CLUSTER ALGEBRAS I: FOUNDATIONS

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To the memory of Sergei Kerov

1. INTRODUCTION

In this paper, we initiate the study of a new class of algebras, which we call *cluster algebras*. Before giving precise definitions, we present some of the main features of these algebras. For any positive integer n, a cluster algebra $\mathcal{A}$ of rank n is a commutative ring with unit and no zero divisors, equipped with a distinguished family of generators called *cluster variables*. The set of cluster variables is the (non-disjoint) union of a distinguished collection of n-subsets called *clusters*. These clusters have the following *exchange property*: for any cluster $\mathbf{x}$ and any element $x \in \mathbf{x}$, there is another cluster obtained from $\mathbf{x}$ by replacing x with an element x' related to x by a *binomial exchange relation*

$$(1.1) \qquad\qquad xx' = M_1 + M_2 \ ,$$

where M_1 and M_2 are two monomials without common divisors in the $n-1$ variables $\mathbf{x} - \{x\}$. Furthermore, any two clusters can be obtained from each other by a sequence of exchanges of this kind.

The prototypical example of a cluster algebra of rank 1 is the coordinate ring $\mathcal{A} = \mathbb{C}[SL_2]$ of the group SL_2, viewed in the following way. Writing a generic element of SL_2 as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we consider the entries a and d as cluster variables, and the entries b and c as scalars. There are just two clusters $\{a\}$ and $\{d\}$, and $\mathcal{A}$ is the algebra over the polynomial ring $\mathbb{C}[b, c]$ generated by the cluster variables a and d subject to the binomial exchange relation

$$ad = 1 + bc \ .$$

Another important incarnation of a cluster algebra of rank 1 is the coordinate ring $\mathcal{A} = \mathbb{C}[SL_3/N]$ of the base affine space of the special linear group SL_3; here N is the maximal unipotent subgroup of SL_3 consisting of all unipotent upper triangular matrices. Using the standard notation $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23})$ for the Plücker coordinates on SL_3/N, we view x_2 and x_{13} as cluster variables; then $\mathcal{A}$ is the algebra over the polynomial ring $\mathbb{C}[x_1, x_3, x_{12}, x_{13}]$ generated by the two cluster variables x_2 and x_{13} subject to the binomial exchange relation

$$x_2 x_{13} = x_1 x_{23} + x_3 x_{12} \ .$$

This form of representing the algebra $\mathbb{C}[SL_3/N]$ is closely related to the choice of a linear basis in it consisting of all monomials in the six Plücker coordinates which are not divisible by $x_2 x_{13}$. This basis was introduced and studied in [9] under the name "canonical basis". As a representation of SL_3, the space $\mathbb{C}[SL_3/N]$ is the multiplicity-free direct sum of all irreducible finite-dimensional representations, and each of the components is spanned by a part of the above basis. Thus, this construction provides a "canonical" basis in every irreducible finite-dimensional representation of SL_3. After Lusztig's work [13], this basis had been recognized as (the classical limit at $q \to 1$ of) the *dual canonical basis*, i.e., the basis in the q-deformed algebra $\mathbb{C}_q[SL_3/N]$ which is dual to Lusztig's canonical basis in the appropriate q-deformed universal enveloping algebra (a.k.a. quantum group). The dual canonical basis in the space $\mathbb{C}[G/N]$ was later constructed explicitly for a few other classical groups G of small rank: for $G = Sp_4$ in [16] and for $G = SL_4$ in [2]. In both cases, $\mathbb{C}[G/N]$ can be seen to be a cluster algebra: there are 6 clusters of size 2 for $G = Sp_4$, and 14 clusters of size 3 for $G = SL_4$.

We conjecture that the above examples can be extensively generalized: for any simply-connected connected semisimple group G, the coordinate rings $\mathbb{C}[G]$ and $\mathbb{C}[G/N]$, as well as coordinate rings of many other interesting varieties related to G, have a natural structure of a cluster algebra. This structure should serve as an algebraic framework for the study of "dual canonical bases" in these coordinate rings and their q-deformations. In particular, we conjecture that all monomials in the variables of any given cluster (the *cluster monomials*) belong to this dual canonical basis.

A particularly nice and well-understood example of a cluster algebra of an arbitrary rank n is the homogeneous coordinate ring $\mathbb{C}[Gr_{2,n+3}]$ of the Grassmannian of 2-dimensional subspaces in $\mathbb{C}^{n+3}$. This ring is generated by the Plücker coordinates $[ij]$, for $1 \le i < j \le n + 3$, subject to the relations

$$[ik][jl] = [ij][kl] + [il][jk] \ ,$$

for all $i < j < k < l$. It is convenient to identify the indices $1, \ldots, n + 3$ with the vertices of a convex $(n + 3)$-gon, and the Plücker coordinates with its sides and diagonals. We view the sides $[12], [23], \ldots, [n + 2, n + 3], [1, n + 3]$ as scalars, and the diagonals as cluster variables. The clusters are the maximal families of pairwise noncrossing diagonals; thus, they are in a natural bijection with the triangulations of this polygon. It is known that the cluster monomials form a linear basis in $\mathbb{C}[Gr_{2,n+3}]$. To be more specific, we note that this ring is naturally identified with the ring of polynomial SL_2-invariants of an $(n+3)$-tuple of points in $\mathbb{C}^2$. Under this isomorphism, the basis of cluster monomials corresponds to the basis considered in [11, 18]. (We are grateful to Bernd Sturmfels for bringing these references to our attention.)

An essential feature of the exchange relations (1.1) is that the right-hand side does not involve subtraction. Recursively applying these relations, one can represent any cluster variable as a *subtraction-free* rational expression in the variables of any given cluster. This positivity property is consistent with a remarkable connection between canonical bases and the theory of total positivity, discovered by G. Lusztig [14, 15]. Generalizing the classical concept of totally positive matrices, he defined totally positive elements in any reductive group G, and proved that all elements of the dual canonical basis in $\mathbb{C}[G]$ take positive values at them.

It was realized in [15, 5] that the natural geometric framework for total positivity is given by *double Bruhat cells*, the intersections of cells of the Bruhat decompositions with respect to two opposite Borel subgroups. Different aspects of total positivity in double Bruhat cells were explored by the authors of the present paper and their collaborators in [1, 3, 4, 5, 6, 7, 12, 17, 20]. The binomial exchange relations of the form (1.1) played a crucial role in these studies. It was the desire to explain the ubiquity of these relations and to place them in a proper context that led us to the concept of cluster algebras. The crucial step in this direction was made in [20], where a family of clusters and exchange relations was explicitly constructed in the coordinate ring of an arbitrary double Bruhat cell. However, this family was not *complete*: in general, some clusters were missing, and not any member of a cluster could be exchanged from it. Thus, we started looking for a natural way to "propagate" exchange relations from one cluster to another. The concept of cluster algebras is the result of this investigation. We conjecture that the coordinate ring of any double Bruhat cell is a cluster algebra.

This article, in which we develop the foundations of the theory, is conceived as the first in a forthcoming series. We attempt to make the exposition elementary and self-contained; in particular, no knowledge of semisimple groups, quantum groups or total positivity is assumed on the part of the reader.

One of the main structural features of cluster algebras established in the present paper is the following *Laurent phenomenon*: any cluster variable x viewed as a rational function in the variables of any given cluster is in fact a Laurent polynomial. This property is quite surprising: in most cases, the numerators of these Laurent polynomials contain a huge number of monomials, and the numerator for x moves into the denominator when we compute the cluster variable x' obtained from x by an exchange (1.1). The magic of the Laurent phenomenon is that, at every stage of this recursive process, a cancellation will inevitably occur, leaving a single monomial in the denominator.

In view of the positivity property discussed above, it is natural to expect that all Laurent polynomials for cluster variables will have *positive* coefficients. This seems to be a rather deep property; our present methods do not provide a proof of it.

On the bright side, it is possible to establish the Laurent phenomenon in many different situations spreading beyond the cluster algebra framework. One such extension is given in Theorem 3.2. By a modification of the method developed here, a large number of additional interesting instances of the Laurent phenomenon are established in a separate paper [8].

The paper is organized as follows. Section 2 contains an axiomatic definition, first examples and the first structural properties of cluster algebras. One of the technical difficulties in setting up the foundations involves the concept of an *exchange graph* whose vertices correspond to clusters, and the edges to exchanges among them. It is convenient to begin by taking the n-regular tree $\mathbb{T}_n$ as our underlying graph. This tree can be viewed as a universal cover for the actual exchange graph, whose appearance is postponed until Section 7.

The Laurent phenomenon is established in Section 3. In Sections 4 and 5, we scrutinize the main definition, obtain useful reformulations, and introduce some important classes of cluster algebras.

Section 6 contains a detailed analysis of cluster algebras of rank 2. This analysis exhibits deep and somewhat mysterious connections between cluster algebras and Kac-Moody algebras. This is just the tip of an iceberg: these connections will be

further explored (for cluster algebras of an arbitrary rank) in the sequel to this paper. The main result of this sequel is a complete classification of cluster algebras of *finite type*, i.e., those with finitely many distinct clusters; cf. Example 7.6. This classification turns out to be yet another instance of the famous Cartan-Killing classification.

2. Main definitions

Let I be a finite set of size n; the standard choice will be $I = [n] = \{1, 2, \ldots, n\}$. Let $\mathbb{T}_n$ denote the *n-regular tree,* whose edges are labeled by the elements of I, so that the n edges emanating from each vertex receive different labels. By a common abuse of notation, we will sometimes denote by $\mathbb{T}_n$ the set of the tree's vertices. We will write $t \overset{i}{\underline{\quad}} t'$ if vertices $t, t' \in \mathbb{T}_n$ are joined by an edge labeled by i.

To each vertex $t \in \mathbb{T}_n$, we will associate a *cluster* of n generators ("variables") $\mathbf{x}(t) = (x_i(t))_{i \in I}$. All these variables will commute with each other and satisfy the following *exchange relations,* for every edge $t \overset{j}{\underline{\quad}} t'$ in $\mathbb{T}_n$:

$$(2.1) \qquad\qquad x_i(t) = x_i(t') \quad \text{for any } i \neq j;$$

$$(2.2) \qquad\qquad x_j(t)\, x_j(t') = M_j(t)(\mathbf{x}(t)) + M_j(t')(\mathbf{x}(t')).$$

Here $M_j(t)$ and $M_j(t')$ are two monomials in the n variables x_i, $i \in I$; we think of these monomials as being associated with the two ends of the edge $t \overset{j}{\underline{\quad}} t'$.

To be more precise, let $\mathbb{P}$ be an abelian group without torsion, written multiplicatively. We call $\mathbb{P}$ the *coefficient group*; a prototypical example is a free abelian group of finite rank. Every monomial $M_j(t)$ in (2.2) will have the form

$$(2.3) \qquad\qquad M_j(t) = p_j(t) \prod_{i \in I} x_i^{b_i},$$

for some coefficient $p_j(t) \in \mathbb{P}$ and some nonnegative integer exponents b_i.

The monomials $M_j(t)$ must satisfy certain conditions (axioms). To state them, we will need a little preparation. Let us write $P \mid Q$ to denote that a polynomial P divides a polynomial Q. Accordingly, $x_i \mid M_j(t)$ means that the monomial $M_j(t)$ contains the variable x_i. For a rational function $F = F(x, y, \ldots)$, the notation $F|_{x \leftarrow g(x,y,\ldots)}$ will denote the result of substituting $g(x, y, \ldots)$ for x into F. To illustrate, if $F(x, y) = xy$, then $F|_{x \leftarrow \frac{y}{x}} = \frac{y^2}{x}$.

Definition 2.1. An *exchange pattern* on $\mathbb{T}_n$ with coefficients in $\mathbb{P}$ is a family of monomials $\mathcal{M} = (M_j(t))_{t \in \mathbb{T}_n,\, j \in I}$ of the form (2.3) satisfying the following four axioms:

$(2.4) \qquad$ If $t \in \mathbb{T}_n$, then $x_j \nmid M_j(t)$.

$(2.5) \qquad$ If $t_1 \overset{j}{\underline{\quad}} t_2$ and $x_i \mid M_j(t_1)$, then $x_i \nmid M_j(t_2)$.

$(2.6) \qquad$ If $t_1 \overset{i}{\underline{\quad}} t_2 \overset{j}{\underline{\quad}} t_3$, then $x_j \mid M_i(t_1)$ if and only if $x_i \mid M_j(t_2)$.

$(2.7) \qquad$ Let $t_1 \overset{i}{\underline{\quad}} t_2 \overset{j}{\underline{\quad}} t_3 \overset{i}{\underline{\quad}} t_4$. Then $\dfrac{M_i(t_3)}{M_i(t_4)} = \dfrac{M_i(t_2)}{M_i(t_1)}\bigg|_{x_j \leftarrow M_0/x_j}$,

$\qquad\qquad$ where $M_0 = (M_j(t_2) + M_j(t_3))|_{x_i=0}$.

We note that in the last axiom, the substitution $x_j \leftarrow \frac{M_0}{x_j}$ is effectively mono-mial, since in the event that neither $M_j(t_2)$ nor $M_j(t_3)$ contain x_i, condition (2.6) requires that both $M_i(t_2)$ and $M_i(t_1)$ do not depend on x_j, thus making the whole substitution irrelevant.

One easily checks that axiom (2.7) is invariant under the "flip" $t_1 \leftrightarrow t_4$, $t_2 \leftrightarrow t_3$, so no restrictions are added if we apply it "backwards". The axioms also imply at once that setting

(2.8) $$M'_j(t) = M_j(t')$$

for every edge $t \overset{j}{\longrightarrow} t'$, we obtain another exchange pattern $\mathcal{M}'$; this gives a natural involution $\mathcal{M} \to \mathcal{M}'$ on the set of all exchange patterns.

Remark 2.2. Informally speaking, axiom (2.7) describes the propagation of an exchange pattern along the edges of $\mathbb{T}_n$. More precisely, let us fix the $2n$ exchange monomials for all edges emanating from a given vertex t. This choice uniquely determines the ratio $M_i(t')/M_i(t'')$ for any vertex t' adjacent to t and any edge $t' \overset{i}{\longrightarrow} t''$ (to see this, take $t_2 = t$ and $t_3 = t'$ in (2.7), and allow i to vary). In view of (2.5), this ratio in turn uniquely determines the exponents of all variables x_k in both monomials $M_i(t')$ and $M_i(t'')$. There remains, however, one degree of freedom in determining the coefficients $p_i(t')$ and $p_i(t'')$ because only their ratio is prescribed by (2.7). In Section 5 we shall introduce an important class of *normalized* exchange patterns for which this degree of freedom disappears, and so the whole pattern is uniquely determined by the $2n$ monomials associated with edges emanating from a given vertex.

Let $\mathbb{ZP}$ denote the group ring of $\mathbb{P}$ with integer coefficients. For an edge $t \overset{k}{\longrightarrow} t'$, we refer to the binomial $P = M_k(t) + M_k(t') \in \mathbb{ZP}[x_i : i \in I]$ as the *exchange polynomial*. We will write $t \overset{}{\underset{P}{\longrightarrow}} t'$ or $t \overset{k}{\underset{P}{\longrightarrow}} t'$ to indicate this fact. Note that, in view of the axiom (2.4), the right-hand side of the exchange relation (2.2) can be written as $P(\mathbf{x}(t))$, which is the same as $P(\mathbf{x}(t'))$.

Let $\mathcal{M}$ be an exchange pattern on $\mathbb{T}_n$ with coefficients in $\mathbb{P}$. Note that since $\mathbb{P}$ is torsion-free, the ring $\mathbb{ZP}$ has no zero divisors. For every vertex $t \in \mathbb{T}_n$, let $\mathcal{F}(t)$ denote the field of rational functions in the cluster variables $x_i(t)$, $i \in I$, with coefficients in $\mathbb{ZP}$. For every edge $t \overset{j}{\underset{P}{\longrightarrow}} t'$, we define a $\mathbb{ZP}$-linear field isomorphism $R_{tt'} : \mathcal{F}(t') \to \mathcal{F}(t)$ by

$$R_{tt'}(x_i(t')) = x_i(t) \quad \text{for } i \neq k;$$

(2.9)

$$R_{tt'}(x_k(t')) = \frac{P(\mathbf{x}(t))}{x_k(t)}.$$

Note that property (2.4) ensures that $R_{t't} = R_{tt'}^{-1}$. The *transition maps* $R_{tt'}$ enable us to identify all the fields $\mathcal{F}(t)$ with each other. We can then view them as a single field $\mathcal{F}$ that contains all the elements $x_i(t)$, for all $t \in \mathbb{T}_n$ and $i \in I$. Inside $\mathcal{F}$, these elements satisfy the exchange relations (2.1)–(2.2).

Definition 2.3. Let $\mathbb{A}$ be a subring with unit in $\mathbb{ZP}$ containing all coefficients $p_i(t)$ for $i \in I$ and $t \in \mathbb{T}_n$. The *cluster algebra* $\mathcal{A} = \mathcal{A}_{\mathbb{A}}(\mathcal{M})$ of rank n over $\mathbb{A}$ associated

with an exchange pattern $\mathcal{M}$ is the $\mathbb{A}$-subalgebra with unit in $\mathcal{F}$ generated by the union of all clusters $\mathbf{x}(t)$, for $t \in \mathbb{T}_n$.

The smallest possible ground ring $\mathbb{A}$ is the subring of $\mathbb{ZP}$ generated by all the coefficients $p_i(t)$; the largest one is $\mathbb{ZP}$ itself. An intermediate choice of $\mathbb{A}$ appears in Proposition 2.6 below.

Since $\mathcal{A}$ is a subring of a field $\mathcal{F}$, it is a commutative ring with no zero divisors. We also note that if $\mathcal{M}'$ is obtained from $\mathcal{M}$ by the involution (2.8), then the cluster algebra $\mathcal{A}_{\mathbb{A}}(\mathcal{M}')$ is naturally identified with $\mathcal{A}_{\mathbb{A}}(\mathcal{M})$.

Example 2.4. Let $n = 1$. The tree $\mathbb{T}_1$ has only one edge $t \overset{1}{\rule{1em}{0.4pt}} t'$. The corresponding cluster algebra $\mathcal{A}$ has two generators $x = x_1(t)$ and $x' = x_1(t')$ satisfying the exchange relation

$$xx' = p + p' \ ,$$

where p and p' are arbitrary elements of the coefficient group $\mathbb{P}$. In the "universal" setting, we take $\mathbb{P}$ to be the free abelian group generated by p and p'. Then the two natural choices for the ground ring $\mathbb{A}$ are the polynomial ring $\mathbb{Z}[p, p']$, and the Laurent polynomial ring $\mathbb{ZP} = \mathbb{Z}[p^{\pm 1}, p'^{\pm 1}]$. All other realizations of $\mathcal{A}$ can be viewed as specializations of the universal one. Despite the seeming triviality of this example, it covers several important algebras: the coordinate ring of each of the varieties SL_2, $Gr_{2,4}$ and SL_3/B (cf. Section 1) is a cluster algebra of rank 1, for an appropriate choice of $\mathbb{P}$, p, p' and $\mathbb{A}$.

Example 2.5. Consider the case $n = 2$. The tree $\mathbb{T}_2$ is shown below:

$$(2.10) \qquad \cdots \overset{1}{\rule{1.5em}{0.4pt}} t_0 \overset{2}{\rule{2em}{0.4pt}} t_1 \overset{1}{\rule{2em}{0.4pt}} t_2 \overset{2}{\rule{2em}{0.4pt}} t_3 \overset{1}{\rule{2em}{0.4pt}} t_4 \overset{2}{\rule{1.5em}{0.4pt}} \cdots .$$

Let us denote the cluster variables as follows:

$$y_1 = x_1(t_0) = x_1(t_1), \qquad y_2 = x_2(t_1) = x_2(t_2), \qquad y_3 = x_1(t_2) = x_1(t_3), \ldots$$

(the above equalities among the cluster variables follow from (2.1)). Then the clusters look like

$$\cdots \overset{1}{\rule{1.5em}{0.4pt}} \overset{y_1, y_0}{\underset{t_0}{\bullet}} \overset{2}{\rule{2em}{0.4pt}} \overset{y_2, y_1}{\underset{t_1}{\bullet}} \overset{1}{\rule{2em}{0.4pt}} \overset{y_3, y_2}{\underset{t_2}{\bullet}} \overset{2}{\rule{2em}{0.4pt}} \overset{y_4, y_3}{\underset{t_3}{\bullet}} \overset{1}{\rule{2em}{0.4pt}} \overset{y_5, y_4}{\underset{t_4}{\bullet}} \overset{2}{\rule{1.5em}{0.4pt}} \cdots .$$

We claim that the exchange relations (2.2) can be written in the following form:

$$(2.11) \qquad \begin{aligned} y_0 y_2 &= q_1 y_1^b + r_1, & y_1 y_3 &= q_2 y_2^c + r_2, \\ y_2 y_4 &= q_3 y_3^b + r_3, & y_3 y_5 &= q_4 y_4^c + r_4, \ldots, \end{aligned}$$

where the integers b and c are either both positive or both equal to 0, and the coefficients q_m and r_m are elements of $\mathbb{P}$ satisfying the relations

$$(2.12) \qquad \begin{aligned} q_0 q_2 r_1^c &= r_0 r_2, & q_1 q_3 r_2^b &= r_1 r_3, \\ q_2 q_4 r_3^c &= r_2 r_4, & q_3 q_5 r_4^b &= r_3 r_5, \ldots. \end{aligned}$$

Furthermore, any such choice of parameters b, c, (q_m), (r_m) results in a well-defined cluster algebra of rank 2.

To prove this, we notice that, in view of (2.4)–(2.5), both monomials $M_2(t_0)$ and $M_2(t_1)$ do not contain the variable x_2, and at most one of them contains x_1. If x_1 enters neither $M_2(t_0)$ nor $M_2(t_1)$, then these two are simply elements of $\mathbb{P}$. But

then (2.6) forces *all* monomials $M_i(t_m)$ to be elements of $\mathbb{P}$, while (2.7) implies that it is possible to give the names q_m and r_m to the two monomials corresponding to each edge $t_m \relbar\joinrel\relbar t_{m+1}$ so that (2.11)–(2.12) hold with $b = c = 0$.

Next, consider the case when precisely one of the monomials $M_2(t_0)$ and $M_2(t_1)$ contains x_1. Applying if necessary the involution (2.8) to our exchange pattern, we may assume that $M_2(t_0) = q_1 x_1^b$ and $M_2(t_1) = r_1$ for some positive integer b and some $q_1, r_1 \in \mathbb{P}$. Thus, the exchange relation associated to the edge $t_0 \overset{2}{\relbar\joinrel\relbar} t_1$ takes the form $y_0 y_2 = q_1 y_1^b + r_1$. By (2.6), we have $M_1(t_1) = q_2 x_2^c$ and $M_1(t_2) = r_2$ for some positive integer c and some $q_2, r_2 \in \mathbb{P}$. Then the exchange relation for the edge $t_1 \overset{1}{\relbar\joinrel\relbar} t_2$ takes the form $y_1 y_3 = q_2 y_2^c + r_2$. At this point, we invoke (2.7):

$$\frac{M_2(t_2)}{M_2(t_3)} = \frac{M_2(t_1)}{M_2(t_0)}\Big|_{x_1 \leftarrow r_2/x_1} = \frac{r_1}{q_1 x_1^b}\Big|_{x_1 \leftarrow r_2/x_1} = \frac{r_1 x_1^b}{q_1 r_2^b} \ .$$

By (2.5), we have $M_2(t_2) = q_3 x_1^b$ and $M_2(t_3) = r_3$ for some $q_3, r_3 \in \mathbb{P}$ satisfying $q_1 q_3 r_2^b = r_1 r_3$. Continuing in the same way, we obtain all relations (2.11)–(2.12).

For fixed b and c, the "universal" coefficient group $\mathbb{P}$ is the multiplicative abelian group generated by the elements q_m and r_m for all $m \in \mathbb{Z}$ subject to the defining relations (2.12). It is easy to see that this is a free abelian group of infinite rank. As a set of its free generators, one can choose any subset of $\{q_m, r_m : m \in \mathbb{Z}\}$ that contains four generators q_0, r_0, q_1, r_1 and precisely one generator from each pair $\{q_m, r_m\}$ for $m \neq 0, 1$.

A nice specialization of this setup is provided by the homogeneous coordinate ring of the Grassmannian $Gr_{2,5}$. Recall (cf. Section 1) that this ring is generated by the Plücker coordinates $[k, l]$, where k and l are distinct elements of the cyclic group $\mathbb{Z}/5\mathbb{Z}$. We shall write $\overline{m} = m \mod 5 \in \mathbb{Z}/5\mathbb{Z}$ for $m \in \mathbb{Z}$, and adopt the convention $[k, l] = [l, k]$; see Figure 1.
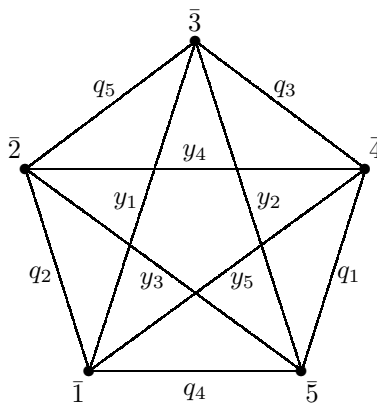

FIGURE 1. The Grassmannian $Gr_{2,5}$

The ideal of relations among the Plücker coordinates is generated by the relations

$$[\overline{m}, \overline{m+2}][\overline{m+1}, \overline{m+3}] = [\overline{m}, \overline{m+1}][\overline{m+2}, \overline{m+3}] + [\overline{m}, \overline{m+3}][\overline{m+1}, \overline{m+2}]$$

for $m \in \mathbb{Z}$. A direct check shows that these relations are a specialization of the relations (2.11), if we set $b = c = 1$, $y_m = [\overline{2m-1}, \overline{2m+1}]$, $q_m = [\overline{2m-2}, \overline{2m+2}]$,

and $r_m = \overline{[2m-2, 2m-1]} \, \overline{[2m+1, 2m+2]} = q_{m-2} q_{m+2}$ for all $m \in \mathbb{Z}$. The coefficient group $\mathbb{P}$ is the multiplicative free abelian group with 5 generators q_m. It is also immediate that the elements q_m and r_m defined in this way satisfy the relations (2.12).

We conclude this section by introducing two important operations on exchange patterns: *restriction* and *direct product*. Let us start with restriction. Let $\mathcal{M}$ be an exchange pattern of rank n with an index set I and coefficient group $\mathbb{P}$. Let J be a subset of size m in I. Let us remove from $\mathbb{T}_n$ all edges labeled by indices in $I - J$, and choose any connected component $\mathbb{T}$ of the resulting graph. This component is naturally identified with $\mathbb{T}_m$. Let $\mathcal{M}'$ denote the restriction of $\mathcal{M}$ to $\mathbb{T}$, i.e., the collection of monomials $M_j(t)$ for all $j \in J$ and $t \in \mathbb{T}$. Then $\mathcal{M}'$ is an exchange pattern on $\mathbb{T}$ whose coefficient group $\mathbb{P}'$ is the direct product of $\mathbb{P}$ with the multiplicative free abelian group with generators x_i, $i \in I - J$. We shall say that $\mathcal{M}'$ is obtained from $\mathcal{M}$ by restriction from I to J. Note that $\mathcal{M}'$ depends on the choice of a connected component $\mathbb{T}$, so there can be several different patterns obtained from $\mathcal{M}$ by restriction from I to J. (We thank the anonymous referee for pointing this out.)

Proposition 2.6. *Let $\mathcal{A} = \mathcal{A}_{\mathbb{A}}(\mathcal{M})$ be a cluster algebra of rank n associated with an exchange pattern $\mathcal{M}$, and let $\mathcal{M}'$ be obtained from $\mathcal{M}$ by restriction from I to J using a connected component $\mathbb{T}$. The $\mathbb{A}$-subalgebra of $\mathcal{A}$ generated by $\bigcup_{t \in \mathbb{T}} \mathbf{x}(t)$ is naturally identified with the cluster algebra $\mathcal{A}_{\mathbb{A}'}(\mathcal{M}')$, where $\mathbb{A}'$ is the polynomial ring $\mathbb{A}[x_i : i \in I - J]$.*

Proof. If $i \in I - J$, then (2.1) implies that $x_i(t)$ stays constant as t varies over $\mathbb{T}$. Therefore, we can identify this variable with the corresponding generator x_i of the coefficient group $\mathbb{P}'$, and the statement follows. $\square$

Let us now consider two exchange patterns $\mathcal{M}_1$ and $\mathcal{M}_2$ of ranks n_1 and n_2, respectively, with index sets I_1 and I_2, and coefficient groups $\mathbb{P}_1$ and $\mathbb{P}_2$. We will construct the exchange pattern $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ (the direct product of $\mathcal{M}_1$ and $\mathcal{M}_2$) of rank $n = n_1 + n_2$, with the index set $I = I_1 \sqcup I_2$, and coefficient group $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$. Consider the tree $\mathbb{T}_n$ whose edges are colored by I, and, for $\nu \in \{1, 2\}$, let $\pi_\nu : \mathbb{T}_n \to \mathbb{T}_{n_\nu}$ be a map with the following property: if $t \overset{i}{\text{---}} t'$ in $\mathbb{T}_n$ and $i \in I_\nu$ (resp., $i \in I - I_\nu = I_{3-\nu}$), then $\pi_\nu(t) \overset{i}{\text{---}} \pi_\nu(t')$ in $\mathbb{T}_{n_\nu}$ (resp., $\pi_\nu(t) = \pi_\nu(t')$). Clearly, such a map π_ν exists and is essentially unique: it is determined by specifying the image of any vertex of $\mathbb{T}_n$. We now introduce the exchange pattern $\mathcal{M}$ on $\mathbb{T}_n$ by setting, for every $t \in \mathbb{T}_n$ and $i \in I_\nu \subset I$, the monomial $M_i(t)$ to be equal to $M_i(\pi_\nu(t))$, the latter monomial coming from the exchange pattern $\mathcal{M}_\nu$. The axioms (2.4)–(2.7) for $\mathcal{M}$ are checked directly.

Proposition 2.7. *Let $\mathcal{A}_1 = \mathcal{A}_{\mathbb{A}_1}(\mathcal{M}_1)$ and $\mathcal{A}_2 = \mathcal{A}_{\mathbb{A}_2}(\mathcal{M}_2)$ be cluster algebras. Let $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ and $\mathbb{A} = \mathbb{A}_1 \otimes \mathbb{A}_2$. Then the cluster algebra $\mathcal{A}_{\mathbb{A}}(\mathcal{M})$ is canonically isomorphic to the tensor product of algebras $\mathcal{A}_1 \otimes \mathcal{A}_2$ (all tensor products are taken over $\mathbb{Z}$).*

Proof. Let us identify each cluster variable $x_i(t)$, for $t \in \mathbb{T}_n$ and $i \in I_1 \subset I$ (resp., $i \in I_2 \subset I$), with $x_i(\pi_1(t)) \otimes 1$ (resp., $1 \otimes x_i(\pi_2(t))$). Under this identification, the exchange relations for the exchange pattern $\mathcal{M}$ become identical to the exchange relations for $\mathcal{M}_1$ and $\mathcal{M}_2$. $\square$

3. The Laurent phenomenon

In this section we prove the following important property of cluster algebras.

Theorem 3.1. *In a cluster algebra, any cluster variable is expressed in terms of any given cluster as a Laurent polynomial with coefficients in the group ring $\mathbb{Z}\mathbb{P}$.*

We conjecture that each of the coefficients in these Laurent polynomials is actually a *nonnegative* integer linear combination of elements in $\mathbb{P}$.

We will obtain Theorem 3.1 as a corollary of a more general result, which applies to more general underlying graphs and more general (not necessarily binomial) exchange polynomials.

Since Theorem 3.1 is trivial for $n = 1$, we shall assume that $n \geq 2$. For every $m \geq 1$, let $\mathbb{T}_{n,m}$ be a tree of the form shown in Figure 2. The tree $\mathbb{T}_{n,m}$ has m vertices of degree n in its "spine" and $m(n-2) + 2$ vertices of degree 1. We label every edge of the tree by an element of an n-element index set I, so that the n edges incident to each vertex on the spine receive different labels. (The reader may wish to think of the tree $\mathbb{T}_{n,m}$ as being part of the n-regular tree $\mathbb{T}_n$ of the cluster-algebra setup.)
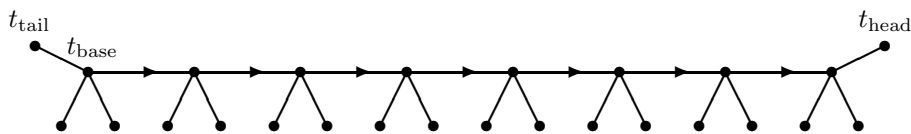

FIGURE 2. The "caterpillar" tree $\mathbb{T}_{n,m}$, for $n = 4$, $m = 8$

We fix two vertices t_{head} and t_{tail} of $\mathbb{T}_{n,m}$ that do not belong to the spine and are connected to its opposite ends. This gives rise to the orientation on the spine: away from t_{tail} and towards t_{head} (see Figure 2).

As before, let $\mathbb{P}$ be an abelian group without torsion, written multiplicatively. Let $\mathbb{Z}_{\geq 0}\mathbb{P}$ denote the additive semigroup generated by $\mathbb{P}$ in the integer group ring $\mathbb{Z}\mathbb{P}$. Assume that a nonzero polynomial P in the variables x_i, $i \in I$, with coefficients in $\mathbb{Z}_{\geq 0}\mathbb{P}$, is associated with every edge $t \relbar\joinrel\relbar t'$ of $\mathbb{T}_{n,m}$. We call P an *exchange polynomial,* and write $t \xrightarrow{\quad} t'$ to describe this situation. Suppose that the exchange polynomials associated with the edges of $\mathbb{T}_{n,m}$ satisfy the following conditions:

(3.1) An exchange polynomial associated with an edge labeled by j does not

depend on x_j, and is not divisible by any x_i, $i \in I$.

(3.2) If $t_0 \xrightarrow[P]{i} t_1 \xrightarrow{j} t_2 \xrightarrow[R]{i} t_3$, then $R = C \cdot (P|_{x_j \leftarrow Q_0/x_j})$, where $Q_0 = Q|_{x_i=0}$,

and C is a Laurent polynomial with coefficients in $\mathbb{Z}_{\geq 0}\mathbb{P}$.

(Note the orientation of the edge $t_1 \to t_2$ in (3.2).)

For every vertex t on the spine, let $\mathcal{P}(t)$ denote the family of n exchange polynomials associated with the edges emanating from t. Also, let $\mathcal{C}$ denote the collection of all Laurent polynomials C that appear in condition (3.2), for all possible choices

of t_0, t_1, t_2, t_3, and let $\mathbb{A} \subset \mathbb{ZP}$ denote the subring with unit generated by all coefficients of the Laurent polynomials from $\mathcal{P}(t_{\mathrm{base}}) \cup \mathcal{C}$, where t_{base} is the vertex on the spine connected with t_{tail}.

As before, we associate a *cluster* $\mathbf{x}(t) = \{x_i(t) \ : \ i \in I\}$ to each vertex $t \in \mathbb{T}_{n,m}$, and consider the field $\mathcal{F}(t)$ of rational functions in these variables with coefficients in $\mathbb{ZP}$. All these fields are identified with each other by the transition isomorphisms $R_{tt'} : \mathcal{F}(t') \to \mathcal{F}(t)$ defined as in (2.9). We then view the fields $\mathcal{F}(t)$ as a single field $\mathcal{F}$ that contains all the elements $x_i(t)$, for $t \in \mathbb{T}_{n,m}$ and $i \in I$. These elements satisfy the exchange relations (2.1) and the following version of (2.2):

$$x_j(t)\, x_j(t') = P(\mathbf{x}(t)) \ ,$$

for any edge $t \ \frac{j}{P}\ t'$ in $\mathbb{T}_{n,m}$.

Theorem 3.2. *If conditions* (3.1)–(3.2) *are satisfied, then each element of the cluster* $\mathbf{x}(t_{\mathrm{head}})$ *is a Laurent polynomial in the cluster* $\mathbf{x}(t_{\mathrm{tail}})$, *with coefficients in the ring* $\mathbb{A}$.

We note that Theorem 3.2 is indeed a generalization of Theorem 3.1, for the following reasons:

- $\mathbb{T}_{n,m}$ is naturally embedded into $\mathbb{T}_n$;
- conditions (3.1)–(3.2) are less restrictive than (2.4)–(2.7);
- the claim being made in Theorem 3.2 about coefficients of the Laurent polynomials is stronger than that of Theorem 3.1, since $\mathbb{A} \subset \mathbb{ZP}$.

Proof. We start with some preparations. We shall write any Laurent polynomial L in the variables $\mathbf{x} = \{x_i : i \in I\}$ in the form

$$L(\mathbf{x}) = \sum_{\alpha \in S} u_\alpha(L) x^\alpha \ ,$$

where all coefficients $u_\alpha(L)$ are nonzero, S is a finite subset of the lattice $\mathbb{Z}^I$ (i.e., the lattice of rank n with coordinates labeled by I), and x^α is the usual shorthand for $\prod_i x_i^{\alpha_i}$. The set S is called the *support* of L and denoted by $S = S(L)$.

Notice that once we fix the collection $\mathcal{C}$, condition (3.2) can be used as a recursive rule for computing $\mathcal{P}(t')$ from $\mathcal{P}(t)$, for any edge $t \xrightarrow{j} t'$ on the spine. It follows that the whole pattern of exchange polynomials is determined by the families of polynomials $\mathcal{P}(t_{\mathrm{base}})$ and $\mathcal{C}$. Moreover, since these polynomials have coefficients in $\mathbb{Z}_{\geq 0}\mathbb{P}$, and the expression for R in (3.2) does not involve subtraction, it follows that the support of any exchange polynomial is uniquely determined by the supports of the polynomials from $\mathcal{P}(t_{\mathrm{base}})$ and $\mathcal{C}$. Note that condition (3.1) can be formulated as a set of restrictions on these supports. In particular, it requires that in the situation of (3.2), the Laurent polynomial C does not depend on x_i and is a *polynomial* in x_j; in other words, every $\alpha \in S(C)$ should have $\alpha_i = 0$ and $\alpha_j \geq 0$.

We now fix a family of supports $S(L)$, for all $L \in \mathcal{P}(t_{\mathrm{base}}) \cup \mathcal{C}$, and assume that this family complies with (3.1). As is common in algebra, we shall view the coefficients $u_\alpha(L)$, for all $L \in \mathcal{P}(t_{\mathrm{base}}) \cup \mathcal{C}$ and $\alpha \in S(L)$, as *indeterminates*. Then all the coefficients in all exchange polynomials become "canonical" (i.e., independent of the choice of $\mathbb{P}$) polynomials in these indeterminates, with positive integer coefficients.

The above discussion shows that it suffices to prove our theorem in the following "universal coefficients" setup: let $\mathbb{P}$ be the free abelian group (written multiplicatively) with generators $u_\alpha(L)$, for all $L \in \mathcal{P}(t_{\text{base}}) \cup \mathcal{C}$ and $\alpha \in S(L)$. Under this assumption, $\mathbb{A}$ is simply the integer polynomial ring in the indeterminates $u_\alpha(L)$.

Recall that we can view all cluster variables $x_i(t)$ as elements of the field $\mathcal{F}(t_{\text{tail}})$ of rational functions in the cluster $x(t_{\text{tail}})$ with coefficients in $\mathbb{Z}\mathbb{P}$. For $t \in \mathbb{T}_{n,m}$, let $\mathcal{L}(t)$ denote the ring of Laurent polynomials in the cluster $\mathbf{x}(t)$, with coefficients in $\mathbb{A}$. We view each $\mathcal{L}(t)$ as a subring of the ambient field $\mathcal{F}(t_{\text{tail}})$.

In this terminology, our goal is to show that the cluster $\mathbf{x}(t_{\text{head}})$ is contained in $\mathcal{L}(t_{\text{tail}})$. We proceed by induction on m, the size of the spine. The claim is trivial for $m = 1$, so let us assume that $m \geq 2$, and furthermore assume that our statement is true for all "caterpillars" with smaller spine.
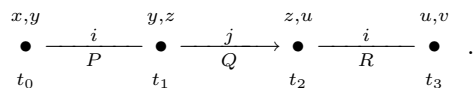
Let us abbreviate $t_0 = t_{\text{tail}}$ and $t_1 = t_{\text{base}}$, and suppose that the path from t_{tail} to t_{head} starts with the following two edges: $t_0 \xrightarrow{\ i\ }_{P} t_1 \xrightarrow{\ j\ }_{Q} t_2$. Let $t_3 \in \mathbb{T}_{n,m}$ be the vertex such that $t_2 \xrightarrow{\ i\ }_{R} t_3$.

The following lemma plays a crucial role in our proof.

Lemma 3.3. *The clusters $\mathbf{x}(t_1)$, $\mathbf{x}(t_2)$, and $\mathbf{x}(t_3)$ are contained in $\mathcal{L}(t_0)$. Furthermore, $\gcd(x_i(t_3), x_i(t_1)) = \gcd(x_j(t_2), x_i(t_1)) = 1$ (as elements of $\mathcal{L}(t_0)$).*

Note that $\mathcal{L}_0 = \mathcal{L}(t_0)$ is a unique factorization domain, so any two elements $x, y \in \mathcal{L}_0$ have a well-defined greatest common divisor $\gcd(x, y)$, which is an element of $\mathcal{L}_0$ defined up to a multiple from the group $\mathcal{L}_0^\times$ of units (that is, invertible elements) of $\mathcal{L}_0$. In our "universal" situation, $\mathcal{L}_0^\times$ consists of Laurent monomials in the cluster $\mathbf{x}(t_0)$ with coefficients ± 1.

Proof. The only element from the clusters $\mathbf{x}(t_1)$, $\mathbf{x}(t_2)$, and $\mathbf{x}(t_3)$ whose inclusion in $\mathcal{L}_0$ is not immediately obvious is $x_i(t_3)$. To simplify the notation, let us denote $x = x_i(t_0)$, $y = x_j(t_0) = x_j(t_1)$, $z = x_i(t_1) = x_i(t_2)$, $u = x_j(t_2) = x_j(t_3)$, and $v = x_i(t_3)$, so that these variables appear in the clusters at $t_0, \ldots, t_3$, as shown below:

$$\begin{array}{ccccccc}
x,y & & y,z & & z,u & & u,v \\
\bullet & \xrightarrow{\ i\ }_{P} & \bullet & \xrightarrow{\ j\ }_{Q} & \bullet & \xrightarrow{\ i\ }_{R} & \bullet \\
t_0 & & t_1 & & t_2 & & t_3
\end{array} \quad .$$

Note that the variables x_k, for $k \notin \{i, j\}$, do not change as we move among the four clusters under consideration. The lemma is then restated as saying that

(3.3) $\qquad\qquad\qquad v \in \mathcal{L}_0;$

(3.4) $\qquad\qquad\qquad \gcd(z, u) = 1 \ (\text{as elements of } \mathcal{L}_0);$

(3.5) $\qquad\qquad\qquad \gcd(z, v) = 1 \ (\text{as elements of } \mathcal{L}_0).$

Another notational convention will be based on the fact that each of the polynomials P, Q, R has a distinguished variable on which it depends, namely x_j for P and R, and x_i for Q. (In view of (3.1), P and R do not depend on x_i, while Q does not depend on x_j.) With this in mind, we will routinely write P, Q, and R as polynomials in one (distinguished) variable. In the same spirit, the notation Q', R', etc., will refer to the partial derivative with respect to the distinguished variable.

We will prove the statements (3.3), (3.4), and (3.5) one by one, in this order.

By (3.2), the polynomial R is given by

$$(3.6) \qquad R(u) = C(u)P\left(\tfrac{Q(0)}{u}\right),$$

where C is an "honest" polynomial in u and a Laurent polynomial in the "mute" variables x_k, $k \notin \{i, j\}$. (Recall that C does not depend on x_i.) We then have:

$$z = \frac{P(y)}{x}\,;$$

$$u = \frac{Q(z)}{y} = \frac{Q\left(\tfrac{P(y)}{x}\right)}{y}\,;$$

$$v = \frac{R(u)}{z} = \frac{R\left(\tfrac{Q(z)}{y}\right)}{z} = \frac{R\left(\tfrac{Q(z)}{y}\right) - R\left(\tfrac{Q(0)}{y}\right)}{z} + \frac{R\left(\tfrac{Q(0)}{y}\right)}{z}\,.$$

Since

$$\frac{R\left(\tfrac{Q(z)}{y}\right) - R\left(\tfrac{Q(0)}{y}\right)}{z} \in \mathcal{L}_0$$

and

$$\frac{R\left(\tfrac{Q(0)}{y}\right)}{z} = \frac{C\left(\tfrac{Q(0)}{y}\right)P(y)}{z} = C\left(\tfrac{Q(0)}{y}\right)x \in \mathcal{L}_0\,,$$

(3.3) follows.

We next prove (3.4). We have

$$u = \frac{Q(z)}{y} \equiv \frac{Q(0)}{y} \bmod z\,.$$

Since x and y are invertible in $\mathcal{L}_0$, we conclude that $\gcd(z, u) = \gcd(P(y), Q(0))$. Now the trouble that we took in passing to universal coefficients finally pays off: since $P(y)$ and $Q(0)$ are nonzero polynomials in the cluster $\mathbf{x}(t_0)$ whose coefficients are distinct generators of the polynomial ring $\mathbb{A}$, it follows that $\gcd(P(y), Q(0)) = 1$, proving (3.4).

It remains to prove (3.5). Let

$$f(z) = R\left(\tfrac{Q(z)}{y}\right).$$

Then

$$v = \frac{f(z) - f(0)}{z} + C\left(\tfrac{Q(0)}{y}\right)x\,.$$

Our goal is to show that $\gcd(z, v) = 1$; to this end, we are going to compute $v \bmod z$ as "explicitly" as possible. We have, $\bmod z$,

$$\frac{f(z) - f(0)}{z} \equiv f'(0) = R'\left(\tfrac{Q(0)}{y}\right) \cdot \tfrac{Q'(0)}{y}\,.$$

Hence

$$v \equiv R'\left(\tfrac{Q(0)}{y}\right) \cdot \tfrac{Q'(0)}{y} + C\left(\tfrac{Q(0)}{y}\right)x \bmod z\,.$$

Note that the right-hand side is a *linear* polynomial in x, whose coefficients are Laurent polynomials in the rest of the variables of the cluster $\mathbf{x}(t_0)$. Thus our claim will follow if we show that $\gcd\left(C\left(\tfrac{Q(0)}{y}\right), P(y)\right) = 1$. This, again, is a

consequence of our "universal coefficients" setup since the coefficients of C, P and Q are distinct generators of the polynomial ring $\mathbb{A}$. □

We can now complete the proof of Theorem 3.2. We need to show that any variable $x = x_k(t_{\text{head}})$ belongs to $\mathcal{L}(t_0)$. Since both t_1 and t_3 are closer to t_{head} than t_0, we can use the inductive assumption to conclude that x belongs to both $\mathcal{L}(t_1)$ and $\mathcal{L}(t_3)$. Since $x \in \mathcal{L}(t_1)$, it follows from (2.1) that x can be written as $x = f/x_i(t_1)^a$ for some $f \in \mathcal{L}(t_0)$ and $a \in \mathbb{Z}_{\geq 0}$. On the other hand, since $x \in \mathcal{L}(t_3)$, it follows from (2.1) and from the inclusion $x_i(t_3) \in \mathcal{L}(t_0)$ guaranteed by Lemma 3.3 that x has the form $x = g/x_j(t_2)^b x_i(t_3)^c$ for some $g \in \mathcal{L}(t_0)$ and some $b, c \in \mathbb{Z}_{\geq 0}$. The inclusion $x \in \mathcal{L}(t_0)$ now follows from the fact that, by the last statement in Lemma 3.3, the denominators in the two obtained expressions for x are coprime in $\mathcal{L}(t_0)$. □

Several examples that can be viewed as applications of Theorem 3.2 are given in [8].

4. Exchange relations: The exponents

Let $\mathcal{M} = (M_j(t)) : t \in \mathbb{T}_n, j \in I)$ be an exchange pattern (see Definition 2.1). In this section we will ignore the coefficients in the monomials $M_j(t)$ and take a closer look at the dynamics of their exponents. (An alternative point of view that the reader may find helpful is to assume that all exchange patterns considered in this section will have all their coefficients $p_j(t)$ equal to 1.) For every edge $t \overset{j}{\text{---}} t'$ in $\mathbb{T}_n$, let us write the ratio $M_j(t)/M_j(t')$ of the corresponding monomials as

$$(4.1) \qquad \frac{M_j(t)}{M_j(t')} = \frac{p_j(t)}{p_j(t')} \prod_i x_i^{b_{ij}(t)} \ ,$$

where $b_{ij}(t) \in \mathbb{Z}$ (cf. (2.3)); we note that ratios of this kind have already appeared in (2.7). Let us denote by $B(t) = (b_{ij}(t))$ the $n \times n$ integer matrix whose entries are the exponents in (4.1). In view of (2.5), the exponents in $M_j(t)$ and $M_j(t')$ are recovered from $B(t)$:

$$(4.2) \qquad \begin{aligned} M_j(t) &= p_j(t) \prod_{i: \ b_{ij}(t)>0} x_i^{b_{ij}(t)}, \\ M_j(t') &= p_j(t') \prod_{i: \ b_{ij}(t)<0} x_i^{-b_{ij}(t)} \ . \end{aligned}$$

Thus, the family of matrices $(B(t))_{t \in \mathbb{T}_n}$ encodes all the exponents in all monomials of an exchange pattern.

We shall describe the conditions on the family of matrices $(B(t))$ imposed by the axioms of an exchange pattern. To do this, we need some preparation.

Definition 4.1. A square integer matrix $B = (b_{ij})$ is called *sign-skew-symmetric* if, for any i and j, either $b_{ij} = b_{ji} = 0$, or else b_{ij} and b_{ji} are of opposite sign; in particular, $b_{ii} = 0$ for all i.

Definition 4.2. Let $B = (b_{ij})$ and $B' = (b'_{ij})$ be square integer matrices of the same size. We say that B' is obtained from B by the *matrix mutation* in direction k

and write $B' = \mu_k(B)$ if

$$
(4.3) \qquad b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\[2mm] b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases}
$$

An immediate check shows that μ_k is *involutive*, i.e., its square is the identity transformation.

Proposition 4.3. *A family of $n \times n$ integer matrices $(B(t))_{t \in \mathbb{T}_n}$ corresponds to an exchange pattern if and only if the following conditions hold:*

(1) $B(t)$ *is sign-skew-symmetric for any $t \in \mathbb{T}_n$.*

(2) *If* $t \xrightarrow{\ k\ } t'$, *then* $B(t') = \mu_k(B(t))$.

Proof. We start with the "only if" part, i.e., we assume that the matrices $B(t)$ are determined by an exchange pattern via (4.1) and check the conditions (1)–(2). The condition $b_{jj}(t) = 0$ follows from (2.4). The remaining part of (1) (dealing with $i \neq j$), follows at once from (2.6). Turning to part (2), the equality $b'_{ik} = -b_{ik}$ is immediate from the definition (4.1). Now suppose that $j \neq k$. In this case, we apply the axiom (2.7) to the edge $t \xrightarrow{\ k\ } t'$ taken together with the two adjacent edges emanating from t and t' and labeled by j. Taking (4.2) into account, we obtain:

$$
(4.4) \qquad \prod_i x_i^{b'_{ij}} = \prod_i x_i^{b_{ij}} \Big|_{x_k \leftarrow M/x_k} ,
$$

where $M = \prod_{i : b_{ik} b_{jk} < 0} x_i^{|b_{ik}|}$. Comparing the exponents of x_k on both sides of (4.4) yields $b'_{kj} = -b_{kj}$. Finally, if $i \neq k$, then comparing the exponents of x_i on both sides of (4.4) gives

$$
b'_{ij} = \begin{cases} b_{ij} & \text{if } b_{ik} b_{jk} \geq 0; \\[2mm] b_{ij} + |b_{ik}|b_{kj} & \text{otherwise.} \end{cases}
$$

To complete the proof of (2), it remains to notice that, in view of the already proven part (1), the condition $b_{ik} b_{jk} \geq 0$ is equivalent to $b_{ik} b_{kj} \leq 0$, which makes the last formula equivalent to (4.3).

To prove the "if" part, it suffices to show that if the matrices $B(t)$ satisfy (1)–(2), then the monomials $M_j(t)$ given by the first equality in (4.2) (with $p_j(t) = 1$) satisfy the axioms of an exchange pattern. This is done by a direct check. $\square$

Since all matrix mutations are involutive, any choice of an initial vertex $t_0 \in \mathbb{T}_n$ and an arbitrary $n \times n$ integer matrix B gives rise to a unique family of integer matrices $B(t)$ satisfying condition (2) in Proposition 4.3 and such that $B(t_0) = B$. Thus, the exponents in all monomials $M_j(t)$ are uniquely determined by a single matrix $B = B(t_0)$. By Proposition 4.3, in order to determine an exchange pattern, B must be such that all matrices obtained from it by a sequence of matrix mutations are sign-skew-symmetric. Verifying that a given matrix B has this property seems to be quite nontrivial in general. Fortunately, there is another restriction on B that is much easier to check, which implies the desired property, and still leaves us with a large class of matrices sufficient for most applications.

Definition 4.4. A square integer matrix $B = (b_{ij})$ is called *skew-symmetrizable* if there exists a diagonal *skew-symmetrizing* matrix D with positive integer diagonal entries d_i such that DB is skew-symmetric, i.e., $d_i b_{ij} = -d_j b_{ji}$ for all i and j.

Proposition 4.5. *For every choice of a vertex $t_0 \in \mathbb{T}_n$ and a skew-symmetrizable matrix B, there exists a unique family of matrices $(B(t))_{t \in \mathbb{T}_n}$ associated with an exchange pattern on $\mathbb{T}_n$ and such that $B(t_0) = B$. Furthermore, all the matrices $B(t)$ are skew-symmetrizable, sharing the same skew-symmetrizing matrix.*

Proof. The proof follows at once from the following two observations:

1. Every skew-symmetrizable matrix B is sign-skew-symmetric.

2. If B is skew-symmetrizable and $B' = \mu_k(B)$, then B' is also skew-symmetrizable, with the same skew-symmetrizing matrix. $\square$

We call an exchange pattern—and the corresponding cluster algebra—*skew-symmetrizable* if all the matrices $B(t)$ given by (4.1) (equivalently, one of them) are skew-symmetrizable. In particular, all cluster algebras of rank $n \leq 2$ are skew-symmetrizable: for $n = 1$ we have $B(t) \equiv (0)$, while for $n = 2$, the calculations in Example 2.5 show that one can take

$$(4.5) \qquad B(t_m) = (-1)^m \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$$

for all $m \in \mathbb{Z}$, in the notation of (2.10)–(2.11).

Remark 4.6. Skew-symmetrizable matrices are closely related to symmetrizable (generalized) Cartan matrices appearing in the theory of Kac-Moody algebras. More generally, to every sign-skew-symmetric matrix $B = (b_{ij})$ we can associate a generalized Cartan matrix $A = A(B) = (a_{ij})$ of the same size by setting

$$(4.6) \qquad a_{ij} = \begin{cases} 2 & \text{if } i = j; \\ -|b_{ij}| & \text{if } i \neq j. \end{cases}$$

There seem to be deep connections between the cluster algebra corresponding to B and the Kac-Moody algebra associated with $A(B)$. We exhibit such a connection for the rank 2 case in Section 6 below. This is however just the tip of an iceberg: a much more detailed analysis will be presented in the sequel to this paper.

In order to show that non-skew-symmetrizable exchange patterns do exist, we conclude this section by exhibiting a 3-parameter family of such patterns of rank 3.

Proposition 4.7. *Let α, β, and γ be three positive integers such that $\alpha\beta\gamma \geq 3$. There exists a unique family of matrices $(B(t))_{t \in \mathbb{T}_3}$ associated with a non-skew-symmetrizable exchange pattern on $\mathbb{T}_3$ and such that the matrix $B(t_0)$ at a given vertex $t_0 \in \mathbb{T}_3$ is equal to*

$$(4.7) \qquad B(\alpha, \beta, \gamma) = \begin{bmatrix} 0 & 2\alpha & -2\alpha\beta \\ -\beta\gamma & 0 & 2\beta \\ \gamma & -\alpha\gamma & 0 \end{bmatrix}.$$

Proof. First of all, the matrix $B(\alpha, \beta, \gamma)$ is sign-skew-symmetric but not skew-symmetrizable. Indeed, any skew-symmetrizable matrix $B = (b_{ij})$ satisfies the equation $b_{12}b_{23}b_{31} = -b_{21}b_{32}b_{13}$. However, this equation for $B(\alpha, \beta, \gamma)$ holds only when $\alpha\beta\gamma$ is equal to 0 or 2.

For the purpose of this proof only, we refer to a 3×3 matrix B as *cyclical* if its entries follow one of the two sign patterns

$$\begin{bmatrix} 0 & + & - \\ - & 0 & + \\ + & - & 0 \end{bmatrix}, \begin{bmatrix} 0 & - & + \\ + & 0 & - \\ - & + & 0 \end{bmatrix} .$$

In particular, $B(\alpha, \beta, \gamma)$ is cyclical; to prove the proposition, it suffices to show that any matrix obtained from it by a sequence of matrix mutations is also cyclical.

The set of cyclical matrices is not stable under matrix mutations. Let us define some subsets of cyclical matrices that behave nicely with respect to matrix mutations. For a 3×3 matrix B, we denote

$$c_1 = |b_{23}b_{32}|, \ c_2 = |b_{13}b_{31}|, \ c_3 = |b_{12}b_{21}|, \ r = |b_{12}b_{23}b_{31}| .$$

For $i \in \{1, 2, 3\}$, we say that B is *i-biased* if we have

$$r > c_i \geq r/2 \geq c_j \geq 6$$

for any $j \in \{1, 2, 3\} \setminus \{i\}$. For $B(\alpha, \beta, \gamma)$, we have

$$r/2 = c_1 = c_2 = c_3 = 2\alpha\beta\gamma \geq 6 ,$$

so it is i-biased for every $i \in \{1, 2, 3\}$.

Our proposition becomes an immediate consequence of the following lemma.

Lemma 4.8. *Suppose B is cyclical and i-biased, and let $j \neq i$. Then $\mu_j(B)$ is cyclical and j-biased.*

Proof. Without loss of generality we can assume that $i = 1$ and $j = 2$. Denote $B' = \mu_2(B)$, and let us write $r' = |b'_{12}b'_{23}b'_{31}|$, $c'_1 = |b'_{23}b'_{32}|$, etc. By (4.3), we have $b'_{12} = -b_{12}$, $b'_{21} = -b_{21}$, $b'_{23} = -b_{23}$, $b'_{32} = -b_{32}$; therefore, $c'_1 = c_1$ and $c'_3 = c_3$. We also have

$$(4.8) \qquad b'_{13} = b_{13} + \frac{|b_{12}|b_{23} + b_{12}|b_{23}|}{2}, \ b'_{31} = b_{31} + \frac{|b_{32}|b_{21} + b_{32}|b_{21}|}{2} .$$

Since B is cyclical, the two summands on the right-hand side of each of the equalities in (4.8) have opposite signs. Note that

$$|b_{12}b_{23}| = \frac{r}{|b_{31}|} \geq \frac{2c_2}{|b_{31}|} = 2|b_{13}|$$

and

$$|b_{32}b_{21}| = \frac{c_1|b_{21}|}{|b_{23}|} \geq \frac{r|b_{21}|}{2|b_{23}|} = \frac{c_3|b_{31}|}{2} \geq 3|b_{31}| .$$

It follows that b'_{13} (resp., b'_{31}) has the opposite sign to b_{13} (resp., b_{31}). Thus, B' is cyclical, and it only remains to show that B' is 2-biased. To this effect, we note that $|b'_{31}| = |b_{32}b_{21}| - |b_{31}|$, and so

$$r' = |b'_{12}b'_{23}b'_{31}| = |b_{12}||b_{23}|(|b_{32}b_{21}| - |b_{31}|) = c_1c_3 - r \geq (r/2) \cdot 6 - r = 2r ;$$

therefore, both $c'_1 = c_1$ and $c'_3 = c_3$ do not exceed $r'/2$. As for c'_2, we have $|b'_{13}| = |b_{12}b_{23}| - |b_{13}|$, and so

$$c'_2 = |b'_{13}b'_{31}| = (|b_{12}b_{23}| - |b_{13}|)(|b_{32}b_{21}| - |b_{31}|) = r'(1 - \frac{c_2}{r}) ;$$

since $c_2/r \leq 1/2$, we conclude that $r' > r'(1 - \frac{c_2}{r}) = c'_2 \geq r'/2$. This completes the proof that B' is 2-biased. Lemma 4.8 and Proposition 4.7 are proved. $\square$

5. Exchange relations: The coefficients

In this section we fix a family of matrices $B(t)$ satisfying the conditions in Proposition 4.3, and discuss possible choices of coefficients $p_j(t)$ that can appear in the corresponding exchange pattern. We start with the following simple characterization.

Proposition 5.1. *Assume that matrices* $(B(t))_{t \in \mathbb{T}_n}$ *satisfy conditions* (1)–(2) *in Proposition 4.3. A family of elements* $p_j(t)$ *of a coefficient group* $\mathbb{P}$ *gives rise, via* (4.2), *to an exchange pattern if and only if they satisfy the following relations, whenever* $t_1 \overset{i}{\relbar\joinrel\relbar} t_2 \overset{j}{\relbar\joinrel\relbar} t_3 \overset{i}{\relbar\joinrel\relbar} t_4$:

$$(5.1) \qquad p_i(t_1)p_i(t_3)p_j(t_3)^{\max(b_{ji}(t_3),0)} = p_i(t_2)p_i(t_4)p_j(t_2)^{\max(b_{ji}(t_2),0)} .$$

Note that $b_{ji}(t_2) = -b_{ji}(t_3)$ by (4.3), so at most one of the $p_j(t_2)$ and $p_j(t_3)$ actually enters (5.1).

Proof. The only axiom of an exchange pattern that involves the coefficients is (2.7), and the relation (5.1) is precisely what it prescribes. $\qquad\square$

First of all, let us mention the trivial solution of (5.1) when all the coefficients $p_i(t)$ are equal to 1.

Moving in the opposite direction, we introduce the *universal coefficient group* (with respect to a fixed family $(B(t))$) as the abelian group $\mathbb{P}$ generated by the elements $p_i(t)$, for all $i \in I$ and $t \in \mathbb{T}_n$, which has (5.1) as the system of defining relations. The torsion-freeness of this group is guaranteed by the following proposition whose straightforward proof is omitted.

Proposition 5.2. *The universal coefficient group* $\mathbb{P}$ *is a free abelian group. More specifically, let* $t_0 \in \mathbb{T}_n$, *and let* S *be a collection of pairs* (i, t) *that contains both* (i, t_0) *and* (i, t) *for any edge* $t_0 \overset{i}{\relbar\joinrel\relbar} t$, *and precisely one of the pairs* (i, t) *and* (i, t') *for each edge* $t \overset{i}{\relbar\joinrel\relbar} t'$ *with* t *and* t' *different from* t_0. *Then* $\{p_i(t) : (i, t) \in S\}$ *is a set of free generators for* $\mathbb{P}$.

We see that, in contrast to (4.3), relations (5.1) leave infinitely many degrees of freedom in determining the coefficients $p_i(t)$ (cf. Remark 2.2).

The rest of this section is devoted to important classes of exchange patterns within which all the coefficients are completely determined by specifying $2n$ of them corresponding to the edges emanating from a given vertex.

Suppose that, in addition to the multiplicative group structure, the coefficient group $\mathbb{P}$ is supplied with a binary operation $\oplus$ that we call *auxiliary addition*. Furthermore, suppose that this operation is commutative, associative, and distributive with respect to multiplication; thus $(\mathbb{P}, \oplus, \cdot)$ is a *semifield*. (By the way, under these assumptions $\mathbb{P}$ is automatically torsion-free as a multiplicative group: indeed, if $p^m = 1$ for some $p \in \mathbb{P}$ and $m \geq 2$, then

$$p = \frac{p^m \oplus p^{m-1} \oplus \cdots \oplus p}{p^{m-1} \oplus p^{m-2} \oplus \cdots \oplus 1} = \frac{p^{m-1} \oplus p^{m-2} \oplus \cdots \oplus 1}{p^{m-1} \oplus p^{m-2} \oplus \cdots \oplus 1} = 1 \ .)$$

Definition 5.3. An exchange pattern and the corresponding cluster algebra are called *normalized* if $\mathbb{P}$ is a semifield and $p_j(t) \oplus p_j(t') = 1$ for any edge $t \overset{j}{\relbar\joinrel\relbar} t'$.

Proposition 5.4. *Fix a vertex $t_0 \in \mathbb{T}_n$ and $2n$ elements q_j and r_j $(j \in I)$ of a semifield $\mathbb{P}$ such that $q_j \oplus r_j = 1$ for all j. Then every family of matrices $B(t)$ satisfying the conditions in Proposition 4.3 gives rise to a unique normalized exchange pattern such that, for every edge $t_0 \overset{j}{\text{---}} t$, we have $p_j(t_0) = q_j$ and $p_j(t) = r_j$. Thus a normalized exchange pattern is completely determined by the $2n$ monomials $M_j(t_0)$ and $M_j(t)$, for all edges $t_0 \overset{j}{\text{---}} t$.*

Proof. In a normalized exchange pattern, the coefficients $p_j(t)$ and $p_j(t')$ corresponding to an edge $t \overset{j}{\text{---}} t'$ are determined by their ratio

$$(5.2) \qquad u_j(t) = \frac{p_j(t)}{p_j(t')}$$

via

$$(5.3) \qquad p_j(t) = \frac{u_j(t)}{1 \oplus u_j(t)}, \quad p_j(t') = \frac{1}{1 \oplus u_j(t)} \ .$$

Clearly, we have

$$(5.4) \qquad u_j(t)u_j(t') = 1$$

for any edge $t \overset{j}{\text{---}} t'$. We can also rewrite the relation (5.1) as follows:

$$(5.5) \qquad u_i(t') = u_i(t)u_j(t)^{\max(b_{ji}(t),0)}(1 \oplus u_j(t))^{-b_{ji}(t)}$$

for any edge $t \overset{j}{\text{---}} t'$ and any $i \neq j$. This form of the relations for the normalized coefficients makes our proposition obvious. $\square$

Remark 5.5. It is natural to ask whether the normalization condition imposes additional multiplicative relations among the coefficients $p_i(t)$ that are not consequences of (5.1). In other words: can a normalized system of coefficients generate the universal coefficient group? In the next section, we present a complete answer to this question in the rank 2 case (see Remark 6.5 below).

One example of a semifield is the multiplicative group $\mathbb{R}_{>0}$ of positive real numbers, $\oplus$ being ordinary addition. However the following example is more important for our purposes.

Example 5.6. Let $\mathbb{P}$ be a free abelian group, written multiplicatively, with a finite set of generators p_i $(i \in I')$, and with auxiliary addition $\oplus$ defined by

$$(5.6) \qquad \prod_i p_i^{a_i} \oplus \prod_i p_i^{b_i} = \prod_i p_i^{\min(a_i,b_i)} \ .$$

Then $\mathbb{P}$ is a semifield; specifically, it is a product of $|I'|$ copies of the *tropical* semifield (see, e.g., [1]). We denote this semifield by $\text{Trop}(p_i : i \in I')$. Note that if all exponents a_i and b_i in (5.6) are nonnegative, then the monomial on the right-hand side is the gcd of the two monomials on the left.

Definition 5.7. We say that an exchange pattern is of *geometric type* if it is normalized, has the coefficient semifield $\mathbb{P} = \text{Trop}(p_i : i \in I')$, and each coefficient $p_j(t)$ is a monomial in the generators p_i with all exponents *nonnegative*.

For an exchange pattern of geometric type, the normalization condition simply means that, for every edge $t \overset{j}{\text{——}} t'$, the two monomials $p_j(t)$ and $p_j(t')$ in the generators p_i are coprime, i.e., have no variable in common.

In all our examples of cluster algebras of geometric origin, including those discussed in the introduction, the exchange patterns turn out to be of geometric type. These patterns have the following useful equivalent description.

Proposition 5.8. *Let* $\mathbb{P} = \text{Trop}(p_i : i \in I')$. *A family of coefficients* $p_j(t) \in \mathbb{P}$ *gives rise to an exchange pattern of geometric type if and only if they are given by*

$$(5.7) \qquad p_j(t) = \prod_{i \in I'} p_i^{\max(c_{ij}(t),0)}$$

for some family of integers $(c_{ij}(t) : t \in \mathbb{T}_n, i \in I', j \in I))$ *satisfying the following property: for every edge* $t \overset{k}{\text{——}} t'$ *in* $\mathbb{T}_n$, *the matrices* $C(t) = (c_{ij}(t)) = (c_{ij})$ *and* $C(t') = (c_{ij}(t')) = (c'_{ij})$ *are related by*

$$(5.8) \qquad c'_{ij} = \begin{cases} -c_{ij} & \text{if } j = k; \\ c_{ij} + \dfrac{|c_{ik}|b_{kj}(t) + c_{ik}|b_{kj}(t)|}{2} & \text{otherwise.} \end{cases}$$

Proof. As in the proof of Proposition 5.4, for every edge $t \overset{j}{\text{——}} t'$ we consider the ratio $u_j(t) = p_j(t)/p_j(t')$. We introduce the matrices $C(t)$ by setting

$$u_j(t) = \prod_i p_i^{c_{ij}(t)} \ .$$

The expression (5.7) then becomes a specialization of the first equality in (5.3), with auxiliary addition given by (5.6). To derive (5.8) from (5.5), first replace i by j and j by k, respectively, then specialize, then pick up the exponent of p_i . $\square$

Comparing (5.8) with (4.3), we see that it is natural to combine a pair of matrices $(B(t), C(t))$ into one rectangular integer matrix $\tilde{B}(t) = (b_{ij})_{i \in I \coprod I', j \in I}$ by setting $b_{ij} = c_{ij}$ for $i \in I'$ and $j \in I$. Then the matrices $\tilde{B}(t)$ for $t \in \mathbb{T}_n$ are related to each other by the same matrix mutation rule (4.3), now applied to any $i \in I \coprod I'$ and $j \in I$. We refer to $B(t)$ as the *principal part* of $\tilde{B}(t)$. Combining Propositions 5.8 and 4.5, we obtain the following corollary.

Corollary 5.9. *Let* $\tilde{B}$ *be an integer matrix with skew-symmetrizable principal part. There is a unique exchange pattern* $\mathcal{M} = \mathcal{M}(\tilde{B})$ *of geometric type such that* $\tilde{B}(t_0) = \tilde{B}$ *at a given vertex* $t_0 \in \mathbb{T}_n$.

In the geometric type case, there is a distinguished choice of ground ring for the corresponding cluster algebra: take $\mathbb{A}$ to be the polynomial ring $\mathbb{Z}[p_i : i \in I']$. In the situation of Corollary 5.9, we will denote the corresponding cluster algebra $\mathcal{A}_{\mathbb{A}}(\mathcal{M})$ simply by $\mathcal{A}(\tilde{B})$. In the notation of Section 2, $\mathcal{A}(\tilde{B})$ is the subring of the ambient field $\mathcal{F}$ generated by cluster variables $x_j(t)$ for all $j \in I$ and $t \in \mathbb{T}_n$ together with the generators p_i $(i \in I')$ of $\mathbb{P}$.

The set I' is allowed to be empty: this simply means that all the coefficients $p_j(t)$ in the corresponding exchange pattern are equal to 1. In this case, we have $\tilde{B}(t) = B(t)$ for all t.

We note that the class of exchange patterns of geometric type (and the corresponding cluster algebras $\mathcal{A}(\tilde{B})$) is stable under the operations of restriction and direct product introduced in Section 2. The restriction from I to a subset J amounts to removing from $\tilde{B}$ the columns labeled by $I - J$; the direct product operation replaces two matrices $\tilde{B}_1$ and $\tilde{B}_2$ by the matrix $\begin{bmatrix} \tilde{B}_1 & 0 \\ 0 & \tilde{B}_2 \end{bmatrix}$.

6. The rank 2 case

In this section, we illustrate the above results and constructions by treating in detail the special case $n = 2$. We label vertices and edges of the tree $\mathbb{T}_2$ as in (2.10). For $m \in \mathbb{Z}$, it will be convenient to denote by $\langle m \rangle$ the element of $\{1, 2\}$ congruent to m modulo 2. Thus, $\mathbb{T}_2$ consists of vertices t_m and edges $t_m \overset{\langle m \rangle}{\longrightarrow} t_{m+1}$ for all $m \in \mathbb{Z}$. We use the notation of Example 2.5, so the clusters are of the form $\mathbf{x}(t_m) = \{y_m, y_{m+1}\}$, and the exchange relations are given by (2.11), with coefficients q_m and r_m satisfying (2.12). More specifically, we have

$$(6.1) \quad x_{\langle m \rangle}(t_m) = y_m, \ x_{\langle m+1 \rangle}(t_m) = y_{m+1}, \ p_{\langle m \rangle}(t_m) = q_{m+1}, \ p_{\langle m+1 \rangle}(t_m) = r_m$$

for $m \in \mathbb{Z}$. Choose $\mathbf{x}(t_1) = \{y_1, y_2\}$ as the *initial* cluster. According to Theorem 3.1, each cluster variable y_m can be expressed as a Laurent polynomial in y_1 and y_2 with coefficients in $\mathbb{ZP}$. Let us write this Laurent polynomial as

$$(6.2) \qquad\qquad y_m = \frac{N_m(y_1, y_2)}{y_1^{d_1(m)} y_2^{d_2(m)}},$$

where $N_m(y_1, y_2) \in \mathbb{ZP}[y_1, y_2]$ is a polynomial not divisible by y_1 or y_2. We will investigate in detail the denominators of these Laurent polynomials.

Recall that for $n = 2$ the matrices $B(t)$ are given by (4.5). Thus, all these matrices have the same associated Cartan matrix

$$(6.3) \qquad\qquad A = A(B(t)) = \begin{bmatrix} 2 & -b \\ -c & 2 \end{bmatrix}$$

(see (4.6)). We will show that the denominators in (6.2) have a nice interpretation in terms of the root system associated to A. Let us recall some basic facts about this root system (cf., e.g., [10]).

Let $Q \cong \mathbb{Z}^2$ be a lattice of rank 2 with a fixed basis $\{\alpha_1, \alpha_2\}$ of *simple roots*. The *Weyl group* $W = W(A)$ is a group of linear transformations of Q generated by two *simple reflections* s_1 and s_2 whose action in the basis of simple roots is given by

$$(6.4) \qquad\qquad s_1 = \begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 1 & 0 \\ c & -1 \end{bmatrix}.$$

Since both s_1 and s_2 are involutions, each element of W is one of the following:

$$w_1(m) = s_1 s_2 s_1 \cdots s_{\langle m \rangle}, \ w_2(m) = s_2 s_1 s_2 \cdots s_{\langle m+1 \rangle} \ ;$$

here both products are of length $m \geq 0$. It is well known that W is finite if and only if $bc \leq 3$; we shall refer to this as the *finite case*. The *Coxeter number* h of W is the order of $s_1 s_2$ in W; it is given by Table 1. In the finite case, W is the dihedral group of order $2h$, and its elements can be listed as follows: $w_1(0) = w_2(0) = e$

(the identity element), $w_1(h) = w_2(h) = w_0$ (the longest element), and $2h - 2$ distinct elements $w_1(m), w_2(m)$ for $0 < m < h$. In the infinite case, all elements $w_1(m)$ and $w_2(m)$ for $m > 0$ are distinct.

TABLE 1. The Coxeter number

bc	0	1	2	3	≥ 4
h	2	3	4	6	∞

A vector $\alpha \in Q$ is a *real root* for A if it is W-conjugate to a simple root. Let Φ denote the set of real roots for A. It is known that $\Phi = \Phi_+ \cup (-\Phi_+)$, where

$$\Phi_+ = \{\alpha = d_1\alpha_1 + d_2\alpha_2 \in \Phi : \ d_1, d_2 \geq 0\}$$

is the set of positive real roots. In the finite case, Φ_+ has cardinality h, and we have

$$\Phi_+ = \{w_1(m)\alpha_{\langle m+1 \rangle} : 0 \leq m < h\} \ .$$

In the infinite case, we have

$$\Phi_+ = \{w_1(m)\alpha_{\langle m+1 \rangle}, \ w_2(m)\alpha_{\langle m+2 \rangle} : m \geq 0\} \ ,$$

with all the roots $w_1(m)\alpha_{\langle m+1 \rangle}$ and $w_2(m)\alpha_{\langle m+2 \rangle}$ distinct.

We will represent the denominators in (6.2) as vectors in the root lattice Q by setting

$$\delta(m) = d_1(m)\alpha_1 + d_2(m)\alpha_2$$

for all $m \in \mathbb{Z}$. In particular, we have $\delta(1) = -\alpha_1$ and $\delta(2) = -\alpha_2$.

Theorem 6.1. *In the rank 2 case (finite or infinite alike), cluster variables are uniquely up to a multiple from $\mathbb{P}$ determined by their denominators in the Laurent expansions with respect to a given cluster. The set of these denominators is naturally identified with $\{-\alpha_1, -\alpha_2\} \cup \Phi_+$. More precisely:*

(i) *In the infinite case, we have*

$$(6.5) \qquad \delta(m+3) = w_1(m)\alpha_{\langle m+1 \rangle}, \quad \delta(-m) = w_2(m)\alpha_{\langle m+2 \rangle} \qquad (m \geq 0) \ .$$

In particular, all y_m for $m \in \mathbb{Z}$ have different denominators $y_1^{d_1(m)} y_2^{d_2(m)}$.

(ii) *In the finite case, we have*

$$(6.6) \qquad\qquad \delta(m+3) = w_1(m)\alpha_{\langle m+1 \rangle} \qquad (h > m \geq 0)$$

and $\delta(m + h + 2) = \delta(m)$ for all $m \in \mathbb{Z}$, so the denominators $y_1^{d_1(m)} y_2^{d_2(m)}$ are periodic with the period $h + 2$. Moreover, $y_{m+h+2}/y_m \in \mathbb{P}$ for $m \in \mathbb{Z}$.

Proof. Before giving a unified proof of Theorem 6.1, we notice that part (ii) can be proved by a direct calculation. We will express all coefficients in terms of r_1, r_2, and q_m for $m \in \mathbb{Z}$ (cf. Proposition 5.2). With the help of `Maple` we find that in each finite case, the elements y_m for $3 \leq m \leq h + 4$ are the following Laurent polynomials in y_1 and y_2.

Type $A_1 \times A_1$: $b = c = 0$.

$$y_3 = \frac{q_2 + r_2}{y_1}, \quad y_4 = \frac{q_3(q_1 + r_1)}{r_1 y_2}, \quad y_5 = \frac{q_4 y_1}{r_2}, \quad y_6 = \frac{r_1 q_5 y_2}{q_1 q_3} \ .$$

Type A_2: $b = c = 1$.

$$y_3 = \frac{q_2 + r_2}{y_1}, \quad y_4 = \frac{q_3(r_2(q_1y_1 + r_1) + r_1q_2y_2)}{r_1y_1y_2},$$

$$y_5 = \frac{q_3q_4(q_1y_1 + r_1)}{r_1y_2}, \quad y_6 = \frac{q_4q_5y_1}{r_2}, \quad y_7 = \frac{r_1q_5q_6y_2}{r_2q_1q_3} \ .$$

Type B_2: $b = 1, c = 2$.

$$y_3 = \frac{q_2y_2^2 + r_2}{y_1}, \quad y_4 = \frac{q_3(r_2(q_1y_1 + r_1) + r_1q_2y_2^2)}{r_1y_1y_2},$$

$$y_5 = \frac{q_3^2q_4(r_2(q_1y_1 + r_1)^2 + r_1^2q_2y_2^2)}{r_1^2y_1y_2^2}, \quad y_6 = \frac{q_3q_4q_5(q_1y_1 + r_1)}{r_1y_2},$$

$$y_7 = \frac{q_4q_5^2q_6y_1}{r_2}, \quad y_8 = \frac{r_1q_5q_6q_7y_2}{r_2q_1q_3} \ .$$

Type C_2: $b = 2, c = 1$.

$$y_3 = \frac{q_2y_2 + r_2}{y_1}, \quad y_4 = \frac{q_3(r_2^2q_1y_1^2 + r_1(q_2y_2 + r_2)^2)}{r_1y_1^2y_2},$$

$$y_5 = \frac{q_3q_4(r_2q_1y_1^2 + r_1(q_2y_2 + r_2))}{r_1y_1y_2}, \quad y_6 = \frac{q_3q_4^2q_5(q_1y_1^2 + r_1)}{r_1y_2},$$

$$y_7 = \frac{q_4q_5q_6y_1}{r_2}, \quad y_8 = \frac{r_1q_5q_6^2q_7y_2}{r_2^2q_1q_3} \ .$$

Type G_2: $b = 1, c = 3$.

$$y_3 = \frac{q_2y_2^3 + r_2}{y_1}, \quad y_4 = \frac{q_3(r_2(q_1y_1 + r_1) + r_1q_2y_2^3)}{r_1y_1y_2},$$

$$y_5 = \frac{q_3^3q_4(r_2^2(q_1y_1 + r_1)^3 + r_1^2q_2y_2^3(r_1q_2y_2^3 + 3r_2q_1y_1 + 2r_1r_2))}{r_1^3y_1^2y_2^3},$$

$$y_6 = \frac{q_3^2q_4q_5(r_2(q_1y_1 + r_1)^2 + r_1^2q_2y_2^3)}{r_1^2y_1y_2^2}, \quad y_7 = \frac{q_3^3q_4^2q_5^3q_6(r_2(q_1y_1 + r_1)^3 + r_1^3q_2y_2^3)}{r_1^3y_1y_2^3},$$

$$y_8 = \frac{q_3q_4q_5^2q_6q_7(q_1y_1 + r_1)}{r_1y_2}, \quad y_9 = \frac{q_4q_5^3q_6^2q_7^3q_8y_1}{r_2}, \quad y_{10} = \frac{r_1q_5q_6q_7^2q_8q_9y_2}{r_2q_1q_3} \ .$$

Type $G_2^\vee$: $b = 3, c = 1$.

$$y_3 = \frac{q_2y_2 + r_2}{y_1}, \quad y_4 = \frac{q_3(r_1(q_2y_2 + r_2)^3 + r_2^3q_1y_1^3)}{r_1y_1^3y_2} \ ,$$

$$y_5 = \frac{q_3q_4(r_1(q_2y_2 + r_2)^2 + r_2^2q_1y_1^3)}{r_1y_1^2y_2} \ ,$$

$$y_6 = \frac{q_3^2q_4^3q_5(r_1^2(q_2y_2 + r_2)^3 + r_2^2q_1y_1^3(r_2q_1y_1^3 + 3r_1q_2y_2 + 2r_1r_2))}{r_1^2y_1^3y_2^2},$$

$$y_7 = \frac{q_3 q_4^2 q_5 q_6 (r_2(q_1 y_1^3 + r_1) + r_1 q_2 y_2)}{r_1 y_1 y_2}, \quad y_8 = \frac{q_3 q_4^3 q_5^2 q_6^3 q_7 (q_1 y_1^3 + r_1)}{r_1 y_2},$$

$$y_9 = \frac{q_4 q_5 q_6^2 q_7 q_8 y_1}{r_2}, \quad y_{10} = \frac{r_1 q_5 q_6^3 q_7^2 q_8^3 q_9 y_2}{r_2^3 q_1 q_3}.$$

By inspection, the exponents of the y variables in the denominators above agree with (6.6). Furthermore, we see that $y_{m+h+2}/y_m \in \mathbb{P}$ for $m = 1, 2$, and therefore for any m.

Now let us give a unified proof of both parts of Theorem 6.1. For $\delta = d_1 \alpha_1 + d_2 \alpha_2 \in Q$, let us denote

$$[\delta]_+ = \max(d_1, 0)\alpha_1 + \max(d_2, 0)\alpha_2.$$

It then follows from the relations (2.11) that

$$(6.7) \qquad \delta(m+1) + \delta(m-1) = \begin{cases} b[\delta(m)]_+ & \text{if } m \text{ is odd}; \\ \\ c[\delta(m)]_+ & \text{if } m \text{ is even}. \end{cases}$$

Starting with $\delta(1) = -\alpha_1$ and $\delta(2) = -\alpha_2$, we use (6.7) to see that

$$\delta(3) = \alpha_1, \quad \delta(4) = b\alpha_1 + \alpha_2 = s_1 \alpha_2,$$

and also

$$\delta(0) = \alpha_2, \quad \delta(-1) = \alpha_1 + c\alpha_2 = s_2 \alpha_1.$$

This proves (6.5) and (6.6) for $m = 0$ and $m = 1$.

Proceeding by induction on m, let us now assume that $m \geq 2$, and that both vectors $\delta(m+1)$ and $\delta(m+2)$ are positive roots given by (6.6); in the finite case, we also assume that $m \leq h$. If m is odd, it follows from (6.7) that

$$\begin{aligned} \delta(m+3) &= b\delta(m+2) - \delta(m+1) = bw_1(m-1)\alpha_1 - w_1(m-2)\alpha_2 \\ &= w_1(m-1)(s_1\alpha_2 - \alpha_2) - w_1(m-2)\alpha_2 \\ &= w_1(m)\alpha_2 - w_1(m-2)(s_2\alpha_2 + \alpha_2) = w_1(m)\alpha_2. \end{aligned}$$

Thus, $\delta(m+3)$ is also given by (6.6). The same argument works with m even, and, in the infinite case, with m negative. This completes the proof of (6.5) and (6.6).

We conclude by a unified proof of the periodicity $\delta(m+h+2) = \delta(m)$ in the finite case. Without loss of generality, we assume that $m = 1$. By the above inductive argument, we know that $\delta(h+3)$ is given by (6.6). Thus, we have $\delta(h+3) = w_1(h)\alpha_{\langle h+1\rangle} = w_0 \alpha_{\langle h+1\rangle}$. It is known—and easy to check—that $w_0 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ if $b = c = 1$ (the only case when h is odd), and $w_0 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ in the other three finite cases. Hence $w_0 \alpha_{\langle h+1\rangle} = -\alpha_1$ in all cases, and we are done. $\square$

In the rest of the section, we discuss the normalized case (see Definition 5.3). In this context, the periodicity property in Theorem 6.1 (ii) can be sharpened considerably.

Proposition 6.2. *In the normalized finite case of a cluster algebra of rank 2, we have $q_{m+h+2} = q_m$, $r_{m+h+2} = r_m$, and $y_{m+h+2} = y_m$ for all $m \in \mathbb{Z}$.*

Proof. We set $u_m = q_m/r_m$ for $m \in \mathbb{Z}$. Then q_m and r_m are recovered from u_m by

$$(6.8) \qquad q_m = \frac{u_m}{1 \oplus u_m}, \quad r_m = \frac{1}{1 \oplus u_m}$$

(cf. (5.3)). To establish the periodicity of q_m and r_m, it suffices to show that $u_{m+h+2} = u_m$. Furthermore, it is enough to check that $u_{h+3} = u_1$ and $u_{h+4} = u_2$ in each of the following four cases: $(b, c) = (0, 0)$ (type $A_1 \times A_1$), $(1, 1)$ (type A_2), $(1, 2)$ (type B_2), and $(1, 3)$ (type G_2). This is done by direct calculation using the relation

$$(6.9) \qquad u_{m+1}u_{m-1} = \begin{cases} (1 \oplus u_m)^c & \text{if } m \text{ is odd;} \\[2mm] (1 \oplus u_m)^b & \text{if } m \text{ is even,} \end{cases}$$

which, in view of (6.1), is a consequence of (5.4) and (5.5). Below are the intermediate steps of this calculation, which in each case expresses u_m for $3 \le m \le h + 2$ as a rational function in u_1 and u_2, and then confirms $u_{h+3} = u_1$ and $u_{h+4} = u_2$.

Type $A_1 \times A_1$: $b = c = 0, h = 2$.

$$u_3 = \frac{1}{u_1}, \quad u_4 = \frac{1}{u_2} \ .$$

Type A_2: $b = c = 1, h = 3$.

$$u_3 = \frac{1 \oplus u_2}{u_1}, \quad u_4 = \frac{1 \oplus u_1 \oplus u_2}{u_1 u_2}, \quad u_5 = \frac{1 \oplus u_1}{u_2} \ .$$

Type B_2: $b = 1, c = 2, h = 4$.

$$u_3 = \frac{1 \oplus u_2}{u_1}, \quad u_4 = \frac{(1 \oplus u_1 \oplus u_2)^2}{u_1^2 u_2}, \quad u_5 = \frac{(1 \oplus u_1)^2 \oplus u_2}{u_1 u_2}, \quad u_6 = \frac{(1 \oplus u_1)^2}{u_2} \ .$$

Type G_2: $b = 1, c = 3, h = 6$.

$$u_3 = \frac{1 \oplus u_2}{u_1}, \quad u_4 = \frac{(1 \oplus u_1 \oplus u_2)^3}{u_1^3 u_2}, \quad u_5 = \frac{(1 \oplus u_1)^3 \oplus u_2(3u_1 \oplus u_2 \oplus 2)}{u_1^2 u_2} \ ,$$

$$u_6 = \frac{((1 \oplus u_1)^2 \oplus u_2)^3}{u_1^3 u_2^2}, \quad u_7 = \frac{(1 \oplus u_1)^3 \oplus u_2}{u_1 u_2}, \quad u_8 = \frac{(1 \oplus u_1)^3}{u_2} \ .$$

To complete the proof of Proposition 6.2, it remains to show that $y_{h+3} = y_1$ and $y_{h+4} = y_2$ in each of the four cases. By Theorem 6.1 (ii), both ratios y_{h+3}/y_1 and y_{h+4}/y_2 belong to $\mathbb{P}$, and we only need to show that these two elements of $\mathbb{P}$ are equal to 1 in the normalized case. The ratios in question were explicitly computed in the course of the proof of Theorem 6.1. We see that it all boils down to the following identities:

$(6.10) \qquad\qquad$ Type $A_1 \times A_1$: $r_1 = q_3$.

$(6.11) \qquad\qquad$ Type A_2: $r_1 = q_3 q_4$.

$(6.12) \qquad\qquad$ Type B_2: $r_1 = q_3 q_4 q_5$, $r_2 = q_4 q_5^2 q_6$.

$(6.13) \qquad\qquad$ Type G_2: $r_1 = q_3 q_4 q_5^2 q_6 q_7$, $r_2 = q_4 q_5^3 q_6^2 q_7^3 q_8$.

All these identities can be proved by a direct calculation (preferably, with a computer): first express both sides in terms of the u_m using (6.8), and then replace each u_m by the above expression in terms of u_1 and u_2. $\qquad\square$

Remark 6.3. Periodicity of the recurrence (6.9) is a very special case of the periodicity phenomenon for Y-system recurrences in the theory of the thermodynamic Bethe ansatz [19]. We plan to address the case of arbitrary rank in a forthcoming paper.

We have shown that in a finite normalized case, any coefficient r_m can be written as a monomial in $q_{m+2}, \ldots, q_{m+h}$. There is a nice uniform way to write down these monomials using the *dual root system* of the system Φ considered above. Recall that the dual root system $\Phi^\vee$ is associated with the transposed Cartan matrix. The simple roots of $\Phi^\vee$ are called *simple coroots* and denoted $\alpha_1^\vee$ and $\alpha_2^\vee$. They generate the *coroot lattice* $Q^\vee$. The Weyl groups of Φ and $\Phi^\vee$ are naturally isomorphic to each other, and it is common to identify them. The same Weyl group W acts on $Q^\vee$ so that the action of simple reflections s_1 and s_2 in the basis of simple coroots is given by the matrices

$$s_1 = \begin{bmatrix} -1 & c \\ 0 & 1 \end{bmatrix} , \quad s_2 = \begin{bmatrix} 1 & 0 \\ b & -1 \end{bmatrix}$$

(cf. (6.4)). By inspection (cf. (6.10)–6.13)), one obtains the following unified expression for r_m.

Proposition 6.4. *For $i \in \{1, 2\}$ and $m \geq 0$, let $c(i, m)$ denote the coefficient of $\alpha_i^\vee$ in the root $w_i(m)\alpha_{\langle i+m \rangle}^\vee$. Then, in every finite case, we have*

$$(6.14) \qquad r_k = \prod_{m=0}^{h-2} q_{k+m+2}^{c(\langle k+1 \rangle, m)}$$

for all $k \in \mathbb{Z}$.

Remark 6.5. The relations (6.14) together with the periodicity relation $q_{m+h+2} = q_m$ imply that in each finite normalized case the subgroup of the coefficient group $\mathbb{P}$ generated by all q_m and r_m is in fact generated by $q_1, \ldots, q_{h+2}$. Thus, this group is dramatically different from the universal coefficient group. One might ask if the normalization condition imposes some relations among $q_1, \ldots, q_{h+2}$. The answer to this question is negative: it is possible to take $\mathbb{P} = \text{Trop}(q_1, \ldots, q_{h+2})$ (cf. Example 5.6), so that multiplicatively, $\mathbb{P}$ is a free abelian group generated by $q_1, \ldots, q_{h+2}$. With the help of (6.14) and the periodicity relation $q_{m+h+2} = q_m$, we can view all r_m and q_m for $m \in \mathbb{Z}$ as elements of $\mathbb{P}$. By the definition of auxiliary addition (see (5.6)), we have $r_m \oplus q_m = 1$ for all m. Hence these coefficients give rise to a normalized exchange pattern (which is in fact of geometric type according to Definition 5.7).

In the infinite case, the situation is very different: there exists a normalized exchange pattern such that the coefficients r_m and q_m generate the universal coefficient group (see Proposition 5.2). To see this, take $\mathbb{P}$ to be the group of formal infinite products $\prod_{m \in \mathbb{Z}} q_m^{c_m}$, where the exponents c_m can be arbitrary integers. Make $\mathbb{P}$ into a semifield by setting

$$\prod_{m \in \mathbb{Z}} q_m^{c_m} \oplus \prod_{m \in \mathbb{Z}} q_m^{c'_m} = \prod_{m \in \mathbb{Z}} q_m^{\min(c_m, c'_m)} .$$

Taking (6.14) as an inspiration, let us define the elements $r_k \in \mathbb{P}$ for all $k \in \mathbb{Z}$ by

$$r_k = \prod_{m \geq 0} \left(q_{k+m+2} q_{k-m-2} \right)^{c(\langle k+1 \rangle, m)} ,$$

with the same exponents as in (6.14). Also, view each q_k as an element of $\mathbb{P}$ via $q_k = \prod_{m \in \mathbb{Z}} q_m^{\delta_{km}}$. A direct check then shows that these coefficients give rise to a normalized exchange pattern (that is, satisfy (2.12)), and also that in $\mathbb{P}$ there are no relations among the elements r_1, r_2, and q_k, for $k \in \mathbb{Z}$.

7. The exchange graph

In this section, we restrict our attention to *normalized* exchange patterns $\mathcal{M} = (M_j(t) : j \in I, t \in \mathbb{T}_n)$. Let $(x_j(t) : j \in I, t \in \mathbb{T}_n)$ be the family of cluster variables associated to $\mathcal{M}$. As in Section 2, we view them as elements of the ambient field $\mathcal{F}$.

A normalized exchange pattern gives rise to a natural equivalence relation on the set of vertices of $\mathbb{T}_n$. Informally speaking, two vertices are equivalent with respect to $\mathcal{M}$ if they have the same clusters and the same exchange relations, up to relabeling of cluster variables. Here is a precise definition.

Definition 7.1. We say that two vertices t and t' in $\mathbb{T}_n$ are $\mathcal{M}$-equivalent if there exists a permutation σ of the index set I such that:

(1) $x_i(t') = x_{\sigma(i)}(t)$ for all $i \in I$;

(2) if $t \xrightarrow{\sigma(j)} t_1$ and $t' \xrightarrow{\quad j \quad} t'_1$ for some $j \in I$, then $M_j(t')(\mathbf{x}(t')) = M_{\sigma(j)}(t)(\mathbf{x}(t))$ and $M_j(t'_1)(\mathbf{x}(t')) = M_{\sigma(j)}(t_1)(\mathbf{x}(t))$.

Remark 7.2. We believe that condition (2) in Definition 7.1 is in fact a consequence of (1). Thus, once a cluster has been fixed as a *set* (forgetting the labels and the cluster's location in the tree $\mathbb{T}_n$), the exchange relations involving its elements are uniquely determined.

In view of Proposition 5.4, a normalized exchange pattern can be defined in terms of a family of integer matrices $B(t)$ together with a family of elements $u_j(t) \in \mathbb{P}$ satisfying relations (5.4)–(5.5). In these terms, property (2) above can be rephrased as saying that $b_{ij}(t') = b_{\sigma(i),\sigma(j)}(t)$ and $u_j(t') = u_{\sigma(j)}(t)$ for all $i, j \in I$.

The following property is an immediate consequence of Proposition 5.4.

Proposition 7.3. *Suppose t and t' are $\mathcal{M}$-equivalent. For every vertex t_1 adjacent to t, there is a unique vertex t'_1 adjacent to t' and equivalent to t_1. In the notation of Definition* 7.1, *if $t \xrightarrow{\sigma(j)} t_1$ for some $j \in I$, then $t' \xrightarrow{\quad j \quad} t'_1$.*

Definition 7.4. The *exchange graph* $\Gamma_{\mathcal{M}}$ associated with a normalized exchange pattern $\mathcal{M}$ has the $\mathcal{M}$-equivalence classes as vertices, joined by an edge if they have adjacent representatives in $\mathbb{T}_n$.

As an immediate corollary of Proposition 7.3, we obtain the following.

Corollary 7.5. *The exchange graph is n-regular, i.e., every vertex has precisely n edges emanating from it.*

Note that passing from $\mathbb{T}_n$ to $\Gamma_{\mathcal{M}}$ may result in losing the coloring of edges by elements of the index set I. This will happen if a permutation σ in Definition 7.1 is nontrivial. (This permutation can be thought of as "discrete monodromy" of our graph.)

Example 7.6. The results in Section 6 (more specifically, Theorem 6.1 and Proposition 6.2) show that, in the rank 2 case, the $\mathcal{M}$-equivalence and the exchange graph can be described as follows. In the infinite case, no two distinct vertices of $\mathbb{T}_2$ are

$\mathcal{M}$-equivalent to each other, so the exchange graph is $\mathbb{T}_2$ itself. In the finite case, t_m and $t_{m'}$ are $\mathcal{M}$-equivalent if and only if $m \equiv m' \bmod (h+2)$, so the exchange graph is a cycle of length $h + 2$. For example, in the type A_2 case, we have $h = 3$, and the graph is a pentagon. Figure 3 shows the corresponding exchange graph, together with its clusters and exchange relations, written with the help of (2.11) and (6.11). The discrete monodromy is present in this special case, as the variables y_1 and y_2 get switched after a full 5-cycle of exchanges:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \to \begin{pmatrix} y_3 \\ y_2 \end{pmatrix} \to \begin{pmatrix} y_3 \\ y_4 \end{pmatrix} \to \begin{pmatrix} y_5 \\ y_4 \end{pmatrix} \to \begin{pmatrix} y_5 \\ y_1 \end{pmatrix} \to \begin{pmatrix} y_2 \\ y_1 \end{pmatrix}.$$

The corresponding cluster algebra can be realized as the homogeneous coordinate ring of the Grassmannian $Gr_{2,5}$; see Example 2.5.
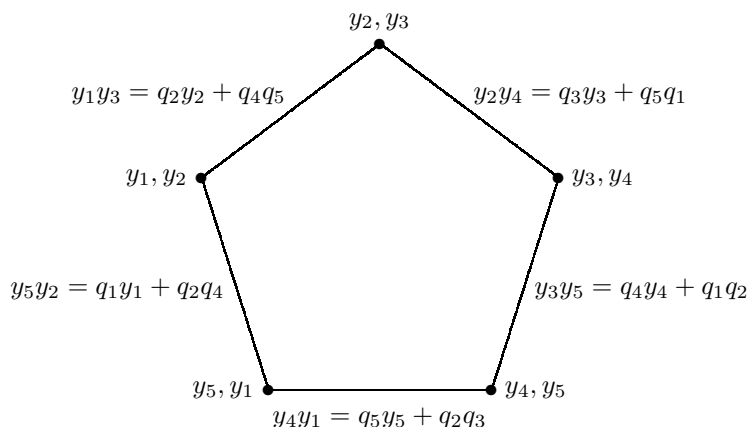

FIGURE 3. The exchange graph for a cluster algebra of type A_2

We see that, in the rank 2 case, the following conditions on a normalized exchange pattern $\mathcal{M}$ are equivalent:

(1) The exchange graph $\Gamma_{\mathcal{M}}$ is finite.

(2) A cluster algebra associated to $\mathcal{M}$ has finitely many distinct cluster variables.

(3) There exists a vertex $t \in \mathbb{T}_2$ such that the Cartan counterpart $A(B(t))$ of the matrix $B(t)$ is a Cartan matrix of finite type, i.e., we have $|b_{12}(t)b_{21}(t)| \leq 3$.

In the sequel to this paper, we shall extend this result to arbitrary rank.

Returning to the general case, we shall now provide some important instances of $\mathcal{M}$-equivalence.

Theorem 7.7. *Suppose $|b_{ij}(t)b_{ji}(t)| \leq 3$ for some vertex $t \in \mathbb{T}_n$ and two distinct indices i and j. Let h be the Coxeter number of the corresponding rank 2 root system, and let $t' \in \mathbb{T}_n$ be a vertex joined with t by a path of length $h + 2$ whose edge labels alternate between i and j. Then t' is $\mathcal{M}$-equivalent to t, so that the path becomes a cycle of length $h + 2$ in the exchange graph.*

Proof. For the sake of convenience, assume that $i = 1$, $j = 2$; thus, the path joining t and t' has the form

$$t = t_1 \overset{1}{\rule{1cm}{0.4pt}} t_2 \overset{2}{\rule{1cm}{0.4pt}} t_3 \overset{1}{\rule{1cm}{0.4pt}} \cdots \overset{\langle h+2 \rangle}{\rule{1cm}{0.4pt}} t_{h+3} = t'$$

(cf. (2.10)), where $\langle m \rangle$ as before stands for the element of $\{1, 2\}$ congruent to m modulo 2. We abbreviate $b_1 = -b_{12}(t_1)$ and $b_2 = b_{21}(t_1)$, and assume (without loss of generality) that b_1 and b_2 are either both positive or both equal to 0. This agrees with the convention (4.5) (also used in Section 6), where b_1 was denoted by b, and b_2 by c. Recall that the value of h is given by Table 1 in Section 6; in particular, h is even unless $b_1 = b_2 = 1$ (the type A_2). We claim that the conditions (1) and (2) in Definition 7.1 hold with the permutation σ equal to identity for h even, and equal to the transposition of indices 1 and 2 in the only case when h is odd. Using the restriction operation on exchange patterns (cf. Proposition 2.6), it suffices to treat the case $n = 3$, so we can take $I = \{1, 2, 3\}$.

In view of the exchange property (2.1), the cluster variable $x_3(t_m)$ does not depend on m. Therefore, verifying condition (1) amounts to checking that $x_i(t') = x_{\sigma(i)}(t)$ for $i \in \{1, 2\}$. Again using restriction, we can assume for the purpose of this checking only that $n = 2$ and $I = \{1, 2\}$, in which case the required property was established in Proposition 6.2.

The same argument shows that the only part of condition (2) that does not follow from Proposition 6.2 is the equalities $u_3(t') = u_3(t)$ and $b_{i3}(t') = b_{\sigma(i),3}(t)$ for $i \in \{1, 2\}$. Let us abbreviate $d_m = b_{\langle m \rangle, 3}(t_m)$ and $u_{m+1} = u_{\langle m \rangle}(t_m)$; the latter notation is chosen to be consistent with the notation in the proof of Proposition 6.2. Iterating (5.5) (with $i = 3$), we obtain

$$(7.1) \qquad \frac{u_3(t')}{u_3(t)} = \prod_{m=1}^{h+2} u_{m+1}^{\max(d_m, 0)} (1 \oplus u_{m+1})^{-d_m} .$$

Thus, we need to show that the product on the right-hand side of (7.1) is equal to 1. To do this, we need some preparation.

Recall that the coefficients u_m satisfy the relation (6.9), which in our present notation can be rewritten as

$$(7.2) \qquad u_m u_{m+2} = (1 \oplus u_{m+1})^{b_{\langle m \rangle}} .$$

On the other hand, the matrix mutation rules (4.3) (with $j = 3$, applied along the edges $t_m \xrightarrow{\langle m \rangle} t_{m+1} \xrightarrow{\langle m+1 \rangle} t_{m+2}$) readily imply the following recurrence for the exponents d_m:

$$(7.3) \qquad d_m + d_{m+2} = b_{\langle m \rangle} \max(d_{m+1}, 0) .$$

The sequence (u_m) is periodic with period $h + 2$, by virtue of Proposition 6.2. Furthermore, the relation (7.3) can be seen, somewhat surprisingly, as a special case of (7.2), for the following version of the tropical semifield: take $\mathbb{Z}$ with the multiplication given by ordinary addition, and the auxiliary addition given by $a \oplus b = \max(a, b)$. This implies in particular that the sequence (d_m) is also periodic with period $h + 2$. We also note that the same periodicity holds for the sequence $(b_{\langle m \rangle})$: when h is even, this is clear from the definition of $\langle m \rangle$; and in the only case when h is odd, we have $b_1 = b_2 = 1$.

To show that the right-hand side of (7.1) is equal to 1, we first treat the case $b_1 = b_2 = 0$. Then $h + 2 = 4$, and we have $u_{m+2} = u_m^{-1}$ and $d_{m+2} = -d_m$ for all m.

It follows that

$$\prod_{m=1}^{h+2} u_{m+1}^{\max(d_m,0)} (1 \oplus u_{m+1})^{-d_m}$$

$$= \prod_{m=1}^{2} u_{m+1}^{\max(d_m,0)} (1 \oplus u_{m+1})^{-d_m} u_{m+1}^{-\max(-d_m,0)} (1 \oplus u_{m+1}^{-1})^{d_m}$$

$$= \prod_{m=1}^{2} u_{m+1}^{d_m} (1 \oplus u_{m+1})^{-d_m} (1 \oplus u_{m+1}^{-1})^{d_m} = 1 \ ,$$

as desired.

In the remaining case when b_1 and b_2 are both positive, we use (7.2) and (7.3) as well as the above-mentioned periodicity, to obtain:

$$\left(\frac{u_3(t')}{u_3(t)} \right)^{b_1 b_2} = \prod_{m=1}^{h+2} u_{m+1}^{b_1 b_2 \max(d_m,0)} (1 \oplus u_{m+1})^{-b_1 b_2 d_m}$$

$$= \prod_{m=1}^{h+2} u_{m+1}^{b_1 b_2 \max(d_m,0)} (u_m u_{m+2})^{-b_{\langle m+1 \rangle} d_m}$$

$$= u_1^{-b_2 d_1} u_2^{b_1 (b_2 \max(d_1,0) - d_2)}$$

$$\times \prod_{m=1}^{h} u_{m+2}^{-b_{\langle m+1 \rangle}(d_m + d_{m+2} - b_{\langle m \rangle} \max(d_{m+1},0))}$$

$$\times u_{h+3}^{b_{\langle h+2 \rangle}(b_{\langle h+1 \rangle} \max(d_{h+2},0) - d_{h+1})} u_{h+4}^{-b_{\langle h+3 \rangle} d_{h+2}}$$

$$= u_1^{-b_2 d_1} u_2^{b_1 d_0} u_{h+3}^{b_{\langle h+4 \rangle} d_{h+3}} u_{h+4}^{-b_{\langle h+3 \rangle} d_{h+2}}$$

$$= u_1^{(b_{\langle h+4 \rangle} - b_2) d_1} u_2^{(b_1 - b_{\langle h+3 \rangle}) d_0} = 1 \ .$$

To conclude that $u_3(t') = u_3(t)$, just recall that the coefficient group $\mathbb{P}$ is assumed to be torsion-free.

To complete the proof of the theorem, it remains to show that $b_{i3}(t') = b_{\sigma(i),3}(t)$ for $i \in \{1,2\}$. But this is equivalent to saying that the sequence (d_m) is periodic with period $h + 2$, which is already proven. $\square$

We conjecture that the $\mathcal{M}$-equivalence relation is generated by its instances described in Theorem 7.7.

Example 7.8. Let $\mathcal{M}(B)$ be an exchange pattern of geometric type associated with the skew-symmetric matrix

$$(7.4) \qquad \tilde{B} = B = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \ ,$$

as prescribed by Corollary 5.9. Thus, the corresponding cluster algebra is of rank 3, and all the coefficients $p_j(t)$ are equal to 1. The exchange graph for this pattern is a "two-layer brick wall" shown in Figure 4. In this figure, distinct cluster variables are associated with regions: there are variables y_m $(m \in \mathbb{Z})$ associated with bounded regions ("bricks"), and two more variables w and z associated with the two unbounded regions. The cluster at any vertex t consists of the three cluster

variables associated to the three regions adjacent to t. The binomial exchange relations are as follows: for $m \in \mathbb{Z}$, we have

$$wy_{2m} = y_{2m-1} + y_{2m+1} \; ,$$

$$y_{2m-1}y_{2m+3} = y_{2m+1}^2 + w \; ,$$

(7.5)
$$y_m y_{m+3} = y_{m+1} y_{m+2} + 1 \; ,$$

$$y_{2m-2} y_{2m+2} = y_{2m}^2 + z \; ,$$

$$y_{2m-1} z = y_{2m-2} + y_{2m} \; .$$

To see all this, we first show that the graph in Figure 4 is a cover of the exchange graph $\Gamma_{\mathcal{M}}$. Pick an initial vertex $t_0 \in \Gamma_{\mathcal{M}}$ with the matrix $B(t_0) = B$ given by (7.4) (abusing notation, we will use the same symbol for a vertex in $\mathbb{T}_3$ and its image in $\Gamma_{\mathcal{M}}$). Denote the cluster variables at t_0 by y_1, y_2, and y_3, so that their order agrees with that of rows and columns of B. Since every principal 2×2 submatrix of B is of type A_2, Theorem 7.7 provides that t_0 is a common vertex of three 5-cycles in $\Gamma_{\mathcal{M}}$. These cycles are depicted in Figure 4 as perimeters of the three bricks surrounding t_0. The variable y_i inside each brick indicates that this brick corresponds to the rank 2 exchange pattern obtained from $\mathcal{M}$ via restriction from $I = [3]$ to $[3] \setminus \{i\}$; equivalently, this means that y_i appears in every cluster on the perimeter of the brick. Now we move from t_0 to an adjacent vertex t_1 with the cluster $\{y_2, y_3, y_4\}$. According to Proposition 4.3,

$$B(t_1) = \mu_1(B) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \; .$$

This matrix differs from B by a simultaneous cyclic permutation of rows and columns. Therefore, we can apply the same construction to t_1, obtaining the three bricks surrounding this vertex. Continuing in the same way, we produce the entire graph in Figure 4. Since this graph is 3-regular, we have covered all the vertices and edges of the exchange graph.
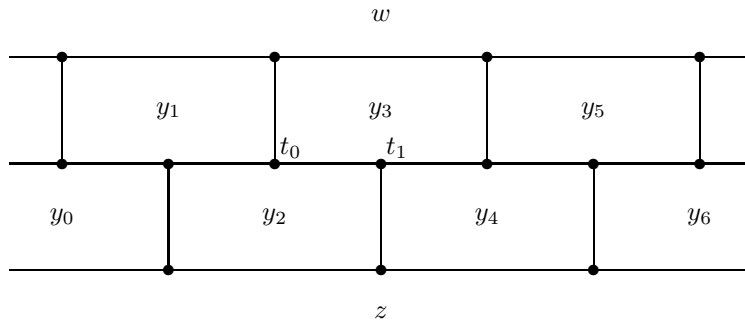

FIGURE 4. The two-layer brick wall

For every vertex t on the median of the wall, the matrix $B(t)$ is obtained from B by a simultaneous permutation of rows and columns. Now take a vertex on the

outer boundary, say the vertex t_2 in Figure 5. Then

$$B(t_2) = \mu_3(B(t_1)) = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \ .$$

Again, for every other vertex on the outer boundary, the corresponding matrix is obtained from $B(t_2)$ by a simultaneous permutation of rows and columns. Substituting the matrices $B(t)$ into (4.2), we generate all the exchange relations (7.5) (cf. Figure 5).
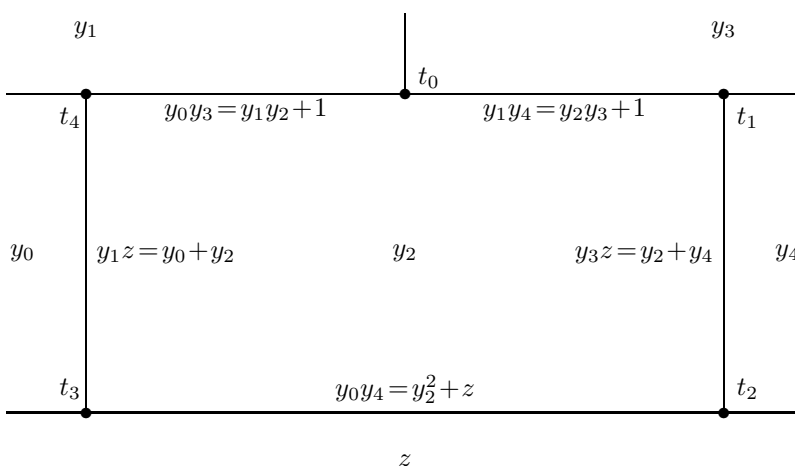

FIGURE 5. Close-up of a brick

To prove that our brick wall is indeed the exchange graph, it remains to show that all the cluster variables y_m, w, and z are distinct; recall that we view them as elements of the ambient field $\mathcal{F}$. Following the methodology of Theorem 6.1, it suffices to show that all of them have different denominators in the Laurent expansion in terms of the initial cluster $\{y_1, y_2, y_3\}$. We write the denominator of a cluster variable y as $y_1^{d_1(y)} y_2^{d_2(y)} y_3^{d_3(y)}$ (cf. (6.2)), and encode it by a vector $\delta(y) = (d_1(y), d_2(y), d_3(y)) \in \mathbb{Z}^3$. In particular, we have

(7.6) $\delta(y_1) = (-1, 0, 0), \ \delta(y_2) = (0, -1, 0), \ \delta(y_3) = (0, 0, -1) \ .$

A moment's reflection shows that the vectors $\delta(y_m)$, $\delta(w)$, and $\delta(z)$ satisfy the "tropical version" of each of the exchange relations (7.5) (with multiplication replaced by vector addition, and addition replaced by component-wise maximum), as long as the two vectors on the right-hand side are different. For example, the first equation in (7.5) will take the form

$$\delta(w) + \delta(y_{2m}) = \max(\delta(y_{2m-1}), \delta(y_{2m+1})) \ ,$$

provided $\delta(y_{2m-1}) \neq \delta(y_{2m+1})$; here max stands for component-wise maximum. All $\delta(y)$'s are uniquely determined from (7.6) by recursive application of these

conditions. As a result, we obtain, for every $m \geq 0$:

$$\delta(w) = (0, 1, 0),$$

$$\delta(y_{1-2m}) = (m-1, m, m), \qquad\qquad \delta(y_{2m+3}) = (m, m, m-1),$$

$$\delta(y_{-2m}) = (m, m, m+1), \qquad\qquad \delta(y_{2m+4}) = (m+1, m, m),$$

$$\delta(z) = (1, 0, 1)$$

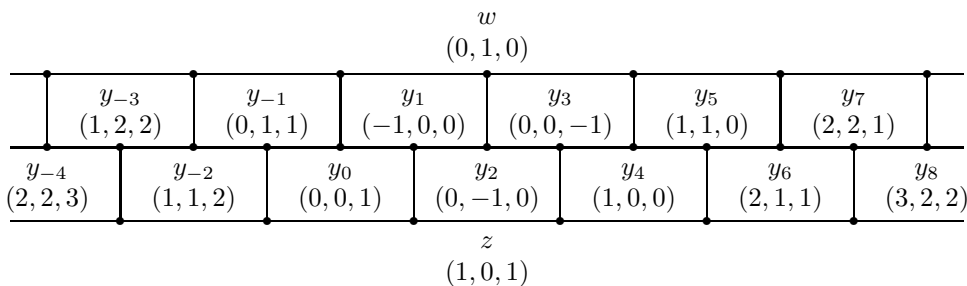(cf. Figure 6, where each vector $\delta(y)$ is shown within the corresponding region).

$$w$$
$$(0, 1, 0)$$

y_{-3} $(1,2,2)$	y_{-1} $(0,1,1)$	y_1 $(-1,0,0)$	y_3 $(0,0,-1)$	y_5 $(1,1,0)$	y_7 $(2,2,1)$

y_{-4} $(2,2,3)$	y_{-2} $(1,1,2)$	y_0 $(0,0,1)$	y_2 $(0,-1,0)$	y_4 $(1,0,0)$	y_6 $(2,1,1)$	y_8 $(3,2,2)$

$$z$$
$$(1, 0, 1)$$

FIGURE 6. The brick wall denominators

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