in [0, \infty) : m(A \cap (x, \infty)) > 0\}$$

is the essential supremum of A*.*

If the gap condition (6.1) *does not hold, then every function* $f \in C(A)$ *from the uniform closure of* $M(\Lambda)$ *on* A *can still be extended analytically throughout the region*

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < r_A\}\,.$$

Proof. Suppose $f \in C(A)$ and suppose there is a sequence $(p_i)_{i=1}^{\infty} \subset M(\Lambda)$ so that

$$\lim_{i \to \infty} \|p_i - f\|_A = 0\,.$$

Then the sequence $(p_i)_{i=1}^{\infty}$ is uniformly Cauchy on A. Therefore, Theorem 5.1 and the definition of r_A yield that $(p_i)_{i=1}^{\infty}$ is uniformly Cauchy on every closed subinterval of $[0, r_A)$. If the gap condition (6.1) holds, then the characterization of the uniform closure of $M(\Lambda)$ on A follows from the results of Clarkson and Erdős [12]. The result of Clarkson and Erdős [12] we need here claims that if the gap condition (6.1) holds and $0 \le a < b < \infty$, then every function $f \in C[a, b]$ from the uniform closure of $M(\Lambda)$ on $[a, b]$ is of the form

$$f(x) = \sum_{i=0}^{\infty} a_i x^{\lambda_i}, \qquad x \in [a, b)\,.$$

If the gap condition (6.1) does not hold, then results of Schwartz [24] yield the theorem. The result of Schwartz [24] we need here claims that for every $0 \le a < b < \infty$, even if the gap condition (6.1) does not hold, every function $f \in C[a, b]$ from the uniform closure of $M(\Lambda)$ on $[a, b]$ can still be extended analytically throughout the region

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < b\}\,.$$

□

Theorem 6.2. *Suppose* $A \subset [0, \infty)$ *is a compact set of positive Lebesgue measure. Then* $M(\Lambda)$ *is dense in* $C(A)$ *if and only if* $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$*.*

Proof. Suppose $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$. Let $f \in C(A)$. By Tietze's Theorem there exists an $\tilde{f} \in C[0,1]$ so that $\tilde{f}(x) = f(x)$ for every $x \in A$. By Müntz's Theorem there is a sequence $(p_i)_{i=1}^{\infty} \subset M(\Lambda)$ so that

$$\lim_{i \to \infty} \|\tilde{f} - p_i\|_{[0,1]} = 0\,.$$

Therefore

$$\lim_{i \to \infty} \|f - p_i\|_A = 0\,,$$

which finishes the trivial part of the theorem. Suppose now that $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. Then Theorem 6.1 yields that $M(\Lambda)$ is not dense in $C(A)$. □

The following surprising theorem shows that if $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, then the pointwise and locally uniform convergence of a sequence $(p_i)_{i=1}^{\infty} \subset M(\Lambda)$ on $[0,b)$ are equivalent. An amusing consequence of this is that if $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, then the set

$$\left\{ f : f(x) = \sum_{i=0}^{\infty} a_i x^{\lambda_i}\,, \quad a_i \in \mathbb{R}\,, \quad x \in [0,b) \right\}$$

is closed under pointwise convergence.

Theorem 6.3. *Suppose $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. Let $A \subset [0,\infty)$ be a set of positive Lebesgue measure, and let r_A be the essential supremum of A defined as in Theorem 6.1. Assume $(p_i)_{i=1}^{\infty} \subset M(\Lambda)$ and*

$$p_i(x) \to f(x)\,, \qquad x \in A\,.$$

Then $(p_i)_{i=1}^{\infty}$ converges uniformly on every closed subinterval of $[0, r_A)$.

This characterizes non-dense Müntz spaces within Müntz spaces, since in any Müntz space $M(\Lambda)$ with $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$ and for any $A \subset [0,\infty)$ of positive Lebesgue measure, there exists a sequence $(p_i)_{i=1}^{\infty} \subset M(\Lambda)$ that converges pointwise on $[0,\infty)$ but not locally uniformly on A. This follows easily from Müntz's Theorem.

Proof of Theorem 6.3. Let $\delta \in (0, r_A)$ be fixed. Egoroff's Theorem and the definition of r_A imply the existence of a set $B \subset A \cap (\delta, \infty)$ of positive Lebesgue measure so that $(p_i)_{i=1}^{\infty}$ converges uniformly on B, hence it is uniformly Cauchy on B. Now Theorem 5.1 yields that $(p_i)_{i=1}^{\infty}$ is uniformly Cauchy on $[0,\delta]$, which proves the theorem. □

Theorem 6.4. *Suppose $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, $A \subset [0,\infty)$ is a set of positive Lebesgue measure, w is a non-negative-valued, integrable weight function on A with $\int_A w > 0$, and $q \in (0,\infty)$. Then $M(\Lambda)$ is not dense in $L_w^q(A)$.*

Moreover, if the gap condition (6.1) holds, then every function $f \in L_w^q(A)$ belonging to the $L_w^q(A)$ closure of $M(\Lambda)$ can be represented as

$$f(x) = \sum_{i=0}^{\infty} a_i x^{\lambda_i}\,, \qquad x \in A \cap [0, r_w)\,,$$

where

$$r_w := \sup \left\{ x \in [0,\infty) : \int_{A \cap (x,\infty)} w(x)\,dx > 0 \right\}.$$

If the gap condition (6.1) *does not hold, then every function* $f \in L^q_w(A)$ *belonging to the* $L^q_w(A)$ *closure of* $M(\Lambda)$ *can still be represented as an analytic function on*

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < r_w\}$$

restricted to A.

Proof. Suppose $f \in L^q_w(A)$ and suppose there is a sequence $(p_i)_{i=1}^\infty \subset M(\Lambda)$ so that

$$\lim_{i\to\infty} \|f - p_i\|_{L^q_w(A)} = 0\,.$$

Minkowski's inequality (if $q \in (0,1)$, then a multiplicative factor $2^{1/q-1}$ is needed) yields that $(p_i)_{i=1}^\infty$ is a Cauchy sequence in $L^q_w(A)$. The assumptions on w imply that for every $\delta \in (0, r_w)$ there exists an $\alpha > 0$ so that

$$B := \{x \in A \cap (\delta, \infty) : w(x) > \alpha\}$$

is of positive Lebesgue measure. Note that

$$\|p\|_{L^q(B)} \le \alpha^{-1}\|p\|_{L^q_w(B)} \le \alpha^{-1}\|p\|_{L^q_w(A)}$$

for every $p \in L^q_w(A)$. Therefore, $(p_i)_{i=1}^\infty$ is a Cauchy sequence in $L^q(B)$. So, by Theorem 5.1, $(p_i)_{i=1}^\infty$ is uniformly Cauchy on $[0,\delta]$. If the gap condition (6.1) holds, then the theorem now follows from results of Clarkson and Erdős [12] (see the end of the proof of Theorem 6.1). If the gap condition (6.1) does not hold, then a result of Schwartz [24] yields the theorem (see also the end of the proof of Theorem 6.1). □

Theorem 6.5. *Suppose* $A \subset [0,1]$ *is a set of positive Lebesgue measure,* w *is a non-negative-valued integrable weight function on* A *with* $\int_A w > 0$, *and* $q \in (0,\infty)$. *Then* $M(\Lambda)$ *is dense in* $L^q_w(A)$ *if and only if* $\sum_{i=1}^\infty 1/\lambda_i = \infty$.

Proof. Suppose $\sum_{i=1}^\infty 1/\lambda_i = \infty$. Let $f \in L^q_w(A)$. It is standard measure theory to show that for every $\varepsilon > 0$, there exists a $g \in C[0,1]$ so that

$$\|f - g\|_{L^q_w(A)} < \frac{\varepsilon}{2}\,.$$

Now Müntz's Theorem implies that there exists a $p \in M(\Lambda)$ so that

$$\|g - p\|_{L^q_w(A)} \le \|g - p\|_A \left(\int_A w\right)^{1/q} < \frac{\varepsilon}{2}\,.$$

Therefore $M(\Lambda)$ is dense in $L^q_w(A)$.

Suppose now that $\sum_{i=1}^\infty 1/\lambda_i < \infty$. Then Theorem 6.4 yields that $M(\Lambda)$ is not dense in $L^q_w(A)$. □

7. Products of Müntz spaces

For

$$\Lambda_j := (\lambda_{i,j})_{i=0}^\infty\,, \qquad 0 = \lambda_{0,j} < \lambda_{1,j} < \lambda_{2,j} < \cdots\,, \qquad j = 1, 2, \dots\,, \tag{7.1}$$

we define the sets

$$M(\Lambda_1, \Lambda_2, \dots, \Lambda_k) := \left\{ p = \prod_{j=1}^k p_j : p_j \in M(\Lambda_j) \right\}.$$

First we prove the following Remez-type inequality for $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$.

Theorem 7.1. *Suppose* (7.1) *holds and*

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_{i,j}} < \infty\,, \qquad j = 1, 2, \dots, k\,. \tag{7.2}$$

Let $s > 0$. Then there exists a constant c depending only on $\Lambda_1, \Lambda_2, \dots, \Lambda_k, s$, and k (and not on ϱ or A) so that

$$\|p\|_{[0,\varrho]} \le c\,\|p\|_A$$

for every $p \in M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$ and for every set $A \subset [\varrho, 1]$ of Lebesgue measure at least s.

Proof. Theorem 5.1 implies that there exist constants $\alpha_j > 0$ depending only on $\Lambda_1, \Lambda_2, \dots, \Lambda_k, s$, and k so that

$$m(\{x \in [y,1] : |p(x)| > \alpha_j^{-1}|p(y)|\}) \ge 1 - y - \frac{s}{2k}$$

for every $p \in M(\Lambda_j)$ and $y \in [0, 1-s]$. Indeed, $\alpha_j := c + 1$ is a suitable choice, where c is the constant in Theorem 5.1 depending only on $\Lambda := \Lambda_j$ and $\tilde{s} := s/(2k)$. Otherwise, if

$$m(\{x \in [y,1] : |p(x)| > (c+1)^{-1}|p(y)|\}) < 1 - y - \frac{s}{2k}$$

for some $p \in M(\Lambda_j)$, then, on one hand, $|p(y)| > 0$ would hold, while on the other hand, Theorem 5.1 with $A := \{x \in [y,1] : |p(x)| \le (c+1)^{-1}|p(y)|\})$ and $m(A) \ge s/(2k)$ would imply that $|p(y)| \le c(c+1)^{-1}|p(y)|$. This is a contradiction, so the existence of the constants α_j with the desired properties are justified.

Now let

$$p \in M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)\,,$$

that is,

$$p = \prod_{j=1}^{k} p_j, \qquad p_j \in M(\Lambda_j)\,.$$

Then, for every $y \in [0, 1-s]$,

$$\begin{aligned} &m(\{x \in [y,1] : |p(x)| > (\alpha_1\alpha_2 \cdots \alpha_k)^{-1}|p(y)|\}) \\ &\qquad \ge m\left(\bigcap_{j=1}^{k} \left\{x \in [y,1] : |p_j(x)| > \alpha_j^{-1}|p_j(y)|\right\}\right) \\ &\qquad \ge 1 - y - k\frac{s}{2k} = 1 - y - \frac{s}{2}\,. \end{aligned}$$

Hence $y \in [0, \inf A]$ and $m(A) \ge s$ imply

$$m(\{x \in A : |p(x)| > (\alpha_1\alpha_2 \cdots \alpha_k)^{-1}|p(y)|\}) \ge \frac{s}{2} > 0\,,$$

and the theorem follows with $c := \alpha_1\alpha_2 \cdots \alpha_k$. □

Theorem 7.1 immediately solves Newman's problem [20].

Corollary 7.2. *Suppose* (7.1) *and* (7.2) *hold and $A \subset [0,1]$ is a set of positive Lebesgue measure. Then $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$ is not dense in $C(A)$. Moreover, if w is a non-negative-valued integrable weight function on A with $\int_A w > 0$ and $q \in (0,\infty)$, then $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$ is not dense in $L_w^q(A)$.*

Our next theorem establishes a Bernstein-type inequality for $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$.

Theorem 7.3. *Suppose* (7.1) *and* (7.2) *hold and*

$$\lambda_{1,j} \geq 1\,, \qquad j = 1, 2, \dots, k\,.$$

Let $s > 0$. Then there exists a constant c depending only on $\Lambda_1, \Lambda_2, \dots, \Lambda_k$, s, and k (and not on ϱ or A) so that

$$\|p'\|_{[0,\varrho]} \leq c\,\|p\|_A$$

for every $p \in M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$ and for every set $A \subset [\varrho, 1]$ of Lebesgue measure at least s.

Proof. Note that a combination of Theorems 5.1 and 3.2 implies that there exist constants $\beta_j > 0$ depending only on $\Lambda_1, \Lambda_2, \dots, \Lambda_k$, s, and k so that

$$m(\{x \in [y,1] : |p(x)| > \beta_j^{-1}|p'(y)|\}) \geq 1 - y - \frac{s}{2k}$$

for every $p \in M(\Lambda_j)$ and $y \in [0, 1-s]$. Indeed, $\beta_j := (c_1 + 1)c_2$ is a suitable choice, where c_1 is the constant in Theorem 5.1 depending only on $\Lambda := \Lambda_j$ and $\tilde{s} := s/(4k)$, while c_2 is a constant depending only on $\Lambda := \Lambda_j$ and $s/(4k)$ so that

$$|p'(y)| \leq c_2\|p\|_{[0,y+s/(4k)]}\,,$$

the existence of which follows from Theorem 3.2 by the scaling $x \mapsto (y + s/(4k))x$. To see that β_j above is a suitable choice, suppose that

$$m(\{x \in [y,1] : |p(x)| > (c_2(c_1+1))^{-1}|p'(y)|\}) < 1 - y - \frac{s}{2k}$$

for some $p \in M(\Lambda_j)$. Then $|p'(y)| > 0$. Let

$$A := \{x \in [y + s/(4k), 1] : |p(x)| \leq ((c_1+1)c_2)^{-1}|p'(y)|\}\,.$$

Then $m(A) \geq s/(2k) - s/(4k) = s/(4k)$, and by Theorem 5.1,

$$\|p\|_{[0,y+s/(4k)]} \leq c_1(c_1+1)^{-1}c_2^{-1}|p'(y)| < c_2^{-1}|p'(y)|\,.$$

This contradicts the choice of c_2. Therefore the existence of the constants β_j with the desired properties are justified.

Let $\alpha_j > 0$, $j = 1, 2, \dots, k$, be chosen as in the proof of Theorem 7.1. Now let

$$p \in M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)\,,$$

that is,

$$p = \prod_{j=1}^{k} p_j\,, \qquad p_j \in M(\Lambda_j)\,.$$

Then, as in the proof of Theorem 7.1, for every $y \in [0, 1-s]$,

$$m\left(\bigcap_{j=1}^{k}\{x \in [y,1] : |p(x)| > \gamma_j^{-1}|(p_1 \cdots p_{j-1}p_j'p_{j+1} \cdots p_k)(y)|\}\right)$$
$$\geq 1 - y - k\frac{s}{2k} = 1 - y - \frac{s}{2}\,,$$

where

$$\gamma_j := \alpha_1 \cdots \alpha_{j-1}\beta_j\alpha_{j+1}\cdots\alpha_k\,.$$

So

$$m\{x \in [y,1] : |p(x)| > c^{-1}|p'(y)|\} \geq 1 - y - \frac{s}{2}\,,$$

where $c := \sum_{j=1}^{k}\gamma_j$. Hence $y \in [0, \inf A]$ and $m(A) \geq s$ imply

$$m(\{x \in A : |p(x)| > c^{-1}|p'(y)|\}) \geq \frac{s}{2} > 0\,,$$

and the theorem follows with $c = \sum_{j=1}^{k}\gamma_j$. □

The following Nikolskii-type inequality is also valid for $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$.

Theorem 7.4. *Suppose* (7.1) *and* (7.2) *hold. Let* $s > 0$ *and* $q \in (0,\infty)$. *Then there exists a constant* c *depending only on* $\Lambda_1, \Lambda_2, \dots, \Lambda_k, s, k, q$, *and* w (*and not on* ϱ *or* A) *so that*

$$\|p\|_{[0,\varrho]}^q \leq c\int_A |p(x)|^q w(x)\,dx$$

for every $p \in M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$, *for every set* $A \subset [\varrho, 1]$ *of Lebesgue measure at least* s, *and for every function* w *measurable and positive a.e. on* $[0,1]$.

Proof. This a straightforward consequence of Theorem 7.1 □

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Abstract. The principal result of this paper is a Remez-type inequality for Müntz polynomials:

$$p(x) := \sum_{i=0}^{n} a_i x^{\lambda_i},$$

or equivalently for Dirichlet sums:

$$P(t) := \sum_{i=0}^{n} a_i e^{-\lambda_i t},$$

where $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$. The most useful form of this inequality states that for every sequence $(\lambda_i)_{i=0}^{\infty}$ satisfying $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, there is a constant c depending only on $\Lambda := (\lambda_i)_{i=0}^{\infty}$ and s (and not on n, ϱ, or A) so that

$$\|p\|_{[0,\varrho]} \le c\, \|p\|_A$$

for every Müntz polynomial p, as above, associated with $(\lambda_i)_{i=0}^{\infty}$, and for every set $A \subset [\varrho, 1]$ of Lebesgue measure at least $s > 0$. Here $\|\cdot\|_A$ denotes the supremum norm on A. This Remez-type inequality allows us to resolve two reasonably long-standing conjectures.

The first conjecture it lets us resolve is due to D. J. Newman and dates from 1978. It asserts that if $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, then the set of products $\{p_1 p_2 : p_1, p_2 \in \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}\}$ is not dense in $C[0,1]$.

The second is a complete extension of Müntz's classical theorem on the denseness of Müntz spaces in $C[0,1]$ to denseness in $C(A)$, where $A \subset [0,\infty)$ is an arbitrary compact set with positive Lebesgue measure. That is, for an arbitrary compact set $A \subset [0,\infty)$ with positive Lebesgue measure, $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is dense in $C(A)$ if and only if $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$.

Several other interesting consequences are also presented.

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