

## LOCAL RANKIN-SELBERG CONVOLUTIONS FOR $GL_n$ : EXPLICIT CONDUCTOR FORMULA

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In this paper,  $F$  denotes a non-Archimedean local field with finite residue field of  $q$  elements. The discrete valuation ring of  $F$  is denoted  $\mathfrak{o}$ , and  $\psi$  is a non-trivial, continuous character of the additive group of  $F$ . We write  $\mathfrak{p}$  for the maximal ideal of  $\mathfrak{o}$  and  $c(\psi)$  for the largest integer  $c$  such that  $\mathfrak{p}^{-c} \subset \text{Ker } \psi$ .

For  $i = 1, 2$ , let  $n_i$  be a positive integer, write  $G_i = GL_{n_i}(F)$ , and let  $\pi_i$  be an irreducible smooth representation of  $G_i$ . To the data  $\pi_1, \pi_2$  and  $\psi$ , Jacquet, Piatetskii-Shapiro and Shalika [14] attach an  $L$ -function  $L(\pi_1 \times \pi_2, s)$  and a local constant  $\varepsilon(\pi_1 \times \pi_2, s, \psi)$ , where  $s$  denotes a complex variable. The alternative approach of Shahidi [19] places these objects in a more general context; either way, they are absolutely central to the study of the local Langlands Conjecture [12].

The  $L$ -function has the form  $L(\pi_1 \times \pi_2, s) = P(q^{-s})^{-1}$ , where  $P(X) \in \mathbb{C}[X]$  satisfies  $P(0) = 1$ . On the other hand,

$$\varepsilon(\pi_1 \times \pi_2, s, \psi) = \varepsilon(\pi_1 \times \pi_2, 0, \psi) q^{-f(\pi_1 \times \pi_2, \psi)s},$$

for some integer  $f(\pi_1 \times \pi_2, \psi)$ . In fact,  $f(\pi_1 \times \pi_2, \psi) = n_1 n_2 c(\psi) + f(\pi_1 \times \pi_2)$ , where  $f(\pi_1 \times \pi_2)$  is independent of  $\psi$ .

There is a full description of the function  $L(\pi_1 \times \pi_2, s)$  in [14] which, at the same time, reduces the study of the local constant to the case where both  $\pi_i$  are supercuspidal. The aim of this paper is to give an *explicit formula for  $f(\pi_1 \times \pi_2)$  when the  $\pi_i$  are supercuspidal*.

This formula (Theorem 6.5 below) contains substantial arithmetic information about the representations  $\pi_i$  and their relationship to each other, in terms of their description as induced representations [5]. As one of the consequences of this formula, we obtain sharp upper and lower bounds for  $f(\pi_1 \times \pi_2)$ , as we shall now explain.

For this purpose, we need to recall the local constant  $\varepsilon(\pi, s, \psi)$  of a supercuspidal representation  $\pi$  of  $GL_n(F)$ , in the sense of Godement and Jacquet [10]. This takes the form

$$\varepsilon(\pi, s, \psi) = \varepsilon(\pi, 0, \psi) q^{-f(\pi, \psi)s},$$

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where  $f(\pi, \psi) = nc(\psi) + f(\pi)$ , for an integer  $f(\pi)$ . One already has an explicit formula for  $f(\pi)$  of the appropriate kind [2] or (6.1.2) below. This shows  $f(\pi) \geq 0$ . In fact, the case  $f(\pi) = 0$  can only arise when  $n = 1$  and  $\pi$  is an unramified quasicharacter of  $F^\times \cong \mathrm{GL}_1(F)$ ; otherwise, we have  $f(\pi) \geq n$ . We say that  $\pi$  has level zero if  $f(\pi) \leq n$ . (An alternative characterization of such representations is given in 6.1 below.)

Another useful fact, derived from the Whittaker model description [13] of  $f(\pi)$ , relates the Godement-Jacquet conductor  $f(\pi)$  to the one above:

$$f(\pi) = f(\pi \times 1_F),$$

where  $1_F$  denotes the trivial representation of  $\mathrm{GL}_1(F)$ .

Let  $\pi_i$  be an irreducible supercuspidal representation of  $G_i$ , as above, and let  $t(\pi_1, \pi_2)$  be the number of unramified quasicharacters  $\chi$  of  $F^\times$  such that  $\pi_2 \otimes \chi \circ \det$  is equivalent to the contragredient  $\check{\pi}_1$  of  $\pi_1$ . In the case  $t(\pi_1, \pi_2) = 0$ , we get

$$n_1 n_2 \leq f(\pi_1 \times \pi_2) \leq n_1 n_2 \max \{f(\pi_1)/n_1, f(\pi_2)/n_2\}.$$

Both bounds are best possible. Indeed, the stated upper bound is the ‘‘expected value’’ for randomly chosen  $\pi_i$ . One can specify precisely when the bound is achieved (6.5 again); for example, it is achieved if  $f(\pi_1)/n_1 \neq f(\pi_2)/n_2$ . The analysis of the general case relies on the fact that one can compare the representations  $\pi_i$  via their inducing data, even when  $n_1 \neq n_2$ : this matter is discussed in detail in [3], §8. The non-negative integer

$$n_1 n_2 \max \{f(\pi_1)/n_1, f(\pi_2)/n_2\} - f(\pi_1 \times \pi_2)$$

may reasonably be thought of as a measure of the closeness of  $\pi_2$  to  $\check{\pi}_1$ . On the other hand, the lower bound on  $f(\pi_1 \times \pi_2)$  is achieved when the  $\pi_i$  both have level zero, and essentially only then.

When  $t = t(\pi_1, \pi_2) > 0$ , we have  $n_1 = n_2 \geq t$ ,  $f(\pi_1) = f(\pi_2)$  and, excluding the trivial case where  $n_1 = n_2 = 1$  and  $f(\pi_1) = f(\pi_2) = 0$ , we obtain

$$n_1^2 - n_1 \leq f(\pi_1 \times \pi_2) \leq n_1 f(\pi_1) - t.$$

Again, these bounds are sharp, being simultaneously attained when  $\pi_1$  has level zero.

As an immediate consequence of the bounds for supercuspidal representations, we get

$$f(\pi_1 \times \pi_2) \geq 0$$

for all irreducible smooth (not necessarily supercuspidal) representations  $\pi_i$ . This bound is achieved when the  $\pi_i$  are unramified principal series representations, and essentially only then (Corollary 6.5). Sharp upper bounds for  $f(\pi_1 \times \pi_2)$  in the general case are given in [4].

It is worthy of note that, in the supercuspidal case, the connection between the conductor  $f(\pi_1 \times \pi_2)$  and the arithmetic inducing data for the  $\pi_i$  is quite close; if one fixes  $(\pi_1, n_1)$ , one can often recover a large portion of the inducing data for  $\pi_1$  from knowledge of  $f(\pi_1 \times \pi_2)$  for certain  $(\pi_2, n_2)$  with  $n_2$  dividing  $n_1$  properly. Viewed in a different light, this indicates that, under a Langlands correspondence, the ramification structure of an irreducible Galois representation is intimately connected with the arithmetic structure of the inducing data for the corresponding supercuspidal representation of  $\mathrm{GL}_n$ .

Our starting point is not the original definition given in [14] of the objects  $L(\pi_1 \times \pi_2, s)$ ,  $\varepsilon(\pi_1 \times \pi_2, s, \psi)$ , but rather the alternative and more general formulation of Shahidi [17], [18], [19] which relates  $L$ -functions and local constants to the theory of Plancherel measure and intertwining operators. When applied to  $\mathrm{GL}_n$  in [18], this approach yields a formula relating  $L$ -functions and the conductor  $f(\check{\pi}_1 \times \pi_2)$  to a certain composition of intertwining operators between representations of  $G = \mathrm{GL}_{n_1+n_2}(F)$  parabolically induced from the Levi subgroup  $G_1 \times G_2$ . However, to extract concrete information, one has to be able to compute the composition. The general theory of types [6], and particularly the explicit existence theorems of [7], provide an effective method of doing this. The connection between conductors and the semisimple types [7] used to compute them is very close indeed; this suggests it will be difficult to avoid the use of types in situations like this.

The intertwining operators in question may be thought of as functions on a certain complex algebraic torus. They are in the first instance defined by integrals convergent only on some analytic open set, but it is known that they admit analytic continuation to meromorphic (in fact rational) functions on the whole torus. Our first observation (1.3) is that the analytically continued intertwining operator can be characterized algebraically, in a manner susceptible to analysis via the theory of types. For  $\mathrm{GL}_n$ , this analysis breaks into two cases, distinguished by the triviality or otherwise of an associated  $L$ -function. In the first case, the existence of the relevant type allows one to compute the desired composition directly from the algebraic description of the operators. This method seems quite general and potentially of wide application; we have therefore given it a rather axiomatic treatment (2.4 below). The case with a non-trivial  $L$ -function is more involved. We have first to relate our algebraic description of the analytically continued intertwining operators to functorial structures attached to certain categories of representations (§3). Then we show (§4 below) that the intertwining operators can be transported across the Hecke algebra isomorphisms of [5]. This reduces the problem of computing them to a very special case in which, as it happens, no computation is necessary.

These reductions, combined with results from [18], give us  $q^{f(\check{\pi}_1 \times \pi_2)}$  expressed as a quotient of (suitably normalized) volumes, one of a compact open subgroup of  $G$  and the other of a compact open subgroup of the Levi subgroup  $G_1 \times G_2$  (Theorems 5.3, 5.4 below). These groups are described explicitly in [7] in terms of the inducing data for the  $\pi_i$ ; we compute the quotient of volumes in §6, using the machinery of [5].

### 1. ANALYTIC CONTINUATION OF INTERTWINING OPERATORS

This section is completely general; therefore,  $G$  here denotes the group of  $F$ -rational points of some connected reductive algebraic group defined over  $F$ . We fix a Levi subgroup  $L$  of  $G$  and a pair  $(P_u, P_\ell)$  of mutually opposite parabolic subgroups of  $G$  with Levi component  $L$ . We write  $N_u, N_\ell$ , respectively, for their unipotent radicals and fix Haar measures  $\mu_u, \mu_\ell$  on  $N_u, N_\ell$ , respectively. We will only actually use the case where the Levi subgroup  $L$  is maximal, but the extra generality costs nothing.

We use the symbol  $\iota$  to denote the functor of *normalized* parabolic induction [8], 3.1, and we abbreviate  $\iota_u = \iota_{P_u}^G$ ; likewise  $\iota_\ell$ . That is, if  $(\sigma, W)$  is a smooth representation of  $L$ , then, e.g.,  $\iota_u(W)$  will be the space  $\mathcal{F}_u(\sigma)$  of smooth functions

$f : G \rightarrow W$  such that

$$f(nmx) = \delta_u^{1/2} \sigma(m) f(x), \quad n \in N_u, \quad m \in L, \quad x \in G,$$

where  $\delta_u$  is the module of the action of  $L$  on  $N_u$ , i.e.,  $\delta_{P_u}$  in the notation of [8]; the action of  $G$  on  $\mathcal{F}_u(\sigma)$  via  $\iota_u(\sigma)$  is that of right translation.

The functor  $\iota$  has a well-known adjoint, namely, normalized Jacquet restriction which we denote here by  $\rho$ . Thus, given smooth representations  $(\pi, V)$ ,  $(\sigma, W)$  of  $G$  and  $L$  respectively, we have an isomorphism

$$(1.0.1) \quad \text{Hom}_G(V, \iota_u(W)) \cong \text{Hom}_L(\rho_u(V), W),$$

which is natural in both  $V$  and  $W$ .

**1.1.** We write  ${}^\circ L$  for the subgroup of  $L$  generated by all compact subgroups of  $L$ . Then  ${}^\circ L$  is open and normal in  $L$  and we have  $L/{}^\circ L \cong \mathbb{Z}^r$ , for some integer  $r \geq 0$  (which is in fact the rank of a maximal  $F$ -split torus in the centre of  $L$ ). We set  $X(L) = \text{Hom}(L/{}^\circ L, \mathbb{C}^\times)$ , although we tend to regard the elements of this group as smooth homomorphisms  $L \rightarrow \mathbb{C}^\times$ . Thus  $X(L)$  is, in a natural way, a complex algebraic torus.

In this subsection we fix an *irreducible* smooth representation  $(\sigma, W)$  of  $L$  and consider the set of representations  $(\sigma_\chi, W)$  where  $\chi$  varies over  $X(L)$ . We will be especially interested in the family of operators

$$A = A_{\ell_u}(\sigma_\chi, \mu_u) : \mathcal{F}_\ell(\sigma_\chi) \longrightarrow \mathcal{F}_u(\sigma_\chi)$$

defined initially by the formula

$$(1.1.1) \quad Af(g) = \int_{N_u} f(ng) d\mu_u(n), \quad f \in \mathcal{F}_\ell(\sigma_\chi),$$

for those characters  $\chi$  for which this integral converges.

For such  $\chi$ , it is clear that  $A \in \text{Hom}_G(\mathcal{F}_\ell(\sigma_\chi), \mathcal{F}_u(\sigma_\chi))$ . Further, there is a certain sense in which the operator  $A_{\ell_u}(\sigma_\chi, \mu_u)$  varies analytically with  $\chi$ ; this will be explained in 1.3 below. Granting this, one then has (see [17], §2 and Theorem 2.2.2):

**Proposition.** *Let  $(\sigma, W)$  be an irreducible, unitary, smooth representation of  $L$ .*

- (i) *The integral (1.1.1) converges, to a holomorphic function, on a non-empty analytic open set  $\mathcal{D}$  in  $X(L)$ .*
- (ii) *The function  $\chi \mapsto A_{\ell_u}(\sigma_\chi, \mu_u)$  admits analytic continuation to a meromorphic function on the torus  $X(L)$ . This function is holomorphic on a non-empty Zariski-open subset  $Y_\sigma$  of  $X(L)$ .*

It is well known that, with  $\sigma$  as in the Proposition, the set of  $\chi$  for which  $\iota_\ell(\sigma_\chi)$  is irreducible contains a non-empty Zariski-open set (cf. [1], 2.11). This is undoubtedly true for general irreducible  $\sigma$ ; indeed, a result of Waldspurger (see Sauvageot [16], Th. 3.2) shows immediately that the set in question is non-empty and the method of “algebraic families” [1] can then be used to prove it is Zariski-open. (We are indebted to an anonymous referee for this comment.) In any case, the result is easy for all representations  $\sigma$  considered here. To avoid a longer digression, we shall simply assume:

**Assumption.** *There exists a non-empty Zariski-open set  $Y'_\sigma$  of  $X(L)$  such that  $\iota_\ell(\sigma_\chi)$  and  $\iota_u(\sigma_\chi)$  are both irreducible for  $\chi \in Y'_\sigma$ .*

In the foregoing, we can interchange the roles of  $P_u$  and  $P_\ell$  to define an operator  $A_{u\ell}(\sigma\chi, \mu_\ell)$ , relative to a choice of Haar measure  $\mu_\ell$  on  $N_\ell$ . Composing, we get

$$(1.1.2) \quad A_{u\ell}(\sigma\chi, \mu_\ell) \circ A_{\ell u}(\sigma\chi, \mu_u) = \Phi(\sigma\chi, \mu_u \otimes \mu_\ell) 1, \quad \chi \in Y'_\sigma,$$

where 1 is the identity operator on  $\mathcal{F}_\ell(\sigma\chi)$  and  $\Phi$  is a scalar-valued function which is defined and holomorphic on some non-empty Zariski-open set. In fact,  $\Phi$  turns out to be a rational function of  $\chi$  which is of substantial arithmetic interest in some generality [17], [18], [19]. In that context the choice of measure  $\mu_u \otimes \mu_\ell$  is significant: see, for example, §5 below.

Our aim, over the next few sections, is to compute the function  $\Phi$  in some particularly important cases. To accomplish this, it will be necessary to give an algebraic characterization of the operators  $A_{u\ell}(\sigma\chi, \mu_\ell)$  which holds in the set  $Y'_\sigma$ . We turn to this now.

**1.2.** This subsection is quite general, so we take  $(\sigma, W)$  to be a smooth representation of  $L$ . We define a canonical map

$$(1.2.1) \quad \mathbf{e}_\sigma = \mathbf{e}_\sigma(\mu_u) : W \longrightarrow \rho_u \iota_\ell(W).$$

To do this, we choose a compact open subgroup  $K_u$  of  $N_u$  and for  $w \in W$ , we define a function  $f_w = f_{w, K_u} \in \mathcal{F}_\ell(\sigma)$  as follows:  $f_w$  is to have support  $P_\ell K_u$  and

$$(1.2.2) \quad f_w(xk) = \mu_u(K_u)^{-1} \sigma(x) \delta_\ell(x)^{1/2} w, \quad x \in P_\ell, k \in K_u.$$

It is obvious that  $f_w \in \mathcal{F}_\ell(\sigma)$ . We then define  $\mathbf{e}_\sigma(w)$  to be the image of  $f_w$  in  $\rho_u \iota_\ell(W)$ .

**Proposition.** *The above definition of  $\mathbf{e}_\sigma$  is independent of the choice of  $K_u$ . Moreover,  $\mathbf{e}_\sigma$  is an injective  $L$ -homomorphism  $W \rightarrow \rho_u \iota_\ell(W)$ .*

*Proof.* Let  $K'$  denote some other compact open subgroup of  $N_u$ , and write  $f'_w = f_{w, K'}$ . We can find a compact open subgroup  $K''$  of  $N_u$  containing both  $K_u$  and  $K'$ ; we write  $f''_w = f_{w, K''}$ . By symmetry, it is enough to show that  $f_w, f''_w$  have the same image in the Jacquet module. However,

$$f''_w(g) = (K'' : K_u)^{-1} \sum_{x \in K_u \setminus K''} f_w(gx)$$

for all  $g \in G$ , so these two functions have the same image, as required.

One checks that  $\mathbf{e}_\sigma$  is an  $L$ -homomorphism. If  $f_w$  has zero image in the Jacquet module, there is a compact open subgroup  $K$  of  $N_u$  such that

$$f_w^K(g) = \int_K f_w(gk) d\mu_u(k) = 0,$$

for all  $g \in G$ . This property is inherited by any compact open subgroup of  $N_u$  containing  $K$ , so we may assume that  $K \supset K_u$ . However, we then have

$$f_w^K(1) = \mu_u(K_u) f_w(1) = w.$$

Hence  $w = 0$ , as required.

The map  $\mathbf{e}_\sigma(\mu_u)$  does depend on the choice of Haar measure  $\mu_u$  on  $N_u$ : indeed, we have the relation

$$\mathbf{e}_\sigma(c\mu_u) = c^{-1} \mathbf{e}_\sigma(\mu_u), \quad c > 0.$$

**1.3.** We return to the context of 1.1; in particular,  $(\sigma, W)$  is unitary, irreducible, and subject to Assumption 1.1. We now recall what it means for the operator-valued function  $A_{\ell\mathfrak{u}}(\sigma\chi, \mu_{\mathfrak{u}})$  to be analytic in  $\chi$  [11], [17], [20].

We fix a minimal  $F$ -Levi subgroup  $L_0 \subset L$  and a special maximal compact subgroup  $K_0$  of  $L_0$  such that  $P_0K_0 = G$ , for every parabolic subgroup  $P_0$  of  $G$  with Levi component  $L_0$ . In particular, we have  $G = P_{\ell}K_0 = P_{\mathfrak{u}}K_0$ ; thus a function  $f \in \mathcal{F}_{\ell}(\sigma\chi)$  is determined by its restriction  $f|_{K_0}$  to  $K_0$ . Moreover, the space

$$\mathcal{F}_{\ell}|_{K_0} = \{f|_{K_0} : f \in \mathcal{F}_{\ell}(\sigma\chi)\}$$

of  $W$ -valued functions on  $K_0$  is *independent of  $\chi$* , and the restriction map  $\mathcal{F}_{\ell}(\sigma\chi) \rightarrow \mathcal{F}_{\ell}|_{K_0}$  is a  $K_0$ -isomorphism. The same remarks apply to  $\mathcal{F}_{\mathfrak{u}}$ . Thus, whenever  $A_{\ell\mathfrak{u}}(\sigma\chi)$  is defined, it gives us a  $K_0$ -homomorphism  $\mathcal{F}_{\ell}|_{K_0} \rightarrow \mathcal{F}_{\mathfrak{u}}|_{K_0}$ . Now let  $K$  be some open subgroup of  $K_0$ . The space  $\mathcal{F}_{\ell}^K|_{K_0}$  of  $K$ -fixed vectors in  $\mathcal{F}_{\ell}|_{K_0}$  is then finite-dimensional, and  $A_{\ell\mathfrak{u}}(\sigma\chi)$  gives a linear map

$$A_{\ell\mathfrak{u}}^K(\sigma\chi) \in \text{Hom}_{\mathbb{C}}(\mathcal{F}_{\ell}^K|_{K_0}, \mathcal{F}_{\mathfrak{u}}^K|_{K_0}).$$

To say that the function  $\chi \mapsto A_{\ell\mathfrak{u}}(\sigma\chi)$  is meromorphic then means that, for every open subgroup  $K$  of  $K_0$ , the function  $\chi \mapsto A_{\ell\mathfrak{u}}^K(\sigma\chi)$  is meromorphic in the usual sense.

**Theorem.** *Let  $(\sigma, W)$  be an irreducible smooth unitary representation of  $L$  satisfying the Assumption of 1.1. Let  $Y_{\sigma}$  be a Zariski-open set in  $X(L)$  on which  $A_{\ell\mathfrak{u}}(\sigma\chi, \mu_{\mathfrak{u}})$  is holomorphic. Let  $K_{\mathfrak{u}}$  be some compact open subgroup of  $N_{\mathfrak{u}}$ , and set  $f_w = f_{w, K_{\mathfrak{u}}} \in \mathcal{F}_{\ell}(\sigma\chi)$ ,  $w \in W$ . We have*

$$(1.3.1) \quad A_{\ell\mathfrak{u}}(\sigma\chi, \mu_{\mathfrak{u}})f_w(1) = w,$$

for every  $\chi \in Y_{\sigma}$  and every  $w \in W$ .

*Proof.* The assertion is independent of the choice of  $K_{\mathfrak{u}}$  used in the definition of the  $f_w$ . The identity (1.3.1) holds when  $\chi$  lies in the domain  $\mathcal{D}$  of convergence of the integral (1.1.1), by a straightforward computation. For general  $\chi \in Y_{\sigma}$  we proceed as follows. We recall (cf. [1], 3.5) that  $K_0$  admits arbitrarily small open normal subgroups  $K$  with the property

$$K = K \cap N_{\ell} \cdot K \cap L \cdot K \cap N_{\mathfrak{u}}.$$

In particular, given  $w$  we can choose  $K$  so that  $K \cap L$  fixes  $w$ .

The process  $f \mapsto f(1)$  gives a holomorphic map  $\mathcal{F}_{\mathfrak{u}}^K \rightarrow W^{K \cap L}$ . Thus  $\chi \mapsto A_{\ell\mathfrak{u}}(\sigma\chi)f_w(1)$  is a meromorphic function on  $X(L)$ ; it takes the constant value  $w$  on  $\mathcal{D}$  and hence on the whole of  $Y_{\sigma}$ .

By the Proposition and Assumption of 1.1, there is a non-empty Zariski-open set  $Z_{\sigma}$  of  $X(L)$  on which  $A_{\ell\mathfrak{u}}(\sigma\chi)$  is defined and  $\iota_{\ell}(\sigma\chi)$ ,  $\iota_{\mathfrak{u}}(\sigma\chi)$  are irreducible. We return to the isomorphism (1.0.1),

$$\text{Hom}_G(\iota_{\ell}(\sigma\chi), \iota_{\mathfrak{u}}(\sigma\chi)) \longrightarrow \text{Hom}_L(\rho_{\mathfrak{u}}\iota_{\ell}(\sigma\chi), \sigma\chi),$$

and denote it  $f \mapsto \bar{f}$ . The identity (1.3.1) then implies

$$(1.3.2) \quad \bar{A}_{\ell\mathfrak{u}}(\sigma\chi, \mu_{\mathfrak{u}}) \circ \mathbf{e}_{\sigma\chi}(\mu_{\mathfrak{u}}) = 1_W, \quad \chi \in Z_{\sigma}.$$

**Proposition.** *The identity (1.3.2) determines  $A_{\ell\mathfrak{u}}(\sigma\chi)$  uniquely. That is, if  $\chi \in Z_{\sigma}$  and  $f \in \text{Hom}_G(\iota_{\ell}(\sigma\chi), \iota_{\mathfrak{u}}(\sigma\chi))$  satisfies  $\bar{f} \circ \mathbf{e}_{\sigma\chi} = 1_W$ , then  $f = A_{\ell\mathfrak{u}}(\sigma\chi)$ .*

It is this simple property on which some cases of our analysis of the intertwining operators  $A$  will be based.

2. TYPES AND COVERS

We continue in the general situation of §1, and recall some basic concepts from [1] and [6]. This will enable us to compute the composite of intertwining operators (1.1.2) in one family of cases. We use the notation of [6], to which we refer for further details.

**2.1.** Let  $\mathfrak{R}(G)$  denote the category of smooth complex representations of  $G$ . We consider the set of pairs  $(L, \pi)$ , where  $L$  is an  $F$ -Levi subgroup of  $G$  and  $\pi$  is an irreducible supercuspidal representation of  $L$ , modulo the relation of *inertial equivalence*: two such pairs  $(L_i, \pi_i)$  are inertially equivalent (in  $G$ ) if  $(L_2, \pi_2)$  is  $G$ -conjugate to  $(L_1, \pi_1 \otimes \chi)$ , for some  $\chi \in X(L_1)$ . We write  $\mathcal{B}(G)$  for the set of inertial equivalence classes of these pairs.

An irreducible smooth representation  $\sigma$  of  $G$  determines an element of  $\mathcal{B}(G)$  as follows. There is a parabolic subgroup  $P$  of  $G$  and an irreducible supercuspidal representation  $\pi$  of a Levi component  $L$  of  $P$  such that  $\sigma$  is equivalent to a composition factor of  $\iota_P^G(\pi)$ . The inertial equivalence class of  $(L, \pi)$  is thereby uniquely determined, and called the *inertial support* of  $\sigma$ .

Fixing  $\mathfrak{s} \in \mathcal{B}(G)$ , we define a full sub-category  $\mathfrak{R}^{\mathfrak{s}}(G)$  of  $\mathfrak{R}(G)$  by demanding that its objects be those smooth representations of  $G$  whose irreducible sub-quotients all have inertial support  $\mathfrak{s}$ . The category  $\mathfrak{R}(G)$  is then the direct product of the sub-categories  $\mathfrak{R}^{\mathfrak{s}}(G)$ ,  $\mathfrak{s} \in \mathcal{B}(G)$ . (In the language of [1], the representation  $\pi$  defines an orbit  $D_\pi = \{\pi \otimes \chi : \chi \in X(L)\}$ , and  $\mathfrak{R}^{\mathfrak{s}}(G)$  is the category denoted  $(\text{Alg } G)(L, D_\pi)$  in [1].)

**2.2.** Now fix a Haar measure on  $G$ . Let  $K$  be a compact open subgroup of  $G$  and  $(\tau, U)$  an irreducible smooth representation of  $K$ . We denote the contragredient of  $(\tau, U)$  by  $(\check{\tau}, U^\vee)$ . We write  $\mathcal{H}(G, \tau)$  for the convolution algebra of  $\text{End}_{\mathbb{C}}(U^\vee)$ -valued functions  $f$  on  $G$  which are compactly supported and which satisfy

$$f(hxk) = \check{\tau}(h) f(x) \check{\tau}(k), \quad h, k \in K, x \in G.$$

Then, for  $(\pi, V) \in \mathfrak{R}(G)$ , the space

$$V_\tau = \text{Hom}_K(U, V)$$

has a canonical left  $\mathcal{H}(G, \tau)$ -module structure [6], §2. Indeed,  $V \mapsto V_\tau$  gives a functor from  $\mathfrak{R}(G)$  to  $\mathcal{H}(G, \tau)\text{-Mod}$ .

Fix  $\mathfrak{s} \in \mathcal{B}(G)$ . The pair  $(K, \tau)$  is an  *$\mathfrak{s}$ -type in  $G$*  if the irreducible representations of  $G$  which contain  $\tau$  are exactly those with inertial support  $\mathfrak{s}$ . We then have [6], §4: *Let  $\mathfrak{s} \in \mathcal{B}(G)$  and let  $(K, \tau)$  be an  $\mathfrak{s}$ -type in  $G$ . A representation  $(\pi, V) \in \mathfrak{R}(G)$  lies in  $\mathfrak{R}^{\mathfrak{s}}(G)$  if and only if  $V$  is generated over  $G$  by its  $\tau$ -isotypic space  $V^\tau$ . The functor*

$$\begin{aligned} \mathfrak{R}^{\mathfrak{s}}(G) &\longrightarrow \mathcal{H}(G, \tau)\text{-Mod}, \\ (\pi, V) &\longmapsto V_\tau, \end{aligned}$$

*is an equivalence of categories.*

**2.3.** We recall some basic constructions described in [6], §8. Suppose now that  $L$  is a Levi subgroup of  $G$ , and we are given an element  $\mathfrak{t} \in \mathcal{B}(L)$ ; this is the  $L$ -inertial equivalence class of some pair  $(M, \pi)$ . The  $G$ -inertial equivalence class of  $(M, \pi)$  is then an element  $\mathfrak{s} \in \mathcal{B}(G)$ . We suppose given a  $\mathfrak{t}$ -type  $(K_L, \tau_L)$  in  $L$ . In particular,  $\mathfrak{R}^{\mathfrak{t}}(L)$  is canonically equivalent to  $\mathcal{H}(L, \tau_L)\text{-Mod}$ . Let  $(K, \tau)$  be

a  $G$ -cover of  $(K_L, \tau_L)$ , as defined in [6], 8.1. In particular,  $K$  is a compact open subgroup of  $G$  and  $\tau$  is an irreducible smooth representation of  $G$ . The pair  $(K, \tau)$  has certain properties, some of which we now recall:

- (i) Let  $P_u$  be a parabolic subgroup of  $G$  with Levi decomposition  $P_u = LN_u$  and opposite  $P_\ell = LN_\ell$ . Then

$$K = K \cap N_\ell \cdot K \cap L \cdot K \cap N_u \quad \text{and} \quad K \cap L = K_L.$$

The representation  $\tau$  is trivial on  $K \cap N_\ell$  and  $K \cap N_u$ , while  $\tau|_{K_L} \cong \tau_L$ .

- (ii) The pair  $(K, \tau)$  is an  $\mathfrak{s}$ -type in  $G$ .
- (iii) There is a canonical algebra homomorphism

$$\mathbf{j}_u : \mathcal{H}(L, \tau_L) \longrightarrow \mathcal{H}(G, \tau)$$

which realizes the induction functor  $\iota_u = \iota_{P_u}^G$ . That is, the diagram

$$(2.3.1) \quad \begin{array}{ccc} \mathfrak{R}^t(L) & \xrightarrow{\iota_u} & \mathfrak{R}(G) \\ \approx \downarrow & & \downarrow \approx \\ \mathcal{H}(L, \tau_L)\text{-Mod} & \xrightarrow{(\mathbf{j}_u)_*} & \mathcal{H}(G, \tau)\text{-Mod} \end{array}$$

commutes, where

$$(\mathbf{j}_u)_* : M \longmapsto \text{Hom}_{\mathcal{H}(L, \tau_L)}(\mathcal{H}(G, \tau), M).$$

In the definition of the module-theoretic induction functor  $(\mathbf{j}_u)_*$ , we view  $\mathcal{H}(G, \tau)$  as a left  $\mathcal{H}(L, \tau_L)$ -module via  $\mathbf{j}_u$ ; the space  $\text{Hom}_{\mathcal{H}(L, \tau_L)}(\mathcal{H}(G, \tau), M)$  becomes a left  $\mathcal{H}(G, \tau)$ -module via the natural right action of  $\mathcal{H}(G, \tau)$  on itself.

There is another useful relation here. The normalized Jacquet functor  $\rho_u$  gives a functor

$$\mathfrak{R}^s(G) \longrightarrow \mathfrak{R}(L) \longrightarrow \mathfrak{R}^t(L),$$

where the second arrow is the canonical projection. We continue to denote this  $\rho_u$ . We get another commutative diagram

$$(2.3.2) \quad \begin{array}{ccc} \mathfrak{R}^s(G) & \xrightarrow{\rho_u} & \mathfrak{R}^t(L) \\ \approx \downarrow & & \downarrow \approx \\ \mathcal{H}(G, \tau)\text{-Mod} & \xrightarrow{\mathbf{j}_u^*} & \mathcal{H}(L, \tau_L)\text{-Mod}, \end{array}$$

where  $\mathbf{j}_u^*$  denotes restriction along  $\mathbf{j}_u$ .

**2.4.** We need a new concept.

**Definition.** Let  $\mathfrak{t} \in \mathcal{B}(L)$ , let  $(K_L, \tau_L)$  be a  $\mathfrak{t}$ -type in  $L$  and  $(K, \tau)$  a  $G$ -cover of  $(K_L, \tau_L)$ . We say that  $(K, \tau)$  is a split cover if, for every choice of parabolic subgroup  $P_u$  with Levi  $L$ , the map  $\mathbf{j}_u$  is an isomorphism of algebras which preserves support of functions:

$$\text{supp}(\mathbf{j}_u(f)) = K \cdot \text{supp} f \cdot K, \quad f \in \mathcal{H}(L, \tau_L).$$

In the presence of a split cover, it is very easy to compute the composition of intertwining operators (1.1.2). We now do this, using the notation of §1.

**Proposition.** *Let  $L$  be an  $F$ -Levi subgroup of  $G$ , let  $\mathfrak{t} \in \mathcal{B}(L)$ , and let  $(K_L, \tau_L)$  be a  $\mathfrak{t}$ -type in  $L$ . Let  $P_{\mathfrak{u}}$  be a parabolic subgroup of  $G$  with Levi decomposition  $P_{\mathfrak{u}} = LN_{\mathfrak{u}}$  and opposite  $P_{\ell} = LN_{\ell}$ . Suppose there exists a split  $G$ -cover  $(K, \tau)$  of  $(K_L, \tau_L)$ . Then:*

- (i) *Let  $\sigma \in \mathfrak{R}^{\mathfrak{t}}(L)$  be irreducible. The induced representation  $\iota_{\mathfrak{u}}(\sigma\chi)$  is irreducible, for every  $\chi \in \mathfrak{X}(L)$ .*
- (ii) *Let  $\mu_{\mathfrak{u}}, \mu_{\ell}$  be the Haar measures on  $N_{\mathfrak{u}}, N_{\ell}$ , respectively, such that  $\mu_{\mathfrak{u}}(K \cap N_{\mathfrak{u}}) = \mu_{\ell}(K \cap N_{\ell}) = 1$ . Then*

$$A_{\mathfrak{u}\ell}(\sigma\chi, \mu_{\ell}) \circ A_{\ell\mathfrak{u}}(\sigma\chi, \mu_{\mathfrak{u}}) = 1,$$

*for every  $\chi \in \mathfrak{X}(L)$ .*

*Proof.* Assertion (i) follows from (2.3.1) and the fact that  $\mathfrak{j}_{\mathfrak{u}}$  is an isomorphism. Thus Assumption 1.1 is satisfied in this case.

As for (ii), we will need the following lemma. This is independent of  $\chi$ , so we drop it from the notation. Write  $W$  for the representation space of  $\sigma$ .

**Lemma.** *For  $w \in W$ , write  $f_w^{\ell} = f_{w, K \cap N_{\mathfrak{u}}}$ , in the notation of §1. The  $\tau$ -isotypic subspace  $\mathcal{F}_{\ell}(\sigma)^{\tau}$  of  $\mathcal{F}_{\ell}(\sigma)$  then consists of the functions  $f_w^{\ell}$ , for  $w$  ranging over  $W^{\tau_L}$ . Moreover, if  $f \in \mathcal{F}_{\ell}(\sigma)^{\tau}$  satisfies  $f(1) = w$ , then  $f = f_w^{\ell}$ .*

*Proof.* The first step is to show that the functions  $f_w^{\ell}$ , for  $w \in W^{\tau_L}$ , lie in  $\mathcal{F}_{\ell}(\sigma)^{\tau}$ . To say that  $w \in W$  lies in  $W^{\tau_L}$  means that there is a  $K_L$ -homomorphism  $\varphi : U \rightarrow W$  with  $w$  in the image of  $\varphi$ ,  $w = \varphi(u)$ , say. One checks that, given such a  $\varphi$ , the map  $u \mapsto f_{\varphi(u)}^{\ell}$  is a  $K_L$ -homomorphism  $\varphi_{\ell} : U \rightarrow \mathcal{F}_{\ell}(\sigma)$ . The function  $f_{\varphi(u)}^{\ell}$  is fixed by the groups  $\sigma(K \cap N_{\ell}), \sigma(K \cap N_{\mathfrak{u}})$ , so  $\varphi_{\ell}$  is in fact a  $K$ -homomorphism  $U \rightarrow \mathcal{F}_{\ell}(\sigma)$  (by condition (i) in 2.3). Thus  $f_w^{\ell} = f_{\varphi(u)}^{\ell} \in \mathcal{F}_{\ell}(\sigma)^{\tau}$ , as desired.

Under the canonical map

$$\mathcal{F}_{\ell}(\sigma) \rightarrow \rho_{\ell}\mathcal{F}_{\ell}(\sigma) \rightarrow \sigma,$$

we have  $f_w^{\ell} \mapsto f_w^{\ell}(1) = w$ . On the other hand, the canonical map  $\mathcal{F}_{\ell}(\sigma) \rightarrow \rho_{\ell}\mathcal{F}_{\ell}(\sigma)$  induces an isomorphism  $\mathcal{F}_{\ell}(\sigma)^{\tau} \cong (\rho_{\ell}\mathcal{F}_{\ell}(\sigma))^{\tau_L}$  [6], 7.9. However, since  $(K_L, \tau_L)$  is a  $\mathfrak{t}$ -type, the  $L$ -subspace of  $\rho_{\ell}\mathcal{F}_{\ell}(\sigma)$  generated by its  $\tau_L$ -isotypic space is the canonical projection of  $\rho_{\ell}\mathcal{F}_{\ell}(\sigma)$  into  $\mathfrak{R}^{\mathfrak{t}}(L)$ . Since  $(K, \tau)$  is a split cover, this projection is just  $\sigma$  (2.3.2). Put another way, the  $\tau_L$ -isotypic subspace of the Jacquet module is just  $W^{\tau_L}$ . The lemma follows.

Returning to the proof of the proposition, let  $\chi$  be such that  $A = A_{\ell\mathfrak{u}}(\sigma\chi)$  is defined. In particular,  $A$  is a  $G$ -homomorphism and so maps  $\mathcal{F}_{\ell}(\sigma\chi)^{\tau}$  to  $\mathcal{F}_{\mathfrak{u}}(\sigma\chi)^{\tau}$ . If we take  $w \in W^{\tau_L}$ , the function  $Af_w^{\ell}$  therefore lies in  $\mathcal{F}_{\mathfrak{u}}(\sigma\chi)^{\tau}$ ; it satisfies  $Af_w^{\ell}(1) = w$ , by 1.3. By the lemma, interchanging the roles of  $P_{\ell}$  and  $P_{\mathfrak{u}}$ , we have  $A_{\ell\mathfrak{u}}(\sigma\chi)f_w^{\ell} = f_w^{\mathfrak{u}}$ .

By symmetry, we get  $A_{\mathfrak{u}\ell}(\sigma\chi)f_w^{\mathfrak{u}} = f_w^{\ell}$ . Thus  $A_{\mathfrak{u}\ell}(\sigma\chi) \circ A_{\ell\mathfrak{u}}(\sigma\chi)$  is the identity wherever it is defined, i.e., at least on a dense open set. The result now follows.

*Comment.* The choice of Haar measures in the proposition is made for convenience when working with covers. In arithmetic applications, the choice of measure is usually dictated by external considerations. In this light, the point of the proposition is that the composite of intertwining operators depends on the (normalized) volumes of the unipotent factors of the split  $G$ -cover attached to the representation  $\sigma$ .

## 3. ADJOINT RELATIONS

This section is concerned with the relation between the functors  $\iota$ ,  $\rho$  and operations of contragredience. Its purpose is to give another description of the map  $\mathbf{e}_\sigma$  of 1.2 in certain situations.

**3.1.** Taking  $G$ ,  $P_\ell = LN_\ell$ , etc., as before, let  $X$  denote the representation space of  $\iota_\ell(\delta_\ell^{1/2})$ . Thus  $X$  consists of the smooth functions  $f : G \rightarrow \mathbb{C}$  which satisfy

$$f(nmx) = \delta_\ell(m)f(x), \quad n \in N_\ell, m \in L, x \in G.$$

Since  $P_\ell \backslash G$  is compact, the space  $X$  admits a positive  $G$ -invariant functional  $I$ , unique up to positive scale; see, for example, [8], 2.4.2.

The space  $C_c^\infty(N_u)$  of compactly supported smooth functions on  $N_u$  embeds in  $X$ , and the restriction of  $I$  to  $C_c^\infty(N_u)$  is a Haar measure  $\mu_u$  on  $N_u$ . Conversely, a Haar measure  $\mu_u$  on  $N_u$  determines a functional  $I$ . We write  $I = I(\mu_u)$  to emphasize this dependence when necessary.

**3.2.** Let  $(\sigma_i, W_i)$ ,  $i = 1, 2$ , be smooth representations of  $L$ , and suppose we have a non-degenerate,  $L$ -invariant bilinear pairing

$$\langle \cdot, \cdot \rangle : W_1 \times W_2 \longrightarrow \mathbb{C}.$$

We use the notation  $(\iota_\ell(\sigma_i), \iota_\ell(W_i))$  for the induced representation. Let  $f_i \in \iota_\ell(W_i)$ . The function

$$x : g \longmapsto \langle f_1(g), f_2(g) \rangle, \quad g \in G,$$

lies in  $X$  and one checks (cf. [8], 2.4.2) that the pairing

$$\langle f_1, f_2 \rangle_\iota = \langle f_1, f_2 \rangle_{\iota, \mu_u} = I(\mu_u)(x)$$

is a non-degenerate  $G$ -invariant pairing of  $\iota_\ell(W_1)$  with  $\iota_\ell(W_2)$ . In particular, if we take  $(\sigma_1, W_1) = (\sigma, W)$ ,  $(\sigma_2, W_2) = (\check{\sigma}, W^\vee)$ , and for  $\langle \cdot, \cdot \rangle$  the canonical pairing, then the associated pairing  $\langle \cdot, \cdot \rangle_\iota$  induces an isomorphism

$$(3.2.1) \quad \iota_\ell(W^\vee) \cong (\iota_\ell(W))^\vee.$$

One checks that this isomorphism is natural in  $W$ .

**3.3.** We recall a parallel result. We fix *admissible* representations  $(\pi_i, V_i)$ ,  $i = 1, 2$ , of  $G$  and suppose we are given a non-degenerate  $G$ -invariant pairing  $\langle \cdot, \cdot \rangle$  between  $V_1$  and  $V_2$ . This pairing determines a non-degenerate  $L$ -invariant pairing  $\langle \cdot, \cdot \rangle_\rho$  of  $\rho_u(V_1)$  with  $\rho_\ell(V_2)$  characterized as follows [8], 4.2.5. As in [8], 1.3, we fix a minimal parabolic subgroup  $P_0$  of  $G$  contained in  $P_u$ ; this gives us a semigroup  $A^-(\epsilon)$  contained in the centre of  $L$ , for any real  $\epsilon$ ,  $0 < \epsilon \leq 1$ . The pairing  $\langle \cdot, \cdot \rangle_\rho$  is then determined uniquely by the following property: *Let  $v_i \in V_i$  and write  $\bar{v}_1, \bar{v}_2$  for the images of  $v_1, v_2$  in  $\rho_u(V_1), \rho_\ell(V_2)$ , respectively. There exists  $\epsilon > 0$  (depending on the  $v_i$ ) such that*

$$(3.3.1) \quad \langle \pi_1(a)v_1, v_2 \rangle = \delta_u^{1/2}(a) \langle \rho_u \pi_1(a) \bar{v}_1, \bar{v}_2 \rangle_\rho, \quad a \in A^-(\epsilon).$$

If  $(\pi, V)$  is an admissible representation of  $G$ , we hence obtain an isomorphism

$$(3.3.2) \quad \rho_\ell(V^\vee) \cong \rho_u(V)^\vee,$$

which, as a further consequence of (3.3.1), is natural in  $V$ .

- Remarks.* (i) The factor  $\delta_u^{1/2}$  on the right hand side of (3.3.1) is explained by the fact that we are using *normalized* Jacquet restriction.  
(ii) The isomorphism (3.3.1) holds for *smooth* representations  $(\pi, V)$ , but the proof in [8] of the existence of the pairing  $\langle \cdot, \cdot \rangle_\rho$  must be modified. We shall not explore this here.

**3.4.** We now consider the effect of the isomorphisms (3.2.1) and (3.3.2) on the adjoint pair  $(\iota_\ell, \rho_\ell)$ . We work in the categories  $\mathcal{A}(G), \mathcal{A}(L)$  of admissible representations of  $G, L$ , respectively; it is useful to note in this context that the functors  $\iota_\ell, \rho_\ell, \iota_u, \rho_u$  all take admissible representations to admissible representations.

**Proposition.** *View  $\rho_u, \rho_\ell$  as functors from  $\mathcal{A}(G)$  to  $\mathcal{A}(L)$ , and  $\iota_\ell, \iota_u$  as functors from  $\mathcal{A}(L)$  to  $\mathcal{A}(G)$ . Then  $(\iota_\ell, \rho_u)$  is an adjoint pair.*

*Proof.* Given admissible representations  $(\pi, V)$  of  $G$  and  $(\sigma, W)$  of  $L$ , we need to produce an isomorphism

$$\eta = \eta(W, V) : \text{Hom}_G(\iota_\ell(W), V) \cong \text{Hom}_L(W, \rho_u(V))$$

in such a way that  $\eta(W, V)$  is natural in both  $W$  and  $V$ . We proceed as follows. Let  $(\pi_i, V_i), i = 1, 2$ , be smooth representations of  $G$ ; for  $\phi \in \text{Hom}_G(V_1, V_2^\vee)$ , define  $\check{\phi} \in \text{Hom}_G(V_2, V_1^\vee)$  by  $\check{\phi}(v_2) : v_1 \mapsto \phi(v_1)(v_2)$ . The map  $\phi \mapsto \check{\phi}$  is an isomorphism  $\text{Hom}_G(V_1, V_2^\vee) \cong \text{Hom}_G(V_2, V_1^\vee)$ , natural in  $V_1$  and  $V_2$ . If the representations  $V_i$  are admissible, the canonical embedding  $V_i \rightarrow V_i^{\vee\vee}$  is an isomorphism. It follows that we have a natural isomorphism

$$(3.4.1) \quad \text{Hom}_G(V_1, V_2) \cong \text{Hom}_G(V_2^\vee, V_1^\vee),$$

which we continue to denote  $\phi \mapsto \check{\phi}$ . Combining this with (3.2.1), we have a natural isomorphism

$$(3.4.2) \quad \text{Hom}_G(\iota_\ell(W), V) \cong \text{Hom}_G(V^\vee, \iota_\ell(W^\vee)).$$

On the other hand, if we combine (3.4.1) with (3.3.2) we obtain a natural isomorphism

$$(3.4.3) \quad \text{Hom}_L(W, \rho_u(V)) \cong \text{Hom}_L(\rho_\ell(V^\vee), W^\vee).$$

Since  $(\rho_\ell, \iota_\ell)$  is an adjoint pair (1.0.1), there is a natural isomorphism of the right hand side of (3.4.2) with that of (3.4.3); our result now follows.

We observe that the map  $\eta$  constructed in the last proof depends on the Haar measure  $\mu_u$ .

**3.5.** We come to the critical observation. Given a smooth representation  $(\sigma, W)$  of  $L$ , we have the natural map

$$\begin{aligned} d(\sigma) : \rho_\ell(\iota_\ell(W)) &\longrightarrow W, \\ \rho_\ell(f) &\longmapsto f(1), \quad f \in \iota_\ell(W). \end{aligned}$$

If  $(\sigma, W)$  is admissible, then we may consider the map

$$d(\sigma)^\vee : W^\vee \rightarrow (\rho_\ell(\iota_\ell(W)))^\vee$$

and, using the isomorphisms of Proposition 3.4, we obtain a map

$$d(\sigma, \mu_u)^\vee : W^\vee \longrightarrow \rho_u(\iota_\ell(W^\vee)).$$

We now prove:

**Proposition.** *Let  $(\sigma, W)$  be an admissible representation of  $L$  and fix a Haar measure  $\mu_u$  on  $N_u$ . Then we have*

$$\mathbf{e}_\sigma(\mu_u) = d(\check{\sigma}, \mu_u)^\vee,$$

that is, in the notation of 3.4,  $\mathbf{e}_\sigma \in \text{Hom}_L(W, \rho_u \iota_\ell(W))$  corresponds to the identity map  $1 \in \text{Hom}_L(W, W)$  under the map  $\eta(W, \iota_\ell(W))$ .

*Proof.* Let  $\langle \cdot, \cdot \rangle_W$  be the canonical pairing of  $W$  with  $W^\vee$ . Set  $\langle \cdot, \cdot \rangle = (\langle \cdot, \cdot \rangle_W)_\iota$  (notation of 3.2) and put  $\mathbf{e}(w) = \mathbf{e}_\sigma(\mu_u)(w)$ . A straightforward diagram chase shows that the assertion  $\mathbf{e}_\sigma(\mu_u) = d(\check{\sigma}, \mu_u)^\vee$  is equivalent to saying that for all  $w \in W$ ,  $F \in \iota_\ell(W^\vee)$  we have

$$\langle \mathbf{e}(w), \overline{F} \rangle_\rho = F(1)(w),$$

where  $\overline{F}$  is the image of  $F$  in  $\rho_\ell \iota_\ell(W^\vee)$ . This is what we will now prove.

To this end, we fix  $w, F$  as above and choose a compact open subgroup  $K$  of  $G$  such that

$$K = K \cap N_\ell \cdot K \cap L \cdot K \cap N_u,$$

with the further properties that  $K \cap L$  fixes  $w$  and  $K \cap N_u$  fixes  $F$ . This is always possible, as in 1.3 above. Set  $(\pi_1, V_1) = \iota_\ell(\sigma, W)$ ,  $(\pi_2, V_2) = \iota_u(\sigma, W)$  and abbreviate  $K_u = K \cap N_u$ . Then since  $V_1^K$  is finite-dimensional, we may pick  $\epsilon > 0$  so that (3.3.1) holds for  $v_2 = F$  and all  $v_1 \in V_1^K$ . If  $a \in A^-(\epsilon)$ , then  $a$  lies in the centre of  $L$  and we have that  $\sigma(a^{-1})w$  is fixed by  $K \cap L$ . It then follows that  $f_{\sigma(a^{-1})w, K_u}$  (notation of §1) lies in  $V_1^K$ . Since  $\mathbf{e}$  is an  $L$ -map, we have

$$\delta_u^{1/2}(a) \langle \mathbf{e}(w), \rho_\ell(F) \rangle_\rho = \langle \pi_1(a) f_{\sigma(a^{-1})w, K_u}, F \rangle.$$

On the other hand, a direct computation shows that

$$\pi_1(a) f_{\sigma(a^{-1})w, K_u} = \delta_u^{1/2}(a) f_{w, aK_u a^{-1}}$$

and by [8], 1.4.3, we may pick  $\epsilon'$ ,  $0 < \epsilon' \leq \epsilon$ , so that  $aK_u a^{-1} \subset K_u$ ,  $a \in A^-(\epsilon')$ . Since  $F$  is invariant under  $K_u$  we see that

$$\langle \pi_1(a) f_{\sigma(a^{-1})w, K_u}, F \rangle = \delta_u^{1/2}(a) F(1)w, \quad a \in A^-(\epsilon'),$$

whence our result.

#### 4. GENERAL LINEAR GROUPS

**4.1.** We now specialize to the case  $G = \text{GL}_n(F)$ , for some  $n \geq 1$ . The group  $X(G)$  then consists of the homomorphisms

$$\chi_s : g \mapsto \|\det g\|^s, \quad g \in G,$$

where  $s \in \mathbb{C}$ . Clearly,  $\chi_s$  only depends on  $q^{-s}$ , where  $q$  is the size of the residue field of  $F$ . We use the notation

$$\pi[s] = \pi \otimes \chi_s, \quad s \in \mathbb{C},$$

where  $\pi$  is an irreducible smooth representation of  $G$ .

We now assume  $n \geq 2$  and fix a maximal proper Levi subgroup  $L$  of  $G$ . We can identify  $L$  with  $\text{GL}_{n_1}(F) \times \text{GL}_{n_2}(F)$ , for positive integers  $n_1, n_2$  such that  $n_1 + n_2 = n$ . We let  $P_u = LN_u$ ,  $P_\ell = LN_\ell$  denote the two parabolic subgroups of  $G$  with Levi component  $L$ . We abbreviate  $G_i = \text{GL}_{n_i}(F)$ ,  $A_i = \mathbb{M}_{n_i}(F)$ .

Let  $\sigma$  denote an irreducible *supercuspidal* representation of  $L$ ; thus  $\sigma = \sigma_1 \otimes \sigma_2$ , where  $\sigma_i$  is an irreducible supercuspidal representation of  $G_i$ ,  $i = 1, 2$ . We return to the problem of computing the composite of intertwining operators (1.1.2) in this particular case.

**Proposition.** *Let  $\sigma = \sigma_1 \otimes \sigma_2$  be an irreducible supercuspidal representation of  $L$ . Let  $\mathfrak{t} \in \mathcal{B}(L)$  be the inertial support of  $\sigma$ . There exists a  $\mathfrak{t}$ -type  $(K_L, \tau_L)$  in  $L$ . The type  $(K_L, \tau_L)$  may be chosen to admit a  $G$ -cover  $(K, \tau)$ . Moreover,*

- (i) *if  $n_1 = n_2$  and  $\sigma_2 \cong \sigma_1[s]$  for some  $s \in \mathbb{C}$ , the cover  $(K, \tau)$  is not split;*
- (ii) *in all other cases,  $(K, \tau)$  may be chosen split.*

*Proof.* We take  $K_L = J_1 \times J_2$ ,  $\tau_L = \lambda_1 \otimes \lambda_2$ , where  $(J_i, \lambda_i)$  is a maximal simple type (see [5], 6.2) occurring in  $\sigma_i$ ; the existence of such is guaranteed by *ibid.* 8.4.1. The existence of the covers is given by [7], 1.5; that the cover is split in case (ii) follows from [7], 8.2. In case (i), on the other hand, it is well known that the representation  $\iota_u(\sigma_1 \otimes \sigma_1[1])$  is reducible; the cover cannot therefore be split (*cf.* (2.3.1)).

*Remark.* The last proposition is only concerned with the existence of the various covers. However, later on it will be essential to know the explicit construction of these as in [7].

**4.2.** In case (ii) of Proposition 4.1, the composition of intertwining operators has already been computed in 2.4. For the remaining case 4.1(i), we temporarily choose Haar measures  $\mu_u, \mu_\ell$  on  $N_u, N_\ell$  satisfying

$$(4.2.1) \quad \mu_u(K \cap N_u) = \mu_\ell(K \cap N_\ell) = 1,$$

where  $(K, \tau)$  is the cover provided by 4.1. As in (1.1.2), we have a function  $\Phi_F(\sigma_1, q^{-s_1}, q^{-s_2})$  such that

$$(4.2.2) \quad \Phi_F(\sigma_1, q^{-s_1}, q^{-s_2}) 1 = A_{u\ell}(\sigma_1[s_1] \otimes \sigma_1[s_2], \mu_\ell) \circ A_{\ell u}(\sigma_1[s_1] \otimes \sigma_1[s_2], \mu_u).$$

To compute this function, we need some fundamental concepts from [5]. First, we have to describe the maximal simple type  $(J_1, \lambda_1)$  appearing in the representation  $\sigma_1$ . Note that, since  $\sigma_2$  is an unramified twist of  $\sigma_1$ , we can take  $J_2 = J_1$ ,  $\lambda_2 = \lambda_1$ . Suppose first that  $\sigma_1$  is not of level zero, i.e., does not admit a fixed vector for the first principal congruence subgroup of a maximal compact subgroup of  $G_1$ . There is then a simple stratum  $[\mathfrak{A}_1, n_1, 0, \beta]$  in  $A_1$  underlying  $\lambda_1$ . If  $\sigma_1$  has level zero, we set  $\beta = 0$ ,  $\mathfrak{A}_1 = \mathbb{M}_{n_1}(\mathfrak{o})$ . In either case, we write  $E$  for the field  $F[\beta]$ .

We next define an integer  $f$  by  $[E:F]f = n_1$ ; there exists an unramified field extension  $\mathbf{k}/E$  inside  $A_1$ , of degree  $f$ , such that  $\mathbf{k}^\times$  normalizes  $\mathfrak{A}_1$ . We view  $\mathbf{k}^\times$  as embedded in  $L = G_1 \times G_2 \subset G$  on the diagonal; we let  $H$  denote the  $G$ -centralizer of  $\mathbf{k}^\times$  and  $M = L \cap H$ . Thus  $H \cong \mathrm{GL}_2(\mathbf{k})$  and  $M \cong \mathbf{k}^\times \times \mathbf{k}^\times$ . Let  $1_{\mathbf{k}}$  denote the trivial character of  $\mathbf{k}^\times$ .

The data  $(H, M, 1_{\mathbf{k}})$  can be treated in exactly the same way as  $(G, L, \sigma_1)$  to define a function  $\Phi_{\mathbf{k}}(1_{\mathbf{k}}, q_{\mathbf{k}}^{-s_1}, q_{\mathbf{k}}^{-s_2})$ , where  $q_{\mathbf{k}}$  is the size of the residue field of  $\mathbf{k}$ . In particular,  $Q_u = P_u \cap H$ ,  $Q_\ell = P_\ell \cap H$  are the two parabolic subgroups of  $H$  with Levi  $M$ . The type in  $M$  corresponding to  $(K_L, \tau_L)$  is the trivial character of the maximal compact subgroup of  $M$ ; the cover corresponding to  $(K, \tau)$  is the trivial character of the standard Iwahori subgroup  $\mathcal{I}$  of  $\mathrm{GL}_2(\mathbf{k})$ . To define  $\Phi_{\mathbf{k}}$ , we choose Haar measures so that  $\mathcal{I} \cap N_u, \mathcal{I} \cap N_\ell$  have measure 1.

We now give the important reduction step in the calculation of  $\Phi_F$ .

**Theorem.** *Let  $\sigma_1$  be an irreducible supercuspidal representation of  $\mathrm{GL}_{n_1}(F)$ , and define  $\Phi_F(\sigma_1, q^{-s_1}, q^{-s_2})$  by (4.2.2). Using the notation above, we have*

$$\Phi_F(\sigma_1, q^{-s_1}, q^{-s_2}) = \Phi_{\mathbf{k}}(1_{\mathbf{k}}, q_{\mathbf{k}}^{-s_1}, q_{\mathbf{k}}^{-s_2}).$$

*Proof.* We use the results of [5], Ch. 7, so we need some information concerning the cover  $(K, \tau)$  provided by Proposition 4.1 in the case  $\sigma_2 = \sigma_1$ . This is straightforward: in the language of [5], there is a simple type  $(J, \lambda)$  with  $J \supset K$  such that  $\lambda$  is induced by  $\tau$ . (Indeed, the constructions of [7] at this point coincide with those of [5], 7.1, 7.2. See [7], sections 1 and 7, for more detail. In fact,  $\tau$  is the representation denoted  $\lambda_P$  in [5].) In particular,  $\mathcal{H}(G, \tau) \cong \mathcal{H}(G, \lambda)$ , and  $\mathcal{H}(G, \lambda)$  is an affine Hecke algebra [5], 5.6.6.

Let  $\mathcal{A}_\tau(G)$  denote the category of admissible representations of  $G$  which are generated by their  $\tau$ -isotypic spaces; thus  $\mathcal{A}_\tau(G)$  is the sub-category of  $\mathfrak{R}^s(G)$  consisting of admissible representations. (We confine attention to admissible representations here simply to avoid some irrelevant technical complications later in the proof.) In particular,  $\mathcal{A}_\tau(G)$  is canonically equivalent to the category  $\mathcal{H}(G, \tau)\text{-Mod}_f$  of  $\mathcal{H}(G, \tau)$ -modules of finite complex dimension [5], 7.5.7, or [6], 4.3 (just as in 2.2). We similarly define  $\mathcal{A}_{\tau_L}(L)$ . Again, this is equivalent to  $\mathcal{H}(L, \tau_L)\text{-Mod}_f$ .

Here, the normalized induction functor  $\iota_u$  takes  $\mathcal{A}_{\tau_L}(L)$  to  $\mathcal{A}_\tau(G)$ . Let  $\mathbf{j}_u : \mathcal{H}(L, \tau_L) \rightarrow \mathcal{H}(G, \tau)$  be the algebra homomorphism which realizes  $\iota_u$  (see 2.3).

There is an exactly parallel set-up for the data  $(H, M, \mathbf{1}_k)$ . We again denote by  $\mathbf{j}_u$  the algebra map  $\mathcal{H}(M, 1_{M \cap I}) \rightarrow \mathcal{H}(H, 1_I)$  which realizes the induction functor  $\iota_u = \iota_{Q_u}^H$  from  $\mathcal{A}_{1_{M \cap I}}(M)$  to  $\mathcal{A}_{1_I}(H)$ .

We similarly define maps  $\mathbf{j}_\ell$  in the two cases.

There is a unique algebra isomorphism

$$\Psi_L : \mathcal{H}(M, 1_{M \cap I}) \xrightarrow{\cong} \mathcal{H}(L, \tau_L)$$

such that, under the corresponding equivalence  $\Psi_L^* : \mathcal{A}_{\tau_L}(L) \cong \mathcal{A}_{1_{M \cap I}}(M)$ , the representation  $\sigma_\chi$  corresponds to  $\chi \upharpoonright M$  (cf. [5], 7.5.12). Now we appeal to [5], 7.6.20: there is a canonical algebra isomorphism

$$\Psi_G : \mathcal{H}(H, 1_I) \xrightarrow{\cong} \mathcal{H}(G, \tau)$$

such that the diagram

$$(4.2.3) \quad \begin{array}{ccc} \mathcal{H}(M, 1_{M \cap I}) & \xrightarrow{\Psi_L} & \mathcal{H}(L, \tau_L) \\ \mathbf{j}_u \downarrow & & \downarrow \mathbf{j}_u \\ \mathcal{H}(H, 1_I) & \xrightarrow{\Psi_G} & \mathcal{H}(G, \tau) \end{array}$$

commutes. (The map  $\Psi_G$  is determined uniquely by this and certain subsidiary properties, recalled in the proof of the next lemma below.) Put another way,  $\Psi_G$  gives a natural isomorphism

$$\iota_u(\Psi_L^*(\pi)) \cong \Psi_G^*(\iota_u(\pi)),$$

for  $\pi \in \mathcal{A}_{\tau_L}(L)$ , where  $\Psi_G^*$  is the equivalence  $\mathcal{A}_\tau(G) \cong \mathcal{A}_{1_I}(H)$  induced by  $\Psi_G$ . Likewise, we have a natural isomorphism

$$(4.2.4) \quad \rho_u(\Psi_G^*(\eta)) \cong \Psi_L^*(\rho_u(\eta)), \quad \eta \in \mathcal{A}_\tau(G).$$

However, by 3.4, the functor  $\rho_u$  has a co-adjoint, namely the induction functor  $\iota_\ell$ . A co-adjoint is uniquely determined up to natural equivalence, so we have a natural isomorphism

$$(4.2.5) \quad \iota_\ell(\Psi_L^*(\pi)) \cong \Psi_G^*(\iota_\ell(\pi)), \quad \pi \in \mathcal{A}_{\tau_L}(L),$$

and a similar relation involving the Jacquet functor  $\rho_\ell$ .

**Lemma.** *The diagram*

$$\begin{array}{ccc} \mathcal{H}(M, 1_{M \cap \mathcal{I}}) & \xrightarrow{\Psi_L} & \mathcal{H}(L, \tau_L) \\ \mathbf{j}_\ell \downarrow & & \downarrow \mathbf{j}_\ell \\ \mathcal{H}(H, 1_{\mathcal{I}}) & \xrightarrow{\Psi_G} & \mathcal{H}(G, \tau) \end{array}$$

*commutes.*

*Proof.* The algebra homomorphisms  $\Psi_L, \Psi_G$  preserve support of functions in the following sense. For example, take  $f \in \mathcal{H}(H, 1_{\mathcal{I}})$ ; the function  $\Psi_G(f)$  then has support  $K \cdot \text{supp}(f) \cdot K$ . (This property, the choice of  $\Psi_L$  and the diagram (4.2.3) together determine  $\Psi_G$  uniquely.)

The maps  $\mathbf{j}_\ell$  have a similar property; to explain this, take the case of the map  $\mathbf{j}_\ell$  attached to  $G$ . The algebra  $\mathcal{H}(L, \tau_L)$  has a sub-algebra  $\mathcal{H}_\ell(L, \tau_L)$  of functions supported on elements of  $L$  which are “ $K \cap N_\ell$ -positive”, in the sense of [6], §6. On these functions,  $\mathbf{j}_\ell$  is given by an explicit formula and, for  $f \in \mathcal{H}_\ell(L, \tau_L)$ ,  $\mathbf{j}_\ell f$  has support  $K \cdot \text{supp } f \cdot K$ . (See [6], §7 for details.)

Thus, if we take  $f \in \mathcal{H}_\ell(M, 1_{\mathcal{I} \cap M})$ , the functions  $\mathbf{j}_\ell \Psi_L(f), \Psi_G \mathbf{j}_\ell(f)$  both have support  $K \cdot \text{supp}(f) \cdot K$ . A  $(K, K)$ -double coset in  $G$  supports at most a one-dimensional space of functions in  $\mathcal{H}(G, \tau)$  ([5], 5.6.6) and the other three algebras in the picture have the analogous property. If this function  $f$  is supported on, say  $m(M \cap \mathcal{I})$ , we thus have a non-zero scalar  $\phi(m)$  such that

$$\mathbf{j}_\ell \Psi_L(f) = \phi(m) \Psi_G \mathbf{j}_\ell(f).$$

The quantity  $\phi(m)$  depends only on  $m(M \cap \mathcal{I})$ ; by [6], 7.1,  $\phi$  is multiplicative in the positive element  $m$ . In other words,  $\phi$  extends (uniquely) to an unramified quasicharacter of  $M$ .

Given an unramified quasicharacter  $\chi$  of  $M$  and  $f \in \mathcal{H}(M, 1_{\mathcal{I} \cap M})$ , define  $f\chi$  by  $f\chi : m \mapsto f(m)\chi(m)$ . We have just shown that

$$\mathbf{j}_\ell \Psi_L(f) = \Psi_G \mathbf{j}_\ell(f\phi).$$

The relation (4.2.5) then forces  $\phi = 1$ .

By 3.5, we now have

$$\mathbf{e}_{\chi|M} = \Psi_L^*(\mathbf{e}_{\sigma\chi}), \quad \chi \in X(L).$$

By Proposition 1.3, this gives us

$$A_{\ell u}(\chi | M) = \Psi_G^*(A_{\ell u}(\sigma\chi)).$$

However, by the lemma, we can interchange the roles of  $P_u$  and  $P_\ell$  without changing  $\Psi_G$ ; therefore

$$A_{u\ell}(\chi | M) = \Psi_G^*(A_{u\ell}(\sigma\chi)).$$

The theorem now follows.

## 5. THE CONDUCTOR-VOLUME RELATION

We continue in the situation of §4. Thus  $G = \mathrm{GL}_n(F)$  and  $L$  is a maximal proper Levi subgroup of  $G$ :  $L = G_1 \times G_2$ ,  $G_i = \mathrm{GL}_{n_i}(F)$ . The parabolic subgroups  $P_u$ ,  $P_\ell$  are as before.

We take a non-trivial character  $\psi$  of the additive group of  $F$  with conductor  $c(\psi)$ : we recall that, by definition,  $c(\psi)$  is the largest integer  $c$  such that  $\mathfrak{p}^{-c} \subset \mathrm{Ker} \psi$ .

**5.1.** *There is a Haar measure  $\tilde{\mu}_L^\psi = \tilde{\mu}_u^\psi \otimes \tilde{\mu}_\ell^\psi$  on  $N_u \times N_\ell$  with the following property. For  $i = 1, 2$ , let  $\pi_i$  be an irreducible, generic, unitary representation of  $G_i$ . Then*

$$\begin{aligned} A_{u\ell}(\pi_1 \left[ \frac{s}{2} \right] \otimes \pi_2 \left[ \frac{-s}{2} \right], \tilde{\mu}_\ell^\psi) \circ A_{\ell u}(\pi_1 \left[ \frac{-s}{2} \right] \otimes \pi_2 \left[ \frac{s}{2} \right], \tilde{\mu}_u^\psi) \\ = q^{-f(\check{\pi}_1 \times \pi_2, \psi)} \frac{L(\pi_1 \times \check{\pi}_2, s)L(\check{\pi}_1 \times \pi_2, -s)}{L(\pi_1 \times \check{\pi}_2, 1+s)L(\check{\pi}_1 \times \pi_2, 1-s)} 1. \end{aligned}$$

For this, see [18], Introduction and Theorem 6.1 (or the more general and systematic account in [19]). For the notion of genericity, see [15].

*Comments.* If we replace  $\tilde{\mu}_L^\psi$  by  $k \tilde{\mu}_L^\psi$ , for some  $k > 0$ , the left hand side of 5.1 gets multiplied by  $k$ . On the other hand, we have

$$f(\check{\pi}_1 \times \pi_2, \psi) = f(\check{\pi}_1 \times \pi_2) + n_1 n_2 c(\psi),$$

where  $f(\check{\pi}_1 \times \pi_2)$  is independent of  $\psi$ . Thus, if we change  $\psi$  to  $\psi'$ , the right hand side of 5.1 gets multiplied by  $q^{n_1 n_2 (c(\psi') - c(\psi))}$ . It follows that  $\tilde{\mu}_L^\psi$  depends only on  $L$  and  $c(\psi)$ , and 5.1 is essentially independent of the choice of  $\psi$ .

**5.2.** The next step is to make the choice of measures more explicit. Let us write  $N_u = 1 + \mathfrak{n}_u$ ,  $N_\ell = 1 + \mathfrak{n}_\ell$ , for vector spaces  $\mathfrak{n}_u$ ,  $\mathfrak{n}_\ell$ . In fact,  $\mathfrak{n}_u \cong N_u$  as topological groups, and likewise for  $N_\ell$ . The pairing

$$\begin{aligned} \mathfrak{n}_u \times \mathfrak{n}_\ell &\longrightarrow \mathbb{C}, \\ (x, y) &\longmapsto \psi(\mathrm{tr}(xy)), \end{aligned}$$

identifies  $\mathfrak{n}_\ell$  with the Pontrjagin dual  $\hat{\mathfrak{n}}_u$  of  $\mathfrak{n}_u$ , so we have an identification of  $\widehat{N}_u$  with  $N_\ell$ . We choose a Haar measure  $\mu_u^\psi$  on  $N_u$  at random, and set

$$(5.2.1) \quad \mu_L^\psi = \mu_u^\psi \otimes \hat{\mu}_u^\psi,$$

where  $\hat{\mu}_u^\psi$  is the measure on  $N_\ell$  dual to  $\mu_u^\psi$ . This is independent of the original choice of  $\mu_u^\psi$ .

**5.3.** We can now establish the first of our desired relations.

**Theorem.** (i) *We have  $\mu_L^\psi = \tilde{\mu}_L^\psi$ , for all non-trivial characters  $\psi$  of  $F$ .*

(ii) *For  $i = 1, 2$ , let  $\sigma_i$  be an irreducible supercuspidal representation of  $G_i = \mathrm{GL}_{n_i}(F)$ . In the case  $n_1 = n_2$ , assume that  $\sigma_2 \not\cong \sigma_1[s]$  for any  $s \in \mathbb{C}$ . Let  $(K, \tau)$  be as in 4.1. Then*

$$(5.3.1) \quad \mu_L^\psi(K \cap N_u \times K \cap N_\ell) = q^{-f(\check{\sigma}_1 \times \sigma_2, \psi)}.$$

*Comment.* The measure appearing in (5.3.1) admits an illuminating interpretation. The  $G$ -cover  $(K, \tau)$  of  $(K_L, \tau_L)$  is attached to a hereditary  $\mathfrak{o}$ -order  $\mathfrak{A}$  in  $\mathbb{M}_n(F)$ ; in the notation of 4.1, this is the order defined by the direct sum ([7], 2.8) of the

lattice chains underlying the factors  $(J_i, \lambda_i)$  of  $(K_L, \tau_L)$ . The unit group of this order admits a decomposition

$$\mathbf{U}(\mathfrak{A}) = \mathbf{U}(\mathfrak{A}) \cap N_\ell \cdot \mathbf{U}(\mathfrak{A}) \cap L \cdot \mathbf{U}(\mathfrak{A}) \cap N_u.$$

If we take  $c(\psi) = -1$ , we get  $\mu_L^\psi(\mathbf{U}(\mathfrak{A}) \cap N_\ell \times \mathbf{U}(\mathfrak{A}) \cap N_u) = 1$ , so

$$q^{f(\check{\sigma}_1 \times \sigma_2, \psi)} = (\mathbf{U}(\mathfrak{A}) \cap N_\ell : K \cap N_\ell) \cdot (\mathbf{U}(\mathfrak{A}) \cap N_u : K \cap N_u).$$

We now prove Theorem 5.3. First, we define  $a(L, \psi) > 0$  by

$$\tilde{\mu}_L^\psi = a(L, \psi) \mu_L^\psi.$$

Next, twisting the  $\sigma_i$  by unramified quasicharacters has no effect on the conductor, so we may assume that the  $\sigma_i$  are both unitary. They are certainly generic [9], so we can use 5.1.

Under the hypotheses of 5.3(ii), we have

$$L(\check{\sigma}_1 \times \sigma_2, s) = L(\sigma_1 \times \check{\sigma}_2, s) = 1,$$

by [14], Proposition 8.1. Combining 5.1 with Propositions 4.1, 2.4, we get

$$(5.3.2) \quad \mu_L^\psi(K \cap N_u \times K \cap N_\ell) = a(L, \psi) q^{-f(\check{\sigma}_1 \times \sigma_2, \psi)}.$$

We calculate the factor  $a(L, \psi)$  relating the measures  $\mu_L^\psi, \tilde{\mu}_L^\psi$ , via a rather different choice of representation  $\sigma$ . Initially, we let  $\sigma_1$  be any irreducible unitary supercuspidal representation of  $G_1$  except that, when  $n_1 = 1$ , we insist that  $\sigma_1$ , which is a quasicharacter of  $F^\times$ , is not unramified. For  $\sigma_2$ , we take the Steinberg representation  $\text{St}(n_2)$  of  $G_2$ . The representation  $\sigma$  admits a type  $(K_L, \tau_L)$  as follows. We take the maximal simple type  $(J_1, \lambda_1)$  occurring in  $\sigma_1$  (as in 4.1), and let  $\mathcal{I}$  denote the standard Iwahori subgroup of  $G_2$ . We put  $K_L = J_1 \times \mathcal{I}$ ,  $\tau_L = \lambda_1 \otimes 1_{\mathcal{I}}$ , where  $1_{\mathcal{I}}$  is the trivial character of  $\mathcal{I}$ . This admits a split  $G$ -cover  $(K, \tau)$ . Assumption 1.1 is therefore valid in this case and the relation (5.3.2) holds in this situation, with the same value of  $a(L, \psi)$ .

The object of this argument is to prove that  $a(L, \psi) = 1$ , so we can make a convenient choice of  $\sigma_1$ . Consider first the case where  $n_1 > 1$ . Here, we assume that  $\sigma_1$  is of level zero; this means it has a fixed vector for the first principal congruence subgroup of  $\text{GL}_{n_1}(\mathfrak{o})$ . We then have [2]

$$f(\check{\sigma}_1, \psi) = f(\check{\sigma}_1 \times \mathbf{1}_F, \psi) = n_1(1 + c(\psi)).$$

However,

$$f(\check{\sigma}_1 \times \text{St}(n_2), \psi) = n_2 f(\check{\sigma}_1 \times \mathbf{1}_F, \psi),$$

by [14], Theorems 3.1 and 8.2. On the other hand, the constructions in [7] give a group  $K$  of the form

$$K = \begin{pmatrix} \text{GL}_{n_1}(\mathfrak{o}) & \mathfrak{o} \\ \mathfrak{p} & \mathcal{I} \end{pmatrix}.$$

Substituting in 5.3.2, we get  $a(L, \psi) = 1$ , as required.

In the case  $n_1 = 1$ , we simply insist that  $\sigma_1$  is not unramified. The argument is straightforward and similar to the first case, so we omit the details. (Observe that the first argument is also valid when  $n_1 = 1$  and  $q \geq 3$ : we can then take for  $\sigma_1$  a tamely ramified quasicharacter which is not unramified.)

This proves both assertions of the theorem.

**5.4.** We now turn to the other case of 4.1, where  $\sigma_2$  is an unramified twist of  $\sigma_1$ . Since twisting by unramified quasicharacters has no effect on the conductor, we may as well take  $\sigma_2 = \sigma_1$ .

**Theorem.** *Let  $\sigma_1$  be an irreducible supercuspidal representation of  $\mathrm{GL}_{n_1}(F)$ . Let  $t(\sigma_1)$  denote the number of unramified characters  $\chi$  of  $F^\times$  such that  $\sigma_1 \otimes \chi \circ \det \cong \sigma_1$ . Let  $(K, \tau)$  be given by 4.1 (with  $\sigma_2 = \sigma_1$ ), and define  $\mu_L^\psi$  by (5.2.1). Then*

$$\mu_L^\psi(K \cap N_u \times K \cap N_\ell) = q^{-(t(\sigma_1) + f(\check{\sigma}_1 \times \sigma_1, \psi))}.$$

*Proof.* If the assertion holds for one choice of  $\psi$ , it holds for all. To simplify the book-keeping, we take  $c(\psi) = 0$ . It will be convenient to start with the special case  $n_1 = n_2 = 1$  and  $\sigma_1 = 1_F$ , the trivial character of  $F^\times$ . In this case,  $K$  is the standard Iwahori subgroup of  $G = \mathrm{GL}_2(F)$ . Thus

$$(5.4.1) \quad \mu_L^\psi(K \cap N_u \times K \cap N_\ell) = q^{-1}.$$

For unramified quasicharacters  $\chi_1, \chi_2$  of  $F^\times$ , we have

$$f(\chi_1 \times \chi_2) = f(\chi_1 \chi_2 \times 1_F) = f(\chi_1 \chi_2) = 0,$$

where the third  $f$  is the normalized Godement-Jacquet conductor. Thus 5.1 and Theorem 5.3(i) give

$$\begin{aligned} A_{u\ell}(1_F \left[ \frac{s}{2} \right] \otimes 1_F \left[ \frac{-s}{2} \right], \tilde{\mu}_\ell^\psi) \circ A_{\ell u}(1_F \left[ \frac{s}{2} \right] \otimes 1_F \left[ \frac{-s}{2} \right], \tilde{\mu}_u^\psi) \\ = \frac{\zeta_F(s) \zeta_F(-s)}{\zeta_F(1+s) \zeta_F(1-s)} 1, \end{aligned}$$

where  $\zeta_F(s) = (1 - q^{-s})^{-1}$ . Now we return to the notation of 4.2; using  $\mathbf{k}$  as base field in place of  $F$ , the special case just done gives

$$\Phi_{\mathbf{k}}(1_{\mathbf{k}}, q_{\mathbf{k}}^{-s/2}, q_{\mathbf{k}}^{s/2}) = q_{\mathbf{k}} \frac{\zeta_{\mathbf{k}}(s) \zeta_{\mathbf{k}}(-s)}{\zeta_{\mathbf{k}}(1+s) \zeta_{\mathbf{k}}(1-s)}.$$

However, by [14], Proposition 8.1, and [5], 6.2.5, we have  $L(\check{\sigma}_1 \times \sigma_1, s) = \zeta_{\mathbf{k}}(s)$  and  $q_{\mathbf{k}} = q^{t(\sigma_1)}$ . Now we can apply Theorem 4.2 to 5.1 to get the result.

## 6. EXPLICIT CONDUCTOR FORMULÆ

We proceed to compute the volumes occurring in the main theorems of §5. This is done in terms of the description of supercuspidal representations in [5]. We must therefore start by recalling a little of this. If  $E/F$  is a finite field extension, we write  $\mathfrak{o}_E$  for the discrete valuation ring in  $E$ ,  $\mathfrak{p}_E$  for the maximal ideal of  $\mathfrak{o}_E$ , etc.

**6.1.** Let  $\sigma$  be an irreducible supercuspidal representation of  $G = \mathrm{GL}_n(F)$ . Via the description [5] of  $\sigma$  as an induced representation, we attach to  $\sigma$  a principal  $\mathfrak{o}$ -order  $\mathfrak{A}$  in  $A = \mathbb{M}_n(F)$ ; an integer  $m \geq 0$  (called the *level of  $\sigma$* ); and an element  $\beta \in A$  such that  $E = F[\beta]$  is a field with  $E^\times$  normalizing  $\mathfrak{A}$ .

First, suppose that  $\sigma$  admits a fixed vector for the group  $1 + \mathfrak{p}\mathbb{M}_n(\mathfrak{o})$ . In this case, we set

$$\mathfrak{A} = \mathbb{M}_n(\mathfrak{o}), \quad m = 0, \quad \beta = 0.$$

Otherwise, there is a maximal simple type occurring in  $\sigma$  and given by a simple stratum  $[\mathfrak{A}, m, 0, \beta]$  in  $A$  [5], 8.4.1. This defines the required objects.

In all cases,  $e$  denotes the  $\mathfrak{o}$ -period of  $\mathfrak{A}$  or, equivalently, the ramification index  $e(E|F)$ . Let  $t(\sigma)$  denote the number of unramified quasicharacters  $\chi$  of  $F^\times$  such that  $\sigma \otimes \chi \circ \det \cong \sigma$ ; we have [5], 6.2.5,

$$(6.1.1) \quad t(\sigma) = n/e.$$

We also recall [2] that the Godement-Jacquet conductor of  $\sigma$  is given by

$$(6.1.2) \quad f(\sigma, \psi) = \begin{cases} c(\psi) & \text{if } n = 1 \text{ and } \sigma \text{ is unramified,} \\ n(1 + c(\psi) + m/e) & \text{otherwise.} \end{cases}$$

Next, suppose that  $\sigma$  does not have level zero. In this case, the simple stratum  $[\mathfrak{A}, m, 0, \beta]$  has a *characteristic polynomial*  $\phi_0(X) \in \mathfrak{k}[X]$  [5], p. 58, where  $\mathfrak{k} = \mathfrak{o}/\mathfrak{p}$ . This is only defined relative to a choice of prime element of  $F$ : we assume this has been made once and for all. The polynomial  $\phi_0(X)$  is a power of a monic irreducible polynomial  $\phi_\sigma(X) \in \mathfrak{k}[X]$ ,  $\phi_\sigma(X) \neq X$ , and  $\phi_\sigma$  is an invariant of  $\sigma$ .

**6.2.** For  $i = 1, 2$ , let  $n_i$  be a positive integer and set  $G_i = \text{GL}_{n_i}(F)$ . Let  $\sigma_i$  be an irreducible supercuspidal representation of  $G_i$ . We use the notation of 6.1, appending subscripts  $i$  as necessary.

We define integers  $e, m$  by

$$(6.2.1) \quad \begin{aligned} e &= \text{lcm}(e_1, e_2), \\ m/e &= \max(m_1/e_1, m_2/e_2). \end{aligned}$$

We set  $n = n_1 + n_2$ ,  $G = \text{GL}_n(F)$ ,  $A = \mathbb{M}_n(F)$ . We identify  $G_1 \times G_2$  with the obvious Levi subgroup  $L$  of  $G$ . As before,  $P_u = LN_u$ ,  $P_\ell = LN_\ell$  are the parabolic subgroups of  $G$  with Levi component  $L$ . It will be convenient to regard  $L$  as the stabilizer of a decomposition of  $F^n$  of the form  $F^n = V_1 \oplus V_2$ .

Suppose first that the  $\sigma_i$  both have level zero. We say that the  $\sigma_i$  are *completely distinct* if one is not an unramified twist of the other.

On the other hand, suppose that  $\sigma_1$ , say, does not have level zero. We then say that the  $\sigma_i$  are *completely distinct* if either  $m_1/e_1 \neq m_2/e_2$  or  $m_1/e_1 = m_2/e_2$  but  $\phi_{\sigma_1} \neq \phi_{\sigma_2}$ .

Observe that, in any case,  $\sigma_2$  cannot be an unramified twist of  $\sigma_1$  when the  $\sigma_i$  are completely distinct.

**6.3.** We need some more notation in the case where the  $\sigma_i$  are not completely distinct, but do not have level zero. In the language of [7], §8, this means that the  $\sigma_i$  admit a *common approximation*. There is no need to recall the full definition of this concept, only enough to compute conductors.

First, the order  $\mathfrak{A}_i$  is determined by a lattice chain  $\mathcal{L}_i$  in  $V_i$ . We can form the direct sum  $\Lambda = \mathcal{L}_1 \oplus \mathcal{L}_2$ , in the sense of [7], 2.8. This is a lattice *sequence* in  $F^n$ ,  $n = n_1 + n_2$ . In particular,  $\Lambda$  is a function  $i \mapsto \Lambda(i)$  from  $\mathbb{Z}$  to the set of  $\mathfrak{o}$ -lattices in  $F^n$  with the periodicity property  $\Lambda(i+e) = \Lambda(i)$ , where  $e = \text{lcm}(e_1, e_2)$  as above. In this case, we have  $m_1/e_1 = m_2/e_2$ , so  $m = m_i e/e_i$ .

A *common approximation* to the  $\sigma_i$  is then an object  $([A, m, 0, \gamma], l, \vartheta)$  where  $l$  is an integer,  $0 \leq l < m$ , and  $\gamma$  is an element of the Levi  $L$  generating a field over  $F$  which stabilizes the lattice sequence  $\Lambda$ . The entry  $\vartheta$  is a certain character of a compact group  $H^{l+1}(\gamma, \Lambda)$  attached to the other data and which occurs in  $\nu_\ell(\sigma_1 \otimes \sigma_2)$ .

We will only ever consider common approximations which are *best possible* in the following sense: if  $([A, m, 0, \gamma'], l', \vartheta')$  is another common approximation, we have

$l \leq l'$ . In this case, we can further assume that the stratum  $[A, m, l, \gamma]$  is simple [7], 5.1. This means either  $\gamma \in F$  or the integer  $r = -k_0(\gamma, A)$  *loc. cit.* satisfies  $l < r \leq m$ . The quantities  $l$  and  $\gamma$ , which are the only ones to concern us, can then be specified explicitly in terms of the inducing data for the  $\sigma_i$ ; for convenience, we have summarized this process in 6.15 below.

**6.4.** We need a sort of “generalized discriminant”. We start with an elementary observation. Let  $U \subset V$  be finite-dimensional  $F$ -vector spaces, and suppose we have linear maps  $f : V \rightarrow V$ ,  $g : V \rightarrow U$  such that the sequence

$$0 \longrightarrow U \longrightarrow V \xrightarrow{f} V \xrightarrow{g} U \longrightarrow 0$$

is exact. There is then a positive constant  $C(f, g)$ , depending only on the maps  $f$  and  $g$ , such that, for any  $\mathfrak{o}$ -lattices  $\mathfrak{u}, \mathfrak{u}'$  in  $U$ ,  $\mathfrak{v}, \mathfrak{v}'$  in  $V$ , fitting into an exact sequence

$$0 \longrightarrow \mathfrak{u} \longrightarrow \mathfrak{v} \xrightarrow{f} \mathfrak{v}' \xrightarrow{g} \mathfrak{u}' \longrightarrow 0,$$

we have

$$\frac{\mu_U(\mathfrak{u}) \mu_V(\mathfrak{v}')}{\mu_V(\mathfrak{v}) \mu_U(\mathfrak{u}')} = C(f, g),$$

where  $\mu_U, \mu_V$  are Haar measures on  $U, V$  respectively. (See [5], 5.1.3, for a more general result of this kind.) The quantity  $C(f, g)$  is clearly independent of the choice of measures.

We now assume given a finite field extension  $E/F$  and an element  $\gamma \in E^\times$  such that  $E = F[\gamma]$ . Write  $A(E) = \text{End}_F(E)$ . Let  $s_\gamma : A(E) \rightarrow E$  be a tame co-restriction [5], 1.3, and write  $a_\gamma$  for the adjoint map  $x \mapsto \gamma x - x\gamma$ ,  $x \in A(E)$ . We have an exact sequence

$$0 \longrightarrow E \longrightarrow A(E) \xrightarrow{a_\gamma} A(E) \xrightarrow{s_\gamma} E \longrightarrow 0.$$

We set

$$(6.4.1) \quad C(\gamma) = C(a_\gamma, s_\gamma) = q^{c(\gamma)},$$

for some integer  $c(\gamma)$ . This number has an interesting arithmetic interpretation, which we give below in 6.13. At the moment, we simply need a rather coarse estimate.

**Lemma.** *With the notation above, suppose that  $E \neq F$ , that  $n = -\nu_E(\gamma) > 0$ , and that the stratum  $[\mathfrak{A}(E), n, 0, \gamma]$  is simple, where  $\mathfrak{A}(E)$  is the unique hereditary  $\mathfrak{o}$ -order in  $A(E)$  normalized by  $E^\times$ . Then  $c(\gamma) > 0$ .*

*Proof.* We apply the above machinery to the exact sequence of [5], 3.1.16, setting the parameter  $m$  there equal to 0. This gives us an exact sequence

$$0 \longrightarrow \mathfrak{p}_E \longrightarrow \mathfrak{J}^1 \xrightarrow{a_\gamma} (\mathfrak{H}^1)^* \xrightarrow{s_\gamma} \mathfrak{o}_E \longrightarrow 0,$$

for  $\mathfrak{o}$ -lattices  $\mathfrak{J}^1, (\mathfrak{H}^1)^*$  in  $A(E)$  (whose definitions need not detain us) satisfying

$$\mathfrak{J}^1 \subset \mathfrak{P}(E) \subset \mathfrak{A}(E) \subset (\mathfrak{H}^1)^*,$$

where  $\mathfrak{P}(E)$  is the radical of  $\mathfrak{A}(E)$ . Thus

$$((\mathfrak{H}^1)^* : \mathfrak{J}^1) \geq (\mathfrak{A}(E) : \mathfrak{P}(E)) > (\mathfrak{o}_E : \mathfrak{p}_E),$$

as required for the lemma.

In the case  $\gamma \in F$ , we of course get  $c(\gamma) = 0$ .

**6.5.** We can now give the desired conductor formulæ.

**Theorem.** For  $i = 1, 2$ , let  $\sigma_i$  be an irreducible supercuspidal representation of  $\mathrm{GL}_{n_i}(F)$ . Define quantities  $m_i, e_i, \beta_i$  as in 6.2, and use the notation (6.2.1), (6.4.1).

- (i) Suppose that  $n_1 = n_2$  and  $\sigma_2 \cong \sigma_1[s]$ , for some  $s \in \mathbb{C}$ . Put  $d = [F[\beta_1]:F]$ . Then

$$f(\check{\sigma}_1 \times \sigma_2) = n_1^2 \left( 1 + \frac{c(\beta_1)}{d^2} \right) - \frac{n_1}{e_1}.$$

In particular,  $f(\check{\sigma}_1 \times \sigma_2) \geq n_1^2 - n_1$ , with equality if the  $\sigma_i$  have level zero. In the case  $d > 1$ , we also have

$$n_1^2 - n_1 < f(\check{\sigma}_1 \times \sigma_2) < n_1^2 \left( 1 + \frac{m_1}{e_1} \right) - \frac{n_1}{e_1}.$$

- (ii) Suppose that the  $\sigma_i$  are completely distinct. Then

$$f(\check{\sigma}_1 \times \sigma_2) = n_1 n_2 \left( 1 + \frac{m}{e} \right).$$

In particular,  $f(\check{\sigma}_1 \times \sigma_2) \geq n_1 n_2$ , with equality if and only if the  $\sigma_i$  have level zero.

- (iii) Suppose that  $\sigma_2$  is not equivalent to an unramified twist of  $\sigma_1$  but that the  $\sigma_i$  are not completely distinct. Let  $([\Lambda, m, 0, \gamma], l, \vartheta)$  be a best common approximation to the  $\sigma_i$ , and assume that the stratum  $[\Lambda, m, l, \gamma]$  is simple. Put  $d = [F[\gamma]:F]$ . Then

$$f(\check{\sigma}_1 \times \sigma_2) = n_1 n_2 \left( 1 + \frac{c(\gamma)}{d^2} + \frac{l}{de} \right) < n_1 n_2 \left( 1 + \frac{m}{e} \right).$$

If either  $d > 1$  or  $l > 0$ , we also have

$$n_1 n_2 < f(\check{\sigma}_1 \times \sigma_2).$$

We remark that the conductor bounds given in the Introduction can be obtained from those of 6.5 via (6.1.2) and (6.2.1).

**Corollary.** For  $i = 1, 2$ , let  $\pi_i$  be an irreducible smooth representation of the group  $\mathrm{GL}_{n_i}(F)$ . Then  $f(\pi_1 \times \pi_2) \geq 0$ . Indeed, the following conditions are equivalent:

- (a)  $f(\pi_1 \times \pi_2) = 0$ ;
- (b) there is a quasicharacter  $\chi$  of  $F^\times$  such that both  $\pi_1 \otimes \chi \circ \det, \pi_2 \otimes \chi^{-1} \circ \det$  are unramified principal series.

When the  $\pi_i$  are supercuspidal, the corollary follows directly from the theorem; to get the general case, one uses the addition formulæ of [14], namely Theorems 3.1, 8.2 and 9.5, together with the observation

$$(6.5.1) \quad f((\pi_1 \otimes \chi \circ \det) \times (\pi_2 \otimes \chi^{-1} \circ \det)) = f(\pi_1 \times \pi_2).$$

The proof of the theorem will occupy most of the remainder of the section. Throughout, we work relative to a character  $\psi$  with  $c(\psi) = -1$ , since this gives the briefest form. Explicitly,

$$f(\check{\sigma}_1 \times \sigma_2, \psi) = f(\check{\sigma}_1 \times \sigma_2) - n_1 n_2.$$

We choose Haar measures  $\mu_u$  on  $N_u$  and  $\mu_\ell$  on  $N_\ell$  satisfying (5.2.1) relative to our choice of  $\psi$ . We take  $(K, \tau)$  as in 4.1; we have to compute  $\mu_u(K \cap N_u)\mu_\ell(K \cap N_\ell)$ . We use the other notation introduced in 6.1–6.4.

**6.6.** We start by dealing with the case where both  $\sigma_i$  have level zero. There is a maximal simple type in  $\sigma_i$  of the form  $(K_i, \tau_i)$ , where  $K_i = \mathrm{GL}_{n_i}(\mathfrak{o})$  and  $\tau_i$  is the inflation of an irreducible cuspidal representation of  $\mathrm{GL}_{n_i}(\mathfrak{o}/\mathfrak{p})$ . The group  $K$  underlying the semisimple type  $(K, \tau)$  then has the form  $K = \mathbf{U}(\mathfrak{A})$ , where  $\mathfrak{A}$  is the hereditary order of block matrices

$$\mathfrak{A} = \begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} \end{pmatrix},$$

in which the  $(i, j)$ -block has dimensions  $n_i \times n_j$  [7], §7. From this, we get immediately that

$$\mu_{\mathfrak{u}}(K \cap N_{\mathfrak{u}}) \mu_{\ell}(K \cap N_{\ell}) = 1.$$

In case (ii) of 6.5, Theorem 5.3(ii) gives us  $f(\check{\sigma}_1 \times \sigma_2, \psi) = 0$ ; in case (i), we get  $f(\check{\sigma}_1 \times \sigma_2, \psi) = -n_1$  from Theorem 5.4, as required.

**6.7.** We now prove the formula in 6.5(ii). Thus we assume that the  $\sigma_i$  are completely distinct. We can exclude the case where both have level zero, since it has been dealt with in 6.6.

We need the lattice sequence  $A$  introduced in 6.2. For  $j \in \mathbb{Z}$ , we write

$$\mathfrak{a}_j(A) = \{x \in A : xA(i) \subset A(i+j), i \in \mathbb{Z}\},$$

consistent with the notation of [7].

We assume, without loss, that  $m_1/e_1 \geq m_2/e_2$ . The construction of  $K$  in [7], §8, gives

$$(6.7.1) \quad K \cap N_{\mathfrak{u}} = 1 + \mathfrak{a}_0(A) \cap \mathfrak{n}_{\mathfrak{u}}, \quad K \cap N_{\ell} = 1 + \mathfrak{a}_{m+1}(A) \cap \mathfrak{n}_{\ell}.$$

Now let us compute the product of volumes of the groups in (6.7.1). The orthogonal complement  $(\mathfrak{a}_0 \cap \mathfrak{n}_{\mathfrak{u}})^{\perp}$  of  $\mathfrak{a}_0 \cap \mathfrak{n}_{\mathfrak{u}}$  is  $\mathfrak{a}_1 \cap \mathfrak{n}_{\ell}$ , by [7], 2.10. Thus

$$\mu_{\mathfrak{u}}(\mathfrak{a}_0 \cap \mathfrak{n}_{\mathfrak{u}}) \mu_{\ell}(\mathfrak{a}_1 \cap \mathfrak{n}_{\ell}) = 1.$$

We now observe that  $\mathfrak{a}_{m+1} \cap \mathfrak{n}_{\ell} = \beta_1^{-1} \mathfrak{a}_1 \cap \mathfrak{n}_{\ell}$  (see [7], 3.7). The product of volumes we seek is therefore

$$(\mathfrak{a}_1 \cap \mathfrak{n}_{\ell} : \beta_1^{-1} \mathfrak{a}_1 \cap \mathfrak{n}_{\ell}) = \|\det \beta_1\|^{-1},$$

where we view  $\beta_1$  as acting on  $\mathfrak{n}_{\ell}$  by right multiplication. The valuation of  $\beta_1$  in  $F[\beta_1]$  is  $-m_1$ , so the required volume is  $q^{-n_1 n_2 m_1 / e_1}$ . In particular,  $f(\check{\sigma}_1 \times \sigma_2, \psi) \geq n_1 n_2$ , with equality if and only if the  $\sigma_i$  both have level zero. This completes the proof of 6.5(ii).

**6.8.** We assume until further notice that  $\sigma_2$  is not an unramified twist of  $\sigma_1$ , and that the  $\sigma_i$  are not completely distinct. Thus we are in case (iii) of Theorem 6.5.

Let us dispose of a trivial case. Let  $\chi$  be a quasicharacter of  $F^{\times}$ , and set  $\sigma'_i = \sigma_i \otimes \chi \circ \det$ . Since we can transfer one-dimensional twists between variables in the conductor, we get

$$f(\check{\sigma}'_1 \times \sigma'_2, \psi) = f(\check{\sigma}_1 \times \sigma_2, \psi).$$

**Lemma.** *Let  $([A, m, 0, \gamma], l, \vartheta)$  be a best common approximation to the  $\sigma_i$ . The following conditions are equivalent:*

- (i) *The element  $\gamma$  lies in  $F$ .*

(ii) *There are completely distinct representations  $\sigma'_i$  and a quasicharacter  $\chi$  of  $F^\times$  such that*

$$\begin{aligned} \sigma_i &= \sigma'_i \otimes \chi \circ \det, \\ f(\sigma'_i, \psi) &< f(\sigma_i, \psi), \end{aligned} \quad i = 1, 2.$$

*When these conditions are satisfied, we have*

$$\begin{aligned} f(\sigma_i, \psi) &= n_i f(\chi, \psi), \\ f(\check{\sigma}'_1 \times \sigma'_2, \psi) &= f(\check{\sigma}'_1 \times \sigma_2, \psi) = n_1 n_2 l / e. \end{aligned}$$

*Proof.* The equivalence of (i) and (ii) follows easily from the definition of best common approximation [7], 8.3 (or use 6.15 below). The next assertion is implied by (6.1.2).

Finally, let  $m'_i$  be the level of the representation  $\sigma'_i$ , as in 6.1, so that  $f(\sigma'_i, \psi) = n_i m'_i / e_i$ . Here we have

$$l/e = \max(m'_1/e_1, m'_2/e_2),$$

and the last assertion of the lemma follows from 6.7.

This deals with the  $d = 1$  case in part (iii) of the theorem.

**6.9.** Let  $([A, m, 0, \gamma], l, \vartheta)$  be a best common approximation to the  $\sigma_i$ . We assume  $\gamma \notin F$ , and  $l > 0$ . We now put  $E = F[\gamma]$  and write  $B$  for the  $A$ -centralizer of  $\gamma$ ; thus  $B \cong \mathbb{M}_{n/d}(E)$ , where  $d = [E:F]$ . We recall that  $e$  denotes the  $F$ -period of the lattice sequence  $A$  as in 6.2.

In this situation, we have to prove:

$$(6.9.1) \quad q^{f(\check{\sigma}'_1 \times \sigma_2, \psi)} = C(\gamma)^{n_1 n_2 / d^2} q^{n_1 n_2 l / de},$$

where  $C(\gamma)$  is defined in 6.4. To do this, we choose a tame co-restriction  $s_\gamma$  on  $A$  relative to  $E/F$ , and write  $a_\gamma$  for the adjoint map  $x \mapsto \gamma x - x\gamma$ ,  $x \in A$ . We have an exact sequence

$$0 \longrightarrow B \longrightarrow A \xrightarrow{a_\gamma} A \xrightarrow{s_\gamma} B \longrightarrow 0$$

which restricts (since  $\gamma \in L$ ) to an exact sequence

$$(6.9.2) \quad 0 \longrightarrow B \cap \mathfrak{n}_u \longrightarrow A \cap \mathfrak{n}_u \xrightarrow{a_\gamma} A \cap \mathfrak{n}_u \xrightarrow{s_\gamma} B \cap \mathfrak{n}_u \longrightarrow 0.$$

**Lemma.** *The quantity  $C(a_\gamma, s_\gamma)$  given by the exact sequence (6.9.2) is*

$$C(a_\gamma, s_\gamma) = C(\gamma)^{n_1 n_2 / d^2}.$$

We prove this at the end of the paragraph. Take the cover  $(K, \tau)$  provided by 4.1 and set

$$K \cap N_u = 1 + \mathfrak{k}_u, \quad K \cap N_\ell = 1 + \mathfrak{k}_\ell,$$

for lattices  $\mathfrak{k}_u, \mathfrak{k}_\ell$  in  $\mathfrak{n}_u, \mathfrak{n}_\ell$ , respectively. Also write

$$\mathfrak{b}_j = \mathfrak{b}_j(A) = \mathfrak{a}_j(A) \cap B, \quad j \in \mathbb{Z},$$

where the lattices  $\mathfrak{a}_j$  are defined in 6.7. The group  $K$  is specified in [7], 8.6 (final display). In our present notation, the Lemma of [7], 6.3, says that the lattices  $\mathfrak{k}_u, \mathfrak{k}_\ell$  fit into an exact sequence

$$(6.9.3) \quad 0 \longrightarrow \mathfrak{b}_0 \cap \mathfrak{n}_u \longrightarrow \mathfrak{k}_u \xrightarrow{a_\gamma} \mathfrak{k}_\ell^\perp \xrightarrow{s_\gamma} \mathfrak{b}_{-l} \cap \mathfrak{n}_u \longrightarrow 0.$$

Applying 5.3, we have

$$q^{-f(\check{\sigma}_1 \times \sigma_2, \psi)} = \mu_{\mathfrak{k}_u} / \mu_{\mathfrak{k}_\ell^\perp}.$$

The considerations of 6.4 show this is equal to

$$C(\gamma)^{-n_1 n_2 / d^2} (\mathfrak{b}_0 \cap \mathfrak{n}_u : \mathfrak{b}_{-l} \cap \mathfrak{n}_u).$$

One computes this index as in 6.7 to get (6.9.1).

This deals with the case  $d > 1$ ,  $l > 0$  in part (iii) of the theorem, once we have proved the lemma above. To do this, we use a device from [5], 1.2. Our underlying vector space  $F^n$  is an  $E$ -vector space; the choice of an  $E$ -basis  $\mathcal{A}$  of  $F^n$  gives a decomposition  $F^n = E \otimes_F V_0$ , where  $V_0$  is the  $F$ -linear span of  $\mathcal{A}$ . This gives an algebra isomorphism  $A = A(E) \otimes_F \text{End}_F(V_0)$  and hence an isomorphism  $A = A(E) \otimes_E B$  of  $(A(E), B)$ -bimodules. Put another way,  $A$  is free as a left  $A(E)$ -module, and an  $E$ -basis  $\mathcal{B}$  of  $B$  provides a left  $A(E)$ -basis of  $A$ , so that

$$A = \bigoplus_{b \in \mathcal{B}} A(E) \otimes b.$$

The tame co-restriction  $s_\gamma : A \rightarrow B$  satisfies

$$s_\gamma : \sum_{b \in \mathcal{B}} x_b \otimes b \mapsto \sum s_\gamma^0(x_b) b,$$

where the  $x_b$  are elements of  $A(E)$  and  $s_\gamma^0$  temporarily denotes a tame co-restriction  $A(E) \rightarrow E$ . Likewise for the adjoint map  $a_\gamma$ . However,  $B$  decomposes as the direct sum of its intersections with  $\mathfrak{n}_u$ ,  $\mathfrak{n}_\ell$  and the  $F$ -algebra  $\text{End}_F(V_1) \oplus \text{End}_F(V_2)$ . Thus the sequence (6.9.2) is effectively a direct sum of  $n_1 n_2 / d^2$  copies of the exact sequence

$$0 \longrightarrow E \longrightarrow A(E) \xrightarrow{a_\gamma^0} A(E) \xrightarrow{s_\gamma^0} E \longrightarrow 0,$$

and the lemma follows.

**6.10.** We take the same notation as in 6.9, except that we now assume  $l = 0$  (and  $\gamma \notin F$ ). The construction of  $K$  and the lattices  $\mathfrak{k}_u$ ,  $\mathfrak{k}_\ell$  is slightly different in this case; see [7], §7. However, there is an exact sequence analogous to (6.9.3) from which we obtain

$$q^{f(\check{\sigma}_1 \times \sigma_2, \psi)} = C(\gamma)^{n_1 n_2 / d^2}.$$

We have now established all the formulæ of part (iii) of the theorem.

**6.11.** We return to part (i) of the theorem; we may as well take  $\sigma_2 = \sigma_1$ . Comparing with the situation of 6.10, we have  $\gamma = \beta_1$  and the group  $K$  is the same as there. The explicit formula now follows from 5.4 and (6.1.1). The strict lower bound follows from Lemma 6.4.

**6.12.** The strict lower bound in (iii) likewise follows from Lemma 6.4. To complete the proof of the theorem, we have only to establish the strict upper bounds in parts (i) and (iii). We first treat (iii).

We return to the notation of 6.9, which is the only case in need of proof. In particular,  $\gamma \notin F$ . We need some notation. Set  $E = F[\gamma]$  and  $d = [E:F]$ . There is a unique hereditary  $\mathfrak{o}$ -order  $\mathfrak{A}(E)$  in  $A(E)$  which is normalized by  $E^\times$ . We denote its radical by  $\mathfrak{P}(E)$ . We put  $k = -\nu_E(\gamma) > 0$  and  $r = -k_0(\gamma, \mathfrak{A}(E))$  (notation of

[5], 1.4). We have  $k \geq r > 0$ , since we have excluded the possibility  $\gamma \in F$ . The integer  $k$  is given by

$$k = me(E|F)/e.$$

We also have  $l < -k_0(\gamma, A) = re(E|F)/e$ . It is therefore enough to prove:

$$(6.12.1) \quad \mathfrak{c}(\gamma) \leq \frac{kd^2}{e(\gamma)} - \frac{rd}{e(\gamma)},$$

where we abbreviate  $e(\gamma) = e(E|F)$ .

We prove (6.12.1) by “induction along  $\gamma$ ”, in a manner reminiscent of many proofs in [5]. Suppose first that  $\gamma$  is minimal over  $F$ ; we have excluded the case  $E = F$ , so we have  $r = k$  [5], 1.4.15. We compute the quantity  $C(\gamma)$  by taking the alternating product of volumes in the exact sequence *loc. cit.*

$$0 \longrightarrow \mathfrak{o}_E \longrightarrow \mathfrak{A}(E) \xrightarrow{a_\gamma} \mathfrak{B}(E)^{-k} \xrightarrow{s_\gamma} \mathfrak{p}_E^{-k} \longrightarrow 0.$$

That is,

$$C(\gamma) = q_E^{-k} q^{d^2 k/e(\gamma)}.$$

The exponent of  $q$  here is  $k(-d/e(\gamma) + d^2/e(\gamma))$ , as required.

In the general case, i.e., where  $\gamma$  is not minimal over  $F$ , we consider the exact sequence [5], 1.4.7:

$$0 \longrightarrow \mathfrak{o}_E \longrightarrow \mathfrak{N}_{-r}(\gamma) \xrightarrow{a_\gamma} \mathfrak{B}(E)^{-r} \xrightarrow{s_\gamma} \mathfrak{p}_E^{-r} \longrightarrow 0.$$

Here,  $\mathfrak{N}_{-r}(\gamma) = \mathfrak{N}_{-r}(\gamma, \mathfrak{A}(E))$ , in the notation of [5], 1.4. We choose a simple stratum  $[\mathfrak{A}(E), k, r, \delta]$  equivalent to  $[\mathfrak{A}(E), k, r, \gamma]$  ([5], 2.4.1). In case  $\delta \in F$ , the element  $\gamma - \delta$  is minimal over  $F$ , so  $r = -\nu_E(\gamma - \delta) < k$  and  $C(\gamma - \delta) = C(\gamma)$ . The result is then given by the first case, so we assume  $\delta \notin F$ . Thus

$$k > -k_0(\delta, \mathfrak{A}(E)) > r.$$

We have  $\mathfrak{N}_{-r}(\gamma) = \mathfrak{N}_{-r}(\delta, \mathfrak{A}(E))$  [5], 2.1.3, so we can compare with the exact sequence

$$0 \longrightarrow \mathfrak{B}_\delta \longrightarrow \mathfrak{N}_{-r}(\gamma) \xrightarrow{a_\delta} \mathfrak{B}(E)^{-r} \xrightarrow{s_\delta} \mathfrak{Q}_\delta^{-r} \longrightarrow 0.$$

Here,  $\mathfrak{B}_\delta$  denotes the intersection of  $\mathfrak{A}(E)$  with the  $A(E)$ -centralizer of  $\delta$  and  $\mathfrak{Q}_\delta$  the radical of  $\mathfrak{B}_\delta$ . We get (using  $\mu$  for any Haar measure on the relevant vector space)

$$\begin{aligned} C(\gamma) &= \frac{\mu(\mathfrak{o}_E)}{\mu(\mathfrak{p}_E^{-r})} \frac{\mu(\mathfrak{B}(E)^{-r})}{\mu(\mathfrak{N}_{-r}(\delta))} \\ &= \frac{\mu(\mathfrak{o}_E)}{\mu(\mathfrak{p}_E^{-r})} \frac{\mu(\mathfrak{Q}_\delta^{-r})}{\mu(\mathfrak{B}_\delta)} C(\delta)^{d^2/d_\delta^2} \\ &= q^{-rd/e(\gamma)} q^{rd^2/d_\delta e(\gamma)} C(\delta)^{d^2/d_\delta^2}, \end{aligned}$$

where  $d_\delta = [F[\delta]:F]$ , etc. Now let  $t = -k_0(\delta, \mathfrak{A}(F[\delta]))$ , so that  $k_0(\delta, \mathfrak{A}(E)) = -te(\gamma)/e(\delta)$ . Note that  $\nu_{F[\delta]}(\delta) = -ke(\delta)/e(\gamma)$ . By inductive hypothesis therefore,

$$\mathfrak{c}(\delta) \leq kd_\delta^2/e(\gamma) - td_\delta/e(\delta).$$

In all,

$$\mathfrak{c}(\gamma) \leq \frac{kd^2}{e(\gamma)} - \frac{rd}{e(\gamma)} + \frac{rd^2}{d_\delta e(\gamma)} - \frac{td^2}{d_\delta e(\delta)}.$$

The result will follow when we show that  $r/e(\gamma) < t/e(\delta)$ . However,

$$te(\gamma)/e(\delta) = -k_0(\delta, \mathfrak{A}(E)) > r,$$

as required.

The strict upper bound in part (i) of the theorem also follows from (6.12.1), so we have now completed the proof.

**6.13.** We make some comments on the discriminant function  $C(\gamma)$ . First, we observe that the arguments of 6.12 can be refined to give a precise determination of  $C(\gamma)$  in terms of the construction of  $\gamma$  from minimal elements as in [5], Ch. 2; see especially Theorem 2.4.1.

We next recall that  $\gamma$  is only determined up to a congruence in the algebra  $A(E)$ . Throughout the arguments above (starting with 6.9), we can replace  $\gamma$  by any element  $\gamma' \in A(E)$  which is sufficiently close to  $\gamma$ , and nothing changes. We can therefore assume that the field extension  $F[\gamma]/F$  is separable.

Abbreviating  $E = F[\gamma]$  as before, we have a canonical projection  $\mathfrak{p} : A(E) \rightarrow E$ , which is orthogonal with respect to the reduced trace pairing  $A(E) \times A(E) \rightarrow F$ . The orthogonal complement of  $E$  is  $A'(E) = a_\gamma(A(E))$ , and  $A(E) = A'(E) \oplus E$ . The adjoint  $a_\gamma$  acts on  $A'(E)$  as an automorphism, and  $s_\gamma = \alpha_0 \mathfrak{p}$ , for some  $\alpha_0 \in E$  such that  $\alpha_0 \mathfrak{o}_E = \mathfrak{p} \mathfrak{p}_E^{-1} \mathfrak{D}_{E/F}^{-1}$ , where  $\mathfrak{D}_{E/F}$  denotes the relative different (cf. [5], 1.3). Thus

$$C(\gamma) = (\mathfrak{p}_E^{e(E|F)-1} : \mathfrak{D}_{E/F})^{-1} \|\det_{A'(E)} a_\gamma\|^{-1}.$$

Of course, if  $\gamma_i$ ,  $1 \leq i \leq d$ , are the conjugates of  $\gamma$  in some splitting field  $E'$  and we extend the absolute value  $\|\cdot\|$  to one on  $E'$ , we have

$$\|\det a_\gamma\| = \prod_{i \neq j} \|\gamma_i - \gamma_j\|.$$

**6.14.** We give a simple application of 6.5, for use in a subsequent paper. For an irreducible smooth representation  $\pi$  of  $\mathrm{GL}_n(F)$  and a quasicharacter  $\chi$  of  $F^\times$ , we use the notation

$$\pi \cdot \chi = \pi \otimes \chi \circ \det.$$

**Proposition.** *For  $i = 1, 2$ , let  $\pi_i$  be an irreducible supercuspidal representation of  $\mathrm{GL}_{n_i}(F)$  with the property*

$$(6.14.1) \quad \pi_i \not\cong \pi_i \cdot \chi$$

for any unramified quasicharacter  $\chi$  of  $F^\times$ .

- (i) *Suppose that we have  $\pi_1 \cdot \chi \not\cong \check{\pi}_2$  for any tamely ramified quasicharacter  $\chi$  of  $F^\times$ . Then*

$$f(\pi_1 \cdot \chi \times \pi_2) = f(\pi_1 \times \pi_2)$$

for every tamely ramified quasicharacter  $\chi$  of  $F^\times$ .

- (ii) *Suppose that  $\pi_1 \cong \check{\pi}_2$  and that  $n_1 = n_2$  is a power of the residual characteristic  $p$  of  $F$ . If  $\chi$  denotes a tamely ramified quasicharacter of  $F^\times$ , we have:*

$$\begin{aligned} f(\pi_1 \cdot \chi \times \pi_2) &= f(\pi_1 \times \pi_2) && \text{if } \chi \text{ is unramified,} \\ f(\pi_1 \cdot \chi \times \pi_2) &= f(\pi_1 \times \pi_2) + 1 && \text{if } \chi \text{ is ramified.} \end{aligned}$$

*Proof.* In part (i), the representation  $\pi_2$  cannot be an unramified twist of  $\check{\pi}_1$ . Consequently, the pair  $(\sigma_1, \sigma_2) = (\check{\pi}_1, \pi_2)$  must satisfy the hypotheses of either 6.5(ii) or 6.5(iii). If both  $\pi_i$  have level zero, they are completely distinct, and the representations  $\pi_1 \cdot \chi, \check{\pi}_2$  have the the same properties. Thus  $f(\pi_1 \cdot \chi \times \pi_2) = f(\pi_1 \times \pi_2) = n_1 n_2$  in this case.

We may therefore assume that at least one of the  $\pi_i$  has positive level and, say,  $f(\pi_1)/n_1 \leq f(\pi_2)/n_2$ . If we are in case 6.5(iii), a best common approximation to  $(\check{\pi}_1, \pi_2)$  is again a best common approximation to  $(\check{\pi}_1 \cdot \chi, \pi_2)$ , and the result follows. In case 6.5(ii), the representations  $\check{\pi}_1 \cdot \chi, \pi_2$  are still completely distinct and  $f(\pi_1 \cdot \chi \times \pi_2) = f(\pi_1 \times \pi_2) = n_2 f(\pi_2)$ .

The first assertion in (ii) is easy: twisting with unramified characters does not change the conductor. The second assertion is trivial if  $n_1 = n_2 = 1$ . We exclude this and assume  $n_1 = n_2 > 1$ . If  $\pi_1$  contains a maximal simple type with underlying simple stratum  $[\mathfrak{A}, m, 0, \beta]$ , say, our hypotheses (6.14.1) and (6.1.1) together imply  $e(F[\beta]|F) = n_1$ . So  $F[\beta]/F$  is totally ramified of degree  $n_1$  which, we recall, is a power of  $p$ . It follows that  $\pi_1 \cdot \chi \not\cong \pi_1$ , for any non-trivial tame quasicharacter  $\chi$  (cf. [5], 5.2.2(iii)). The result then follows on comparing 6.5(i) with 6.5(iii).

**6.15.** As promised in 6.3, we give a method for finding the quantities  $l$  and  $\gamma$  appearing in the best common approximation to a given pair of (not completely distinct) supercuspidal representations. The description is simpler than in [7], §8, since we are here only concerned with a special case. The results of [3], §8, show that the outline here is equivalent to that of [7].

We recapitulate the notation. For  $i = 1, 2$ ,  $\sigma_i$  is an irreducible supercuspidal representation of  $\mathrm{GL}_{n_i}(F)$ ; this contains a maximal simple type with underlying simple stratum  $[\mathfrak{A}_i, m_i, 0, \beta_i]$  in  $\mathbb{M}_{n_i}(F)$ . We have  $e_i = e(\mathfrak{A}_i|\mathfrak{o}) = e(F[\beta_i]|F)$ ; we put  $e = \mathrm{lcm}(e_1, e_2)$ . The simple type in  $\sigma_i$  contains a simple character  $\theta_i \in \mathcal{C}(\mathfrak{A}_i, 0, \beta_i)$ , to use the notation of [5], Ch. 3.

Let  $f_i$  denote the residual degree  $f(F[\beta_i]|F)$ , and set  $f = \mathrm{lcm}(f_1, f_2)$ . Let  $V_0$  be an  $F$ -vector space of dimension  $ef$  and let  $\mathfrak{A}_0$  be a principal  $\mathfrak{o}$ -order in  $\mathrm{End}_F(V_0)$  of period  $e$ . There are then  $F$ -algebra embeddings  $\varphi_i$  of the fields  $F[\beta_i]$  in  $\mathrm{End}_F(V_0)$  such that  $\mathfrak{A}_0$  is normalized by both  $\varphi_i(F[\beta_i])^\times$ . The strata  $[\mathfrak{A}_0, m, 0, \varphi_i(\beta_i)]$  are both simple.

Next, we use the canonical bijections  $\mathcal{C}(\mathfrak{A}_i, 0, \beta_i) \rightarrow \mathcal{C}(\mathfrak{A}_0, 0, \varphi_i(\beta_i))$  of [5], 3.6.14, induced by the embeddings  $\varphi_i$ . Write  $\theta'_i$  for the image of  $\theta_i$  under this map,  $i = 1, 2$ . We obtain  $l$  as the least non-negative integer  $k$  such that the characters  $\theta'_i \mid H^{k+1}(\varphi(\beta_i), \mathfrak{A}_0)$  intertwine in  $\mathrm{Aut}_F(V_0)$ . (This implies they are conjugate in  $\mathrm{Aut}_F(V_0)$ , by [5], 3.5.11.) We then choose a simple stratum  $[\mathfrak{A}_0, m, l, \gamma_0]$  equivalent to, say,  $[\mathfrak{A}_0, m, l, \varphi_1(\beta_1)]$ . The element  $\gamma$  may then be taken as  $\varphi_0(\gamma_0)$ , for some embedding  $\varphi_0$  of  $F[\gamma_0]$  in  $A$  such that  $\varphi_0(F[\gamma_0])^\times$  normalizes  $A$  and is contained in  $L$ .

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