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ON AN n-MANIFOLD IN $\mathbf{C}^n$ NEAR AN ELLIPTIC COMPLEX TANGENT

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§1. Introduction

In this paper, we will be concerned with the local biholomorphic properties of a real n-manifold M in $\mathbf{C}^n$. At a generic point, such a manifold basically has the nature of the standard $\mathbf{R}^n$ in $\mathbf{C}^n$. Near a complex tangent, however, the consideration can be much more complicated and the manifold may acquire a non-trivial local hull of holomorphy and many other biholomorphic invariants. The study of such a problem was first carried out in a celebrated paper of E. Bishop [BIS] where, for each sufficiently non-degenerate complex tangent, he attached a biholomorphic invariant λ, called the Bishop invariant. When the complex tangent is elliptic, i.e., when $0 \leq \lambda < \frac{1}{2}$ (for a more precise definition, see §2), he showed the existence of families of complex analytic disks with boundary on M that shrink down to the locus of points in M with complex tangents. In particular, using the well-known continuity principle, one sees that the image $\widetilde{M}$ of such families is contained in the holomorphic hull of the manifold. At the time, he asked whether $\widetilde{M}$ gives precisely the local holomorphic hull of M, as well as certain uniqueness properties of the attached disks. He also proposed the problem of determining the fine structure of $\widetilde{M}$ near such complex tangents.

Later, there appeared a sequence of papers concerning the smooth character of $\widetilde{M}$ in case $M \subset \mathbf{C}^2$. Here we would like to mention, in particular, the famous theorem proved by Kenig-Webster in their deep work [KW1] which states that the local hull of holomorphy $\widetilde{M}$ near an elliptic complex tangent is a smooth Levi flat hypersurface with $M \subset \mathbf{C}^2$ as part of its smooth boundary. In another important paper of Moser-Webster [MW], a systematic normal form theory was employed for the understanding of the local biholomorphic invariants of M in case M is real analytic. When the Bishop invariant $\lambda \neq 0$, their method works in any dimension and even for some hyperbolic complex tangents; but it breaks down for complex tangents with $\lambda = 0$. Among other things, they showed that M can be biholomorphically mapped into the affine space

$$\mathcal{A} = \{(z_1, z', z_n) \in \mathbf{C} \times \mathbf{C}^{n-2} \times \mathbf{C} : \ \mathrm{Im}(z') = 0, \mathrm{Im}(z_n) = 0\}$$

near an elliptic complex tangent with $\lambda \neq 0$. Therefore, it follows easily that $\widetilde{M}$ is a real analytic $(n+1)$-manifold with real analytic boundary $M \subset \mathbf{C}^n$ for $\lambda \neq 0$. For

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the exceptional case $\lambda = 0$, Moser [MOS] showed that if M is formally equivalent to $M_0 = \{(z_1, z_2) \in \mathbf{C}^2 : z_2 = |z_1|^2\}$, then M is biholomorphically equivalent to M_0 and thus the holomorphic hull is real analytic up to the complex tangent point. When $M \subset \mathbf{C}^2$, Krantz and the author [HK] were recently able to prove the real analyticity of $\widetilde{M}$ across the boundary M near any elliptic complex tangent, by making use of the smoothness result of [KW1].

Efforts in [BG], [KW1], [MW], [MOS] and [HK] provide a fairly good understanding of the smooth and analytic structure of $\widetilde{M}$ in case $M \subset \mathbf{C}^2$, though it is still unknown how many biholomorphic invariants an analytic elliptic complex tangent with $\lambda = 0$ can have (see [MW] and [MOS]). In the case of higher dimensions, several features make the problem much more difficult. First, the situation that the complex tangents are no longer isolated complicates the study of the attached analytic disks. Also, the "expected hull" $\widetilde{M}$ has real dimension only $n + 1$, and therefore is far from being a hypersurface in $\mathbf{C}^n$. This makes it difficult to prove the uniqueness of the attached analytic disks and to verify that the object obtained is precisely the holomorphic hull of M. In [KW2], Kenig-Webster showed that near an elliptic complex tangent p under study, for each ℓ, there is a real $(n+1)$-manifold $\widetilde{M_\ell}$ with a small neighborhood of p in M as part of its C^ℓ-smooth boundary, which is foliated by families of analytic disks shrinking down to the complex tangents. But it remained open whether all the $\widetilde{M_\ell}$ coincide and give the local hull of holomorphy of M.

In this work, we present a fairly complete description of the local hull of holomorphy $\widetilde{M}$ for both smooth and real analytic M in $\mathbf{C}^n$ for any $n \geq 2$. When M is merely smooth, we will verify that all the Levi-flat submanifolds $\widetilde{M_\ell}$ which were constructed in the deep work of [KW2] are indeed the same near the elliptic complex tangent p under study, and give the local holomorphic hull of M. In fact, we will construct a suitable Stein neighborhood basis for a certain small neighborhood of p in $\widetilde{M_\ell}$ and prove the uniqueness property of the attached disks. This result can be immediately applied with the work in [KW2] to prove that the local hull $\widetilde{M}$ of M is smooth up to M, as was already known in the $\mathbf{C}^2$ case ([KW1]). When M is real analytic, using the above mentioned smooth character of $\widetilde{M}$, we will show that $\widetilde{M}$ is real analytic across M, as was already known in the $\mathbf{C}^2$ case ([MW], [MOS] and [HK]) or in the $\mathbf{C}^n$ case but with $\lambda \neq 0$ ([MW]). Equivalently, we show that the formal process flattening M is indeed convergent even for the exceptional case where the Bishop invariant λ vanishes at the point under study. Our approach in this case is motivated by the one that appeared in [HU], [HK] for the study of two-dimensional real analytic elliptic tangents.

We now proceed to present the main theorems of the paper, in whose statement we will use some terminology which will be defined in §2. Here we only recall that, for a subset $E \subset \mathbf{C}^n$, the holomorphic hull of E is defined to be the intersection of all pseudoconvex domains containing E. Write Δ for the unit disk in $\mathbf{C}^1$. Let ϕ be a holomorphic mapping from Δ into $\mathbf{C}^n$, which is continuous up to $\overline{\Delta}$. ϕ is said to be a (complex) analytic disk attached to E, if $\phi(\partial\Delta) \subset E$.

Theorem. (A) *Let M be a real analytic n-manifold in $\mathbf{C}^n$ with an elliptic complex tangent at a point p. For a sufficiently small $\epsilon > 0$ and near p, the holomorphic hull of $M \cap \{z : |z-p| < \epsilon\}$ is then a real analytic $(n+1)$-submanifold with a small neighborhood of p in M as part of its real analytic boundary. Moreover, M can be* *biholomorphically mapped into the affine space*

$$\mathcal{A} = \{(z_1, z', z_n) \in \mathbf{C} \times \mathbf{C}^{n-2} \times \mathbf{C} : \ \mathrm{Im}(z') = 0, \mathrm{Im}(z_n) = 0\}.$$

(B) *When M is merely smooth, for a sufficient small $\epsilon > 0$, the holomorphic hull $\widetilde{M}$ of $M \cap \{z : \ |z - p| < \epsilon\}$ is a C^∞-smooth Levi-flat $(n+1)$-submanifold near p, which has a small neighborhood of p in M as part of its C^∞ boundary. Near p, $\widetilde{M}$ is foliated by pairwise disjoint embedded complex analytic disks shrinking down to the locus of points in M with complex tangents. Moreover, every complex analytic disk attached to M and close to p has image inside $\widetilde{M}$, and is thus a reparametrization of a leaf in $\widetilde{M}$.*

Remark. We are only concerned with the so-called *local hull of holomorphy* $\widetilde{M}$ of M near p, which is defined in this paper as the intersection of all pseudoconvex neighborhoods of $M \cap \{z : \ |z - p| < \epsilon\}$ for a sufficiently small ϵ. Notice that we did not claim in the theorems that the whole $\widetilde{M}$ is exactly formed by the disjoint union of the attached analytic disks. Instead, what we will prove is that *a certain sufficiently small neighborhood* of p in $\widetilde{M}$ is filled in by the images of the pairwise disjoint complex analytic disks attached to M. Also, the regularity of the local hull of holomorphy of M is understood as that in a sufficiently small neighborhood of p in $\mathbf{C}^n$.

Results obtained in the present paper, together with the previous work in [HW], [BG], [KW1], [KW2], [MW], [MOS] and [HK], provide a complete solution to a question originating in the work of Bishop [BIS].

Since Levi-flat manifolds can be viewed as manifolds with certain mean curvature 0, the existence and regularity problem of the non-trivial holomorphic hull $\widetilde{M}$ of M is also interpreted by some people as the local Plateau problem. Indeed, as is well known, once one puts M in the right position, this problem can be described by a certain degenerate second order elliptic equation, called the Levi equation (see [DG], [BG], [ST], for example). Hence, Theorem (A) and Theorem (B) provide immediately the existence and the boundary regularity for solutions of the Levi equation near elliptic complex tangents.

The method used in this work is partially based on the analysis of complex analytic disks attached to M. Indeed, after the work of Bishop [BIS], the method of complex analytic disks has also been used in working on many other problems both in complex analysis and geometry. Besides the aforementioned work in [KW1], [KW2], [MW], [HK], we also wish to mention the interesting work in [ALX1], [ALX2], [BRT], [BG], [BK], [DS], [ELI], [FO1], [FO2], [GRO], [LE], [TR] and [TU], to name a few.

A few words about the organization: In §2, we first set up some notation and give some definition. Then we state a dependence result of Riemann mappings with respect to certain parameters. In §3, we give a proof of Theorem (B), using the construction result established in [KW2]. In §4, we prove Theorem (A), assuming Proposition 4.1 concerning the existence of a family of nicely attached disks. In §5, we give the proof of Proposition 4.1 and thus complete the proof of Theorem (A). We also include two appendices in the paper. The first one is concerned with the formal flattening of real analytic elliptic points. The second appendix is on the application of our main theorems to the convergence proof of the formal solutions of a useful functional equation.

§2. Notations, definitions and preliminaries

In all that follows, we let $M \subset \mathbf{C}^n$ be a smooth submanifold of real dimension n $(n \geq 2)$. We say that $p \in M$ is a totally real point if $\mathrm{T}_p^{(1,0)}M = \mathrm{T}_pM \cap \sqrt{-1}\mathrm{T}_pM = \{0\}$, where we use T_pM to denote the real tangent space of M at p. According to the polynomial approximation theorem of Hörmander-Wermer and Nirenberg-Wells, it is known that the local hull of holomorphy of M near such a point contains no new point. Also, in the real analytic case, M does not support any other biholomorphic invariant here. Hence, we assume that $p \in M$ is a point with complex tangent $\mathrm{T}_p^{(1,0)}M$ of complex dimension 1. Then, after a linear change of coordinates, we can assume that $p = 0$ and the affine space $\{(z_1, x_2, \cdots, x_{n-1}, 0) \in \mathbf{C}^n\}$ is the tangent space of M at 0. Here and in what follows, we use $z = (z_1, z', z_n) = x + \sqrt{-1}y$ for the coordinates in $\mathbf{C}^n$. (For instance, we write x' for $(x_2, \cdots, x_{n-1})$.) However, we will write u for x_n. Near 0, M can then be regarded as the graph of the following functions:

$$\begin{aligned} z_n &= F(z_1, \overline{z_1}, x'), \\ y_\alpha &= f_\alpha(z_1, \overline{z_1}, x') = \overline{f_\alpha(z_1, \overline{z_1}, x')}, \end{aligned}$$

where α has range from 2 to $n-1$, and F and f_α start with quadratic terms.

Write the quadratic term of F involving only z_1 and $\overline{z_1}$ as

$$q_0^*(z_1, \overline{z_1}) = az_1^2 + bz_1\overline{z_1} + c\overline{z_1^2},$$

where $a,\ b,\ c$ are constant. In case $q_0^* \not\equiv 0$, i.e., when the contact order of $\mathrm{T}_0^{(1,0)}M$ with M is two, after a holomorphic change of coordinates, we can further assume that $a = c = \lambda \geq 0$ and $b = 1$ (when $\lambda = \infty$, q_0 is understood as $z_1^2 + \overline{z_1^2}$). According to Bishop [BIS] or Moser-Webster [MW], $0 \in M$ is called an *elliptic complex tangent* point if $\lambda < 1/2$. λ is a biholomorphic invariant, called the *Bishop invariant* of M at the origin.

From now on, we always let $0 \in M$ be an elliptic complex tangent point. Then, after a holomorphic change of coordinates, we can further assume that the above f_α in the defining equations of M has vanishing order at least three at the origin ([BIS], [MW], etc.). Hence, one can assume in what follows that M is already defined near 0 by equations of the following form:

$$\begin{aligned} z_n &= F(z_1, \overline{z_1}, x') = q_0(z_1, \overline{z_1}, x') + F^*(z_1, \overline{z_1}, x'), \\ y_\alpha &= f_\alpha(z_1, \overline{z_1}, x') = \overline{f_\alpha(z_1, \overline{z_1}, x')}, \end{aligned} \tag{2.1}$$

where $q_0(z_1, \overline{z_1}, x') = z_1\overline{z_1} + (\lambda(x')z_1^2 + \overline{\lambda(x')z_1^2})$ with $0 \leq |\lambda(x')| < 1/2$, $\lambda(0) \in [0, 1/2)$, and $f_\alpha,\ F^* = O(\|z\|^3)$. We point out that $\lambda(0)$ is the Bishop invariant of M at 0.

Write $\mathcal{P}$ for the locus of complex tangent points in M near 0. Then $\mathcal{P} \subset M$ is a totally real submanifold of real dimension $n-2$, which is defined by (2.1) and the following equation ([KW2]):

$$\frac{\partial\left(z_n - F(z_1, \overline{z_1}, x'), y_\alpha - f_\alpha(z_1, \overline{z_1}, x'), \overline{z_n - F(z_1, \overline{z_1}, x')}\right)}{\partial(z_1, z_\alpha, z_n)} = 0.$$

By the implicit function theorem, one sees that $\mathcal{P}$ is parametrized by

$$\mathcal{P}(x') = (P(x'), x_\alpha + \sqrt{-1}f_\alpha(P(x'), \overline{P(x')}, x'), F(P(x'), \overline{P(x')}, x')). \tag{2.2}$$

Here $P(x')$ is a certain smooth function with $P(0) = dP(0) = 0$, and $P(x')$ is real analytic when M is real analytic. We point out that any point in $\mathcal{P}$ is also an elliptic point of M, once it is sufficiently close to 0.

In case M is real analytic, we notice that P can be extended holomorphically to an open subset of $0 \in \mathbf{C}^n$. Moreover, we notice that the transformation in Proposition 2.1 of [KW2] can be made biholomorphic near the origin. Hence, in this case, the arguments in Proposition 2.1 of [KW2] can also be used to prove the following (see Appendix I):

Proposition I.A. *For each $\ell \gg 1$, there is a biholomorphic change of coordinates preserving the origin such that in the new coordinates, M can be defined by an equation of the form* (2.1) *but with the following extra properties:*
(a) $\lambda(x')$ *is real analytic in x' for $x' \approx 0$;*
(b) $F^* = O(|z_1|^3)$*; and*
(c) $f_\alpha = o(|z_1|^{\ell-1})$, $\mathrm{Im}(F^*) = o(|z_1|^{\ell})$.

We next state a folklore lemma. To this aim, we need to set up some notation, which will be used throughout the paper. Let $M_0 \subset \mathcal{A}$ be an n-manifold defined by equations of the form: $u = F_0(z_1, \overline{z_1}, x') = q_0(z_1, \overline{z}_1, x') + k(z_1, \overline{z_1}, x')$, $y_j = 0$ for $j \geq 2$. Here $q_0 = |z_1|^2 + 2\mathrm{Re}(\lambda(x')z_1^2)$ with $\lambda(x')$ smooth in x', $\lambda(0) \in [0, 1/2)$ and the real-valued $k(z_1, \overline{z_1}, x') = O(|z_1|^3)$ for $|x'|, |z_1| \ll 1$. (Namely, $|k(z_1, \overline{z_1}, x')| \lesssim C(x')|z_1|^3$ for some constant $C(x')$ depending on x'.) Write $I_\epsilon = (-\epsilon, \epsilon) \subset \mathbf{R}$ and $I_\epsilon^+ = (0, \epsilon)$ with $1 \gg \epsilon > 0$. Let S^1 denote the unit circle in $\mathbf{C}$. For each fixed $r \in I_\epsilon$ and $x' \in \mathcal{O}$, a small neighborhood of 0 in $\mathbf{R}^{n-2}$, we let $D_{(x',r)}$ denote the domain

$$D_{(x',r)} \equiv \{z_1 \in \mathbf{C}^1 : q_0(z_1, \overline{z_1}, x') + \frac{1}{r^2} k(rz_1, r\overline{z_1}, x') < 1\}.$$

When $|x'|, |r|$ are sufficiently small, by the ellipticity of M_0 at 0 (i.e., $\lambda(0) \in [0, 1/2)$), it is easy to see that $D_{(x',r)}$ is simply connected. Let $\sigma(\xi, x', r)$ be a conformal mapping from Δ to $D_{(x',r)}$. Assume in advance that $\sigma(0, x', r) = 0$ and $\sigma'_\xi(0, x', r) > 0$ to make our choice of $\sigma(\xi, x', r)$ unique.

We remark that up to reparametrizations, all analytic disks attached to M_0, near $(0, x', 0)$, are given by

$$(r\sigma(\xi, x', r), x', r^2)$$

which fill in the following domain in $\mathcal{A}$ and shrink down to $\{0\} \times \mathcal{O} \times \{0\}$ as $r \to 0$:

$$\widetilde{M_0} = \{(z_1, x', u) \in \mathcal{A} : \ |z_1|, \|x'\| \ll 1, \ u \geq q_0(z_1, \overline{z}_1, x') + k(z_1, \overline{z_1}, x')\}. \tag{2.3}$$

For convenience of notation, in all that follows, we use R to stand for a large constant, which might be different in different contexts.

Lemma 2.1. *Let M_0 and $\sigma(\xi, x', r)$ be defined as above and write $F_0 = q_0(z_1, \overline{z_1}, x') + k(z_1, \overline{z_1}, x')$ as before. If $\lambda(x')$, k are real analytic in (z_1, x'), then after shrinking $\mathcal{O}$ and ϵ, $\sigma(\xi, x', r)$ is real analytic over $\overline{\Delta} \times \mathcal{O} \times I_\epsilon$.*

Proof of Lemma 2.1. The result seems standard in the literature. However, due to the lack of a cogent reference, we sketch some details for the convenience of the reader.

Write $\tau = (x', r)$ and write $\sigma^{-1}(\cdot, \tau)$ for the inverse of $\sigma(\cdot, \tau)$. We first notice that $\sigma(\cdot, 0)$ is biholomorphic from $\overline{\Delta}$ to $\overline{D_0}$. Hence, by considering $\sigma^{-1}(\sigma(\xi, \tau), 0)$ instead of $\sigma(\xi, \tau)$, we can assume, without loss of generality, that D_τ is defined by an equation of the form: $|\xi|^2-1+\rho(\xi,\overline{\xi},\tau)=0$. Here, ρ is real analytic near $S^1\times\{0\}$ and $\rho=O(\tau)$. Now, we seek the real-valued function $U(\xi,\tau)$, defined over $S^1\times\{\tau:\ \|\tau\|\ll 1\}$, such that for $\xi\in S^1$, $\sigma(\xi,\tau)=\xi\,(1+U(\xi,\tau)+i\mathcal{H}(U(\cdot,\tau))(\xi))$, where $\mathcal{H}$ is the standard Hilbert transform. Then, it can be verified directly that $U(\xi,\tau)$ satisfies an equation of the following form: $U=F(\tau,\xi,U,\mathcal{H}U)$, where $F=O(\|\tau\|+|U^2|+|\mathcal{H}U|^2)$ and $F(\cdot)$ is real analytic in its variables. Hence, applying the analytic version of the implicit function theorem [Dei], say in the real-valued function space $C^{1/2}(S^1)$ with the standard Banach norm $\|\cdot\|_{1/2}$, to solve for U and then returning to σ, we can conclude that $\sigma=\sum_\alpha\sigma_\alpha(\xi)\tau^\alpha$ with $\sigma_\alpha\in \mathrm{Hol}(\Delta)\cap C^{1/2}(\overline{\Delta})$ and $\|\sigma_\alpha\|_{1/2}\equiv\sup_{\xi_1,\xi_2\in\partial\Delta}\{|\sigma_\alpha(\xi_1)|+\frac{|\sigma_\alpha(\xi_1)-\sigma_\alpha(\xi_2)|}{|\xi_1-\xi_2|^{1/2}}\}\lesssim R^{\|\alpha\|}$. Hence, it is clear that $\sigma\in C^\omega(\Delta\times\mathcal{O}\times I_\epsilon)$.

Next, fix a point $\xi_0\in S^1$ and write $q_0=\sigma(\xi_0,0)$. We like to show that σ is real analytic over $\{\xi\in\overline{\Delta}:\ |\xi-\xi_0|<\epsilon\}\times\{\tau:\ \|\tau\|<\epsilon\}$, after shrinking ϵ. This can be done by applying the reflection principle as follows:

Write $S=\{(\xi,\tau)\in\mathbf{C}^1\times\mathbf{R}^{n-1}:\ \xi\approx q_0,\xi\in\partial D_\tau,\ \|\tau\|\ll 1\}$. Then the hypothesis indicates that S is a real analytic totally real submanifold of maximal dimension in $\mathbf{C}^n$. Therefore, there is a biholomorphic map $\psi(\xi,z)$ taking an open piece of S near $(q_0,0)$ into $\mathbf{R}^n\subset\mathbf{C}^n$. For $(\xi,\tau)\approx(\xi_0,0)$, write $S'=\{(\xi,\tau):\ \xi\in S^1,\ \ \tau\in\mathbf{R}^{n-1}\ \}$. Then it can be biholomorphically mapped into $\mathbf{R}^n$ by $\phi(\xi,z)=(\phi_1(\xi),z)$, where ϕ_1 is a conformal map from Δ to the upper half plane with $\phi_1(\xi_0)=0$.

Define $\Sigma(\xi,z)=\psi(\sum_\alpha\sigma_\alpha(\xi)z^\alpha,z)$ for $\xi(\approx\xi_0)\in\overline{\Delta}$ and $\|z\|\ll 1$. Then the above obtained analytic dependence of σ on τ implies that by making ϵ sufficiently small,

$$\begin{aligned}\Sigma\in \mathrm{Hol}(&\{\xi:\xi\in\Delta,\ |\xi-\xi_0|\ll 1\}\times\{z\in\mathbf{C}^{n-1}:\ \|z\|<\epsilon\})\\ &\cap C(\{\xi:|\xi|\le 1,|\xi-\xi_0|\ll 1\}\times\{z\in\mathbf{C}^{n-1}:\ \|z\|<\epsilon\}).\end{aligned}$$

For $(\xi,z)\approx 0$, we define $\Sigma^*(\xi,z)$ by $\Sigma\circ\phi^{-1}(\xi,z)$ for $\mathrm{Im}(\xi)>0$ (thus for $\mathrm{Im}(\xi)>\|\mathrm{Im}(z)\|$); and by $\overline{\Sigma\circ\phi^{-1}(\overline{(\xi,z)})}$ for $-\mathrm{Im}(\xi)>0$ (thus also for $-\mathrm{Im}(\xi)>\|\mathrm{Im}(z)\|$). Then, it is easy to see, by the edge of the wedge theorem, that Σ^* extends holomorphically to a neighborhood of $(0,0)\in\mathbf{C}^n$. The proof of Lemma 2.1 thus follows easily, too. □

§3. Uniqueness of the analytic disks attached to M— Proof of Theorem (B)

In this section, we will present a proof of Theorem (B), which will be used in the proof of Theorem (A). Let M be as given by (2.1). Our starting point is the following deep result already proved in [KW2]. We mention that when M is real analytic, this result also follows from our construction in §5 (see Remark 5.3):

Theorem 3.1 (Kenig-Webster). *Let M be a smooth n-manifold as defined in (2.1) with 0 being an elliptic complex tangent point. For each large integer ℓ, there then exists an n-manifold $M_{0,\ell}\subset\mathcal{A}=\{(z_1,x',u)\}$ defined by equations of the form: $u=(q_0^*+h)(z_1,\overline{z_1},x')$, $y_j=0$ $(j\ge 2)$, and there exists a mapping T_ℓ from $\widetilde{M_{0,\ell}}=\{(z_1,x',u)\in\mathcal{A}:\ |z|\ll 1,\ u\ge(q_0^*+h)(z_1,\overline{z_1},x')\}$ into $\mathbf{C}^n$ such that the following four properties hold (here $q_0^*=|z_1|^2+2\mathrm{Re}(\lambda^*(x')z_1^2)$ with $|\lambda^*|<1/2$ and $\lambda^*(x')$ smooth in x', $h(z_1,\overline{z_1},x')=\overline{h(z_1,\overline{z_1},x')}\in O(|z_1|^2)\cap o(\|z\|^2)$ is smooth over $\{z_1\in\mathbf{C}^1:\ |z_1|\ll 1\}\times\mathcal{O}$ with $\mathcal{O}$ a certain small neighborhood of 0 in $\mathbf{R}^{n-2}$):*

(a) $T_\ell(0, x', 0) \in \mathcal{P}$ *where, as before,* $\mathcal{P}$ *is the locus of the complex tangent points in* M*, and* $T_\ell = \mathrm{id} + o(\|z\|)$*;*

(b) $T_\ell(M_{0,\ell}) \subset M$ *and* $T_\ell(M_{0,\ell})$ *contains an open subset* $(0 \in)$ E *of* $\mathcal{P}$*;*

(c) T_ℓ *is a* C^ℓ*-smooth diffeomorphism from* $\widetilde{M_{0,\ell}}$ *and* C^∞*-smooth over* $\widetilde{M_{0,\ell}} \setminus \{0, x', 0\}$*;*

(d) $\overline{\partial}_{z_1} T_\ell \equiv 0$.

We *fix* a sufficiently large ℓ, and let $M_{0,\ell}$, $\widetilde{M_{0,\ell}}$, and T_ℓ be constructed as above. Write $\widetilde{M_\ell} = T_\ell(\widetilde{M_{0,\ell}})$. Since T_ℓ is holomorphic when restricted to each analytic disk attached to $M_{0,\ell}$, we see that near 0, $\widetilde{M_\ell}$ is also a C^ℓ-smooth Levi-flat $(n+1)$-manifold foliated by embedded analytic disks shrinking down to the complex tangents of M. Notice that $\widetilde{M_\ell}$ is C^∞ away from $\mathcal{P}$, by Theorem 3.1 (c).

For $0 < \delta_1 \ll \delta_2 \ll \delta_3 \ll 1$, $\widetilde{M_\ell} \cap \{z : \|z\| < \delta_1\}$ is clearly contained in the holomorphic hull of $M \cap \{z : \|z\| < \delta_2\}$ (by the disk filling property of $\widetilde{M_\ell}$ and the *continuity principle*); and the holomorphic hull of $M \cap \{z : \|z\| < \delta_2\}$ is contained in the holomorphic hull of $\widetilde{M_\ell} \cap \{z : \|z\| \le \delta_3\}$. In terms of Theorem 3.1, to complete the proof of Theorem (B), it therefore suffices for us to prove the following uniqueness property of the attached disks and to show that $\widetilde{M_\ell} \cap \{z : \|z\| < \epsilon\}$ is holomorphically convex for $\epsilon \ll 1$.

Indeed, assuming the following Theorem 3.2, we clearly only need to show that $\widetilde{M_\ell}$ is smooth near 0 to complete the proof of Theorem (B). By Theorem 3.1, it suffices to verify that $\widetilde{M_\ell}$ is smooth at any point $z(\approx 0) \in \mathcal{P}$. Since we know that z is also an elliptic complex tangent of M, for any large integer N, by Theorem 3.1, M bounds a Levi-flat C^N-smooth $(n+1)$-manifold E_N with M near z as part of its C^N-smooth boundary. Also, E_N is foliated by holomorphic disks shrinking down to $\mathcal{P}$ near z. Applying again the following Theorem 3.2, we see that E_N must coincide with $\widetilde{M_\ell}$ near z. Thus we showed that $\widetilde{M_\ell}$ is of class C^N near z for any N. Thus it is easy to see that $\widetilde{M}$ is smooth at 0.

Theorem 3.2. *There exists a certain small number* ϵ_0 *such that for each bounded non-constant holomorphic map* ϕ *from* Δ *into* $\mathbf{C}^n$*, if* $\phi(\Delta) \subset\subset \{\|z\| < \epsilon_0\}$ *and* $\lim_{\tau \to \xi \in \partial\Delta} \phi(\tau) \subset M$ *for all* $\xi \in \partial\Delta$*, then* $\phi(\Delta) \subset \widetilde{M_\ell}$*. Therefore* ϕ *must be a reparametrization of a leaf inside* $\widetilde{M_\ell}$*. Moreover,* $\widetilde{M_\ell} \cap \{\|z\| \le \epsilon_0\}$ *is holomorphically convex.*

Notice that Theorem 3.2 also indicates that for any sufficiently small pseudoconvex neighborhood Ω of p in $\mathbf{C}^n$, $\widetilde{M_\ell} \cap \Omega$ is also holomorphically convex.

To prove Theorem 3.2, we would like to find a sequence of piecewise strongly pseudoconvex neighborhood systems of $\widetilde{M_\ell}$ with slow decay on its Levi eigenvalues, so that its image under the $\overline{\partial}$-flat extension of the CR mapping T_ℓ is also pseudoconvex. We mention that in case $n = 2$, Theorem 3.2 was obtained in [KW1] and [BG] (see, for example, Proposition 4.3 of [KW1]).

Proof of Theorem 3.2. Returning to the map $T_\ell(z_1, x', u)$, we can first easily construct its C^ℓ-extension T^e_ℓ to a neighborhood of $\widetilde{M_{0,\ell}}$, which is $\overline{\partial}$-flat to the order of $\ell - 1$ along $\widetilde{M_{0,\ell}}$:

Indeed, for each $(z_1, x', u)(\approx 0) \in \widetilde{M_{0,\ell}}$, we let

$$\mathcal{J}_\beta T_\ell = \frac{\partial^{|\beta|} T_\ell}{\partial^{\beta_1} z_1 \partial^{\beta_2} x' \partial^{\alpha} u}$$

($\|\beta\| = \|(\beta_1, \beta_2, \alpha)\| \le \ell$) and

$$\mathcal{T}_{z_0}(T_\ell) = \sum_{\|\beta\| \le \ell} \frac{(z - z_0)^\beta}{\beta!} \mathcal{J}_\beta T_\ell(z_0), \quad z = (z_1, z', z_n),\ z_0 \in \widetilde{M_{0,\ell}}.$$

Then, by noting $T_\ell \in C^\ell(\widetilde{M_{0,\ell}})$ and $\overline{\partial}_{z_1} T_\ell(z_1, x', u) \equiv 0$, it follows easily that $\|\partial^\beta \mathcal{T}_{z_0}(T_\ell)(z_1) - \mathcal{J}_\beta T_\ell(z_1)\| = o(\|z_1 - z_0\|)^{\ell - |\beta|}$ and $\|D^\gamma \overline{\partial}^{\beta^*} \mathcal{T}_{z_0}(T_\ell)(z_1) - 0\| = o(\|z - z_0\|^{\ell - \|\gamma\| - 1})$ (where $\|\beta^*\| = 1$), for any $z_1,\ z_0 \in \widetilde{M}_{0,\ell}$. Thus, by a standard use of the Whitney extension theorem (see [Bog], for example), we can obtain a C^ℓ-extension T_ℓ^e of T_ℓ to a neighborhood of $\widetilde{M_{0,\ell}}$ in $\mathbf{C}^n$ with $\|D_W^\gamma \overline{\partial}_W^{\beta^*} T_\ell^e(W)\| = o(\delta_0^{\ell - |\gamma| - 1}(W))$ where $\|\beta^*\| = 1$ and we write $\delta_0(W)$ for the distance from W to $\widetilde{M}_{0,\ell}$.

From Theorem 3.1 (a), it is clear that $\mathrm{d}T_\ell^e|_0 = \mathrm{id}$. Hence T_ℓ^e is a C^ℓ diffeomorphism near the origin. Moreover, $\|D_W^\gamma \overline{\partial}_W^{\beta^*} (T_\ell^e)^{-1}(W)\| = o(\delta_{+1}^{\ell - |\gamma| - 1}(W))$, where $\|\beta^*\| = 1$ and $\delta_{+1}(W)$ denotes the distance from W to $\widetilde{M}_\ell$.

For each small t, write $T_\ell^e = (\phi_1, \cdots, \phi_n)$ and $(T_\ell^e)^{-1} = (\psi_1, \cdots, \psi_n) = \mathrm{id} + o(|W|)$.

Now, for each small $t > 0$, let $\Omega_t^* = \{z \in \mathbf{C}^n : \rho_{j,t}^{*\pm} = \pm(y_j) + t(\|z\|^2 - 1) < 0,\ j = 2, \cdots, n,\ \rho_{1,t}^* = -u + (q_0^* + h)(z_1, \overline{z_1}, x') + 2(y_2^2 + \cdots + y_n^2) + t(\|z\|^2 - 1) < 0\}$. Here, $y_j = \mathrm{Im}(z_j)$ and $q_0^* + h$ is as in the statement of Theorem 3.1.

Write $\Omega_t = T_\ell^e(\Omega_t^*)$. Then $\{\Omega_t\}_t$ is a decreasing continuous sequence as $t \to 0^+$, and for a sufficiently small $\epsilon > 0$, $\bigcap_{1 \gg t > 0} \Omega_t \cap \{z :\ \|z\| < \epsilon\} = \widetilde{M}_\ell \cap \{z :\ \|z\| < \epsilon\}$.

Notice that for each $t > 0$, Ω_t is defined by $\rho_{j,t}^{\pm}(z) = \rho_{j,t}^{*\pm} \circ (T_\ell^e)^{-1}(z) < 0$ ($j = 2, \cdots, n$) and $\rho_{1,t}(z) = \rho_{1,t}^* \circ (T_\ell^e)^{-1}(z) < 0$. Also notice that for each $W \in \Omega_t^*$, $\delta_0(W) \lesssim O(t)$.

We now let $z \in \overline{\Omega_t}$ with $\|z\| \le \epsilon$. It is then clear by our construction, that $\delta_{+1}(z) \approx \delta_0((T_\ell^e)^{-1}(z)) \lesssim O(t)$. On the other hand, it is clear that for $j \ge 2$,

$$\begin{aligned}\partial\overline{\partial}\rho_{j,t}^{\pm}(z) &= \partial\overline{\partial}(\pm \mathrm{Im}(\psi_j) + t(\|\psi\|^2 - 1)) \\ &= t \sum_{j=1}^n \partial\psi_j(z) \wedge \overline{\partial\psi_j(z)} + \sum_{i,j} o(|\delta_{+1}(z)|^{\ell-2}) dz_i \wedge \overline{dz_j}.\end{aligned}$$

Hence, using the fact that $\partial\psi_j = dz_j + \sum_k O(\epsilon) dz_k$, we see that

$$\partial\overline{\partial}\rho_{j,t}^{\pm}(z) = t \sum_{j=1}^n dz_j \wedge \overline{dz_j} + \sum_{i,j} (tO(\epsilon) + o(|\delta_{+1}(z)|^{\ell-2})) dz_i \wedge \overline{dz_j}.$$

So, each eigenvalue of $\partial\overline{\partial}\rho_{j,t}^{\pm}(z)$ is of the quantity $(1 + O(\epsilon))t + o(t) \ge \frac{1}{2}t$, once $\ell > 4$, and ϵ, t are sufficiently small. Hence, we conclude that $\rho_{j,t}^{\pm}(z)$ are strongly plurisubharmonic in $\overline{\Omega_t} \cap \{\|z\| \le \epsilon\}$. Similarly, a direct computation shows that

$$\begin{aligned}\partial\overline{\partial}\rho_{1,t}(z) &= \sum \partial\psi_j \wedge \overline{\partial\psi_j} + \sum_{j,k}(O(\epsilon+t)+o(t^{\ell-2}))dz_j \wedge \overline{dz_k} \\ &= \sum_{j=1}^{n} dz_j \wedge \overline{dz_j} + \sum_{j,k}(O(\epsilon+t)+o(t^{\ell-2}))dz_j \wedge \overline{dz_k}.\end{aligned}$$

Therefore, when ϵ and t are sufficiently small, $\rho_{1,t}$ becomes strongly plurisubharmonic over $\overline{\Omega_t} \cap \{\|z\| \le \epsilon\}$, too. This implies that there are two small (fixed) positive numbers t_0 and ϵ_0 such that when $0 < t \le t_0$ and $0 < \epsilon \le \epsilon_0$, $\Omega_t \cap \{\|z\| < \epsilon\}$ are pseudoconvex defined by functions $\{\rho^{\pm}_{j,t}, \rho_{1,t}, \|z\|^2 - \epsilon^2\}$, which are strongly plurisubharmonic over $\overline{\Omega_t} \cap \{\|z\| \le \epsilon\}$.

Thus, we proved that $\widetilde{M}_\ell \cap \{z : \|z\| \le \epsilon\}$ is holomorphically convex for any $\epsilon \le \epsilon_0$. To see the uniqueness of the attached disks, we let ϕ be a bounded holomorphic map from Δ into $\mathbf{C}^n$ such that $\phi(\Delta) \subset\subset \Omega_{t_0} \cap \{z : \|z\| < \epsilon_0\}$ and $\lim_{\xi\to\partial\Delta} \phi(\xi) \subset M$. Then, if $\phi(\Delta) \not\subset \widetilde{M}_\ell$, there would be a $t^* < t_0$ and a certain $\xi_0 \in \Delta$ such that $\phi(\Delta) \subset \overline{\Omega_{t^*}}$ and $\phi(\xi_0) \in \partial\Omega_{t^*}$. By what we just proved and by applying the maximum principle to $\rho^{\pm}_{j,t^*}(\phi(\xi))$, $\rho_{1,t^*}(\phi(\xi))$, we see that either $\rho^{\pm}_{j,t^*}(\phi(\xi)) \equiv 0$ for a certain j or $\rho_{1,t^*}(\phi(\xi)) \equiv 0$. This is a contradiction; for ϕ is (almost) attached to M. The proof of Theorem 3.2 is now complete. □

To conclude this section, we remark that when M in Theorem (B) is merely C^ℓ for some $\ell > 7$, then the result of Kenig-Webster [KW2] and Theorem 3.2 shows that the holomorphic hull $\widetilde{M}$ of M near p is at least in the smoothness class $C^{\frac{\ell-7}{3}}$. In [HU], it was shown that in case $n = 2$, $\widetilde{M}$ is of $C^{\frac{\ell-2}{2}}$ near the elliptic complex tangent p (see also Remark 5.3). The following example indicates that the smoothness of $\widetilde{M}$ can be only about half of that of the original manifold M:

Example. Let $M = \{(z,w) \in \mathbf{C}^2 : \ w = |z|^2 + |z|z^\ell\}$. Then M is of class C^ℓ near the elliptic point 0 and the analytic disks $\phi(\xi, r) = (r\xi, r^2 + r^{\ell+1}\xi^\ell)$ are attached to M. The holomorphic hull $\widetilde{M}$ of M is the set $\{(z = x + \sqrt{-1}y, u + iv)\}$ with $z = r\xi$, $u = r^2 + r\mathrm{Re}(z^\ell)$, and $v = r\mathrm{Im}(z^\ell)$. (Here $\xi \in \overline{\Delta}$ and $0 \le r \ll 1$.) Now, regard $\widetilde{M}$ as the graph of the function $v = v(x, y, u)$ over $\pi(\widetilde{M})$ and let $\ell \equiv 1 \bmod 4$. Here π is the projection to (z, u)-space. Then along $x = 0$, $v(0, y, u) = u^{1/2}y^\ell$ for (y, u) with $u \ge y^2$. Hence, we see that for $k > \frac{\ell+1}{2}$, $\frac{\partial^k v(x,y,u)}{\partial u^k}|_{(x,y,u)=(0,y,y^2)}$ is unbounded as $y \to 0$. Therefore, we see that $\widetilde{M}$ is at most $C^{\frac{\ell+1}{2}}$ near 0.

§4. Proof of Theorem (A)— Assuming Proposition 4.1

In this and the next sections, we further let M be real analytic near 0. As pointed out in §2 (Proposition I.A), we can assume that M is defined by (2.1) with $f_\alpha = o(|z_1|^m)$, $F^* = O(|z_1|^3)$, $\mathrm{Im}(F^*) = o(|z_1|^m)$ and $\lambda(x')$ being real analytic. Here $m > 10$ is a certain fixed large integer. We will give a detailed proof that the local hull of holomorphy of M is real analytic near 0 and M can be biholomorphically transformed into $\mathcal{A}$. We will employ ideas from [HK] or [HU].

Still let π be the natural projection from $\mathbf{C}^n$ into $\mathcal{A}$, which sends each point (z_1, z', z_n) to (z_1, x', u). Write $M_0 = \pi(M)$, whose defining function clearly has the properties described in §2. Let $\widetilde{M_0}$ be as defined in (2.3). As in §2, we write all the analytic disks attached to M_0 as $(r\sigma(\xi,x',r),x',r^2)$ (up to reparametrizations), where by Lemma 2.1, $\sigma(\xi,x',r)$ depends real analytically on (ξ,x',r) and holomorphically on ξ. We recall that $\sigma(0,x',r)=0$ and $\sigma'_\xi(0,x',r)>0$.

Our starting point is the following result, whose proof is the content of §5:

Proposition 4.1. *There exists a family of analytic disks*

$$\phi(\xi,x',r)=\left(r\sigma(\xi,x',r)(1+\psi_1^*(\xi,x',r)),x_\alpha+\psi_\alpha^*(\xi,x',r),r^2+\psi_n^*(\xi,x',r)\right),$$

which are attached to M (i.e., $\phi(\partial\Delta,x',r)\subset M$) such that the following hold:

(I) *After shrinking $\mathcal{O}$ and ϵ, $\phi(\xi,x',r)\in C^m(\overline{\Delta}\times\mathcal{O}\times I_\epsilon)$. (Here and in the following, the fixed integer m (>10) is as mentioned above.)*

(II) *For each j and for $(x',r)\approx(o,0)$, $\xi\in\Delta$, $\psi_j^*(\xi,x',r)$ depends real analytically on (ξ,x',r), holomorphically on ξ. Moreover $\psi_j^*=o(|r|^{m-2})$.*

Next, write $\widetilde{M_m}=\bigcup_{0\le r\ll 1,x'\in\mathcal{O}}\phi(\overline{\Delta},x',r)$. Define $\Phi:\widetilde{M_0}\to\widetilde{M_m}$ by $\Phi(0,x',0)=(0,x',0)$ and by

$$\Phi(z_1,x',u)=\phi(\xi(z_1,x',u),x',u^{\frac{1}{2}}) \tag{4.1}$$

for $u>0$, where ξ is uniquely determined by the equations: $z_1=r\sigma(\xi,x',r)$; $r=u^{\frac{1}{2}}$.

Lemma 4.2. $\Phi\in C^2(\widetilde{M_0})\cap C^\omega(\widetilde{M_0}\setminus M_0)$. *Also* $\Phi=\mathrm{id}+o(\|z\|)$.

Proof of Lemma 4.2. First, from the definition, Φ is obviously continuous over $\widetilde{M_0}$. Let $(z_1,x',u)\in\widetilde{M_0}\setminus M_0$. Then $u>0$. Moreover, $\Phi(z_1,x',u)=\phi(\xi(z_1,x',u),x',u)$, where $\xi=\sigma^{-1}(\frac{z_1}{r},x',r)$ and $r=\sqrt{u}$. (Here, as before, we write $\sigma^{-1}(\cdot,x',r)$ for the inverse of $\sigma(\cdot,x',r)$.) Using Proposition 4.1, one sees that $\Phi(z_1,x',u)=(z_1,x',u)+u^{\frac{m}{2}-1}\Phi^*(\xi(\frac{z_1}{\sqrt{u}},x',\sqrt{u}),x',\sqrt{u})$, where Φ^* is a certain function holomorphic in ξ and C^m in $(\xi,x',r)\in\overline{\Delta}\times\mathcal{O}\times I_\epsilon$. Since $m>10$, it is easy to see that $D^\alpha\Phi(z_1,x',u)$ is uniformly bounded for $\|\alpha\|\le 3$ and for $(z_1,x',u)\in\widetilde{M_0}\setminus M_0$ near 0. Hence, Φ is twice differentiable near 0. Clearly, Φ is real analytic away from points in M_0 by Proposition 4.1 (II). ☐

Since $\widetilde{M_m}$ is the image of $\widetilde{M_0}$ under the twice differentiable diffeomorphism Φ, it is also a twice differentiable $(n+1)$-manifold with a small open piece of 0 in M as part of its boundary. Also we mention that Lemma 4.2 indicates that $\widetilde{M_m}\setminus M$ is real analytic. Notice that $\widetilde{M_m}$ is foliated by a continuous family of the analytic disks shrinking down to the complex tangents, and notice that $\widetilde{M_m}$ occupies an open piece of M containing 0. By Theorem 3.2, we easily see that $\widetilde{M_m}$ coincides with the local hull of holomorphy of M near 0. Thus it is smooth near 0, by Theorem (B). In the following, we write $\widetilde{M}$ for $\widetilde{M_m}$ to simplify the notation.

Notice that $\widetilde{M}$ is tangent to $\widetilde{M_0}$ at 0 by Lemma 4.2. From the simple fact that $\pi(M)=M_0$ and $\pi(\widetilde{M})\cap\widetilde{M_0}\setminus M_0\ne\emptyset$, it follows easily that $\pi(\widetilde{M})=\widetilde{M_0}$ and thus $\pi|_{\widetilde{M}}$ is a diffeomorphism from $\widetilde{M}$ to $\widetilde{M_0}$. Therefore, $\widetilde{M}$ can thus be written as the graph of certain functions $\{v_\alpha,v\}$ over $\widetilde{M_0}$. Namely,

$$\widetilde{M}=\left\{\left(z_1,x_\alpha+iv_\alpha(z_1,x',u),u+iv(z_1,x',u)\right):(z_1,x',u)\in\widetilde{M_0}\right\}.$$

Using the smooth character of $\widetilde{M}$, it is clear that $v_\alpha(z_1,x',u)$, $v(z_1,x',u)\in C^\infty(\widetilde{M_0})$. Also, by Lemma 4.2 they are analytic over $\widetilde{M_0}\setminus M$. We will show in the following that $v_\alpha(z_1, x', u)$ and $v(z_1, x', u)$ have convergent Taylor series expansions near the origin. Namely, $v_\alpha(z_1, x', u)$ and $v(z_1, x', u)$ are real analytic near 0. This will complete the proof of Theorem (A).

Proof of Theorem (A). We let

$$z = (z_1, z_\alpha, z_n) = (z_1, x_\alpha + iv_\alpha(z_1, x', u), u + iv(z_1, x', u)) \in \widetilde{M} \setminus M.$$

We first note, by Lemma 4.2 and the definition of Φ, that there is a unique $(\xi, x', r) \in \Delta \times \mathcal{O} \times I_\epsilon^+$ such that

$$z_1 = \phi_1(\xi, x', r),\ z_\alpha = \phi_\alpha(\xi, x', r),\ u + iv = \phi_n(\xi, x', r).$$

Hence,

$$z_1 = r\sigma(\xi, x', r)\big(1 + \psi_1^*\big); \tag{4.2}$$

$$z_\alpha = x_\alpha + \psi_\alpha^*(\xi, x', r); \tag{4.3}$$

$$u = \mathrm{Re}\phi_n = r^2 + \mathrm{Re}\psi_n^*(\xi, x', r) = r^2(1 + o(r)); \tag{4.4}$$

$$v = \mathrm{Im}\phi_n = \mathrm{Im}\psi_n^*(\xi, x', r),\ v_\alpha = \mathrm{Im}\psi_\alpha^*(\xi, x', r). \tag{4.5}$$

Here we mention again, by Proposition 4.1, that $\psi_j^*(\xi, x', r)$ are holomorphic in $\xi \in \Delta$ and real analytic in (ξ, x', r). From (4.4), it is clear that $u > 0$. Our next goal is to solve v_α and v as functions of (z_1, x', u) from the above equations:

Lemma 4.3. *There exist certain functions $h_j(\eta_1, \eta_2, x')$ $(j = 2, \cdots, n)$, which are real analytic in (η_1, η_2, x') near the origin, such that when $\pi(z)$ $(\in \widetilde{M_0} \setminus M_0)$ stays in the region*

$$\{(z_1, x', u) \in \mathcal{A} :\ \sqrt{u},\ \|x'\|,\ |z_1/\sqrt{u}| \text{ are sufficiently small}\},$$

then $v(z_1, x', u) = h_n(\sqrt{u}, z_1/\sqrt{u}, x')$ and $v_\alpha(z_1, x', u) = h_\alpha(\sqrt{u}, z_1/\sqrt{u}, x')$ $(\alpha < n)$.

Proof of Lemma 4.3. From (4.4), Proposition 4.1 (II) and noting $u > 0$, we obtain

$$u^{\frac{1}{2}} = r + r^3 h(\xi, x', r), \tag{4.5$'$}$$

where $h(\xi, x', r)$ is a certain function jointly real analytic in (ξ, x', r) for (ξ, x', r) near the origin. When $|r| \ll 1$ and $\|x'\| \ll 1$, applying the implicit function theorem to (4.5)$'$, we see that there exists a certain function $g(\eta_1, \xi, x')$ which is real analytic near the origin and $g = o(|\eta_1|)$ such that

$$r = \eta_1 \cdot (1 + g(\eta_1, \xi, x')) \equiv \widetilde{g}(\eta_1, \xi, x'),$$

where η_1 is identified with $u^{1/2}$. Thus by (4.2), we see that

$$z_1 = \eta_1(1 + g(\eta_1, \xi, x'))\sigma(\xi, x', \widetilde{g}(\eta_1, \xi, x'))(1 + \psi^*{}_1(\xi, x', \widetilde{g}(\eta_1, \xi, x'))),$$

with $\eta_1 = \sqrt{u}$. Write $\eta_2 = z_1/\eta_1 = z_1/u^{\frac{1}{2}}$. We then have

$$\eta_2 = (1 + g(\eta_1, \xi, x'))\sigma(\xi, x', \widetilde{g}(\eta_1, \xi, x'))(1 + \psi^*{}_1(\xi, x', \widetilde{g}(\eta_1, \xi, x'))). \tag{4.6}$$

Regard the right hand side of the above as a function $\eta_2(\eta_1, \xi, x')$ in (η_1, ξ, x'). Then, we notice that when $\eta_1, \xi, x' \approx 0$, we have (a) $\eta_2(\eta_1, \xi, x')$ is real analytic in ξ, x' and η_1; (b) $\eta_2(0, 0, 0) = 0$; (c) $g(\eta_1, \xi, x')|_0$, $\psi_1^*(\xi, x', \widetilde{g}(\eta_1, \xi, x'))|_0 = 0$; $d_\xi(1 + g(\eta_1, \xi, x'))|_0, d_\xi\psi^*{}_1(\xi, x', \widetilde{g}(\eta_1, \xi, x'))|_0 = 0$; and (d) $\sigma(\xi, x', \widetilde{g}(\eta_1, \xi, x')) = c\xi + o(|\xi|) + O(|x'| + |\eta_1|)$ with c a certain positive constant (as $(\xi, \eta_1, x') \to 0$).

Hence, the implicit function theorem can be easily applied again to (4.6) to conclude that for $|\eta_1|, |\xi|, \|x'\|, |\eta_2| \ll 1$,

$$\xi = \hat{f}(\eta_1, \eta_2, x') \tag{4.7}$$

where $\hat{f}$ is a certain real analytic function defined near 0 with $\hat{f}|_0 = 0$, and η_1 and η_2 are identified with $\sqrt{u}$ and $z_1/\sqrt{u}$, respectively. Hence, we have that

$$r = \widetilde{g}(\eta_1, \hat{f}(\eta_1, \eta_2, x'), x') \equiv \hat{g}(\eta_1, \eta_2, x'), \tag{4.8}$$

where $\hat{g}$ is also real analytic in (η_1, η_2, x') near 0, $\eta_1 = \sqrt{u}$ and $\eta_2 = z_1/\sqrt{u}$.

Letting

$$h_n(\eta_1, \eta_2, x') = \mathrm{Im}\phi_n(\hat{f}(\eta_1, \eta_2, x'), x', \hat{g}(\eta_1, \eta_2, x')),$$

we then see from (4.5), (4.7) and (4.8) that when $\sqrt{u}, \|x'\|, |z_1/\sqrt{u}| \ll 1$,

$$v(z_1, x', u) = h_n(\sqrt{u}, z_1/\sqrt{u}, x'),$$

where h_n is clearly real analytic in (η_1, η_2, x') for (η_1, η_2, x') close to the origin.

Similarly, for each $\alpha \in [2, n-1]$, we can find a certain real analytic function $h_\alpha(\eta_1, \eta_2, x')$ in (η_1, η_2, x') defined near the origin such that $v_\alpha = h_\alpha(\sqrt{u}, z_1/\sqrt{u}, x')$. $\square$

Now, for η_1 real, expand

$$h_n(\eta_1, \eta_2, x') = \mathrm{Im}\phi_n(\hat{f}(\eta_1, \eta_2, x'), x', \hat{g}(\eta_1, \eta_2, x')) = \sum_{i,j,s \geq 0, \alpha} a_{ijs,\alpha} \eta_1^i \eta_2^j \overline{\eta}_2^s x'^\alpha$$

with $|a_{ijs,\alpha}| \lesssim R^{i+j+s+\|\alpha\|}$.

Notice that when $0 < u < \epsilon^2$, $\|x'\| < \epsilon$ and $|z_1|/u^{\frac{1}{2}} < \epsilon$ with $0 < \epsilon \ll 1$, it can be easily seen that $(z_1, x', u) \in \widetilde{M_0} \setminus M_0$. Therefore, in terms of Lemma 4.3, we have that

$$v(z_1, x', u) = h_n(\sqrt{u}, z_1/\sqrt{u}, x') = \sum_{i,j,s,\alpha} a_{ijs,\alpha} u^{\frac{1}{2}(i-j-s)} z_1^j \overline{z_1}^s x'^\alpha.$$

However since $v(z_1, x', u)$ is C^∞ near the origin, we see, in particular, that

$$\frac{\partial^{j+s+\|\alpha\|} v(z_1, x', u)}{\partial z_1^j \partial \overline{z_1}^s \partial x'^\alpha}\Big|_{(0,x',u)}$$

is C^∞ in (x', u), as long as $0 \leq u \ll 1$ and $x' \approx 0$.

Meanwhile, for $0 < u \ll 1$, one clearly has

$$\frac{\partial^{j+s+\|\alpha\|} v(z_1, x', u)}{\partial z_1^j \partial \overline{z_1}^s \partial x'^\alpha}\Big|_{(0,0,u)} = \sum_{i=0}^{\infty} j!s!\alpha! a_{ijs,\alpha} u^{\frac{1}{2}(i-j-s)} .$$

This implies that $a_{ijs,\alpha} = 0$ when $(1/2)(i-j-s)$ is not a non-negative integer; for the left hand side of the above is smooth when $0 \leq u \ll 1$. Thus

$$v(z_1, x', u) = \sum_{i,j,s,\alpha} a_{ijs,\alpha} u^{\frac{1}{2}(i-j-s)} z_1^j \overline{z_1}^s x'^\alpha = \sum_{\tau,j,s,\|\alpha\| \geq 0} a_{2\tau+j+s,j,s,\alpha} u^\tau z_1^j \overline{z_1}^s x'^\alpha$$

when $0 < u < \epsilon^2$, $\|x'\| < \epsilon$, and $|z_1| < \epsilon u^{\frac{1}{2}}$.

On the other hand,

$$|a_{2\tau+j+s,j,s,\alpha}| \lesssim R^{2\tau+j+s+j+s+\|\alpha\|} \lesssim (R^2)^{\tau+j+s+\|\alpha\|}.$$

Thus we conclude that

$$\widetilde{v}(z_1, x, u) = \sum_{\tau,j,s,\alpha} a_{2\tau+j+s,j,s,\alpha} u^\tau z_1^j \overline{z_1}^s x'^\alpha$$

is a real analytic function in (z_1, u, x') near the origin. Also $\widetilde{v}(z_1, x', u) \equiv v(z_1, x', u)$ when $0 < u < \epsilon^2$, $\|x'\| < \epsilon$, and $|z_1| < \epsilon u^{1/2}$. Notice that $v(z_1, x', u)$ is real analytic on $\widetilde{M_0} \setminus M_0$ and is C^∞ on $\widetilde{M_0}$. By the unique continuation property of real analytic functions, it follows that $\widetilde{v}(z_1, x', u) \equiv v(z_1, x', u)$ for all $z, x', u \approx 0$ with $(z_1, x', u) \in \widetilde{M_0}$. Similarly, we can prove that v_α are real analytic near 0, too.

At last, we see that $\widetilde{M}$ is a real analytic submanifold across part of its real analytic boundary M near 0.

Moreover, using the graph of the real analytic extension of the map

$$(z_1, x' + \sqrt{-1}v_\alpha, u + \sqrt{-1}v(z_1, x', u)) \text{ near } 0,$$

we obtain a real analytic $(n+1)$-submanifold E containing $\widetilde{M}$. One can see easily that E is a generic submanifold with real analytic CR-subbundle $T^{(1,0)}E$ of real dimension two. Now, after choosing suitable analytic coordinates, we can define the Levi-form $\mathcal{L}$ of E near 0 which is also real analytic (see [Bog] for related concepts). Noting that $\widetilde{M}$ is foliated by analytic disks, we see that $\mathcal{L}$ vanishes identically on $\widetilde{M}$ and thus $\mathcal{L}$ vanishes identically on a neighborhood of 0. That is, E is a Levi-flat generic submanifold of real codimension $n-1$. Applying the Frobinus theorem, we see that M near 0 can be holomorphically transformed into the affine space $\mathcal{A}$. The proof of Theorem (A) is complete, assuming Proposition 4.1. □

§5. Singular Bishop equations— Proof of Proposition 4.1

In this section, we still assume that M is real analytic. We will present a proof of Proposition 4.1. This leads us to the study of the singular Bishop equation near 0. We would like to mention that by using the Hilbert transform along a variable curve and by using a much refined Picard iteration technique appearing in [KW1], [KW2], one can prove the existence of the family of analytic disks attached to M. Then, by applying the implicit function theorem around each fixed disk as was done in [KW1], [KW2], one can see the nice dependence on the real parameter (x', r) for $0 < r \ll 1$. (See, for example, [KW1, Propositions 3.3, 3.4].) However, Proposition 4.1 requires the analytic dependence of the disks with respect to (x', r) for r in a neighborhood of $r = 0$. Our idea to do this is to use a perturbation method. (See [HU] or [HK] for certain related arguments in the case of complex dimension two.) As mentioned in §4, for the fixed integer $m > 10$, we can assume that M is defined by (2.1) but with the following extra properties:

(a) $F^* = k(z_1, \overline{z_1}, x') + \sqrt{-1}h(z_1, \overline{z_1}, x') = O(|z_1|^3)$, $\lambda(x')$, and f_α are real analytic functions;

(b) $h = \mathrm{Im}(F^*(z_1, \overline{z_1}, x')) = o(|z_1|^m)$, and $f_\alpha(z_1, \overline{z_1}, x') = o(|z_1|^m)$. Here, $|z_1| \ll 1$ and $x' \in \mathcal{O}$.

As in §2, we write

$$M_0 = \{(z_1, x', u) \in \mathcal{A} : u = q_0(z_1, \overline{z_1}, x') + k(z_1, \overline{z_1}, x')\}$$

and

$$\widetilde{M_0} = \{(z_1, x', u) \in \mathcal{A} : u > q_0(z_1, \overline{z_1}, x') + k(z_1, \overline{z_1}, x')\}.$$

Also, we know that $(r\sigma(\xi, x', r), x', r^2)$ forms a family of analytic disks attached to M_0.

Proof of Proposition 4.1. We will seek a suitable family of functions

$$\{\mathcal{F}_1(\xi, x', r), \cdots, \mathcal{F}_n(\xi, x', r)\}$$

from $\overline{\Delta} \times \mathcal{O} \times I_\epsilon$ to $\mathbf{C}^n$, which are holomorphic in $\xi \in \Delta$ for each fixed r and x', so that

$$\Phi(\xi, x', r) = (r\sigma(\xi, x', r)(1 + \mathcal{F}_1(\xi, x', r)), \mathcal{F}_\alpha(\xi, x', r), \mathcal{F}_n(\xi, x', r))$$

provides the desired family of analytic disks attached to M, as in Proposition 4.1.

Therefore, we need to solve the following system of equations:

$$\begin{aligned}\mathcal{F}_n(\xi, x', r) = q_0(\mathcal{F}_1^*(\xi, x', r), \overline{\mathcal{F}_1^*(\xi, x', r)}, \mathrm{Re}(\mathcal{F}(\xi, x', r))) + k(\mathcal{F}_1^*, \overline{\mathcal{F}}_1^*, \mathrm{Re}(\mathcal{F}))\\ + \sqrt{-1}h(\mathcal{F}_1^*, \overline{\mathcal{F}_1^*}, \mathrm{Re}(\mathcal{F})),\end{aligned} \tag{5.1}$$

$$\mathrm{Im}(\mathcal{F}_\alpha) = f_\alpha(\mathcal{F}_1^*, \overline{\mathcal{F}_1^*}, \mathrm{Re}(\mathcal{F})). \tag{5.2}$$

Here $\mathcal{F}_1^* = r\sigma(\xi, x', r)(1 + \mathcal{F}_1(\xi, x', r))$ and $\mathcal{F} = (\mathcal{F}_2, \cdots, \mathcal{F}_{n-1})$. In what follows, we write $\mathcal{X} = \mathrm{Re}(\mathcal{F})$ and $\mathcal{Y} = \mathrm{Im}(\mathcal{F})$. We also notice that (5.2) is a normal Bishop equation, while (5.1) becomes degenerate when $r \to 0$. Let N be a certain fixed large integer. Write $(E_{N,k}, \|\cdot\|_{N,1/2})$ for the real-valued function space $C^{N,1/2}(S^1) \times \cdots \times C^{N,1/2}(S^1)$ equipped with the standard Banach norm $\|\cdot\|_{N,1/2}$, where the product takes k times. For simplicity, we write $\|\cdot\|_N$ for $\|\cdot\|_{N,1/2}$, in what follows. And for a complex-valued function g, $\|g\|_N := \sqrt{\|\mathrm{Re}g\|_N^2 + \|\mathrm{Im}g\|_N^2}$.

We impose a normalization condition on $\mathcal{F}$ such that the harmonic extension (in ξ-variable) of $\mathcal{X}$ to Δ takes the value x' at the origin. Then (5.2) reduces to the equation: $\mathcal{X} = -\mathcal{H}(\mathcal{Y}) + x'$, and thus $\mathcal{X} = -\mathcal{H}(f(\mathcal{F}_1^*, \overline{\mathcal{F}_1^*}, \mathcal{X})) + x'$. Notice that $\mathcal{F}_1^* = r\sigma(\xi, x', r)(1 + \mathcal{F}_1)$ and $f(z_1, \overline{z_1}, x') = o(|z_1|^m)$. We see that there is a real-valued vector function $G = (G_2, \cdots, G_{n-1})$, which is real analytic in $(A_1, B'; \xi, x', r) \in U^* \times \widetilde{U} \times S^1 \times \mathcal{O} \times I_\epsilon$, such that equation (5.2) can be further reduced to the following form:

$$\mathcal{X} - x' = r^m \mathcal{H}(G(\mathcal{F}_1, \mathcal{X}; \xi, x', r)). \tag{5.3}$$

Here U^* is a certain small open subset in $\mathbf{C}^1$ containing 0 and $\widetilde{U}$ is a certain small open neighborhood of 0 in $\mathbf{R}^{n-2}$. Here and in what follows, for a function g over S^1 with parameter (x', r), $\mathcal{H}(g(\xi, r, x')) := (\mathcal{H}(g(\cdot, r, x')))(\xi)$.

We next simplify equation (5.1). Dividing it by r^2, we get

$$\begin{aligned}\frac{1}{r^2}\mathcal{F}_n(\xi, x', r) = q_0(\sigma(1 + \mathcal{F}_1), \overline{\sigma}(1 + \overline{\mathcal{F}_1}), \mathcal{X})\\ + \frac{1}{r^2}k(\mathcal{F}_1^*, \overline{\mathcal{F}_1^*}, \mathcal{X}) + \frac{\sqrt{-1}}{r^2}h(\mathcal{F}_1^*, \overline{\mathcal{F}_1^*}, \mathcal{X}).\end{aligned} \tag{5.3$'$}$$

Linearizing the right hand side of (5.3)$'$ at the point $(\mathcal{F}_1, \mathcal{X}, r) = (o, x', r)$, one obtains

$$\frac{1}{r^2}\mathcal{F}_n = \Omega_0 + \Omega_1 + \Omega_2.$$

Here Ω_0, Ω_1 are given by

$$\Omega_0 = q_0(\sigma, \overline{\sigma}, x') + \frac{1}{r^2} k(r\sigma, r\overline{\sigma}, x') = 1,$$

$$\Omega_1 = 2\mathrm{Re}\left((\sigma \frac{\partial q_0}{\partial z_1}(\sigma, \overline{\sigma}, x') + \frac{\sigma}{r}\frac{\partial k}{\partial z_1}(r\sigma, r\overline{\sigma}, x'))\mathcal{F}_1 \right);$$

and Ω_2 is the right hand side of $(5.3)'$ with Ω_0 and Ω_1 being taken away. By Lemma 2.1, there clearly exists a C^ω-function Λ_0 in $(\xi, x', r; A_1, B') \in S^1 \times \mathcal{O} \times I_\epsilon \times U^* \times \widetilde{U}$ such that $\Omega_2(\mathcal{F}_1, \mathcal{X}; x', r)(\xi) = \Lambda_0(\xi, x', r; \mathcal{F}_1, \mathcal{X})$, where $\Lambda_0(\xi, x', r; A_1, B') = \sum_{\alpha=2}^{n-1} a_\alpha(\xi, x', r)(B_\alpha - x_\alpha) + \Lambda_0^*(\xi, x', r; A_1, B')$ with $\Lambda_0^* = o(\|B' - x'\|) + o(|A_1|) + o(|r|^{m-2})$ and $a_\alpha(\xi, x', r)$ certain real analytic functions in (ξ, x', r). Here, U^* is a certain small neighborhood of 0 in $\mathbf{C}^1$ and $\widetilde{U}$ is a small neighborhood of 0 in $\mathbf{R}^{n-2}$. Since $(5.3)'$ is coupled with (5.3), making use of (5.3), we get

$$\sum_\alpha a_\alpha(\xi, x', r)(\mathcal{X}_\alpha - x_\alpha) = r^m \sum_{\alpha=2}^{n-1} a_\alpha(\xi, x', r)\mathcal{H}(G_\alpha(\mathcal{F}_1, \mathcal{X}; \xi, x', r)).$$

Hence, we can reformulate (without changing the solutions to the system (5.3), $(5.3)'$) Ω_2, the non-linear operator with variables in $(\mathcal{F}_1, \mathcal{X}; x', r)$ and with value in the complex-valued $C^{N,1/2}(S^1)$-space, by

$$(5.3)'^* \qquad \Omega_2(\mathcal{F}_1, \mathcal{X}; x', r)(\xi) = r^m \sum_{\alpha=2}^{n-1} a_\alpha(\xi, x', r)\mathcal{H}(G_\alpha(\mathcal{F}_1, \mathcal{X}; \xi, x', r)) + \Lambda_0^*(\mathcal{F}_1, \mathcal{X}; \xi, x', r).$$

Apparently, for each fixed $x' \approx 0$, as $\|\mathcal{F}_1\|_N, |r|, \|\mathcal{X} - x'\|_N \to 0$, making use of the boundedness of the Hilbert transform acting on the Banach space $C^{N,1/2}$ with the standard norm, the right hand side of $(5.3)'^*$ gives the following asymptotic property of Ω_2:

$$(5.3)'' \qquad \Omega_2 = o(\|\mathcal{X} - x'\|_N) + o(\|\mathcal{F}_1\|_N) + o(|r|^{m-2}).$$

Write

$$c(\xi, x', r) = 2\sigma(\xi, x', r)\left(\frac{\partial q_0}{\partial z_1} + \frac{1}{r}\frac{\partial k}{\partial z_1}(r\sigma, r\overline{\sigma}, x')\right).$$

Hence, to solve the system (5.1), (5.2), it suffices to solve (5.3) coupled with the following:

$$\mathrm{Re}\,(c(\xi, x', r)\mathcal{F}_1) + \Omega_2(\mathcal{F}_1, \mathcal{X}; x', r)(\xi) + 1 = \frac{1}{r^2}\mathcal{F}_n(\xi, x', r).$$

Here and in what follows, Ω_2 is given by $(5.3)'^*$. Notice that $\frac{1}{r^2}\mathcal{F}_n$ is only required to be holomorphic in ξ. The solutions of (5.3) and the following equation (5.4) easily yield solutions for the system (5.1), (5.2):

$$(5.4) \qquad \mathrm{Re}(c(\xi, x', r)\mathcal{F}_1) + \mathrm{Re}(\Omega_2(\mathcal{F}_1, \mathcal{X}; x', r)(\xi)) = -\mathcal{H}(\mathrm{Im}\Omega_2(\mathcal{F}_1, \mathcal{X}; x', r))(\xi).$$

Remark that the high order vanishing property of h implies that $\|\mathrm{Im}(\Omega_2)\|_N \lesssim r^{m-2}$, after the existence and dependence property of the solution $(\mathcal{F}_1, \mathcal{X})$ is established.

Lemma 5.1. *With the above notation, one has* $c(\xi, x', r) \neq 0$ *and* $\mathrm{Ind}_{S^1} c(\xi, x', r) = 0$ *for* $|r| \ll 1$, $\xi \in S^1$ *and* $x' \in \mathcal{O}$. *Hence, there exists a positive function* $d(\xi, x', r) \in C^\omega(S^1 \times \mathcal{O} \times I_\epsilon)$ *such that* $d^*(\xi, x', r) = d(\xi, x', r)c(\xi, x', r)$ *has a holomorphic extension to* Δ *for each fixed* r *and* $x' \in \mathcal{O}$. *Moreover,* $d^*(\xi, x', r)$ *is real analytic in* (ξ, x', r).

Proof of Lemma 5.1. We first see that

$$\begin{aligned}\mathrm{Re} c(\xi, x', r) &= 2\mathrm{Re}\left(\sigma(\xi, x', r)\frac{\partial q_0}{\partial z_1}(\sigma, \overline{\sigma}, x') + \frac{\sigma}{r}\frac{\partial k(r\sigma, r\overline{\sigma}, x')}{\partial z_1}\right) \\ &= 2\mathrm{Re}\left(|\sigma|^2 + 2\lambda(x')\sigma^2 + O(|r|\|\sigma\|^2)\right) > 0,\end{aligned}$$

when r is sufficiently small.

Now we can simply let $d(\xi, x', r) = \frac{1}{|c|}\mathrm{e}^{\mathcal{H}(i \log \frac{c}{|c|}(\xi, x', r))}$ and $d^*(\xi, x', r) = d \cdot c$. Then d and d^* possess the properties as imposed in the lemma. ∎

Returning to equation (5.4), we have

$$\begin{aligned}\mathrm{Re}\,(d^*(\xi, x', r)\mathcal{F}_1) = &-d(\xi, x', r)\mathrm{Re}\Omega_2(\mathcal{F}_1, \mathcal{X}; x', r) \\ &- d(\xi, x', r)\mathcal{H}(\mathrm{Im}(\Omega_2(\mathcal{F}_1, \mathcal{X}; x', r))).\end{aligned}$$

Let $\widetilde{\mathcal{F}_1} = d^*(r, \xi)\mathcal{F}_1 \equiv U(\xi, x', r) + \sqrt{-1}\mathcal{H}(U(\xi, x', r))$. Then we obtain

$$\begin{aligned}U(\xi, x', r) = &-d(\xi, x', r)\mathrm{Re}\left(\Omega_2\left(\frac{U + \sqrt{-1}\mathcal{H}(U)}{d^*(\xi, x', r)}, \mathcal{X}; x', r\right)\right) \\ &- d(\xi, x', r)\mathcal{H}\left(\mathrm{Im}\left(\Omega_2\left(\frac{U + \sqrt{-1}\mathcal{H}U}{d^*(\xi, x', r)}, \mathcal{X}; x', r\right)\right)\right).\end{aligned} \tag{5.4$'$}$$

We will now solve $\vec{U}(x', r) = (U(x', r), \mathcal{X}(x', r) - x')$ from equation (5.3) and equation (5.4)$'$. To this aim, we let

$$\begin{aligned}\Lambda_1(\vec{U}; x', r) &= (-d(\xi, x', r)\mathrm{Re}\Omega_2(\mathcal{F}_1, \mathcal{X}; x', r), 0)\,; \\ \Lambda_2(\vec{U}; x', r) &= -d(\xi, x', r)\left(\mathcal{H}\left(\mathrm{Im}\Omega_2(\mathcal{F}_1, \mathcal{X}; x', r)\right), 0\right); \\ \Lambda_3(\vec{U}; x', r) &= -r^m\left(0, \mathcal{H}(G(\mathcal{F}_1, \mathcal{X}; \xi, x', r))\right).\end{aligned}$$

Here, as defined before, $\mathcal{F}_1 = (U + \sqrt{-1}\mathcal{H}(U))/d^*$. Then we obtain

$$\vec{U} = \Lambda_1(\vec{U}; x', r) + \Lambda_2(\vec{U}; x', r) + \Lambda_3(\vec{U}; x', r). \tag{5.5}$$

We are going to apply the implicit function theorem to (5.5) to obtain a solution $\vec{U}$ that is C^ω in the variable (x', r). To this end, we write N for the same index as before.

Write

$$B^{N,1/2}_{\epsilon,k} = \{\phi \in E_{N,k} : \|\phi\|_N < \epsilon\}.$$

Consider the operator

$$\Lambda : B^{N,1/2}_{\epsilon,n-1} \times \mathcal{O} \times I_\epsilon \to E_{N,n-1},$$

$$\Lambda(\vec{U}; x', r) = \Lambda_1(\vec{U}; x', r) + \Lambda_2(\vec{U}; x', r) + \Lambda_3(\vec{U}; x', r).$$

By the boundedness of the Hilbert transform acting on the Banach space $E_{N,1}$, we easily see that Λ is a well-defined smooth operator when $\epsilon \ll 1$. In fact, we have (see [Dei] for the definition of the related concept):

Lemma 5.2. *For sufficiently small ϵ, $\mathcal{O}$ and r, Λ is an analytic map from $B^{N,1/2}_{\epsilon,n-1}\times \mathcal{O}\times I_\epsilon$ into $E_{N,n-1}(S^1)$.*

Proof of Lemma 5.2. First notice that the Hilbert transform is bounded, linear and hence analytic. It is self-evident that Λ must be real analytic, because it is formed from the compositions and other basic operations of real analytic functions and mappings between Banach spaces [Dei]. Indeed, $\Lambda(\vec{U};x',r)$ is clearly a finite sum of non-linear operators of the following forms: $F_1(\xi,x',r;\vec{U},\mathcal{H}(\vec{U}))$, $c_1(\xi,x',r)\mathcal{H}(F_3(\xi,x',r;\vec{U},\mathcal{H}(\vec{U})))$, and $c_2\mathcal{H}(c_3\mathcal{H}(F_4(\xi,x',r;\vec{U},\mathcal{H}(\vec{U}))))$, where F_j and c_j are real-valued real analytic vector or scale functions in their arguments. □

We notice that $\Lambda(0)=0$. Meanwhile, since $\|\mathcal{F}_1\|_N\approx\|U\|_N$, from (5.3), (5.3)$''$, as well as the definition of Λ, it follows quite clearly that $\Lambda'_{\vec{U}}|_0=0$. Thus, from the real analytic version of the implicit function theorem in Banach spaces (see [Dei]), (5.5) can be uniquely solved. Moreover, the solution $\vec{U}(x',r)$ depends C^ω on the parameter $(x',r)\approx 0$; i.e., $\vec{U}(x',r)=\sum_{\alpha,j}\vec{U}_{\alpha,j}x'^{\alpha}r^j$ with $\vec{U}_{\alpha,j}\in C^{N,1/2}(S^1)$ and $\|\vec{U}_{\alpha,j}\|_N\lesssim R^{\|\alpha\|+j}$. Hence, $\vec{U}(\xi,x',r)=\vec{U}(x',r)(\xi)$ belongs to $C^{N,1/2}(S^1\times\mathcal{O}\times I_\epsilon)$. Also, by (5.3)$''$, (5.4)$'$, (5.3) (which now indicates that $\|\mathcal{X}-x'\|_N\lesssim r^m$) and the previously mentioned fact $\|\mathrm{Im}(\Omega)\|_N\lesssim r^{m-2}$, it holds that

$$\|U(\xi,x',r)\|_N\le\epsilon\|U\|_N+O(r^{m-2}),\ \text{and hence}\ \|U(\xi,x',r)\|_N\le C\cdot\frac{1}{1-\epsilon}r^{m-2}.$$

Having obtained $(U,\mathcal{X})$, we can then get $\mathcal{F}_1$ and $\mathcal{F}$ by holomorphically extending to Δ in ξ-variable the functions $(U+\sqrt{-1}\mathcal{H}(U))/d^*(\xi,x',r)$ and $\mathcal{X}+\sqrt{-1}\mathcal{H}(\mathcal{X})$ (by the Cauchy integral), respectively. From the ways equations (5.4)$'$ and (5.3) were derived, it is self-evident that the $(\mathcal{F}_1,\mathcal{F})$ constructed in such a manner yields a solution to the system (5.1), (5.2). Namely, after substituting the just obtained $(\mathcal{F}_1,\mathcal{F})$ into (5.1) and (5.2), (5.2) becomes the identity and the right hand side of (5.1), apriori defined for $\xi\in\mathcal{S}^1$, extends to the holomorphic function $\mathcal{F}_n$ over Δ in ξ.

Next, by the maximum principle, we easily see that $|\mathcal{F}_1|\lesssim r^{m-2}$ and $|\mathcal{F}-x'|<r^m$ hold over $\overline{\Delta}$ uniformly for (x',r) sufficiently close to $(o,0)$.

Lastly, for $\xi\in\overline{\Delta}$, let

$$(\phi_1,\phi_\alpha,\phi_n)(\xi)=\left(r\sigma(\xi,x',r)(1+\mathcal{F}_1(\xi,x',r)),\mathcal{F}_\alpha(\xi,x',r),\mathcal{F}_n(\xi,x',r)\right).$$

Then, one has:

$$\phi_1(\xi,x',r)=r\sigma(\xi,x',r)(1+\psi_1^*(\xi,x',r)),\ \phi_\alpha(\xi,x',r)=x'+\psi_\alpha^*(\xi,x',r)$$

with $\psi_1^*(\xi,x',r)=O(r^{m-2})$ and $\psi_\alpha^*(\xi,x',r)=O(r^m)$. Also write $\phi_n=r^2+\psi_n^*(\xi,x',r)$. Notice that $q_0+k\big|_{(r\sigma,\overline{r\sigma},x')}=r^2$ and

$$\begin{aligned}&F\left(r\sigma(\xi,x',r)(1+\mathcal{F}_1),\overline{r\sigma(\xi,x',r)(1+\mathcal{F}_1)},\mathrm{Re}(\mathcal{F})\right)\\&=F(r\sigma(\xi,x',r),\overline{r\sigma(\xi,x',r)},x')+O(|r^2\sigma\mathcal{F}_1|)+O(\|\mathcal{X}-x'\|)\\&=r^2+h(r\sigma(\xi,x',r),\overline{r\sigma(\xi,x',r)},x')+O(|r|^m)=r^2+O(r^m),\end{aligned}$$

where $F=q_0+k+\sqrt{-1}h$. We also see that $\psi_n^*(\xi,x',r)=O(r^m)$.

Moreover, from the analytic dependence of $\vec{U}(x', r)$ on (x', r), some standard properties of the Hilbert transform, and the Cauchy estimates for holomorphic functions, it is quite easy to conclude that ψ_j^*'s depend real analytically on $(\xi, x', r) \in \Delta \times \mathcal{O} \times I_\epsilon$ and $C^{N,1/2}$ over $\overline{\Delta} \times \mathcal{O} \times I_\epsilon$ when $\mathcal{O}$ and ϵ are sufficiently small. Therefore, the family ϕ just obtained satisfies the property as in Proposition 4.1, if we take $N > m$. The proof of Proposition 4.1 is finally complete. □

Remark 5.3. Let ϕ be obtained as above, and let Φ be defined as in (4.1). It can be further shown that $\Phi \in C^{\frac{m-2}{2}}(\widetilde{M_0}) \cap C^\omega(\widetilde{M_0} \setminus \{(0, x', 0)\})$. Also $\Phi = (\mathrm{id} + o(\|z\|))$ is holomorphic in z_1.

Appendix I: Formal flattening of M

In this appendix, we will show that a real analytic non-degenerate elliptic point can be flattened to any order, which was used in the proof of Theorem (A). This result follows implicitly from the work of Kenig-Webster [KW2]. However, due to its importance to this paper and for convenience of the reader, we include a detailed proof.

Proposition I.A. *Let $M \subset \mathbf{C}^n$ be a real analytic n-submanifold and let $p \in M$ be a non-degenerate elliptic point. Then, for any positive integer ℓ, there is a biholomorphic change of coordinates sending p to 0 such that, in the new coordinates, M can be defined by an equation of the form as in (2.1) but with the following extra properties:*

(a) $\lambda(x')$ *is real analytic in* x'*;*
(b) $F^* = O(|z_1|^3)$*; and*
(c) $f_\alpha = O(|z_1|^{\ell-1})$, $\mathrm{Im}(F^*) = o(|z_1|^\ell)$.

Proof of Proposition I.A. As mentioned in §2, after a holomorphic change of coordinates, we can assume that $p = 0$ and, near 0, M is given by an equation of the following form:

$$\begin{aligned} z_n &= F = q_0(z_1, \overline{z_1}, x') + F^*(z_1, \overline{z_1}, x'), \\ y_\alpha &= f_\alpha(z_1, \overline{z_1}, x') = \overline{f_\alpha}(\overline{z_1}, z_1, x'), \end{aligned} \tag{I.1}$$

where $q_0 = z_1\overline{z_1} + \lambda(x')z_1^2 + \overline{\lambda(x')z_1^2}$, $|\lambda(x')| < 1/2$, $0 \le \lambda(0) < 1/2$, $F^* = k(z_1, \overline{z_1}, x') + \sqrt{-1}h(z_1, \overline{z}_1, x')$, and $f_\alpha(z_1, \overline{z_1}, x')$ have vanishing order at least 3 at 0.

Let $\mathcal{P}$ be the locus of complex tangents of M near 0, as introduced in (2.2). Then P is real analytic in $x' \approx 0$. In particular, we see that P can be extended holomorphically to an open subset of $0 \in \mathbf{C}^n$. We now let:

$$z_1 = {z_1}^* + P({z'}^*), \quad z' = {z'}^* + \sqrt{-1}f(P({z'}^*), \overline{P(\overline{z'^*})}, {z'}^*),$$
$$z_n = z_n^* + F(P(z'^*), \overline{P(\overline{z'^*})}, z'^*).$$

Applying the inverse of the above transform to M, then in the new coordinates, $\mathcal{P}$ is given by the ${x'}^*$-(affine) space; and M is given by an equation of the form: $z_n^* = F$ and $y_\alpha^* = f_\alpha$ with $F = b_1({x'}^*)z_1^* + O(|z_1^*|^2)$ and $f_\alpha = O(|z_1^*|)$. In what follows, when there is no risk of causing confusion, we still write z for z^*. Write $F = b_1(x')z_1 + b_{21}(x')z_1^2 + \overline{b_{12}(x')z_1{}^2} + b_{11}(x')z_1\overline{z_1} + O(|z_1|^3)$ and $f = \mathrm{Im}(c_0(x')z_1) + o(|z_1|)$, where

$b_{ij}(x')$ are real analytic in x'. We now make another change of variables:

$$z_1^* = z_1,\ z'^* = z' - c_1(z')z_1, \text{ and } z_n^* = z_n - b_1(z')z_1.$$

Then in the new coordinates, we see that M is now given by an equation of the form: $z_n = F$ and $y_\alpha = f_\alpha$ with $F = b_{21}(x')z_1^2 + \overline{b_{12}(x')z_1{}^2} + b_{11}(x')z_1\overline{z_1} + O(|z_1|^3)$ and $f = O(|z_1|^2)$. By the ellipticity, we have $b_{11}(0) \neq 0$. Applying the transformation $(z_1^*, z'^*, z_n^*) = (z_1, z', \frac{z_n}{b_{11}(z')} - \frac{b_{21}(z')}{b_{11}(z')}z_1^2 + (b_{12}(z')/\overline{b_{11}(\overline{z}')})z_1^2)$, we can see that in the new coordinates, $z_n = F = q_0 + k + ih = |z_1|^2 + (\lambda(x')z_1^2 + \overline{\lambda(x')z_1^2}) + O(|z_1|^3)$ with $q_0 = |z_1|^2 + (\lambda(x')z_1^2 + \overline{\lambda(x')z_1^2})$, and $y_\alpha = f_\alpha = O(|z_1|^2)$, where $\lambda(x') = b_{12}(x')/\overline{b_{11}(x')}$ is real analytic near 0. Moreover, by applying the transform $(z_1, z', z_n) \to (cz_1, z', |c|^2 z_n)$ with a suitable non-zero constant c, we can further assume, without loss of generality, that $\lambda(0) \geq 0$. (By the ellipticity, $\lambda(0) < 1/2$.) This completes the proofs of (a) and (b) in Proposition I.A.

Next, we will show that for any given $\ell > 3$, we can further find a biholomorphic mapping defined near the origin, which will send M to a submanifold defined by an equation of the form $z_n = F = q_0 + k + ih = z_1\overline{z_1} + 2\mathrm{Re}(\lambda(x')z_1^2) + F^*$, $y_\alpha = f_\alpha$ with the following properties described as in Proposition I.A: $f_\alpha = O(|z_1|^{\ell-1})$ and $h = \mathrm{Im}(F^*) = o(|z_1|^\ell)$. Here $\lambda(x')$ is as obtained above and $F^* = O(|z_1|^3)$.

To this aim, we will seek the required transformation given in the following way:

$$\text{(I.2)} \quad z_1^* = z_1, \quad z'^* = z' + \sum_{j=2}^{\ell-1} C^{(j)}(z_1, z_n; z'), \text{ and } z_n^* = z_n + \sum_{j=3}^{\ell} B^{(j)}(z_1, z_n; z'),$$

where $C^{(j)}$ and $B^{(j)}$ are weighted homogeneous polynomials in z_1 and z_n of degree j with coefficients in the germs of holomorphic functions defined near a small neighborhood $\mathcal{O}$ of $0 \in \mathbf{R}^{n-2}$. Here, we assign the weights 1 and 2 to z_1 and z_n, respectively. Namely, a polynomial of the form $z_1^i z_n^j$ has weighted degree $i + 2j$.

Also, we impose the following normalization condition:

$$\text{(I.2)}' \qquad \mathrm{Re}C^{(j)}(0, u; x') = \mathrm{Re}B^{(j)}(0, u; x') = 0.$$

We of course expect that after the above change of variables, M is mapped to a submanifold given as follows:

$$\text{(I.3)} \qquad \begin{aligned} z_n^* &= q_0(z_1^*, \overline{z_1^*}, x'^*) + k^*(z_1^*, \overline{z_1^*}, x'^*) + \sqrt{-1}h^*(z_1^*, \overline{z_1^*}, x'^*), \\ y_\alpha^* &= f_\alpha^*(z_1^*, \overline{z_1^*}, x'^*) = \overline{f_\alpha}^*(\overline{z_1^*}, z_1^*, x'^*), \end{aligned}$$

where $k^* = O(|z_1^*|^3)$, $h^*(z_1^*, \overline{z_1^*}, x'^*) = o(|z_1^*|^\ell)$ and $f^*(z_1^*, \overline{z_1^*}, x'^*) = o(|z_1^*|^{\ell-1})$.

Substituting (I.2) into (I.3) and collecting terms of weighted degree k, we see that we only need to solve the following functional equations for $k \leq \ell$:

$$\text{(I.4)} \qquad \begin{aligned} \mathrm{Im}B^{(k)}(z_1, q_0(z_1, \overline{z_1}, x'), x') &= -h^{(k)}(z_1, \overline{z_1}; x') + G_1^{(k)}(z_1, \overline{z}_1; x'), \\ \mathrm{Im}C^{(k-1)}(z_1, q_0(z_1, \overline{z_1}, x'), x') &= -f^{(k-1)}(z_1, \overline{z_1}, x') + G_2^{(k-1)}(z_1, \overline{z_1}; x'), \end{aligned}$$

where $h = \sum_{k\geq 3} h^{(k)}(z_1, \overline{z_1}, x')$, $f = \sum_{k\geq 2} f^{(k)}(z_1, \overline{z_1}, x')$ with $h^{(k)}$ and $f^{(k)}$ being weighted homogeneous polynomials in z_1 and $\overline{z_1}$ of degree k, whose coefficients are real analytic in $x' \in \mathcal{O}$. $G_1^{(k)}$ and $G_2^{(k-1)}$ are finitely contributed by $B^{(\sigma)}$ and $C^{(\sigma-1)}$ for $\sigma \leq k - 1$. Notice that when $k = 3$, we have the initial condition that $G_1^{(k)} = G_2^{(k-1)} = 0$. Indeed, by the following Lemma I.B, we can inductively and uniquely solve (I.4) with the normalization condition (I.2)$'$. This obviously completes the proof the proposition. ∎

Lemma I.B. *For any homogeneous polynomial $G^{(s)}(z_1,\overline{z_1},x')$ of degree $s\geq 1$ in $(z_1,\overline{z_1})$ with coefficients real analytic in $x'\in\mathcal{O}$, a small open neighborhood of 0 in $\mathbf{R}^{n-2}$, there exists a unique weighted homogeneous polynomial $X^{(s)}(z_1,u,x')$ in (z_1,u) with coefficients real analytic in $x'\in\mathcal{O}$ such that*

$$\text{(I.5)}\qquad \mathrm{Im}X^{(s)}(z_1,q_0(z_1,\overline{z_1},x'),x')=G^{(s)}(z_1,\overline{z_1},x'),\quad \mathrm{Re}X^{(s)}(0,u,x')\equiv 0.$$

Proof of Lemma I.B. The proof of this lemma is similar to that in [KW2, Proposition 1.1].

Write $\mathcal{R}(\mathcal{O})$ for the ring of real analytic functions in $x'\in\mathcal{O}$. Let

$$\mathcal{D}_1^{(s)}=\{G(z_1,\overline{z_1},x'):\ G(z_1,\overline{z_1},x')=\sum_{i+j=s}a_{ij}(x')z_1^i\overline{z_1^j}\},$$

where $a_{ij}(x')=\overline{a_{ji}}(x')\in\mathcal{R}(\mathcal{O})$ are real analytic functions in $x'\in\mathcal{O}$; and let

$$\mathcal{D}_2^{(s)}=\{F(z_1,u,x'):\ F(z_1,u,x')=\sum_{i+2j=s}a_{ij}(x')z_1^iu^j\},$$

where the function $\sqrt{-1}a_{0s/2}(x')\in\mathcal{R}(\mathcal{O})$ takes the real value when s is even.

Then, denote the subsets of $D_1^{(s)}$ and $D_2^{(s)}$, whose coefficients $a_{ij}(x')$ are constant, by $D_1^{(s)}(0)$ and $D_2^{(s)}(0)$, respectively. Then, they can be viewed as $\mathbf{R}$-vector spaces. Moreover, $\dim_{\mathbf{R}}D_1^{(s)}(0)=\dim_{\mathbf{R}}D_2^{(s)}(0)=s+1$. Meanwhile, by using the maximum principle and noting that the level set of $q_0(z_1,\overline{z_1},x')=r^2$ for each fixed $x'\in\mathcal{O}$ is the boundary of an analytic disk, one sees that $g(z_1,z_n)\equiv 0$ if

$$\mathrm{Im}g(z_1,q_0(z_1,\overline{z_1},x'))\equiv 0.$$

Here $g\in\mathcal{D}_2^{(s)}(0)$ and $\mathrm{Re}g(0,u)=0$. In fact, if the last condition holds, then for each r, one sees that $\mathrm{Im}g(r\sigma_0(\xi,x',r),r^2)\equiv 0$ for every $\xi\in\partial\Delta$. Here we use σ_0 to denote the conformal mapping from Δ to the domain $D_0=\{\xi\in\mathbf{C}^1:\quad q_0(\xi,\overline{\xi},x')<1\}$. Thus, it follows that $g(z_1,r^2)\equiv 0$ for $(z_1,x',r^2)\in\widetilde{M_0}$. Hence one obtains $g\equiv 0$.

Therefore, for each fixed $x'\approx 0$, we can conclude that the $\mathbf{R}$-linear operator $\mathcal{I}(x'):\ \mathcal{D}_2^{(s)}(0)\to\ \mathcal{D}_1^{(s)}(0)$, which sends each $g\in\mathcal{D}_2^{(s)}(0)$ to $\mathrm{Im}(g(z_1,q_0(z_1,\overline{z_1},x')))$, is one-to-one. Since the two vector spaces over $\mathbf{R}$ involved here have the same finite dimension, it therefore follows that $\mathcal{I}(x')$ is an $\mathbf{R}$-linear isomorphism.

Now, we can use this fact to show that the map $\mathcal{I}:\ \mathcal{D}_2^{(s)}\to\ \mathcal{D}_1^{(s)}$, which sends each $g\in\mathcal{D}_2^{(s)}$ to $\mathcal{I}(g)=\mathrm{Im}(g(z_1,q_0(z_1,\overline{z_1},x'),x'))$, is onto. Notice that $\mathcal{I}$ is linear over the ring of real-valued real analytic functions in x'.

Indeed, we can explain it as follows: Write $z_1=x_1+\sqrt{-1}y_1$. Then $\mathcal{D}_1^{(s)}$ is actually the collection of functions of the form: $h=\sum_{i+j=s}a_{ij}(x')x_1^iy_1^j$ with $a_{ij}(x')$ real-valued and real analytic in $x'\in\mathcal{O}$. Set $e_j=x_1^{j-1}y_1^{s-j+1}$ for $j=1,\cdots,s+1$. Then $\{e_j\}$ gives a basis for $\mathcal{D}_1^{(s)}$. We can also choose a basis $\{\widetilde{e_1},\cdots,\widetilde{e_{s+1}}\}$ of $\mathcal{D}_2^{(s)}$, which has constant coefficients. Say, when s is even, we let $\widetilde{e_1}=\sqrt{-1}u^{s/2}$, $\widetilde{e_2}=z_1^2u^{s/2-1}$, $\widetilde{e_3}=\sqrt{-1}z_1^2u^{s/2-1}$, $\cdots,\widetilde{e_{s+1}}=\sqrt{-1}z_1^s$. Notice that these basis are also the vector space basis of $\mathbf{R}$-vector spaces $\mathcal{D}_1^{(s)}(0)$ and $\mathcal{D}_2^{(s)}(0)$, respectively. Clearly, for each element g in $\mathcal{D}_1^{(s)}$, there is a unique decomposition $g=\sum g_j(x')e_j$ with $g_j(x')$ *real-valued.* Also, for an element $\phi\in D_2^{(s)}$, we have $\phi=\sum\phi_j(x')\widetilde{e_j}$ with $\phi_j(x')$ *real-valued.* Write $\mathcal{I}(\widetilde{e_i})=\sum_j b_{ij}(x')e_j$, where $b_{ij}(x')$ are *real-valued* real analytic functions in x'. Then, since *for each fixed* x', $\mathcal{I}(x')$ (defined as above) is an $\mathbf{R}$-linear isomorphism from the $\mathbf{R}$-vector space $\mathcal{D}_2^{(s)}(0)$ to the $\mathbf{R}$-vector space $\mathcal{D}_1^{(s)}(0)$, its matrix with respect to the above mentioned basis, which is obviously given by $(b_{ij}(x'))$, must be invertible. Now, for any $g = \sum g_j e_j \in \mathcal{D}_1^{(s)}$, let $\phi = (g_1, \cdots, g_{s+1})(b_{ij}(x'))^{-1}(\widetilde{e_1}, \cdots, \widetilde{e_{s+1}})^{\mathrm{t}}$. Since g_j, b_{ij} are real-valued, one verifies easily that $\phi \in \mathcal{D}_2^{(s)}$ and $\mathcal{I}(\phi) = g$. The proof of Lemma I.B is now complete. □

Appendix II. A functional equation along M

In this appendix, we present an application of Theorem (A) to the convergence proof of a formal power series coming from a certain functional equation, which may be useful in certain other studies (see, for example, Corollary II.B).

Theorem II.A. *Let M be a real analytic n-manifold in $\mathbf{C}^n$. Let $p \in M$ be an elliptic complex tangent point. For any given real-valued real analytic function $G(z, \overline{z})$ near p, there is a holomorphic function $X(z)$ near p such that*

$$\mathrm{Im}(X(z)) = G(z, \overline{z}), \quad \textit{for } z \in M. \tag{II.1}$$

Proof of Theorem II.A. By Theorem (A), we can clearly assume that M is already flattened. Namely, we can assume that $p = 0$ and M is defined by an equation of the form

$$u = q_0(z_1, \overline{z_1}, x') + k(z_1, \overline{z_1}, x'),\ y_j = 0,\ j \geq 2,$$

where $k = o(\|z_1\|^2)$. Also, we impose the normalization condition

$$\mathrm{Re}(X(0, u, x')) \equiv 0 \tag{II.1$'$}$$

so that (II.1) will be uniquely solved. We may assume that $G(0) = 0$.

Assign the weights 1 and 2 to z_1 and u, respectively, and write $G(z_1, \overline{z_1}, x') = G((z_1, x', q_0+k), \overline{(z_1, x', q_0+k)})$. Write $X^{(s)}(z_1, u, x')$ (respectively, $G^{(s)}(z_1, \overline{z_1}, x')$) for the weighted homogeneous polynomial in (z_1, u) (respectively, in $(z_1, \overline{z_1})$) of degree s in the expansion of $X(z_1, u, x')$ (respectively, $G(z_1, z_1, x')$) near the origin. Then collecting terms in (II.1) of weighted degree s, we obtain

$$\mathrm{Im}(X^{(s)}(z_1, q_0(z_1, \overline{z_1}, x'), x')) = G^{(s)}(z_1, \overline{z_1}, x') + G^{*(s)}(z_1, \overline{z_1}, x'), \tag{II.2}$$

where $G^{*(s)}$ is contributed by $X^{(\tau)}$ and $G^{(\tau-1)}$ with $\tau \leq s-1$. When $s = 0$, we see that $X^{(0)} = iG(0, 0, x')$, and when $s = 1$, we have the initial condition $G^{*(s)} = 0$.

Now, applying Lemma I.B to (II.2) with the normalization (II.1)$'$, one sees that $X^{(s)}$ can be inductively and uniquely solved, and completely determined by $X^{(\tau)}$ and $G^{(\tau+1)}$ for $\tau < s$. However, this dependence is very complicated and does not seem to be directly usable to get the Cauchy estimates for $\{X^{(s)}\}$. Our method to prove the convergence is to get the Cauchy estimates by showing the sequence of the finite expansions of X oscillates to a well-defined function. See also [MW] and [MOS] for some other methods to handle the convergence when $\lambda \neq 0$.

To this aim, we write $X_k(z_1, u; x') = \sum_{s=0}^{k} X^{(s)}(z_1, u, x')$. Then for $z \in M$,

$$\mathrm{Im}X_k(z_1, u; x') = G(z_1, \overline{z_1}, x') + O(|z_1|^{k+1}). \tag{II.3}$$

Let $\sigma(\xi, x', r)$ be as before. Consider the following equation:

$$\mathrm{Im}\widetilde{X}(z_1, u; z') = G(z_1, \overline{z_1}, x'), \quad (z_1, x', u) \in M, \tag{II.4}$$

with the normalization condition $\mathrm{Re}\widetilde{X}(0, u; x') = 0$. Here, $\widetilde{X}$ is only required to be defined over $\widetilde{M_0}$ and holomorphic along each analytic disk $\phi = (\phi_1, \cdots, \phi_n) =$

$(r\sigma, x', r^2)$ attached to M, i.e., $\widetilde{X} \circ \phi(\xi, x', r)$ is holomorphic in ξ for each given (x', r). Then

$$\mathrm{Im}\widetilde{X}(\phi(\xi, x', r)) = G(\phi_1(\xi, x', r), \overline{\phi_1(\xi, x', r)}, x')$$

for $\xi \in \partial\Delta$. Thus, $\widetilde{X}(\phi(\xi, x', r)) = \mathcal{S}(G(\phi_1(\cdot, r, x'), \overline{\phi_1(\cdot, x', r)}, x'))(\xi)$ for $\xi \in \Delta$. Here $\mathcal{S} = \sqrt{-1}(\mathrm{id} + \sqrt{-1}\mathcal{H})$ is the Schwartz transform. Let

$$X^*(\xi, r, x') = \mathcal{S}\left(G(\phi_1(\cdot, x', r), \overline{\phi_1(\cdot, x', r)}, x')\right)(\xi).$$

Then $X^*(\xi, r, x')$ is holomorphic in ξ on $\overline{\Delta}$. Also, by Lemma 2.1 and some basic properties of the Hilbert transform, it follows easily that $X^*(\xi, r, x')$ is jointly real analytic in (ξ, r, x'). Hence

$$X^* = \sum_{i,j,\|\alpha\| \geq 0} a_{ij\alpha} x'^{\alpha} \xi^i r^j$$

for $|r|, |\xi|, \|x'\| \ll 1$.

Now, for $r = u^{1/2}$ and $\xi = \sigma^{-1}(z_1/\sqrt{u}, x', \sqrt{u})$, we obtain the following expansion:

$$\begin{aligned}\widetilde{X}(z_1, u; x') &= X^*\left(\sigma^{-1}(\frac{z_1}{\sqrt{u}}, x', \sqrt{u}), \sqrt{u}, x'\right) \\ &= \sum_{i,j,\|\alpha\| \geq 0} \widetilde{a}_{ij\alpha} (z_1/\sqrt{u})^i \sqrt{u}^j x'^{\alpha}\end{aligned}$$

with $|\widetilde{a}_{ij\alpha}| \lesssim R^{i+j+\|\alpha\|}$, where $|z_1/\sqrt{u}|, |\sqrt{u}|, \|x'\| < \epsilon_0 \ll 1$.

On the other hand, by (II.3), we notice that on M,

$$\mathrm{Im}X_k(z_1, u; x') = G(z_1, \overline{z_1}, x') + O(|z_1|^{k+1}).$$

Hence, for $(z_1, x, u) \in M$, $\mathrm{Im}\left(X_k(z_1, u; x') - \widetilde{X}(z_1, u; x')\right) = O(|z_1|^{k+1})$.

Thus, by the same argument as above, involving the use of the Schwartz transformation formula, we see that for $(\xi, r, x') \in \overline{\Delta} \times I_\epsilon \times \mathcal{O}$,

$$X_k(\phi(\xi, x', r)) - \widetilde{X}(\phi(\xi, x', r)) = r^{(k+1)} g^*(\xi, x', r) + C_k(x', r),$$

where g^* is real analytic, holomorphic in ξ for $|\xi| < 1$, and $\mathrm{Re}(g^*(0, x', r)) \equiv 0$. Here, $C_k(x', r) = \mathrm{Re}\left(X_k(\phi(0, r, x')) - \widetilde{X}(\phi(0, x', r))\right) = \mathrm{Re}(X_k(0, r; x')) \equiv 0$, by the previously mentioned normalization condition. Notice that $(\xi, x', r) = (\sigma^{-1}(z_1/\sqrt{u}, x', \sqrt{u}), x', \sqrt{u}) := \psi(z_1/\sqrt{u}, x', \sqrt{u})$. Therefore, when

$$|z_1/\sqrt{u}|,\ |\sqrt{u}|, \|x'\| \ll 1,$$

we obtain

$$\begin{aligned}X_k(z_1, u; x') &= \sum_{ij,\|\alpha\| \geq 0} \widetilde{a}_{ij\alpha} (z_1/\sqrt{u})^i \sqrt{u}^j x'^{\alpha} + u^{\frac{k+1}{2}} g_k^{**}(\psi(z_1/\sqrt{u}, x', u)) \\ &= \sum_{ji,\|\alpha\| \geq 0} \widetilde{a}_{ij\alpha} x'^{\alpha} z_1^i u^{\frac{j-i}{2}} + u^{\frac{k+1}{2}} g_k^{**}(\psi(z_1/\sqrt{u}, x', u)),\end{aligned}$$

for some holomorphic function g_k^{**} depending on k.

Now, we have for each $k_0 \ll k$ that

$$\frac{\partial^{k_0} X_k}{\partial z_1^{k_0}}|_{z_1=0} = \sum_{j\alpha} k_0! \widetilde{a}_{k_0 j\alpha} (x')^{\alpha} u^{(j-k_0)/2} + o(u^{(k-k_0)/2}).$$

Also notice that X_k is a polynomial in (z_1, u) with coefficients analytic in x'. Thus, the following can be easily verified:

(i) when $j < k_0$, $\widetilde{a}_{k_0 j\alpha} = 0$; and

(ii) for $k > j > k_0$, if $j - k_0$ is not even, then $\widetilde{a}_{k_0 j\alpha} = 0$.

Since k can be made arbitrarily large, we conclude that at least formally, we have

$$X = \sum_{i,j(=2\ell+i),\alpha} \widetilde{a}_{i,j(=2\ell+i),\alpha} x'^{\alpha} z_1^i u^{\ell}.$$

Notice that $|\widetilde{a}_{i,j(=2\ell+i),\alpha}| \lesssim R^{i+2\ell+i+\|\alpha\|} \lesssim (R^2)^{i+\ell+j}$. We see that the above is actually a convergent power series in (z_1, x', u). Hence, (II.1) with the normalization condition in (II.1)$'$ has a unique convergent solution. The proof of Theorem II.A is complete. $\square$

To conclude, we give the following, which also follows from the Moser-Webster [MW] normal form in case $\lambda \neq 0$:

Corollary II.B. *Let M be a real analytic n-manifold in $\mathbf{C}^n$. Let p be an elliptic complex tangent. Then M near p can be biholomorphically mapped into the semi-Heisenberg submanifold $H_0 = \{(z_1, \cdots, z_n) \in \mathbf{C}^n : \ \mathrm{Re}(z_n) = |z_1|^2, \mathrm{Im}(z_j) = 0, \ j = 2, \cdots, n-1\}$.*

Proof of Corollary II.B. By Theorem (A), we can assume that $p = 0$ and M near 0 is defined by

$$\text{(II.5)} \qquad u = z_1\overline{z_1} + \mathrm{Re}F_0(z_1, x') + F(z_1, \overline{z_1}, x'), \ y_\alpha = 0, \ y_n = 0.$$

Here $F_0(z_1, x') = O(\|z_1\|^2)$ and $F(z_1, \overline{z_1}, x') = \sum_{i+j\geq 3; i,j\neq 0} a_{ij}(x') z_1^i \overline{z_1}^j$ with $a_{ij}(x') = \overline{a_{ji}(x')}$.

We will seek a real analytic function $f(z_1, u, x')$ which is holomorphic in z_1 with $f(z_1, u, x) = \sum_{l=1}^{\infty} f_l(z_1, x') u^l = o(\|z\|^2)$ such that

$$\text{(II.6)} \qquad u - \mathrm{Re}F_0(z_1, x') + \mathrm{Re}f(z_1, u, x') = |z_1|^2.$$

Therefore, if we set $z_1^* = z_1$, $z_\alpha^* = z_\alpha$ and $z_n^* = z_n - F_0(z_1, z') + f(z_1, z_n, z')$, then the holomorphic transformation $z \to z^*$ maps M into H_0. Substituting (II.5) into (II.6), we see that to obtain f, it suffices to solve the functional equation:

$$\mathrm{Im}(if(z_1, u, x')) = -F(z_1, \overline{z_1}, x'),$$

where u is given in (II.5). Applying Theorem II.A, we see the proof of the Corollary. $\square$

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