

MATRIX A_p WEIGHTS VIA S -FUNCTIONS

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0. INTRODUCTION

The statement of the problem. In this paper we study weighted norm inequalities with matrix valued weights. Namely, let W be a $d \times d$ matrix weight, i.e. a L^1 -function whose values are selfadjoint nonnegative $d \times d$ matrices. We suppose that the weight W is defined on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ or the real line \mathbb{R} .

Let $L^p = L^p(\mathbb{C}^d)$ be the space of all measurable vector functions on \mathbb{T} whose \mathbb{C}^d -norm is summable to the power of p . Let $H^p = H^p(\mathbb{C}^d)$ be the corresponding Hardy space of analytic functions, and let P_+ be projection in L^p onto H^p annihilating antianalytic functions in L^p which vanish at the origin. Let H denote the Hilbert transform, $H = -iP_+ + i(I - P_+)$. Clearly, $P_+, P_- \stackrel{\text{def}}{=} I - P_+$, and H act as usual operators P_+, P_- , and H coordinatewise if we fix for all $t \in \mathbb{T}$ the same coordinate basis in \mathbb{C}^d .

We are interested in the conditions on W under which the following weighted norm inequality for operator H holds (say for all $f \in L^p \cap L^\infty$):

$$\int_{\mathbb{T}} (W(t)^{2/p} Hf(t), Hf(t))^{p/2} dm(t) \leq C \int_{\mathbb{T}} (W(t)^{2/p} f(t), f(t))^{p/2} dm(t).$$

Clearly this inequality is equivalent to the same inequality for P_+ (with another constant). Here m denotes Lebesgue measure on \mathbb{T} , and $W(t)^{2/p}$ stands for the operator power of nonnegative selfadjoint operator $W(t)$. If we define a weighted space $L^p(W)$ as the space of all measurable \mathbb{C}^d -valued functions on \mathbb{T} satisfying

$$\|f\|_{L^p(W)}^p \stackrel{\text{def}}{=} \int_{\mathbb{T}} (W(t)^{2/p} f(t), f(t))^{p/2} dt < \infty$$

(of course we should factorize it over the subspace of functions of norm 0), then the last inequality means that H (or, equivalently, P_+) is a bounded operator in $L^p(W)$. The notation is chosen in such a way that $L^p(W)$ becomes a familiar weighted L^p space in the scalar situation $d = 1$. We consider the range $(1, \infty)$ of p .

In the scalar-valued setting $d = 1$ the answer is given by the famous Hunt–Muckenhoupt–Wheeden theorem which says that the Muckenhoupt condition (A_p)

$$\sup_I \left(\frac{1}{|I|} \int_I W \right)^{1/p} \cdot \left(\frac{1}{|I|} \int_I W^{-q/p} \right)^{1/q} < \infty$$

Received by the editors May 24, 1996 and, in revised form, December 2, 1996.

1991 *Mathematics Subject Classification.* Primary 42B20, 42A50, 47B35.

Key words and phrases. A_p weights, matrix-functions, area integrals, singular integrals, Carleson measures, Triebel-Lizorkin spaces.

This work was partially supported by National Science Foundation Grant DMS9622936.

($q = p/(p - 1)$; supremum is taken over all intervals) is necessary and sufficient in this case.

The main purpose of this paper is to give a matrix analog of (A_p) and to prove that it is necessary and sufficient for the matrix weighted inequality to hold.

In [TV1] the matrix (A_2) condition appeared:

$$\sup_I \left\| \left(\frac{1}{|I|} \int_I W \right)^{1/2} \cdot \left(\frac{1}{|I|} \int_I W^{-1} \right)^{1/2} \right\| < \infty.$$

Here the power $1/2$ stands for the operator power of nonnegative operators—averages of W in our case. Later S. Treil came up with the conjecture of what the matrix (A_p) should look like.

The matrix (A_p) condition does not have such a simple form if p is not 2. It cannot be obtained by the combination of averagings and raising to operator powers. It requires the analysis of the geometric meaning of the Muckenhoupt condition (A_p) , even though there is no geometry in \mathbb{C}^1 . We explain the matrix (A_p) in the next section. Now let us remark that very few scalar methods are available in the vector situation.

The difficulties. As an illustration of what kind of difficulties one can encounter with the vector case, let us present two very simple examples. It is trivial in the scalar case that if we have an integral operator in $L^p(W)$ with positive (scalar) kernel, and we know that an operator with a bigger kernel is bounded, then the original operator is bounded too. This statement (even for scalar kernels) does not hold for weighted L^p spaces with matrix weights. Certainly if $W(t)$ can be diagonalized by *the same* basis for each $t \in \mathbb{T}$, we do not have any difficulties. But this is not the case we are interested in. Another difficulty comes from the fact that while any nonzero nonnegative operator is invertible in \mathbb{C}^1 , this is quite false in \mathbb{C}^d . For example, suppose we meet the expression (we do meet such expressions while working with matrix weights) $(A(I+B)^{-1}x, (I+B)^{-1}x)$, where A and B are nonnegative operators, $B \leq I$, and x is a vector in \mathbb{C}^d . If A, B, x are numbers, the estimate from below $\frac{1}{4}A|x|^2$ would follow. But no estimate $\delta\|x\|^2$, $\delta > 0$, exists for operators.

The main idea. Nazarov and Treil in [NT] overcome all the difficulties by using the Bellman function approach (see below). But first a simple idea which was successfully used in [TV1] comes into play. Roughly speaking the idea is as follows. The first step is to choose a good basis in $L^p(W)$. It should be good in the sense that:

- 1) the coefficient space with respect to this basis should be relatively simple;
- 2) the matrix of the operator H (or P_+) in this coefficient space should also be relatively simple (desirably as close to a diagonal matrix as possible).

Let h_I stand for a Haar function. It turns out that matrix (A_p) is equivalent to the fact that $\{h_I \mathbb{C}^d\}_{I \in \mathcal{D}}$ forms an unconditional basis in $L^p(W)$. In the scalar situation this is well known (see [FJW]), and the coefficient space is the weighted Triebel-Lizorkin space $f_p^{0,2}$ (see [FJW] and below for the notation).

The Bellman function. The main tool in [NT] is the use of the Bellman function. Here is what it is. The desired estimates involve the averages of functions and

weights over dyadic intervals. Consider these averages as “variables”. Generally, the estimates one encounters will have the form

$$\sum_{I \subset J} a_I \leq CF_J,$$

where a_I is a certain (depending on the problem) function of our “variables” (read “averages”) corresponding to I . Now try to build a pretty concave function B (let us call it the Bellman function of the problem) of our “variables” in such away that:

- 1) for all $I \in \mathcal{D}$, $B(I) \leq CF_I$,
- 2) $B(I) - 1/2(B(I_-) + B(I_+)) \geq |a_I|$.

Here $B(I)$ stands for a value of function B on “variables” (i.e. averages) corresponding to I . The second requirement above is the strong concavity condition. After function B is built, the desired estimate is obtained just by repeatedly applying the concavity. Actually it happens very often that the existence of B is equivalent to the singular integral estimate under consideration.

The function B is closely related to the Burkholder function from [B1], [B2], [B3], [Str], Section 6.3.15, constructed for obtaining the best constants in scalar martingale and singular integral estimates.

The Bellman functions have been used for decades in Control Theory. They can be interpreted as the supremum of price over different plans of distributing the resources. By nature such functions are convex. In our dyadic estimates we have averages (or integrals) over J and we wish to show that whatever the distribution of them over the dyadic sons, grandsons, et cetera is, the price will not exceed CF_I provided that the price for distributing from any dyadic I to its dyadic “sons” I_- and I_+ is a_I .

Carleson measures and A_∞ . In the present paper another approach to proving matrix weighted inequalities is suggested. The observation is that even though many scalar methods are not available now, there is at least one which still manages to survive. We mean *the good λ -inequality*, which should be more rigorously called *the relative distributional inequality* (see [St], pp. 206–209). Let us also remind the reader about the role of A_∞ weights. A positive function cannot be much bigger than its average over I on a large portion of I . If we change the word “bigger” to “smaller”, we get the class of (A_∞) functions (see various definitions of (A_∞) on pp. 196–197, and 218 of [St]). In the classical case (see [St], pp. 200–205) the class (A_∞) comes into play as follows. The singular integral operator is “factorized” through the maximal operator M in the sense that the weighted estimate $\|Hf\|_W \leq C\|f\|_W$ splits into estimates $\|Hf\|_W \leq C\|Mf\|_W$ and $\|Mf\|_W \leq C\|f\|_W$.

The proof of the first estimate uses *the good λ - inequality*, which, roughly speaking, amounts to the fact that $|H| \leq CM$ up to a “telescopically graded” set. More precisely, $W(\{|Hf| > 2\lambda\} \setminus \{Mf < \delta\lambda\}) \leq \frac{1}{2}W(\{|Hf| > \lambda\})$ for any $\lambda > 0$ if $\delta > 0$ is small. This property follows from the (A_∞) property of W .

To some extent, but in a more mysterious and strange way, the same two ideas (splitting the problem into two inequalities, where one of them amounts to the (A_∞) property of weight and uses *the good λ -inequality*) work in this paper.

But we will really factorize our operator, $H = E_U^* \cdot D \cdot E_W$, where $U = W^{-q/p}$, $q = p/(p - 1)$, and where E is an embedding operator. It will embed to a (defined below) Triebel-Lizorkin space $f_p^{02}(\mathbb{C}^d)$. D is an almost diagonal operator in the sequence space $f_p^{02}(\mathbb{C}^d)$. Those readers who preferred to read the main body of

this article before reading the introduction may recognize the geometric meaning of U : $U^{1/q}$ defines the metric which is conjugate to the metric defined by $W^{1/p}$. In this factorization, the (A_∞) property of the weight W will be used to prove the boundedness of the first factor E_W (and not of the second “factor” as it happened in the classical case treated on pp. 201–207 of [St]). Actually the property of the matrix weight W corresponding to (A_∞) will be called $(A_{p,\infty})$. For matrix weights there is a whole spectrum of $(A_{p,\infty})$ classes graded by $p \in (1, \infty)$ —they all coincide with the usual (A_∞) in the scalar situation.

So, as we will see, the boundedness of H follows from $W \in (A_{p,\infty}), U \in (A_{q,\infty})$. Together, these two properties of W are equivalent to the matrix (A_p) condition imposed on W .

In proving this equivalence we come across a Carleson measure characterization of (scalar) (A_∞) weights obtained in [FKP], [Bu1], and [Bu2]. For example, the matrix $(A_{2,\infty})$ condition is equivalent to the requirement that the following measure in the disc is a Carleson measure:

$$\mu \stackrel{\text{def}}{=} \sum_{I \in \mathcal{D}} \|\langle W \rangle_I^{-1/2} (\langle W \rangle_{I_-} - \langle W \rangle_{I_+}) \langle W \rangle_I^{-1/2}\| \cdot |I| \cdot \delta_{c_I}.$$

Here c_I denotes the center of the Carleson box built on the arc I : $Q_I \stackrel{\text{def}}{=} \{z \in \mathbb{D} : 1 - |I| \leq |z| < 1, z/|z| \in I\}$. This idea from [FKP], [Bu1], [Bu2] that (A_∞) is equivalent to a certain Carleson measure condition for the measures built by the averages of W , turns out to be extremely useful in [TV1], [TV2], [TV3] for proving that E_W and E_U are the correct embeddings. This is not surprising because Carleson measure is what embeddings are about.

Even though we do not know such a neat characterization of $(A_{p,\infty})$ if p is not 2, we managed to prove that $(A_{p,\infty})$ is equivalent to the embedding we need. This is done in Section 3, after explaining the matrix (A_p) condition in Section 1, and the matrix $(A_{p,\infty})$ condition in Section 2. Then we prove the boundedness of the Hilbert transform H in Sections 4 and 5. To do this we will need the second ingredient from the classical theory: we use a certain variant of *the good λ -inequality*, the idea of which one can trace back to the works of Burkholder and Gundy [BG] and Axler, Chang, and Sarason [AChS].

The latter work is especially important because of its operator theory motivation.

Operator theory motivation. There is a part of the theory of singular integrals which treats the Hardy spaces in \mathbb{R}^n . The passage from \mathbb{R}^1 (or \mathbb{T}) to \mathbb{R}^n makes the theory immensely richer. At the same time the theory of vector-valued Hardy spaces on \mathbb{T} was developing for the needs of the spectral theory of operators (see [Nik]), because the dilation theory of linear contractions (see [Nik]) reduces questions about bounded operators in Hilbert space to function theoretic questions in a vector Hardy space. Even a finite-dimensional case is known to be much richer than the scalar case (see [Nik] again). So the increase in dimension in this direction also enriches the theory. The connection with singular integrals becomes manifest if one considers Hankel and Toeplitz operators, given by formulae $H_F f \stackrel{\text{def}}{=} P_-(Ff) = (I - P_+)(Ff)$ and $T_F f \stackrel{\text{def}}{=} P_+(Ff)$, where F is a $d \times d$ matrix function and $f \in H^p(\mathbb{C}^d)$. One such problem which is very difficult already in the scalar case $d = 1$ was considered in [AChS], and then a similar problem was considered in [S1], [S2]. It is closely related to a two-weight estimate for the Hilbert transform which is still open (see e.g. [TVZ], and the literature cited there).

There is another classical problem for Toeplitz operators which leads to a weighted estimate (with one weight) for the Hilbert transform. In fact, the invertibility of the Toeplitz operator T_F on H^p is equivalent (see [Si]) to the factorization $F = G_1^* \cdot G_2$, where G_1, G_2 are $d \times d$ outer matrix functions such that the following estimate holds:

$$(0.1) \quad \int_{\mathbb{T}} (V(t)P_+f(t), P_+f(t))dm(t) \leq C \int_{\mathbb{T}} (W(t)f(t), f(t))dm(t),$$

where we denote $W = G_1G_1^*$, $V = (G_2^{-1})^*G_2^{-1}$.

On first glance the matrix weights $W = G_1G_1^*$, $V = (G_2^{-1})^*G_2^{-1}$ seem to be different, but the invertibility of T_F implies easily $F^{-1} \in L_{d \times d}^\infty$, which means that the matrix weights V, W are equivalent in the sense that there exists a constant C such that for all $e \in \mathbb{C}^d$ and for almost all $t \in \mathbb{T}$

$$\frac{1}{C}(V(t)e, e) \leq (W(t)e, e) \leq C(V(t)e, e).$$

This is how one can come to the matrix weighted norm inequality considered in this paper.

As far as we know, the first results about the matrix weight inequality were obtained by Steven Blum [B11], [B12], who noticed that if the matrix weight W is assumed to be appropriately “smooth”, then it can be diagonalized by a “smooth” unitary matrix function; and the operator of multiplication on this unitary matrix function commutes with P_+ up to a compact term (because of “smoothness”). This approach leads to the pointwise diagonalization of the estimate under consideration and so to the corresponding scalar problem.

In the present work the matrix function W is a priori arbitrary. Rather than doing pointwise diagonalization (which is not available now) we prefer to come to global (almost) diagonalization of our operator in the weighted space $L^p(W)$.

Notation. The symbol f_p^{02} denotes the sequences $s = \{s_I\}_{I \in \mathcal{D}}$ enumerated by the set of dyadic intervals \mathcal{D} such that

$$\|s\|_{f_p^{02}}^p = \int \left(\sum_{I \ni x} |s_I|^2 \frac{1}{|I|} \right)^{p/2} dx < \infty.$$

Similarly one can introduce $f_p^{02}(\mathbb{C}^d)$ by just replacing numbers s_I by \mathbb{C}^d vectors s_I and $|\cdot|$ by $\|\cdot\|$. One can consider $f_p^{02}(w)$, which is a weighted Triebel-Lizorkin space. This amounts to changing the integration dx to $w(x)dx$. As to the vector weighted norm one just replaces $\|s_I\|^2$ by $\|W^{1/p}(x)s_I\|^2$ to obtain the definition of $f_p^{02}(\mathbb{C}^d, W)$, where $W(x)$ is now a positive selfadjoint operator (matrix).

1. MATRIX A_p CONDITION

Let $\|\cdot\|$ denote the standard Hilbert norm in \mathbb{C}^d , and let (\cdot, \cdot) be the scalar product in \mathbb{C}^d . If ρ denotes a norm in \mathbb{C}^d , we can notice that $\rho(x) \asymp_d \|Ax\|$, where A is a positive selfadjoint operator and constants depend only on the dimension d . In fact, let us consider the ellipsoid of the largest volume contained in the ρ -unit ball $\{x : \rho(x) \leq 1\}$, which is a convex subset of \mathbb{C}^d . Clearly, $\{x : \|Ax\| \leq 1\} \subset \{x : \rho(x) \leq 1\} \subset C_d \{x : \|Ax\| \leq 1\}$, where A is a positive matrix mapping the ellipsoid onto the standard Hilbert ball.

If we have a family ρ_t of norms and consider $L^p(\rho) = \{f : (\int \rho_t(f)^p dt)^{1/p} < \infty\}$, then we can also introduce an equivalent norm $\|f\|_{L^p(\rho)} \asymp (\int \|A(t)f\|^p dt)^{1/p}$, with $A(t)$ as above constructed for ρ_t .

In weighted matrix inequalities one deals usually with $L^p(W)$ spaces, where $W(t)$ is matrix function with positive (a.e.) $W(t) : \mathbb{C}^d \rightarrow \mathbb{C}^d$, and $\|f\|_{L^p(W)} \stackrel{\text{def}}{=} (\int \|W(t)^{1/p} f\|^p)^{1/p}$.

As we see one can consider $L^p(\rho)$ as well and reduce it to $L^p(W)$ by setting $A(t) = W^{1/p}(t)$.

Sometimes it is more convenient to use the language of ρ and sometimes that of W .

Recall that the dual metric ρ^* is given by

$$\rho^*(x) = \sup_{y \neq 0} \frac{|(x, y)|}{\rho(y)}.$$

We can identify $(L^p(\rho))^*$ with $L^q(\rho^*)$, $\frac{1}{p} + \frac{1}{q} = 1$, and $(L^p(W))^*$ with $L^q(U)$, where the $W^{1/p}$ -metric should be dual to the $U^{1/q}$ -metric, that is,

$$W^{1/p}(t) = U^{-1/q}(t), \quad \text{for a.e. } t.$$

To introduce the analog of A_p for matrix weights we use the language of metrics ρ and their averagings. Consider the norms (I is any interval)

$$\rho_{p,I}(x) \stackrel{\text{def}}{=} \left(\frac{1}{I} \int_I \rho_t^p(x) dt \right)^{1/p}.$$

The unit ball again can be “reduced” to an ellipsoid, that is, there exists a positive A_I such that

$$\rho_{p,I}(x) \asymp \|A_I x\|.$$

Let us reduce the dual metric as well:

$$\rho_{q,I}^*(x) \asymp \|B_I x\|.$$

Proposition 1.1. 1) $|(x, y)| \leq \rho_{p,I}(x) \rho_{q,I}^*(y)$.

2) $(\rho_{p,I})^* \leq \rho_{q,I}^*$.

Proof. 1) Obvious:

$$|(x, y)| = \frac{1}{I} \int_I |(x, y)| \leq \frac{1}{I} \int_I \rho_t(x) \rho_t^*(y) \leq \rho_{p,I}(x) \rho_{q,I}^*(y).$$

2) This is a paraphrase of the first assertion.

Definition. If we have an opposite inequality

$$\rho_{q,I}^* \leq C(\rho_{p,I})^*, \quad \forall I,$$

then ρ is called an A_p -metric. If $\rho_t(x) = \|W^{1/p}(t)x\|$ and ρ is an A_p -metric, then W is called an A_p (matrix) weight.

Proposition 1.2. *The following assertions are equivalent:*

- 1) ρ is an A_p -metric;
- 2) $\rho_{p,I} \leq C(\rho_{q,I}^*)^*$;

3) if $\rho_{p,I}(x) \asymp \|A_I x\|$, $\rho_{q,I}^*(x) \asymp \|B_I x\|$, then $B_I^2 \asymp A_I^{-2}$ in the sense of selfadjoint operators.

Proof. Obvious.

Remark. For $p = q = 2$ the operators A_I and B_I can be calculated if we use the language of matrix weight W ,

$$A_I = \langle W \rangle_I^{1/2}, \quad B_I = \langle W^{-1} \rangle_I^{1/2}.$$

$$\langle W \rangle_I^{-1} \leq \langle W^{-1} \rangle_I.$$

The A_2 -condition obtained in [TV1], [TV2] had a form

$$\langle W^{-1} \rangle_I \leq C \langle W \rangle_I^{-1},$$

where inequality is understood in the sense of selfadjoint operators.

2. MATRIX A_∞ CONDITION

Let us introduce the $A_{p,\infty}$ condition on metric weight which serves as the analog of the A_∞ condition for scalar weights. Its definition goes exactly along the lines of the definition of A_p , except $\rho_{q,I}^*$ is replaced by

$$\rho_{o,I}^*(x) \stackrel{\text{def}}{=} e^{\frac{1}{I} \int_I \log \rho_t^*(x) dt}.$$

Proposition 2.1. $(\rho_{p,I})^* \leq \rho_{o,I}^*$.

Proof. The inequality can be rewritten as

$$(\rho_{o,I}^*)^* \leq \rho_{p,I}.$$

Then it follows from two elementary observations:

$$\rho_{o,I} \leq \rho_{p,I},$$

which is just Jensen's inequality, and

$$(2.1) \quad (\rho_{o,I}^*)^* \leq \rho_{o,I}.$$

This last inequality follows from

$$\log |(x, y)| - \frac{1}{I} \int_I \log \rho_t^*(y) dt \leq \frac{1}{I} \int_I \log \rho_t(x) dt,$$

and this is the integration of the logarithm of $|(x, y)| \leq \rho_t^*(y) \rho_t(x)$.

Definition. If we have an opposite inequality

$$(2.2) \quad \rho_{o,I}^* \leq C(\rho_{p,I})^*, \quad \forall I$$

which is equivalent to

$$(2.3) \quad \rho_{p,I} \leq C(\rho_{o,I}^*)^*, \quad \forall I,$$

then ρ is called an $A_{p,\infty}$ -metric. If $\rho_t(x) = \|W^{1/p}(t)x\|$ and ρ is an $A_{p,\infty}$ -metric, then W is called an $A_{p,\infty}$ matrix weight.

Remark. By Jensen's inequality $\rho_{o,I}^* \leq \rho_{q,I}^*$ and thus $A_p \Rightarrow A_{p,\infty}$.

Let B_ρ denote the unit ball in the metric ρ .

Proposition 2.2. *The following assertions are equivalent:*

- 1) $\rho \in A_{p,\infty}$;

2) $\frac{1}{I} \int_I \log \text{vol } B_{\rho_t} dt \leq C + \log \text{vol } B_{\rho_{p,I}}$ for all I .

In particular, $W \in A_{p,\infty}$ if and only if

$$\det A_I \leq C e^{\frac{1}{I} \int_I \log \det W(t)^{1/p} dt}, \quad \forall I,$$

where A_I is a positive definite operator reducing $\rho_{p,I}$ in the previous sense, namely

$$\rho_{p,I}(x) \asymp \|A_I x\|.$$

Proof. 1) \Rightarrow 2). It is convenient to use the language of matrices. So let $\rho_t(x) \asymp \|A(t)x\|$ and $\rho_{p,I}(x) \asymp \|A_I x\|$. Let $\{e_i\}_{i=1}^d$ be an orthonormal basis of eigenvectors of A_I . We apply 1) to each of these vectors:

$$\frac{1}{I} \int_I \log \|A(t)^{-1} e_i\| \leq C + \log \|A_I^{-1} e_i\|, \quad i = 1, \dots, d.$$

Use the following elementary observation: for any positive definite $d \times d$ matrix Q and any orthonormal basis $\{e_i\}_{i=1}^d$ in \mathbb{C}^d , $\det Q \leq \prod_{i=1}^d (Qe_i, e_i) \leq \prod_{i=1}^d \|Qe_i\|$. This inequality is just the consequence of arithmetic mean-geometric mean inequality. Summing up the inequalities above we get

$$\frac{1}{I} \int_I \log \det A(t)^{-1} dt \leq C + \log \det A_I^{-1},$$

which means

$$\frac{1}{I} \int_I \log \text{vol } B_{\rho_t} \leq C + \log \text{vol } B_{\rho_{p,I}}.$$

2) \Rightarrow 1). Let g be a unit vector. Choose $g_1(t) \equiv g, g_2(t), \dots, g_d(t)$ to be unit vectors such that for $C(t) \stackrel{\text{def}}{=} A_I^{-1} A(t)$ we have $\det C(t)^{-1} \geq C_d \prod_{i=1}^d \|C(t)^{-1} g_i(t)\|$. We know that

$$\begin{aligned} \frac{1}{I} \int_I \|C(t) g_i(t)\|^p &\leq C_{d,p} \sum_{j=1}^d \frac{1}{I} \int_I |(C^*(t) e_j, g_j(t))|^p \\ &\leq C_{d,p} \sum_{j=1}^d \|A(t) A_I^{-1} e_j\|^p \asymp C_{d,p} \sum_{j=1}^d \rho_{p,I}^p(A_I^{-1} e_j) \\ &< C'_{d,p} < \infty. \end{aligned}$$

In particular,

$$\sum_{i=1}^d \frac{1}{I} \int_I \log^+ \|C(t) g_i(t)\| dt \leq \frac{1}{p} C'_{d,p}.$$

We are given $\frac{1}{I} \int_I \log \det C(t)^{-1} dt \leq C$ which implies

$$\begin{aligned} \frac{1}{I} \int_I \log^+ \|C(t)^{-1} g\| dt &\leq C + \sum_{i=1}^d \frac{1}{I} \int_I \log^- \|C(t)^{-1} g_i(t)\| \\ &\leq C + \sum_{i=1}^d \frac{1}{I} \int_I \log^+ \|C(t) g_i(t)\| dt \leq C + \frac{1}{p} C'_{d,p} = C'' < \infty. \end{aligned}$$

This implies

$$\frac{1}{I} \int_I \log \|A(t)^{-1}g\| dt \leq C'' + \log \|A_I^{-1}g\|.$$

The matrix $A_{2,\infty}$ condition can be reformulated (see [TV1]) in terms of the Carleson property of a certain measure built using $\|W_I^{-1/2}(W_{I_1} - W_{I_2})W_I^{-1/2}\|$ exactly as it is done for scalar A_∞ in Fefferman, Kenig and Pipher’s work [FKP]. The same should probably be true for $A_{p,\infty}$ if some A_I -built measure is used.

3. EMBEDDING THEOREM

In this section we deal with $\rho \in A_{p,\infty}$. We consider the “reducing” selfadjoint operators A_I :

$$(3.1) \quad \left(\frac{1}{I} \int_I \rho_t(x)^p dt\right)^{1/p} \asymp \|A_I x\|.$$

Then the $A_{p,\infty}$ condition can be written as $(\rho_{p,I})^* \geq c\rho_{o,I}^*$, that is,

$$(3.2) \quad \forall J, \quad \frac{1}{J} \int_J \log \rho_t^*(x) dt \leq C + \log \|A_J^{-1}x\|,$$

with the inverse inequality being always valid as well with $C = 0$ (see Proposition 2.1).

Lemma 3.1. *Let $\rho \in A_{p,\infty}$, and given a dyadic I let \mathcal{F}_k denote the family of dyadic J , $J \subset I$, such that*

$$\|A_J^{-1}A_I\| \geq e^k.$$

Then

$$(3.3) \quad \left| \bigcup_{J \in \mathcal{F}_k} J \right| \leq \frac{c}{k} |I|.$$

Proof. Let $\{e_i\}_{i=1}^d$ be a standard basis in \mathbb{C}^d . Denote by \mathcal{F}_k^i the set of dyadic intervals with

$$\|A_J^{-1}A_I e_i\| \geq e^k.$$

First use (3.2) to conclude that

$$(3.4) \quad \frac{1}{I} \int_I \log \rho_t^*(A_I e_i) \leq C.$$

Then use the inverse to (3.2) to conclude that for $J \in \mathcal{F}_k^i$

$$(3.5) \quad \frac{1}{J} \int_J \log^+ \rho_t^*(A_I, e_i) \geq \log \|A_J^{-1}A_I e_i\| \geq k.$$

If we could replace \log by \log^+ in (3.4) we would be done by standard argument. To do this replacement let us notice that

$$\begin{aligned} \frac{1}{I} \int_I \log^+ \rho_t^*(A_I e_i) &= \frac{1}{I} \int_I \log \rho_t^*(A_I e_i) + \frac{1}{I} \int_I \log^+ \frac{1}{\rho_t^*(A_I e_i)} \\ &\leq C + \frac{1}{pI} \int_I \log^+ \frac{1}{(\rho_t^*(A_I e_i))^p} \leq C + \frac{1}{pI} \int_I \log^+ (\rho_t(A_I^{-1} e_i))^p \\ &\leq C + \frac{1}{pI} \int_I \rho_t^p(A_I^{-1} e_i) \leq C + \frac{1}{p} (C \|e_i\|)^p \leq C'. \end{aligned}$$

In this chain we used an elementary inequality

$$\rho_t(A_I^{-1} e_i) \rho_t^*(A_I e_i) \geq |(e_i, e_i)| = 1.$$

The lemma is proved.

Remark. It is needless to say that for the scalar situation all $A_{p,\infty}$ just coincide with A_∞ . Lemma 3.1 reflects a characteristic property of A_∞ functions: the set where the function is much smaller than its average is small.

Let h_J denote a standard Haar function normalized in $L^2(dt)$.

Before stating the theorem we need to introduce a new type of weighted Triebel-Lizorkin spaces. Given a family of linear operators $\{C_J\}_{J \in D}$ we consider the space $f_p^{02}(\mathbb{C}^d, \{C_J\})$ to consist of sequences $\{s\}_{J \in D}$ such that $\{C_J s_J\} \in f_p^{02}$.

The following is the embedding theorem which gives the name to the section. As always $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 3.2. *Let $\rho \in A_{p,\infty}$ and let $\{A_J\}$ be its reducing operators as in (3.1). Let $s_J = \int_J g h_J$. Then the following inequality holds:*

$$\|\{s_J\}\|_{f_q^{02}(\{A_J^{-1}\})} \leq C \|g\|_{L^q(\rho^*)}.$$

Remarks. 1) Even in the simplest case $p = q = 2$ and the scalar weight $w \in A_\infty$ this result seems to be rather new (although we did not check all the literature). It amounts to the inequality

$$w \in A_\infty \Rightarrow \sum_{J \in D} \langle w \rangle_J^{-1} |(g, h_J)|^2 \leq C \|g\|_{L^2(\frac{1}{w})}^2,$$

which was mentioned in [TV1], [TVZ]. Notice that when $w \in A_2$ the left part is equivalent to

$$\begin{aligned} \sum \langle w^{-1} \rangle_J |(g, h_J)|^2 &= \int \sum_{J \ni x} |(g, h_J)|^2 \frac{1}{|J|} w^{-1}(x) dx \\ &= \|\{(g, h_J)\}\|_{f_2^{02}(w^{-1})}^2 \asymp \|g\|_{L^2(w^{-1})}^2. \end{aligned}$$

2) There are interesting general relationships between

$$f_p^{02}(\mathbb{C}^d, \rho) \stackrel{\text{def}}{=} \left\{ \{s_I\} : \int \left(\sum_{I \ni t} \rho_t^2(s_I) \frac{1}{|I|} \right)^{p/2} dt < \infty \right\}$$

and

$$f_p^{02}(\mathbb{C}^d, \{C_I\}) = \left\{ \{s_I\} : \int \left(\sum_{I \ni t} \|C_I s_I\|^2 \frac{1}{|I|} \right)^{p/2} dt < \infty \right\}.$$

Quite naturally one should choose C_I reducing ρ_t :

$$(3.6) \quad \left(\frac{1}{I} \int_I \rho_t^p(x) dt \right)^{1/p} \asymp \|C_I x\|,$$

and then

$$f_p^{02}(\mathbb{C}^d, \{C_I\}) = \left\{ \{s_I\} : \int \left(\sum_{I \ni t} \left(\frac{1}{I} \int_I \rho_s^p(s_I) ds \right)^{2/p} \frac{1}{|I|} \right)^{p/2} dt < \infty \right\}.$$

It is clear that for $p = 2$ these spaces coincide. It will be proved elsewhere that for $p \geq 2$, $\rho \in A_{p,\infty}$ they also coincide. Clearly, many questions appear: what about $p < 2$?, what if we change exponent p to another one in (3.6)?, etc.

Notice that if $W \in A_p$ these spaces coincide.

3) The space $f_q^{02}(\mathbb{C}^d, \{A_J^{-1}\})$ is not of the type considered in 3). The difference is that now $C_J = A_J^{-1}$ are not generated as reducing operators as it was in (3.6). They are inverses to reducing operators.

Notice that if $W \in A_p$, then inverses to reducing operators are also reducing for the dual weight $U = W^{-q/p}$ and in this particular case we are going to prove in the next section that

$$(3.7) \quad L^p(W) \approx f_p^{02}(\mathbb{C}^d, \{A_J\}),$$

where A_J are reducing operators introduced in (3.1).

Proof of Theorem 3.2. Fix $g \in L^q(\rho^*)$ and a test function $f \in L^p$. We need to show that

$$(3.8) \quad \sum_{J \in D} \left| \left(\int g h_J, A_J^{-1} \int f h_J \right) \right| \leq C \|f\|_{L^p} \|g\|_{L^q(\rho^*)}.$$

The sum on the left can be rewritten as

$$\int \left(\sum_{J \ni t} \left| \left(\int g h_J, A_J^{-1} \int f h_J \right) \right| \frac{1}{|J|} \right) dt \stackrel{\text{def}}{=} \int S(t) dt.$$

Let us consider a nonnegative function $h(t)$ and introduce

$$S_{h(t)}(t) = \sum_{\substack{J \ni t \\ |J| \leq h(t)}} \left| \left(\int g h_J, A_J^{-1} \int f h_J \right) \right| \frac{1}{|J|}.$$

$S_{h(t)}(t) \leq S(t)$ but

$$\int S_{h(t)}(t) dt \geq a \int S(t) dt$$

if the function $h(t)$ has the following property:

$$(3.9) \quad \forall J \in D \quad |\{t \in J : h(t) \geq |J|\}| \geq a|J|.$$

We are going to choose now a function h satisfying (3.9) and such that

$$(3.10) \quad S_{h(t)}(t) \leq B (M \|f\|^{p^*})^{1/p^*} (t) (M (\rho^*(g))^{q^*})^{1/q^*} (t)$$

for certain $B < \infty$, $1 < p_* < p$, $1 < q_* < q$. M denotes the maximal function. After (3.10) we are done:

$$\begin{aligned} \int S(t)dt &\leq a^{-1} \int S_{h(t)}(t)dt \leq a^{-1} \int (M\|f\|^{p_*})^{1/p_*}(t) \cdot (M\rho^*(g)^{q_*})^{1/q_*}(t) \\ &\leq a^{-1} \left(\int (M\|f\|^{p_*})^{p/p_*} \right)^{1/p} \left(\int (M\rho^*(g)^{q_*})^{q/q_*} \right)^{1/q} \\ &\leq C(a, p/p_*, q/q_*) \left(\int \|f\|^p \right)^{1/p} \left(\int \rho^*(g)^q \right)^{1/q} \\ &= C(a, p/p_*, q/q_*) \|f\|_{L^p} \cdot \|g\|_{L^q(\rho^*)}, \end{aligned}$$

which is what we need.

To choose $h(t)$ satisfying (3.9) and (3.10) we follow the algorithm below.

Fix a dyadic interval I . Fix a large k and delete from I all the dyadic intervals for which

$$\|A_I A_J^{-1}\| \geq e^k.$$

By Lemma 3.1 the set E_I which rests has the property that

$$|E_I| \geq \frac{1}{2}|I|.$$

Let us introduce

$$\begin{aligned} t \in I, \quad S^{A_I^{-1}}(g)(t) &= \left(\sum_{\substack{J \ni t \\ J \subseteq I}} \|A_I^{-1} \int g h_J\|^2 \frac{1}{|J|} \right)^{1/2}; \\ t \in I, \quad S^{A_I A_J^{-1}}(f)(t) &= \left(\sum_{\substack{J \ni t \\ J \subseteq I}} \|A_I A_J^{-1} \int f h_J\|^2 \frac{1}{|J|} \right)^{1/2}. \end{aligned}$$

We are interested in proving that there are relatively large subsets of I on which these S -functions are smaller than maximal functions from (3.10).

To find such sets it is sufficient to estimate any averages of these S -functions. Fix $\alpha > 1$ and very close to 1. Then

$$\begin{aligned} \frac{1}{I} \int_I \left(S^{A_I^{-1}}(g)(t) \right)^\alpha dt &\leq C(\alpha) \frac{1}{I} \int \|A_I^{-1} g \cdot \chi_I\|^\alpha dt \\ &= C(\alpha) \frac{1}{I} \int_I \|A_I^{-1} g\|^\alpha dt. \end{aligned}$$

The first inequality follows from the classical L^α estimate of the S -function by the L^α norm of the function when $\alpha > 1$ (see e.g. [St]). To continue the estimate we use that we are in the space of finite dimension to write $\{e_i\}$ is a standard basis of \mathbb{C}^d :

$$\|A_I^{-1} g(t)\| \leq C_d \sum_{i=1}^d |(A_I^{-1} g(t), e_i)| \leq C_d \sum_{i=1}^d \rho_t^*(g) \rho_t(A_I^{-1} e_i).$$

Thus

$$(3.11) \quad \begin{aligned} \|S^{A_I^{-1}}(g)\|_{L^\alpha(\frac{dt}{t})} &\leq C(\alpha, d) \sum_{i=1}^d \left(\frac{1}{I} \int_I \rho_t^{(p+\varepsilon)\alpha}(A_I^{-1}e_i) \right)^{\frac{1}{(p+\varepsilon)\alpha}} \\ &\cdot \left(\frac{1}{I} \int_I \rho_t^*(g)^{(q-\delta)\alpha} \right)^{\frac{1}{(q-\delta)\alpha}}. \end{aligned}$$

Here $(p + \varepsilon)^{-1} + (q - \delta)^{-1} = 1$, and ε is chosen to be very small to guarantee $q - \delta > 1$. The choice of ε and α is also dictated by the observation that the positive function $t \rightarrow \rho_t^p(x)$ is an A_∞ -function if $\rho \in A_{p,\infty}$.

In fact, inequalities (2.1) and (2.3) imply that

$$(3.12) \quad \frac{1}{I} \int_I \rho_t^p(x) dt \leq C e^{\frac{1}{I} \int_I \log \rho_t^p(x) dt},$$

which is equivalent to A_∞ (see [GCRF] and [St], p. 218). As $\rho_t^p(x)$ is (uniformly in x) in A_∞ it satisfies (uniformly in x) a reverse Hölder inequality. In particular, there exist a $\bar{p} > p$ such that

$$(3.13) \quad \forall I, \forall x, \left(\frac{1}{I} \int_I \rho_t^{\bar{p}}(x) dt \right)^{1/\bar{p}} \leq C \left(\frac{1}{I} \int_I \rho_t^p(x) dt \right)^{1/p}.$$

Now we choose α close to 1 and ε close to 0 to guarantee that

$$(p + \varepsilon)\alpha \leq \bar{p}.$$

In particular, (3.13) implies

$$(3.14) \quad \begin{aligned} \left(\frac{1}{I} \int_I \rho_t^{(p+\varepsilon)\alpha}(A_I^{-1}e_i) dt \right)^{\frac{1}{(p+\varepsilon)\alpha}} &\leq C \left(\frac{1}{I} \int_I \rho_t^p(A_I^{-1}e_i) \right)^{1/p} \\ &\leq C' \|A_I A_I^{-1}e_i\| = C', \end{aligned}$$

where we use the definition (3.1) of A_I .

Now we can continue (3.11) using (3.14):

$$(3.15) \quad \|S^{A_I^{-1}}(g)\|_{L^\alpha(I, \frac{dt}{t})} \leq C(\alpha, d) \cdot d \cdot C' \cdot \inf_{t \in I} [M(\rho^*(g)^{q_*})(t)]^{\frac{1}{q_*}},$$

where $q_* = (q - \delta) \cdot \alpha \in (1, q)$ if $\delta = \delta(\varepsilon)$ and α are chosen properly.

This last inequality ensures that for any $\tau, \tau \in (0, 1)$, we can find the subset $E(\tau, I) \subset I$, $|E(\tau, I)| \geq (1 - \tau^\alpha)|I|$ such that

$$(3.16) \quad t \in E(\tau, I) \Rightarrow S^{A_I^{-1}}(g)(t) \leq \frac{C''(\alpha, d)}{\tau} \inf_{t \in I} [M(\rho^*(g)^{q_*})]^{1/q_*}.$$

This is the main estimate of our first S -function.

Now we will give a similar estimate for the second S -function $S^{A_I A_J^{-1}}(f)(t)$. Again choose $\alpha > 1$ and close to 1 to estimate $\frac{1}{|E_I|} \int_{E_I} \left(S^{A_I A_J^{-1}}(f)(t) \right)^\alpha dt$. For $t \in E_I$ all $J \ni t, J \subset I$ have the property

$$\|A_I A_J^{-1}\| \leq e^k$$

and so

$$t \in E_I, \quad S^{A_I A_J^{-1}}(f)(t) \leq S^I(f)(t) \stackrel{\text{def}}{=} \left(\sum_{\substack{J \ni t \\ J \subset I}} \left\| \int f h_J \right\|^2 \frac{1}{|J|} \right)^{1/2}.$$

Thus

$$\frac{1}{|E_I|} \int_{E_I} \left(S^{A_I A_J^{-1}}(f)(t) \right)^\alpha dt \leq \frac{2}{I} \int_I (S^I(f)(t))^\alpha dt \leq C(\alpha) \frac{1}{I} \int_I \|f\|^\alpha dt.$$

Choosing $1 < \alpha < p$ and denoting it by p_* we get

$$\|S^{A_I A_J^{-1}}(f)\|_{L^\alpha(E_I, \frac{dt}{|E_I|})} \leq C(\alpha) \inf_{t \in I} (M\|f\|^{p_*})^{1/p_*}(t).$$

As before it implies for any $\tau \in (0, 1)$ the existence of $e(\tau, I) \subset E_I$, $|e(\tau, I)| \geq (1 - \tau^\alpha)|E_I| \geq \frac{1}{2}(1 - \tau^\alpha)|I|$ such that

$$(3.17) \quad t \in e(\tau, I) \Rightarrow S^{A_I A_J^{-1}}(f)(t) \leq \frac{C(\alpha)}{\tau} \inf_{t \in I} (M\|f\|^{p_*})^{1/p_*}.$$

Now choose τ so small that

$$(3.18) \quad |e(\tau, I) \cap E(\tau, I)| \geq \frac{1}{4}|I|.$$

We are ready to choose the height function $h(t)$ satisfying (3.9) and (3.10). Put

$$(3.19) \quad h(t) \stackrel{\text{def}}{=} \sup \left\{ h : S_h(t) \stackrel{\text{def}}{=} \sum_{\substack{J \ni t \\ |J| \leq h}} \frac{1}{|J|} \left| \left(\int g h_J, A_J^{-1} \int f h_J \right) \right| \leq B(M\|f\|^{p_*})^{1/p_*}(t) (M\rho^*(g)^{q_*}(t)) \right\},$$

where p_*, q_* appeared in (3.17) and (3.16) and where $B = \frac{C''(\alpha, d)C(\alpha)}{\tau^2}$, with constants taken from (3.16) and (3.17).

Then (3.10) is fulfilled automatically. To check (3.9) let us fix any $I \in D$ and let us consider for $t \in I$

$$S_I(t) \stackrel{\text{def}}{=} \sum_{\substack{J \ni t \\ J \subset I}} \frac{1}{|J|} \left| \left(\int g h_J, A_J^{-1} \int f h_J \right) \right|.$$

Notice that

$$S_I(t) = \sum_{\substack{J \ni t \\ J \subset I}} \left| \left(A_I^{-1} \int g h_J, A_I A_J^{-1} \int f h_J \right) \right| \leq S^{A_I^{-1}}(g)(t) S^{A_I A_J^{-1}}(f)(t).$$

Thus (3.16) and (3.17) applied for $t \in e(\tau, I) \cap E(\tau, I)$ show that on at least a quarter of I (see (3.18))

$$S_I(t) \leq B(M\|f\|^{p_*})^{1/p_*}(t) (M\rho^*(g)^{q_*})^{1/q_*}(t).$$

Thus on a quarter of I the function h just defined in (3.19) is greater than $|I|$. Because I was arbitrary, (3.9) holds and Theorem 3.2 is finished.

4. T_{\pm} THEOREM

Theorem 3.2 means that the operator $\sum A_J^{-1} h_J \otimes h_J$ given by

$$g \rightarrow \sum_{J \in D} A_J^{-1} \left(\int g h_J \right) \cdot h_J$$

is bounded as $L^q(\rho^*) \rightarrow f_q^{02}$. If we use the language of matrix functions, meaning that $\rho_t(x) = \|W^{1/p}(t)x\|$, $\rho_t^*(x) = \|U^{1/q}(t)x\|$, where $U^{1/q} = W^{-1/p}$, we can make a change of variable $g(t) = W^{1/p}(t)G(t)$ now having $G \in L^q$ with no weight. The operator now can be written in the form $\sum A_J^{-1} h_J \otimes W^{1/p} h_J$. If A_J is given by (3.1), the sum above is proved to be a bounded operator from L^q to f_q^{02} . The same proof certainly shows that operators $\sum \pm A_J^{-1} h_J \otimes W^{1/p} h_J$ are uniformly bounded.

All these claims are true under the condition that ρ (or W) belongs to $A_{p,\infty}$.

Now if ρ (or W) belongs to A_p one can say more. We summarize it in two theorems.

Theorem 4.1. *Let $\rho \in A_p$. Then if A_J are from (3.1) and $s_J = \int f h_J$ we have*

$$(4.1) \quad \|\{s_J\}\|_{f_p^{02}(\{A_J\})} \asymp \|f\|_{L^p(\rho)}.$$

Proof. It follows immediately from the definition that $\rho \in A_p \Rightarrow \rho \in A_{p,\infty}$ and $\rho \in A_p \Rightarrow \rho^* \in A_q \Rightarrow \rho^* \in A_{q,\infty}$.

Let us also introduce the reducing operators for ρ^* :

$$\left(\frac{1}{J} \int_J \rho_t^*(x)^q dt \right)^{1/q} \asymp \|C_J x\|.$$

Let us apply Theorem 3.2 to $\rho^* \in A_{q,\infty}$ to obtain

$$\left\| \left\{ \int f h_J \right\} \right\|_{f_p^{02}(\{C_J^{-1}\})} \leq C \|f\|_{L^p(\rho)}.$$

Notice that $\rho \in A_p$ means that $C_J^{-2} \asymp A_J^2$ and so we get

$$\left\| \left\{ \int f h_J \right\} \right\|_{f_p^{02}(\{A_J\})} \leq C \|f\|_{L^p(\rho)}.$$

To obtain the inverse inequality we use duality and again Theorem 3.2 for $\rho \in A_{p,\infty}$ to obtain

$$\begin{aligned} \|f\|_{L^p(\rho)} &= \sup_{\|g\|_{L^q(\rho^*)} \leq 1} |\langle f, g \rangle| = \sup \left| \sum \left(\int f h_J, \int g h_J \right) \right| \\ &= \sup \left| \sum \left(A_J \int f h_J, A_J^{-1} \int g h_J \right) \right| \\ &\leq \left\| \left\{ \int f h_J \right\} \right\|_{f_p^{02}(\{A_J\})} \cdot \left\| \left\{ \int g h_J \right\} \right\|_{f_q^{02}(\{A_J^{-1}\})} \\ &\leq C \left\| \left\{ \int f h_J \right\} \right\|_{f_p^{02}(\{A_J\})}. \end{aligned}$$

The last inequality is Theorem 3.2. We are done.

Remark. It seems to us that this theorem gives new information about $L^p(w)$, $w \in A_p$, even in the scalar case. The reader should notice that we organized a canonical isomorphism of $L^p(w)$ and f_p^{02} . This is not the classical isomorphism $f \rightarrow \{(f, h_J)\}_{J \in D}$ between $L^p(w)$ and $f_p^{02}(w)$.

Corollary 4.2. *Operators*

$$T_{\pm} \stackrel{\text{def}}{=} \sum \pm h_J \otimes h_J$$

defined by the formula

$$T_{\pm} \varphi = \sum_{J \in D} \pm h_J \cdot \int \varphi h_J$$

are uniformly bounded in $L^p(\rho)$ if and only if $\rho \in A_p$.

Proof. The “if” part follows immediately from Theorem 4.1. In fact, we just use the equivalent norm $\|\cdot\|_{f_p^{02}(\{A_J\})}$ in $L^p(\rho)$ which does not change if φ is replaced by $T_{\pm} \varphi$ because $\int T_{\pm} \varphi \cdot h_J = \pm \int \varphi h_J$. The “only if” part follows from the uniform boundedness of rank one operators $h_J \otimes h_J$. It follows exactly as the first part of Theorem 5.1 below. So we send the reader to this coming proof.

The next result is an analog of the classical isomorphism $L^p(w) \approx f_p^{02}(w)$.

Theorem 4.3. *Let $\rho \in A_p$. Let $s_J = \int f h_J$. Then*

$$(4.2) \quad \|\{s_J\}\|_{f_p^{02}(\rho)} \asymp \|f\|_{L^p(\rho)}.$$

Proof. This follows easily from the corollary. In fact, we know that

$$\|f\|_{L^p(\rho)}^p \asymp \int \rho_t^p \left(\sum_{J \in D} \pm h_J \cdot \int f h_J \right) dt.$$

We integrate over the signs and use the Khintchin theorem which is valid here. We obtain

$$\begin{aligned} \|f\|_{L^p(\rho)}^p &\asymp \int \left(\sum_{J \ni t} \rho_t^2 \left(\int f h_J \right) \frac{1}{|J|} \right)^{p/2} dt \\ &= \|\{s_J\}\|_{f_p^{02}(\rho)}^p. \end{aligned}$$

Remark and question. In particular, if $\rho \in A_p$ and A_J are reducing operators for ρ considered in (3.1), we see from (4.1) and (4.2) the following equivalence of norms in sequence spaces:

$$(4.3) \quad \|\{s_J\}\|_{f_p^{02}(\rho)} \asymp \|\{s_J\}\|_{f_p^{02}(\{A_J\})}.$$

But it has already been mentioned in the remarks to Theorem 3.2 that this equivalence does not require $\rho \in A_p$. For $p = 2$ it holds always. For $p \geq 2$, $\rho \in A_{p,\infty}$ would be sufficient for (4.3). This can be proved along the lines of Theorem 3.2.

We would like to ask for the criterion for (4.3) to hold.

5. HILBERT TRANSFORM

Theorem 4.1 and Corollary 4.2 allow one to show that any Calderon Zygmund operator with $T1 = T^*1 = 0$ is bounded in $L^p(\rho)$ if $\rho \in A_p$. But still the reduction of any T to T_\pm is a very technical and difficult task. In this section we show that at least for the Hilbert transform one can give a relatively easy and straightforward proof very similar to the one of Corollary 4.2.

Theorem 5.1. *Let $\rho \in A_p$. Then*

$$(5.1) \quad \|P_+f\|_{L^p(\rho)} \leq C\|f\|_{L^p(\rho)}.$$

Conversely if (5.1) holds, then $\rho \in A_p$.

Before proving Theorem 5.1 let us prove two simple lemmas. Consider the Poisson kernel $P_I = \frac{|I|}{x^2 + |I|^2}$ and define $k_{I,q} = P_I^{q/2} \cdot |I|^{q/2-1}$ ($1 < q < \infty$). Then $\int k_{I,q} \asymp 1$ and here is the first lemma.

Lemma 5.2. *If u is a scalar A_q weight, then $\int uk_{I,q} \leq C \frac{1}{|I|} \int_I u$.*

Proof. We can write $k_{I,q} \asymp \sum_{k=1}^\infty \varepsilon_k \frac{1}{|2^k I|} \chi_{2^k I}$, where $\varepsilon_k \asymp (2^{-k})^{q/p}$. Now if $u \in A_q$ let $w = u^{-p/q} \in A_p$ and we can write the following chain of inequalities:

$$\begin{aligned} (w_I)^{q/p} \left(\int k_{I,q} u \right) &\leq \sum_{k=1}^\infty (w_I)^{q/p} u_{2^k I} \varepsilon_k \\ &\leq \sum_{k=1}^\infty \frac{(w_I)^{q/p}}{(w_{2^k I})^{q/p}} \varepsilon_k = \sum_{k=1}^\infty \left(\frac{w(I)}{w(2^k I)} \right)^{q/p} 2^{k \cdot \frac{q}{p}} \varepsilon_k \\ &\leq C \sum_{k=1}^\infty \left(\frac{w(I)}{w(2^k I)} \right)^{q/p} \leq C \sum_{k=1}^\infty (1 + \varepsilon)^{-k} \leq C'. \end{aligned}$$

The last inequality follows from the doubling property of A_p weights. We get $\int k_{I,q} u \leq C(w_I)^{-q/p} \leq C' u_I$ by using the A_p condition again.

Lemma 5.3. *If W is a matrix A_p weight, then $t \rightarrow \|W^{1/p}x\|^p$ is a scalar A_p weight for any vector x .*

Proof. It is convenient to use the language of $\rho_t(x) = \|W^{1/p}(t)x\|^p$. Fix x and choose $y(x)$ in such a way that $(\rho_{p,I})(x) = \sup_y \frac{|(x,y)|}{(\rho_{p,I})^*(y)} \leq 2 \frac{|(x,y(x))|}{(\rho_{p,I})^*(y(x))}$. Now

$$\begin{aligned} \left(\frac{1}{|I|} \int_I \rho_t^p(x) \right)^{1/p} \left(\frac{1}{|I|} \int_I \rho_t^{-q}(x) \right)^{1/q} &\leq \rho_{p,I}(x) \left(\frac{1}{|I|} \int_I \left(\frac{\rho_t^*(y(x))}{|(x,y(x))|} \right)^q \right)^{1/q} \\ &\leq \frac{\rho_{p,I}(x) \rho_{q,I}^*(y(x))}{|(x,y(x))|} \leq C \frac{\rho_{p,I}(x) (\rho_{p,I})^*(y(x))}{|(x,y(x))|} \leq 2C. \end{aligned}$$

Proof of Theorem 5.1. (5.1) $\Rightarrow \rho \in A_p$.

Operator P_+ has the kernel $\frac{1}{1-\bar{\zeta}z}$. We are given that it is a bounded operator. Fix an a in the unit disk and consider the operator with the kernel

$$\frac{1}{1-\bar{\zeta}z} \left[1 - \left(\frac{\bar{\zeta}-a}{1-\bar{a}\zeta} \right) \frac{z-a}{1-\bar{a}z} \right].$$

Obviously it is a bounded operator in $L^p(\rho)$ too, actually it is given by the formula $P_+ - b_a P_+ \bar{b}_a$, where $b_a(\zeta) = \frac{\zeta-a}{1-\bar{a}z}$. Notice that pointwise multiplication by b_a does not change the norm in $L^p(\rho)$. Now let us make the simplification in the kernel to see that it equals $\frac{1-|a|^2}{(1-\bar{\zeta}a)(1-\bar{a}z)} = k_a(z) \otimes k_a(\zeta)$, where $k_a(\zeta) = \frac{(1-|a|^2)^{1/2}}{1-\bar{a}\zeta}$ is a normalized Szegő kernel in L^2 . Notice that

$$|k_a(\zeta)|^2 = P_a(\zeta),$$

where $P_a(\zeta)$ denotes the Poisson kernel.

Now let us use the fact (which we just derived from the boundedness of P_+) that $k_a \otimes k_a$ are uniformly bounded. Let $\varphi(t) \in L^q$, $\psi \in L^p$, and $\int \|\varphi\|^q dt \leq 1$, $\int \|\psi\|^p dt \leq 1$. Then

$$\left| \left(\int k_a(t) W^{1/p}(t) \varphi(t) dt, \int k_a(s) U^{1/q}(s) \psi(s) ds \right) \right| \leq C$$

uniformly in a . Denoting $k_a^{(p)} = (1-|a|^2)^{1/2-1/p} k_a$ and $k_a^{(q)} = (1-|a|^2)^{1/2-1/q} k_a$, we have

$$\int |k_a^{(p)}(t)|^p \left\| W^{1/p}(t) \int k_a^{(q)}(s) U^{1/q}(s) \psi(s) ds \right\|^p \leq C.$$

Denoting $z = \int k_a^{(q)}(s) U^{1/q}(s) \psi(s) ds$ we conclude that

$$\left(\int |k_a^{(p)}|^p \|W^{1/p}(t)z\|^p \right)^{1/p} \leq C \|\psi\|_{L^p}.$$

This is an averaging because $\int |k_a^{(p)}|^p \approx 1$. Denote it by $\rho_{p,a}$:

$$\rho_{p,a}(z) \leq C \|\psi\|_{L^p}.$$

We want to show that

$$(\rho_{q,a}^*)^*(z) \geq C \inf \left\{ \|\psi\|_{L^p} : z = \int k_a^{(q)}(s) U^{1/q}(s) \psi(s) ds \right\}.$$

But

$$\begin{aligned} (\rho_{q,a}^*)^*(z) &= \max_y |(y, z)| / \left(\int \|U^{1/q}(s)y\|^q |k_a^{(q)}(s)|^q \right)^{1/q} \\ &= \max_y |\langle k_a^{(q)}(s) U^{1/q}(s)y, \psi(s) \rangle| / \|k_a^{(q)}(s) U^{1/q}(s)y\|_{L^q} \\ &= \inf \left\{ \|\psi\|_{L^p} : z = \int k_a^{(q)}(s) U^{1/q}(s) \psi(s) ds \right\}. \end{aligned}$$

Thus

$$\rho_{p,a}(z) \leq C(\rho_{q,a}^*)^*(z).$$

But $|k_a^{(p)}|^p \geq C(p) \frac{\chi_{I_a}}{|I_a|}$, where I_a is the interval of length $1 - |a|$ centered at $a/|a|$.

Thus

$$\rho_{p,I} \leq C(\rho_{q,I}^*)^*.$$

Now let us prove that $\rho \in A_p \Rightarrow (5.1)$.

Let ρ be given by $W^{1/p}$ and ρ^* by $U^{1/q} = W^{-1/p}$. We have to prove that

$$f \rightarrow W^{1/p} P_+ W^{-1/p} f$$

is a bounded operator as $L^p \rightarrow L^p$. Fix $g \in L^q$. We need to show

$$\int |(\nabla P_+ W^{-1/p} f(z), \nabla P_+ W^{1/p} g(z))| \delta(z) dA(z) \leq C \|f\|_p \|g\|_q,$$

where $\delta(z)$ means the distance to the boundary. In other words, introducing a Stolz cone Γ_t , and

$$S(t) \stackrel{\text{def}}{=} \int_{\Gamma_t} |(\nabla P_+ W^{-1/p} f(z), \nabla P_+ W^{1/p} g(z))| dA(z),$$

one needs to prove that

$$(5.2) \quad \int S(t) dt \leq C \|f\|_p \|g\|_q.$$

We follow closely the lines of the proof of Theorem 3.2. Let us consider a nonnegative function $h(t)$ and

$$S_{h(t)}(t) = \int_{\Gamma_{t,h(t)}} |(\nabla P_+ W^{-1/p} f(z), \nabla P_+ W^{1/p} g(z))| dA(z),$$

where

$$\Gamma_{t,h(t)} = \Gamma_t \cap \{z : \delta(z) \leq h(t)\}.$$

As in Section 3

$$(5.3) \quad \int S_{h(t)} dt \asymp \int S(t) dt$$

if the function $h(t)$ has the following property:

$$(5.4) \quad \forall J \quad |\{t \in J : h(t) \geq |J|\}| \geq a|J|.$$

Let us choose h to be maximal such that

$$(5.5) \quad S_{h(t)}(t) \leq B(M\|f\|^{p_*})^{1/p_*}(t) (M\|g\|^{q_*})^{1/q_*}(t),$$

where $B, p_* \in (1, p), q_* \in (1, q)$ will be chosen later.

If this h satisfies (5.4), then (5.3) and (5.2) imply what we need.

To choose B, p_*, q_* and to prove that h satisfies (5.4) we follow the algorithm below.

Fix an interval I . Consider $f_1 = f \cdot \chi_{2I}, g_1 = g \cdot \chi_{2I}$ and $f_2 = f - f_1, g_2 = g - g_1$. Let us introduce A_I as in (3.1):

$$t \in I, \quad S^{A_I}(f_i)(t) = \left(\int_{\Gamma_{t,|I|}} \|\nabla P_+ A_I W^{-1/p} f_i\|^2 dA(z) \right)^{1/2}, \quad i = 1, 2;$$

$$t \in I, \quad S^{A_I^{-1}}(g_i)(t) = \left(\int_{\Gamma_{t,|I|}} \|\nabla P_+ A_I^{-1} W^{1/p} g_i\|^2 dA(z) \right)^{1/2}, \quad i = 1, 2.$$

Then exactly as in Section 3 we choose $\alpha > 1$, α close to 1, and

$$\begin{aligned} \|S^{A_I}(f_1)\|_{L^\alpha(I, \frac{dt}{|I|})} &\leq C(\alpha, d) \left(\frac{1}{|I|} \int_{2I} \|f\|^{(p-\delta)\alpha} \right)^{\frac{1}{(p-\delta)\alpha}} \\ &\quad \cdot \sum_{i=1}^d \left(\frac{1}{|I|} \int_I \|W^{-1/p}(t)A_I e_i\|^{(q+\varepsilon)\alpha} dt \right)^{\frac{1}{(q+\varepsilon)\alpha}}. \end{aligned}$$

We can now use the fact that the function $t \rightarrow \|U^{1/q}(t)x\| = \|W^{-1/p}(t)x\|$ is uniformly in A_∞ to conclude that we have a uniform inverse Hölder inequality with $\bar{q} > q$. Now choosing ε and α to satisfy $(q + \varepsilon)\alpha < \bar{q}$ we conclude that the last integral is bounded by $\left(\frac{1}{|I|} \int_I \|W^{-1/p}A_I e_i\|^q dt \right)^{1/q}$. But $W \in A_p$, so A_I^{-1} reduces averages of the metric ρ^* given by $W^{-1/p} = U^{1/q}$. In other words,

$$(5.6) \quad \left(\frac{1}{|I|} \int_I \|W^{-1/p}x\|^q \right)^{1/q} \asymp \|A_I^{-1}x\|.$$

Thus finally

$$\|S^{A_I}(f_1)\|_{L^\alpha(I, \frac{dt}{|I|})} \leq C'(\alpha, d) \inf_{t \in I} (M\|f\|^{p_*})^{1/p_*(t)},$$

where $p_* = (p - \delta(\varepsilon))\alpha$ is ensured to be in $(1, p)$. Similarly

$$\|S^{A_I^{-1}}(g_1)\|_{L^\alpha(I, \frac{dt}{|I|})} \leq C'(\alpha, d) \inf_{t \in I} (M\|g\|^{q_*})^{1/q_*(t)}.$$

Now let us estimate $S^{A_I}(f_2)(t), S^{A_I^{-1}}(g_2)(t)$, for $t \in I$.

Let C_I be a center of the square Q_I built on I . It is easy to see that for $t \in I$ we have

$$(5.7) \quad \left(\int_{\Gamma_{t,|I|}} \|(\nabla P_+ A_I W^{-1/p}(t)f_2(t))(z)\|^2 \right)^{1/2} \leq a \int P_{C_I}(t) \|A_I W^{-1/p}(t)f_2(t)\| dt,$$

where P_{C_I} denotes the Poisson kernel. Let δ be small, $p - \delta > 1$, and let $(p - \delta)^{-1} + (q + \varepsilon)^{-1} = 1$. We can continue the estimate as follows ($k_a^{(r)} \stackrel{\text{def}}{=} (1 - |a|^2)^{\frac{1}{2} - \frac{1}{r}} \cdot k_a$):

$$\begin{aligned} \int P_{C_I} \|A_I W^{-1/p} f_2\| &\leq \sum_{i=1}^d \int \|W^{-1/p} A_I e_i\| \|f_2\| |k_{C_I}|^2(t) \\ &\leq C(p, q) \sum_{i=1}^d \int \|W^{-1/p} A_I e_i\| \|f_2\| k_{C_I}^{(q+\varepsilon)} k_{C_I}^{(p-\delta)} \\ (5.8) \quad &\leq C(p, q) \sum_{i=1}^d \left(\int \|W^{-1/p} A_I e_i\|^{q+\varepsilon} |k_{C_I}^{(q+\varepsilon)}|^{q+\varepsilon} \right)^{1/(q+\varepsilon)} \\ &\quad \cdot \left(\int \|f_2\|^{p-\delta} |k_{C_I}^{(p-\delta)}|^{p-\delta} \right)^{1/(p-\delta)}. \end{aligned}$$

Notice that $|k_{C_I}^{(q+\varepsilon)}|^{q+\varepsilon} \leq C|k_{C_I}^{(q)}|^q \stackrel{\text{def}}{=} Ck_{I,q}$.

As $t \rightarrow \|W^{-1/p}(t)x\|^q$ is uniformly A_q weight (see Lemma 5.3) we can conclude that there exists a small ε such that $t \rightarrow \|W^{-1/p}(t)x\|^{q+\varepsilon}$ is uniformly A_q weight by a classical inverse Hölder inequality. Lemma 5.2 claims that if u is a scalar weight in A_q , then $\int u k_{I,q} \leq C \frac{1}{I} \int_I u$. Combining this knowledge we conclude that

$$\begin{aligned} & \sum_{i=1}^d \left(\int \|W^{-1/p} A_I e_i\|^{q+\varepsilon} |k_{C_I}^{(q+\varepsilon)}|^{q+\varepsilon} \right)^{1/(q+\varepsilon)} \\ & \leq C \sum_{i=1}^d \left(\frac{1}{I} \int_I \|W^{-1/p} A_I e_i\|^{q+\varepsilon} \right)^{1/(q+\varepsilon)} \leq C \sum_{i=1}^d \left(\frac{1}{I} \int_I \|W^{-1/p} A_I e_i\|^q \right)^{1/q}. \end{aligned}$$

The last inequality is another application of inverse Hölder inequality.

Finally, using (5.6) we estimate the last sum by a constant.

Combining this with (5.7) and (5.8) we get ($p_* = p - \delta$)

$$t \in I \Rightarrow S^{A_I}(f_2)(t) \leq C(p, q, d) (M\|f\|^{p_*})^{1/p_*}(t).$$

Similarly

$$t \in I \Rightarrow S^{A_I^{-1}}(g_2)(t) \leq C(p, q, d) (M\|g\|^{q_*})^{1/q_*}(t).$$

Combining with our previous estimates for $S^{A_I}(f_1)$, $S^{A_I^{-1}}(g_1)$ we get

$$\|S^{A_I}(f)\|_{L^\alpha(I, \frac{dt}{|I|})} \leq C(p, q, \alpha, d) (M\|f\|^{p_*})^{1/p_*}(t),$$

$$\|S^{A_I^{-1}}(g)\|_{L^\alpha(I, \frac{dt}{|I|})} \leq C(p, q, \alpha, d) (M\|g\|^{q_*})^{1/q_*}(t).$$

In particular, on a quarter of I

$$\begin{aligned} S_I(t) & \stackrel{\text{def}}{=} \int_{\Gamma_{t,|I|}} |(\nabla P_+ W^{-1/p} f(z), \nabla P_+ W^{1/p} g(z))| dA(z) \\ & = \int_{\Gamma_{t,|I|}} |(A_I \nabla P_+ W^{-1/p} f, A_I^{-1} \nabla P_+ W^{1/p} g)| dA(z) \\ & = \int_{\Gamma_{t,|I|}} |(\nabla P_+ A_I W^{-1/p} f, \nabla P_+ A_I^{-1} W^{1/p} g)| dA(z) \\ & \leq S^{A_I}(f)(t) S^{A_I^{-1}}(g)(t) \leq C^2(p, q, \alpha, d) \\ & \quad \cdot (M\|f\|^{p_*})^{1/p_*}(t) (M\|g\|^{q_*})^{1/q_*}(t). \end{aligned}$$

For such t , $h(t) \geq |I|$, if we choose in the definition of h to have $B = C^2(p, q, \alpha, d)$ and p_*, q_* as above. Thus (5.4) is satisfied and the Theorem is proved.

ACKNOWLEDGEMENTS

The author is grateful to the participants of the Analysis Seminar at Michigan State University, and especially to Sheldon Axler, Michael Frazier, Fedja Nazarov, Serguei Treil and Dechao Zheng for fruitful discussions and constant enthusiasm. The author is also grateful to Robert Fefferman and Peter Jones for valuable discussions.

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