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KAZHDAN-LUSZTIG POLYNOMIALS AND CHARACTER FORMULAE FOR THE LIE SUPERALGEBRA $\mathfrak{gl}(m|n)$

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1. Introduction

The problem of computing the characters of the finite dimensional irreducible representations of the Lie superalgebra $\mathfrak{gl}(m|n)$ over $\mathbb{C}$ was raised originally by V. Kac in 1977 [Ka2], [Ka3]. Kac proved that the finite dimensional universal highest weight modules, known nowadays as *Kac modules*, are irreducible for so-called *typical* highest weights, and gave a formula for their characters. After that, there were several conjectures and partial results dealing with atypical highest weights [BL], [BR], [Sg1], [JHKT1], [JHKT2], [HKJ], [KaW], [PS2], before the complete solution to the problem was given by V. Serganova [S2], [S3] in 1995 using a mixture of algebraic and geometric techniques.

In this article, we present a different, purely algebraic solution of the problem. One consequence is a proof of a conjecture made by van der Jeugt and Zhang [JZ], which is apparently closely related to the conjectures made in [HKJ]. In particular the composition multiplicities of the Kac modules are all either 0 or 1, a fact which does not seem to follow easily from Serganova's formula since that involves certain alternating sums. We also formulate for the first time a conjecture for the characters of the infinite dimensional irreducible representations in the analogue of category $\mathcal{O}$ for the Lie superalgebra $\mathfrak{gl}(m|n)$.

Inspired by ideas of Lascoux, Leclerc and Thibon [LLT], our approach is to relate the finite dimensional representation theory of $\mathfrak{gl}(m|n)$ to the structure of the module

$$\mathscr{E}^{m|n} := \bigwedge^m \mathscr{V}^* \otimes \bigwedge^n \mathscr{V},$$

where $\mathscr{V}$ denotes the natural representation of the quantized enveloping algebra $U_q(\mathfrak{gl}_\infty)$. By work of Lusztig [L, Chapter 27], the module $\mathscr{E}^{m|n}$ possesses a canonical basis $\{U_\lambda\}$ and a dual canonical basis $\{L_\lambda\}$ (see Theorems 3.6 and 3.13), which for the purpose of this introduction we parametrize via (4.4) by the set $X^+(m|n)$ of dominant integral weights for $\mathfrak{gl}(m|n)$. The entries of the transition matrices between these bases and the natural monomial basis $\{K_\lambda\}$ of $\mathscr{E}^{m|n}$ define polynomials $u_{\mu,\lambda}(q)$ and $l_{\mu,\lambda}(q)$ for each $\mu, \lambda \in X^+(m|n)$:

$$U_\lambda = \sum_{\mu \in X^+(m|n)} u_{\mu,\lambda}(q) K_\mu, \qquad L_\lambda = \sum_{\mu \in X^+(m|n)} l_{\mu,\lambda}(q) K_\mu.$$

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The remarkable thing is that it is quite easy to compute these polynomials explicitly, because all the $\mathfrak{sl}_2$-strings in the crystal graph underlying the module $\mathscr{E}^{m|n}$ are of length ≤ 2; see Corollary 3.39 for the explicit formulae. The main result of the article shows that the polynomials $u_{\mu,\lambda}(q)$ evaluated at $q=1$ compute the composition multiplicities of the Kac modules; see Theorem 4.37 and (4.39). Moreover, the polynomials $l_{\mu,\lambda}(-q^{-1})$ coincide with the Kazhdan-Lusztig polynomials $K_{\lambda,\mu}(q)$ defined originally by Serganova [S2], [S3], and so can be used to compute Ext's between Kac modules and irreducible modules; see Theorem 4.51 and Corollary 4.52.

The module $\mathscr{E}^{m|n}$ is a summand of the tensor space

$$\mathscr{T}^{m|n} := \bigotimes^m \mathscr{V}^* \otimes \bigotimes^n \mathscr{V}.$$

The latter also possesses a canonical basis $\{T_\lambda\}$ and a dual canonical basis $\{L_\lambda\}$ (see Theorems 2.17 and 2.23), which we parametrize via (4.4) by the set $X(m|n)$ of *all* integral weights for $\mathfrak{gl}(m|n)$. For $\lambda \in X^+(m|n) \subset X(m|n)$, the elements L_λ here coincide with the images of the elements with the same name in the previous paragraph under the embedding $\mathscr{E}^{m|n} \hookrightarrow \mathscr{T}^{m|n}$. The entries of the transition matrices between these bases and the natural monomial basis $\{M_\lambda\}$ of $\mathscr{T}^{m|n}$ give us polynomials $t_{\mu,\lambda}(q)$ and $l_{\mu,\lambda}(q)$ for each $\mu, \lambda \in X(m|n)$:

$$T_\lambda = \sum_{\mu \in X(m|n)} t_{\mu,\lambda}(q) M_\mu, \qquad L_\lambda = \sum_{\mu \in X(m|n)} l_{\mu,\lambda}(q) M_\mu.$$

These should be viewed as the true combinatorial analogues for $\mathfrak{gl}(m|n)$ of the Kazhdan-Lusztig polynomials of [KL], [Deo]. We explain an explicit algorithm to compute $t_{\mu,\lambda}(q)$ in §2-j, and conjecture, based on calculations with this algorithm, that our polynomials share the positivity properties of the usual Kazhdan-Lusztig polynomials; see Conjecture 2.28. We conjecture moreover that the polynomials $t_{\mu,\lambda}(q)$ evaluated at $q=1$ compute the composition multiplicities of the Verma modules in category $\mathcal{O}$; see Conjecture 4.32 and (4.34). This conjecture is true in the case $m=0$ by the original Kazhdan-Lusztig conjecture [KL] for the Lie algebra $\mathfrak{gl}(n)$ proved in [BB], [BrK]; see Theorem 4.31(i).

Returning to finite dimensional representations, let us now formulate the conjecture of van der Jeugt and Zhang proved here precisely, to give the flavor of the combinatorics that arises. So let $\mathfrak{g}$ denote the Lie superalgebra $\mathfrak{gl}(m|n)$ over $\mathbb{C}$, labeling rows and columns of matrices in $\mathfrak{g}$ by the ordered index set $I(m|n) = \{-m, \dots, -1, 1, \dots, n\}$. We work always with the standard choices $\mathfrak{h}$ and $\mathfrak{b}$ of Cartan and Borel subalgebras, consisting of diagonal and upper triangular matrices, respectively. For $i \in I(m|n)$, let $\delta_i \in \mathfrak{h}^*$ denote the function picking out the ith diagonal entry of a diagonal matrix. Put a symmetric bilinear form $(.|.)$ on $\mathfrak{h}^*$ by setting $(\delta_i|\delta_j) = 1$ if $i=j>0$, -1 if $i=j<0$, and 0 otherwise. Let $W \cong S_m \times S_n$ denote the Weyl group associated to $\mathfrak{g}_{\bar{0}}$, acting naturally on $\mathfrak{h}^*$. We also need the dot action of W on $\mathfrak{h}^*$: $w \cdot \lambda = w(\lambda+\rho) - \rho$ where $\rho = -\sum_{i \in I(m|n)} i\delta_i$.

Let $X(m|n) \subset \mathfrak{h}^*$ denote the set of all $\mathbb{Z}$-linear combinations of the weights $\{\delta_i\}_{i \in I(m|n)}$, and let $X^+(m|n) \subset X(m|n)$ denote the dominant integral weights, namely, the weights $\lambda = \sum_{i \in I(m|n)} \lambda_i \delta_i \in X(m|n)$ with $\lambda_{-m} \geq \cdots \geq \lambda_{-1}, \lambda_1 \geq \cdots \geq \lambda_n$. Associated to $\lambda \in X^+(m|n)$, we have the Kac module $K(\lambda)$ and its unique irreducible quotient $L(\lambda)$; see §4-a. We should note that there is no loss of generality in restricting our attention to integral weights, since an arbitrary

finite dimensional irreducible representation of $\mathfrak{gl}(m|n)$ is either typical or can be obtained from $L(\lambda)$ for some $\lambda \in X^+(m|n)$ by tensoring with a one dimensional representation.

Main Theorem. *Let $\lambda \in X^+(m|n)$. Let r be maximal such that there exist $-m \leq i_1 < \cdots < i_r < 0 < j_r < \cdots < j_1 \leq n$ with $(\lambda+\rho|\delta_{i_s}-\delta_{j_s}) = 0$ for each $s = 1, \ldots, r$. Let $(k_1, \ldots, k_r)$ be the lexicographically smallest tuple of strictly positive integers such that for all $\theta = (\theta_1, \ldots, \theta_r) \in \{0,1\}^r$, $\lambda + \sum_{s=1}^r \theta_s k_s(\delta_{i_s} - \delta_{j_s})$ is conjugate under the dot action of W to a dominant weight, denoted $\mathtt{R}_\theta(\lambda) \in X^+(m|n)$. Then, for each $\mu \in X^+(m|n)$,*

$$[K(\mu) : L(\lambda)] = \begin{cases} 1 & \text{if } \mu = \mathtt{R}_\theta(\lambda) \text{ for some } \theta = (\theta_1, \ldots, \theta_r) \in \{0,1\}^r, \\ 0 & \text{otherwise.} \end{cases}$$

To prove the Main Theorem, we work with a different family of modules $\{U(\lambda)\}_{\lambda \in X^+(m|n)}$ called *indecomposable tilting modules*, following the general framework developed by Soergel [So2] and extended to Lie superalgebras in [B2]. The problem of computing the multiplicities of Kac modules in indecomposable tilting modules is, roughly speaking, transpose to the problem of computing the composition multiplicities of Kac modules; see (4.16) for the precise relationship (a twisted BGG reciprocity). The main step in the proof gives an explicit inductive construction of the $U(\lambda)$'s starting from the typical case, when $U(\lambda) = K(\lambda)$, and applying certain special translation functors that arise from tensoring with the natural module and its dual. Actually, we see eventually that the indecomposable tilting modules in this finite dimensional setting *coincide* with the indecomposable projectives (also injectives), but they are parametrized by highest weight rather than by their irreducible quotients. Though one could just as well choose to work with the latter more familiar labeling, the alternate parameterization seems to be the one that emerges naturally when considering canonical bases. There are also indecomposable tilting modules denoted $\{T(\lambda)\}_{\lambda \in X(m|n)}$ in category $\mathcal{O}$, where again they seem to correspond most directly to the canonical basis.

We now explain how the remainder of the article is organized. In sections 2 and 3, we give the construction and properties of the canonical bases of the modules $\mathscr{T}^{m|n}$ and $\mathscr{E}^{m|n}$ from a purely combinatorial standpoint. Then in section 4 we describe the representation theory of $\mathfrak{gl}(m|n)$, working in two natural categories $\mathcal{O}_{m|n}$ and $\mathcal{F}_{m|n}$ whose Grothendieck groups are identified with the spaces $\mathscr{T}^{m|n}$ and $\mathscr{E}^{m|n}$, respectively. In sections 2 and 3 we work exclusively in a ρ-shifted notation which is more convenient for the combinatorics, replacing the set $X(m|n)$ of weights with the set $\mathbb{Z}^{m|n}$ of functions $I(m|n) \to \mathbb{Z}$. See (4.4) for the rule to translate between the two notations.

2. Tensor algebra

In this section, we define and study the canonical basis of the tensor space $\mathscr{T}^{m|n}$. We will work throughout over the field $\mathbb{Q}(q)$ of rational functions, where q is an indeterminate.

§2-a. **Combinatorial notation.** For $m, n \geq 0$, let $S_{m|n}$ denote the symmetric group $S_m \times S_n$ acting on the set $I(m|n) = \{-m, \ldots, -1, 1, \ldots, n\}$ so that S_m permutes $\{-m, \ldots, -1\}$ and S_n permutes $\{1, \ldots, n\}$. Thus $S_{m|n}$ is generated by

the basic transpositions

$$s_{-m+1} = (-m \;\; -m{+}1), \dots, s_{-1} = (-2 \;\; -1), s_1 = (1 \;\; 2), \dots, s_{n-1} = (n{-}1 \;\; n).$$

Let $\mathbb{Z}^{m|n}$ be the set of all functions $I(m|n) \to \mathbb{Z}$. We call $f \in \mathbb{Z}^{m|n}$ *antidominant* if $f(-m) \geq \cdots \geq f(-1), f(1) \leq \cdots \leq f(n)$. Note that $S_{m|n}$ acts on the right on $\mathbb{Z}^{m|n}$ by composition of functions, and every $f \in \mathbb{Z}^{m|n}$ is conjugate under this action to a unique antidominant function. We also have the 'flip' $\omega : \mathbb{Z}^{m|n} \to \mathbb{Z}^{n|m}$, where $\omega(f)$ is the function $I(n|m) \to \mathbb{Z}, i \mapsto f(-i)$.

Let P denote the free abelian group on basis $\{\varepsilon_a \mid a \in \mathbb{Z}\}$ endowed with a symmetric bilinear form $(.,.)$ for which the ε_a form an orthonormal basis. We view P as the integral weight lattice associated to the Lie algebra $\mathfrak{gl}_\infty$. The *simple roots* are the elements $\varepsilon_a - \varepsilon_{a+1} \in P$ for $a \in \mathbb{Z}$. The *dominance ordering* on P is defined by $\mu \leq \nu$ if $(\nu - \mu)$ is an $\mathbb{N}$-linear combination of simple roots (here and later $\mathbb{N} = \{0, 1, 2, \dots\}$). Equivalently, $\mu \leq \nu$ if

$$\sum_{b \leq a} (\mu, \varepsilon_b) \leq \sum_{b \leq a} (\nu, \varepsilon_b) \tag{2.1}$$

for all $a \in \mathbb{Z}$ with equality for $a \gg 0$.

For $f \in \mathbb{Z}^{m|n}$ and $j \in I(m|n)$, define

$$\mathrm{wt}(f) := \sum_{i \in I(m|n)} \mathrm{sgn}(i) \varepsilon_{f(i)}, \qquad \mathrm{wt}_j(f) := \sum_{j \leq i \in I(m|n)} \mathrm{sgn}(i) \varepsilon_{f(i)}, \tag{2.2}$$

where $\mathrm{sgn}(i) \in \{\pm 1\}$ denotes the sign of i. The *degree of atypicality* of $f \in \mathbb{Z}^{m|n}$ is defined to be

$$\# f := \frac{1}{2} \left(m + n - \sum_{a \in \mathbb{Z}} \big| (\mathrm{wt}(f), \varepsilon_a) \big| \right). \tag{2.3}$$

If $\# f = 0$, then f is called *typical*. So f is typical if and only if

$$\{f(-m), \dots, f(-1)\} \cap \{f(1), \dots, f(n)\} = \varnothing.$$

§2-b. **Bruhat ordering.** Introduce a partial ordering on $\mathbb{Z}^{m|n}$ by declaring that $g \preceq f$ if $\mathrm{wt}(g) = \mathrm{wt}(f)$ and $\mathrm{wt}_j(g) \leq \mathrm{wt}_j(f)$ for all $j \in I(m|n)$. It is immediate that if $g \preceq f$ then $\# g = \# f$. Using (2.1), we see that $g \preceq f$ if and only if

$$\sum_{\substack{j \leq i \in I(m|n) \\ g(i) \leq a}} \mathrm{sgn}(i) \leq \sum_{\substack{j \leq i \in I(m|n) \\ f(i) \leq a}} \mathrm{sgn}(i) \tag{2.4}$$

for all $a \in \mathbb{Z}$ and $j \in I(m|n)$, with equality if either $a \gg 0$ or $j = -m$. From this, one gets in particular that $g \preceq f$ if and only if $\omega(g) \preceq \omega(f)$. In proofs, it will be convenient to have a shorthand for the sums appearing in the inequality (2.4), so for $f \in \mathbb{Z}^{m|n}, a \in \mathbb{Z}$ and $j \in I(m|n)$ we abbreviate

$$\#(f, a, j) = \sum_{\substack{j \leq i \in I(m|n) \\ f(i) \leq a}} \mathrm{sgn}(i).$$

Thus, $g \preceq f$ if and only if $\#(g, a, j) \leq \#(f, a, j)$ for all $a \in \mathbb{Z}$ and $j \in I(m|n)$, with equality for $a \gg 0$ or $j = -m$.

Here is another description of the partial order. Let $d_i \in \mathbb{Z}^{m|n}$ be the function $j \mapsto \mathrm{sgn}(i)\delta_{i,j}$, for each $i \in I(m|n)$. Write $f \downarrow g$ if one of the following holds:

(1) $g = f - d_i + d_j$ for some $-m \le i \le -1, 1 \le j \le n$ such that $f(i) = f(j)$;
(2) $g = f \cdot (i\ \ j)$ for some $1 \le i < j \le n$ such that $f(i) > f(j)$;
(3) $g = f \cdot (i\ \ j)$ for some $-m \le i < j \le -1$ such that $f(i) < f(j)$.

Then:

Lemma 2.5. *$f \succeq g$ if and only if there is a sequence $h_1, \dots, h_r \in \mathbb{Z}^{m|n}$ such that $f = h_1 \downarrow \cdots \downarrow h_r = g$.*

Proof. ($\Leftarrow$) Obvious.

($\Rightarrow$) We show by induction on $(m+n)$ that if $f \succ g$ are neighbors in the ordering, then $f \downarrow g$. The case $m+n = 0$ is vacuous, so suppose $m+n > 0$. Replacing f, g by $\omega(f), \omega(g)$ if necessary, we may assume in fact that $n > 0$. If $f(n) = g(n)$, then we are done by induction, so we may assume that $a = f(n) < g(n) = b$. We consider two cases.

Case one: there exists $0 < i < n$ with $a < f(i) \le b$. Pick the greatest such i, so each $f(j)$ for $j = i+1, \dots, n$ is either $\le a$ or $> b$, and set $c = f(i)$. We claim that $f \succ f \cdot (i\ \ n) \succeq g$, whence $f \downarrow g$ as required since f and g are neighbors. For $i < j$ and $a \le d < c$, we have that $\#(f \cdot (i\ \ n), d, j) = \#(f, d, j) - 1$, while $\#(f \cdot (i\ n), d, j) = \#(f, d, j)$ for all other j, d. Therefore to prove the claim, we just need to show that $\#(f, d, j) > \#(g, d, j)$ for each $i < j$ and each $a \le d < c$. But by the choice of i, we have that $\#(f, d, j) = \#(f, b, j) \ge \#(g, b, j) > \#(g, d, j)$ since $g(n) = b$.

Case two: each $f(j)$ for $j = 1, \dots, n$ is either $\le a$ or $> b$. From $(\mathrm{wt}(g), \varepsilon_b) = (\mathrm{wt}(f), \varepsilon_b) \le 0$, we deduce that there must exist $-m \le i < 0$ with $g(i) = b$. Take the greatest such i. Now we claim that $f \succeq g + d_i - d_n \succ g$, so again $f \downarrow g$ as they are neighbors. To prove the claim, note that $\#(g + d_i - d_n, d, j) = \#(g, d, j)$ unless $j > i$ and $d = b - 1$, while $\#(g + d_i - d_n, b-1, j) = \#(g, b-1, j) + 1$ for $j > i$. Therefore we need to show that $\#(f, b-1, j) > \#(g, b-1, j)$ for each $j > i$. Now observe that $\#(f, b-1, j) \ge \#(f, b, j) \ge \#(g, b, j) > \#(g, b-1, j)$. $\square$

For example, writing elements of $\mathbb{Z}^{2|2}$ as tuples,

$$(1, 2|2, 1) \downarrow (1, 2|1, 2) \downarrow (1, 3|1, 3) \downarrow (3, 1|1, 3).$$

It is worth pointing out that $f \in \mathbb{Z}^{m|n}$ is minimal with respect to the ordering just defined if and only if f is typical and antidominant.

§2-c. **The quantum group.** Recall that the quantum integer associated to $n \ge 0$ is $[n] := (q^n - q^{-n})/(q - q^{-1})$ and the quantum factorial is $[n]! := [n][n-1] \dots [2][1]$. Let $- : \mathbb{Q}(q) \to \mathbb{Q}(q)$ be the field automorphism induced by $q \mapsto q^{-1}$. We will call an additive map $f : V \to W$ between $\mathbb{Q}(q)$-vector spaces *antilinear* if $f(cv) = \overline{c} f(v)$ for all $c \in \mathbb{Q}(q), v \in V$.

Let $\mathscr{U}$ denote the quantum group $U_q(\mathfrak{gl}_\infty)$. By definition, this is the $\mathbb{Q}(q)$-algebra on generators E_a, F_a, K_a, K_a^{-1} $(a \in \mathbb{Z})$ subject to relations

$$\begin{aligned} K_a K_a^{-1} &= K_a^{-1} K_a = 1, \\ K_a K_b &= K_b K_a, \\ K_a E_b K_a^{-1} &= q^{(\varepsilon_a, \varepsilon_b - \varepsilon_{b+1})} E_b, \\ K_a F_b K_a^{-1} &= q^{(\varepsilon_a, \varepsilon_{b+1} - \varepsilon_b)} F_b, \end{aligned}$$

$$\begin{aligned}
E_aF_b - F_bE_a &= \delta_{a,b}\frac{K_{a,a+1}-K_{a+1,a}}{q-q^{-1}}, \\
E_aE_b &= E_bE_a && \text{if } |a-b|>1, \\
E_a^2E_b + E_bE_a^2 &= (q+q^{-1})E_aE_bE_a && \text{if } |a-b|=1, \\
F_aF_b &= F_bF_a && \text{if } |a-b|>1, \\
F_a^2F_b + F_bF_a^2 &= (q+q^{-1})F_aF_bF_a && \text{if } |a-b|=1.
\end{aligned}$$

Here, for any $a,b\in\mathbb{Z}$, $K_{a,b}$ denotes $K_aK_b^{-1}$. Also introduce the *divided powers* $F_a^{(r)} := F_a^r/[r]!$ and $E_a^{(r)} := E_a^r/[r]!$. We have the bar involution on $\mathscr{U}$, namely, the unique antilinear automorphism such that $\overline{E_a}=E_a, \overline{F_a}=F_a, \overline{K_a}=K_a^{-1}$.

We regard $\mathscr{U}$ as a Hopf algebra with comultiplication $\Delta:\mathscr{U}\to\mathscr{U}\otimes\mathscr{U}$ defined on generators by

$$\begin{aligned}
\Delta(E_a) &= 1\otimes E_a + E_a\otimes K_{a+1,a}, \\
\Delta(F_a) &= K_{a,a+1}\otimes F_a + F_a\otimes 1, \\
\Delta(K_a) &= K_a\otimes K_a.
\end{aligned}$$

This is the comultiplication from Kashiwara [K2], and is different from the one in Lusztig's book [L]. The counit ε is defined by $\varepsilon(E_a)=\varepsilon(F_a)=0$, $\varepsilon(K_a)=1$, the antipode S by $S(E_a)=-E_aK_{a,a+1}, S(F_a)=-K_{a+1,a}F_a, S(K_a)=K_a^{-1}$.

§2-d. **The space $\mathscr{T}^{m|n}$.** Let $\mathscr{V}$ be the natural $\mathscr{U}$-module, with basis $\{v_a\}_{a\in\mathbb{Z}}$ and action defined by

$$K_av_b = q^{\delta_{a,b}}v_b, \qquad E_av_b=\delta_{a+1,b}v_a, \qquad F_av_b=\delta_{a,b}v_{a+1}.$$

Let $\mathscr{W}=\mathscr{V}^*$ be the dual $\mathscr{U}$-module, with basis $\{w_a\}_{a\in\mathbb{Z}}$ related to the basis of $\mathscr{V}$ by $\langle w_a,v_b\rangle=(-q)^{-a}\delta_{a,b}$. The action of $\mathscr{U}$ on $\mathscr{W}$ satisfies

$$K_aw_b = q^{-\delta_{a,b}}w_b, \qquad E_aw_b=\delta_{a,b}w_{a+1}, \qquad F_aw_b=\delta_{a+1,b}w_a.$$

Let $\mathscr{T}^{m|n} := \mathscr{W}^{\otimes m}\otimes\mathscr{V}^{\otimes n}$, viewed as a $\mathscr{U}$-module in the natural way. Recall that $\mathbb{Z}^{m|n}$ denotes the set of all functions $I(m|n)\to\mathbb{Z}$. For $f\in\mathbb{Z}^{m|n}$, we let

$$M_f = w_{f(-m)}\otimes\cdots\otimes w_{f(-1)}\otimes v_{f(1)}\otimes\cdots\otimes v_{f(n)}.$$

The vectors $\{M_f\}_{f\in\mathbb{Z}^{m|n}}$ give a basis for $\mathscr{T}^{m|n}$. A vector v in a $\mathscr{U}$-module M is said to be of *weight* $\nu\in P$ if $K_av=q^{(\nu,\varepsilon_a)}v$ for all $a\in\mathbb{Z}$. The weight of the vector M_f is $\mathrm{wt}(f)$, as defined in (2.2).

We will often work with a completion $\widehat{\mathscr{T}}^{m|n}$ of $\mathscr{T}^{m|n}$. To define this formally, let $\mathbb{Z}^{m|n}_{\le d}$ denote the set of all $f\in\mathbb{Z}^{m|n}$ with $f(i)\le d$ for all $i\in I(m|n)$. Let $\mathscr{T}^{m|n}_{\le d}$ denote the subspace of $\mathscr{T}^{m|n}$ spanned by $\{M_f\}_{f\in\mathbb{Z}^{m|n}_{\le d}}$, and let $\pi_{\le d}:\mathscr{T}^{m|n}\to\mathscr{T}^{m|n}_{\le d}$ denote projection along the basis. The filtration $(\ker\pi_{\le d})_{d\in\mathbb{Z}}$ induces a topology on the abelian group $\mathscr{T}^{m|n}$; see [Bou, Chapter III, §2.5]. Let

$$\widehat{\mathscr{T}}^{m|n} = \varprojlim \mathscr{T}^{m|n}_{\le d}$$

denote the corresponding completion, and identify $\mathscr{T}^{m|n}$ with its image in $\widehat{\mathscr{T}}^{m|n}$. The projections $\pi_{\le d}$ extend by continuity to give maps $\pi_{\le d}:\widehat{\mathscr{T}}^{m|n}\to\mathscr{T}^{m|n}_{\le d}$. As usual, we will view elements of $\widehat{\mathscr{T}}^{m|n}$ as infinite $\mathbb{Q}(q)$-linear combinations of the basis elements $\{M_f\}_{f\in\mathbb{Z}^{m|n}}$ whose projections onto each $\mathscr{T}^{m|n}_{\le d}$ are finite sums. A

homomorphism $\theta : \mathscr{T}^{m|n} \to \widehat{\mathscr{T}}^{m|n}$ of abelian groups satisfying the compatibility condition

$$\pi_{\leq d}(u) = 0 \textit{ implies } \pi_{\leq d}(\theta(u)) = 0 \textit{ for all } u \in \mathscr{T}^{m|n} \textit{ and all } d \gg 0$$

is automatically continuous, hence extends uniquely to a continuous endomorphism of $\widehat{\mathscr{T}}^{m|n}$. In particular, the action of $\mathscr{U}$ lifts uniquely to a continuous action on $\widehat{\mathscr{T}}^{m|n}$, since E_a, F_a and K_a commute with $\pi_{\leq d}$ for all $d > a$.

§2-e. **The Iwahori-Hecke algebra.** Associated to the symmetric group $S_{m|n}$ we have the Iwahori-Hecke algebra $\mathscr{H}_{m|n}$. This is defined as the $\mathbb{Q}(q)$-algebra on generators $H_{-m+1}, \ldots, H_{-1}, H_1, \ldots, H_{n-1}$ subject to relations

$$\begin{aligned} &H_i^2 = 1 - (q - q^{-1})H_i, \\ &H_i H_{i+1} H_i = H_{i+1} H_i H_{i+1}, \\ &H_i H_j = H_j H_i \quad \text{if } |i - j| > 1. \end{aligned}$$

For $x \in S_{m|n}$, we have the corresponding element $H_x \in \mathscr{H}_{m|n}$, where $H_x = H_{i_1} \cdots H_{i_r}$ if $x = s_{i_1} \cdots s_{i_r}$ is a reduced expression for x. The bar involution on $\mathscr{H}_{m|n}$ is the unique antilinear automorphism such that $\overline{H_x} = H_{x^{-1}}^{-1}$, in particular $\overline{H_i} = H_i + (q - q^{-1})$.

We define a linear right action of $\mathscr{H}_{m|n}$ on $\mathscr{T}^{m|n}$ by the formulae

$$M_f H_i = \begin{cases} M_{f \cdot s_i} & \text{if } f \prec f \cdot s_i, \\ q^{-1} M_f & \text{if } f = f \cdot s_i, \\ M_{f \cdot s_i} - (q - q^{-1}) M_f & \text{if } f \succ f \cdot s_i. \end{cases}$$

Since the action of $\mathscr{H}_{m|n}$ commutes with all $\pi_{\leq d}$, it lifts by continuity to $\widehat{\mathscr{T}}^{m|n}$. As is well known (see e.g. [D2]), the actions of $\mathscr{U}$ and $\mathscr{H}_{m|n}$ on $\mathscr{T}^{m|n}$ commute with one another, hence the actions on the completion $\widehat{\mathscr{T}}^{m|n}$ also commute.

§2-f. **Some (anti)automorphisms.** Let $\sigma, \tau : \mathscr{U} \to \mathscr{U}$ be the antiautomorphisms and $\omega : \mathscr{U} \to \mathscr{U}$ be the automorphism defined by

$$\begin{aligned} \sigma(E_a) &= E_{-1-a}, & \sigma(F_a) &= F_{-1-a}, & \sigma(K_a) &= K_{-a}, \\ \tau(E_a) &= q^{-1} K_{a+1,a} F_a, & \tau(F_a) &= q E_a K_{a,a+1}, & \tau(K_a) &= K_a, \\ \omega(E_a) &= F_a, & \omega(F_a) &= E_a, & \omega(K_a) &= K_a^{-1}. \end{aligned}$$

Let $\tau : \mathscr{H}_{m|n} \to \mathscr{H}_{m|n}$ be the antiautomorphism and $\omega : \mathscr{H}_{m|n} \to \mathscr{H}_{n|m}$ be the isomorphism defined by $\tau(H_i) = H_i$ and $\omega(H_i) = H_{-i}$ for $i \in I(m-1|n-1)$. Introduce the linear map

$$\omega : \mathscr{T}^{m|n} \to \mathscr{T}^{n|m}, \qquad M_f \mapsto M_{\omega(f)}, \tag{2.6}$$

where $\omega(f)$ is as in §2-a. Note that ω extends by continuity to a linear map $\widehat{\mathscr{T}}^{m|n} \to \widehat{\mathscr{T}}^{n|m}$. Next let $(.,.)_{\mathscr{T}}$ be the symmetric bilinear form on $\mathscr{T}^{m|n}$ defined by

$$(M_f, M_g)_{\mathscr{T}} = \delta_{f,g} \tag{2.7}$$

for $f, g \in \mathbb{Z}^{m|n}$. Finally, define an antilinear map

$$\sigma : \mathscr{T}^{m|n} \to \mathscr{T}^{m|n}, \qquad M_f \mapsto M_{-f}. \tag{2.8}$$

The form $(.,.)_{\mathscr{T}}$ and the map σ do not extend to the completion.

Lemma 2.9. (i) *$\omega(XuH) = \omega(X)\omega(u)\omega(H)$ for all $X \in \mathscr{U}, H \in \mathscr{H}_{m|n}$ and $u \in \widehat{\mathscr{T}}^{m|n}$.*

(ii) *$(XuH, v)_{\mathscr{T}} = (u, \tau(X)v\tau(H))_{\mathscr{T}}$ for all $X \in \mathscr{U}, H \in \mathscr{H}_{m|n}$ and $u, v \in \mathscr{T}^{m|n}$.*

(iii) *$\sigma(XuH) = \tau(\overline{\sigma(X)})\sigma(u)\overline{H}$ for all $X \in \mathscr{U}, H \in \mathscr{H}_{m|n}$ and $u \in \mathscr{T}^{m|n}$.*

Proof. These are all checked directly for $\mathscr{H}_{m|n}$. To prove them for $\mathscr{U}$, one first checks that τ and $- \circ \sigma$ are coalgebra automorphisms and ω is a coalgebra antiautomorphism of $\mathscr{U}$. Hence it suffices to check (i)–(iii) when $m + n = 1$. □

§2-g. **Generation.** We proceed to prove that $\widehat{\mathscr{T}}^{m|n}$ is generated as a topological $\mathscr{U}$-module by the vectors M_f for *typical* $f \in \mathbb{Z}^{m|n}$.

Lemma 2.10. *Suppose that $f \in \mathbb{Z}^{m|n}$ and $1 \le i_1 < \cdots < i_r \le n$ are such that $f(i_1) = \cdots = f(i_r) = a + 1$ and $f(j) \neq a, a + 1$ for all $j \in \{i_1, i_1 + 1, \dots, n\} - \{i_1, \dots, i_r\}$. Let f' be the function with $f'(i_1) = \cdots = f'(i_r) = a$ and $f'(j) = f(j)$ for all $j \neq i_1, \dots, i_r$. Then, for any $g \preceq f$,*

$$E_a^{(r)} M_g \in \delta_{f,g} M_{f'} + \sum_{g' \prec f'} \mathbb{Z}[q, q^{-1}] M_{g'}.$$

Proof. Take $g \preceq f$. Recall the definition of $d_j \in \mathbb{Z}^{m|n}$ from §2-a. Note that $E_a^{(r)} M_g$ is a linear combination of $M_{g'}$'s where $g' = g - d_{j_1} - \cdots - d_{j_r}$ for $j_1 < \cdots < j_r \in I(m|n)$ such that

$$g(j_s) = \begin{cases} a & \text{if } j_s < 0, \\ a + 1 & \text{if } j_s > 0. \end{cases}$$

Let us show that for such a g', we have that $g' \preceq f'$. By (2.4), we need to show that $\#(g', b, j) \le \#(f', b, j)$ for all $b \in \mathbb{Z}$ and $j \in I(m|n)$. Since $g \preceq f$, we know that $\#(g, b, j) \le \#(f, b, j)$. So we are done except possibly for $b = a$. Suppose then that $\#(g', a, j) > \#(f', a, j)$ for some j. Say $i_1, \dots, i_s < j \le i_{s+1}, \dots, i_r$ and $j_1, \dots, j_t < j \le j_{t+1}, \dots, j_r$. Then,

$$\begin{aligned} \#(f', a, j) < \#(g', a, j) &= \#(g, a, j) + (r - t) \le \#(f, a, j) + (r - t) \\ &= \#(f', a, j) - (r - s) + (r - t) = \#(f', a, j) + s - t. \end{aligned}$$

Hence, we must have that $s > t$. This implies in particular that $j > 0$, and using this we get that

$$\#(g, a + 1, j) \ge \#(g', a, j) > \#(f', a, j) = \#(f, a + 1, j),$$

which is a contradiction. So indeed we must have that $g' \preceq f'$. Finally suppose that $g' = f'$. The assumption that $f(j) \neq a, a+1$ for $j \in \{i_1, i_1+1, \dots, n\} - \{i_1, \dots, i_r\}$ means that we must have $j_1 \le i_1, \dots, j_r \le i_r$. Hence, $f \preceq g$. Since we started with the assumption that $g \preceq f$, we therefore have $g = f$ which completes the proof. □

Twisting with ω using Lemma 2.9(i), we also have the analogous statement for $F_a^{(r)}$:

Lemma 2.11. *Suppose that $f \in \mathbb{Z}^{m|n}$ and $-m \le i_r < \cdots < i_1 \le -1$ are such that $f(i_1) = \cdots = f(i_r) = a + 1$ and $f(j) \neq a, a + 1$ for all $j \in \{-m, 1 - m, \dots, i_1\} - \{i_1, \dots, i_r\}$. Let f' be the function with $f'(i_1) = \cdots = f'(i_r) = a$ and $f'(j) = f(j)$ for all $j \neq i_1, \dots, i_r$. Then, for any $g \preceq f$,*

$$F_a^{(r)} M_g \in \delta_{f,g} M_{f'} + \sum_{g' \prec f'} \mathbb{Z}[q, q^{-1}] M_{g'}.$$

Theorem 2.12. *We can write each M_f as a (possibly infinite) $\mathbb{Z}[q,q^{-1}]$-linear combination of terms of the form $F_{a_1}^{(r_1)}\cdots F_{a_s}^{(r_s)}M_g$ for $a_1,\dots,a_s\in\mathbb{Z}, r_1,\dots,r_s\geq 1$ and* typical *$g\in\mathbb{Z}^{m|n}$.*

Proof. To prove the theorem, it suffices to show for each $d\in\mathbb{Z}$ and $f\in\mathbb{Z}^{m|n}$ that we can write M_f as a finite linear combination of terms of the form $F_{a_1}^{(r_1)}\cdots F_{a_s}^{(r_s)}M_g$ for typical $g\in\mathbb{Z}^{m|n}$ modulo $\ker\pi_{\leq d}$. So fix $d\in\mathbb{Z}$ and $f\in\mathbb{Z}^{m|n}$. There are only finitely many $g\preceq f$ with $\pi_{\leq d}M_g\neq 0$. So proceeding by induction on the dominance ordering, we may assume that every M_g with $g\prec f$ can be expressed as a finite linear combination of terms of the form $F_{a_1}^{(r_1)}\cdots F_{a_s}^{(r_s)}M_g$ for typical $g\in\mathbb{Z}^{m|n}$ modulo $\ker\pi_{\leq d}$.

Let $\{a_1<a_2<\cdots<a_s\}=\{f(-m),\dots,f(-1)\}$ and let $r_t=\#\{i\in I(m|0)\mid f(i)=a_t\}$ for each $t=1,\dots,s$. Choose $k\gg 0$ so that every element of the set $\{f(-m)+k,\dots,f(-1)+k\}$ exceeds every element of the set $\{f(1),\dots,f(n)\}$. Define $g\in\mathbb{Z}^{m|n}$ by

$$g(i)=\begin{cases} f(i) & \text{if } i>0,\\ f(i)+k & \text{if } i<0.\end{cases}$$

Note that g is typical by the choice of k. Now consider

$$F_{a_s}^{(r_s)}\cdots F_{a_s+k-1}^{(r_s)}F_{a_{s-1}}^{(r_{s-1})}\cdots F_{a_{s-1}+k-1}^{(r_{s-1})}\cdots F_{a_1}^{(r_1)}\cdots F_{a_1+k-1}^{(r_1)}M_g.$$

One checks using Lemma 2.11 that this equals M_f plus a $\mathbb{Z}[q,q^{-1}]$-linear combination of M_h's with $h\prec f$. So we are done by the induction hypothesis. □

Corollary 2.13. *Suppose $\theta:\widehat{\mathscr{T}}^{m|n}\to\widehat{\mathscr{T}}^{m|n}$ is a continuous $\mathscr{U},\mathscr{H}_{m|n}$-bimodule endomorphism fixing M_f for all typical antidominant $f\in\mathbb{Z}^{m|n}$. Then θ is the identity map.*

Proof. If f is antidominant, then $M_fH_x=M_{f\cdot x}$ for all $x\in S_{m|n}$. So for typical antidominant f we have that $\theta(M_{f\cdot x})=\theta(M_fH_x)=\theta(M_f)H_x=M_fH_x=M_{f\cdot x}$. This shows that θ fixes M_g for *all* typical $g\in\mathbb{Z}^{m|n}$. Now using the continuity of θ and Theorem 2.12, we get that θ fixes all M_f. Hence by continuity again, θ is the identity map. □

§2-h. **Canonical bases.** We now follow ideas of Lusztig [L, Chapter 27] to define a canonical topological basis of $\widehat{\mathscr{T}}^{m|n}$. We should note that in *loc. cit.*, Lusztig only considers finite dimensional quantum groups, but the techniques generalize to our situation on passing to the completion. The first step in the construction is to introduce a bar involution on the space $\widehat{\mathscr{T}}^{m|n}$ that is compatible with the bar involutions on $\mathscr{U}$ and on $\mathscr{H}_{m|n}$. The definition of this in Lusztig's work involves the quasi-R-matrix associated to $\mathscr{U}$. One gets from [L, §27.3] a bar involution $-:\widehat{\mathscr{T}}^{m|n}\to\widehat{\mathscr{T}}^{m|n}$ that satisfies property (iv), hence (i), in the theorem below, and that is compatible with the bar involution on $\mathscr{U}$. One then checks easily using Lusztig's definition that it is also compatible with the bar involution on $\mathscr{H}_{m|n}$, giving the existence half of the proof of the theorem. We will sketch a direct construction of the bar involution on $\widehat{\mathscr{T}}^{m|n}$ below, independent of Lusztig's work.

Theorem 2.14. *There exists a unique continuous, antilinear map $-:\widehat{\mathscr{T}}^{m|n}\to\widehat{\mathscr{T}}^{m|n}$ such that*

(i) $\overline{M_f}=M_f$ *for all typical antidominant $f\in\mathbb{Z}^{m|n}$;*

(ii) $\overline{XuH} = \overline{X}\,\overline{u}\,\overline{H}$ *for all* $X \in \mathscr{U}, u \in \widehat{\mathscr{T}}^{m|n}$, $H \in \mathscr{H}_{m|n}$.

Moreover,

(iii) *bar is an involution;*
(iv) $\overline{M_f} = M_f + (*)$ *where* $(*)$ *is a (possibly infinite)* $\mathbb{Z}[q, q^{-1}]$*-linear combination of* M_g*'s for* $g \prec f$*;*
(v) $\overline{\omega(u)} = \omega(\overline{u})$ *for all* $u \in \widehat{\mathscr{T}}^{m|n}$.

Proof. Let us first explain how to prove the uniqueness, and also the fact that bar is necessarily an involution. Suppose that we are given two continuous antilinear maps $-, \sim : \widehat{\mathscr{T}}^{m|n} \to \widehat{\mathscr{T}}^{m|n}$ satisfying properties (i) and (ii). Then we can consider the composite map

$$\varphi : \widehat{\mathscr{T}}^{m|n} \to \widehat{\mathscr{T}}^{m|n}, u \mapsto \widetilde{\overline{u}}.$$

This is a continuous $\mathscr{U}, \mathscr{H}_{m|n}$-bimodule endomorphism of $\widehat{\mathscr{T}}^{m|n}$ fixing M_f for all typical, antidominant f. Hence, φ is the identity map by Corollary 2.13. In particular, this gives that $\overline{\overline{u}} = u$ and $\widetilde{\widetilde{u}} = u$ for each $u \in \widehat{\mathscr{T}}^{m|n}$. Finally, applying $\sim$ to both sides of the equation $\widetilde{\overline{u}} = u$ gives that $\overline{u} = \widetilde{u}$ for all u, whence uniqueness.

To get existence without appealing directly to Lusztig's work, we need a little more notation. For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$, let $|\lambda|$ denote the sum of its parts, $r(\lambda)$ denote the total number of non-zero parts, and $r_s(\lambda)$ denote the total number of parts equal to s. Also let

$$p(\lambda) = (1-q^2)^{r(\lambda)}(-q)^{-|\lambda|} \prod_{s \geq 1} q^{r_s(\lambda)(r_s(\lambda)-1)/2}[r_s(\lambda)]!.$$

Finally, for $a \in \mathbb{Z}$ and a partition λ, let $a_\lambda^{m|n} \in \mathbb{Z}^{m|n}$ denote the function $i \mapsto a + \lambda_{|i|}$. Let $D_\lambda^{m|n}$ denote the set of minimal length $\mathrm{stab}_{S_{m|n}}(a_\lambda^{m|n}) \backslash S_{m|n}$-coset representatives. For example, if $\lambda = (2,1,1)$, then

$$\begin{aligned} M_{0_\lambda^{3|2}} &= w_1 \otimes w_1 \otimes w_2 \otimes v_2 \otimes v_1, \\ D_\lambda^{3|2} &= \{1, s_{-1}, s_{-1}s_{-2}, s_1, s_{-1}s_1, s_{-1}s_{-2}s_1\}. \end{aligned}$$

Now define $- : \mathscr{T}^{m|n} \to \widehat{\mathscr{T}}^{m|n}$ to be the unique antilinear map satisfying the following properties:

(B1) For $a \in \mathbb{Z}$, $\overline{w_a^{\otimes m} \otimes v_a^{\otimes n}} = \displaystyle\sum_{\substack{\lambda \text{ with} \\ r(\lambda) \leq m,n}} p(\lambda) \Big[\sum_{x \in D_\lambda^{m|n}} q^{\ell(x)} M_{a_\lambda^{m|n} \cdot x} \Big]$.
(B2) If $x \in S_{m|n}$ is the unique element of minimal length such that $f \cdot x$ is antidominant, then $\overline{M_f} = \overline{M_{f \cdot x}} H_x^{-1}$.
(B3) If all elements of $\{f(-m), \ldots, f(-k-1), f(l+1), \ldots, f(n)\}$ are strictly greater than all elements of $\{f(-k), \ldots, f(-1), f(1), \ldots, f(l)\}$ for some $1 \leq k \leq m, 1 \leq l \leq n$, then

$$\overline{M_f} = \sum_{\substack{g \in I(m-k|n-l) \\ h \in I(k|l)}} a_g b_h w_{g(k-m)} \otimes \cdots \otimes w_{g(-1)} \otimes M_h \otimes v_{g(1)} \otimes \cdots \otimes v_{g(n-l)},$$

where

$$\overline{w_{f(-m)} \otimes \cdots \otimes w_{f(-k-1)} \otimes v_{f(l+1)} \otimes \cdots \otimes v_{f(n)}} = \sum_{g \in I(m-k|n-l)} a_g M_g,$$

$$\overline{w_{f(-k)} \otimes \cdots \otimes w_{f(-1)} \otimes v_{f(1)} \otimes \cdots \otimes v_{f(l)}} = \sum_{h \in I(k|l)} b_h M_h.$$

The following property is a consequence of (B1), and is useful in inductive arguments. We omit the proof.

(B1)′ For $m, n > 0$ and $a \in \mathbb{Z}$,

$$\overline{w_a^{\otimes m} \otimes v_a^{\otimes n}} = w_a \otimes \overline{w_a^{\otimes(m-1)} \otimes v_a^{\otimes n}} + q^{m-n}(q-q^{-1}) \sum_{l \geq 1} (-q)^{1-l} w_{a+l} \otimes \overline{w_a^{\otimes(m-1)} \otimes v_a^{\otimes(n-1)}} \otimes v_{a+l} X,$$

where $X = 1 + qH_{n-1}^{-1} + \cdots + q^{n-1} H_{n-1}^{-1} \cdots H_1^{-1}$.

For example:

$$\overline{w_a^{\otimes m}} = w_a^{\otimes m}, \qquad \overline{v_a^{\otimes n}} = v_a^{\otimes n},$$

$$\overline{w_a \otimes v_a} = w_a \otimes v_a + (q-q^{-1}) \sum_{l \geq 1} (-q)^{1-l} w_{a+l} \otimes v_{a+l}.$$

Now one checks easily that the map $- : \mathscr{T}^{m|n} \to \widehat{\mathscr{T}}^{m|n}$ just defined satisfies (iv), hence (i), and (v). In particular, (iv) implies that bar is continuous, so it extends uniquely to a continuous antilinear map $- : \widehat{\mathscr{T}}^{m|n} \to \widehat{\mathscr{T}}^{m|n}$. One finally needs to show that it satisfies (ii). This is done by a lengthy—but elementary—verification, using (B1), (B1)′, (B2) and (B3) directly. We omit the details. □

Now we appeal to the following general lemma originating in [KL], also used implicitly in [L, §27.3]. See [D1, 1.2] for a short proof.

Lemma 2.15. *Let $(I, \preceq)$ be a partially ordered set with the property that $\{j \in I \mid j \preceq i\}$ is finite for all $i \in I$. Suppose that M is a $\mathbb{Q}(q)$-vector space with basis $\{u_i\}_{i \in I}$ equipped with an antilinear involution $- : M \to M$ such that $\overline{u_i} = u_i + (*)$ for each $i \in I$, where $(*)$ is a $\mathbb{Z}[q, q^{-1}]$-linear combination of u_j's for $j \prec i$. Then there exist unique bases $\{x_i\}_{i \in I}$, $\{y_i\}_{i \in I}$ for M such that*

(i) *$\overline{x_i} = x_i$ and $\overline{y_i} = y_i$;*
(ii) *$x_i \in u_i + \sum_{j \in I} q\mathbb{Z}[q] u_j$ and $y_i \in u_i + \sum_{j \in I} q^{-1}\mathbb{Z}[q^{-1}] u_j$,*

for each $i \in I$. Moreover, the coefficient of u_j in x_i (resp. y_i) is zero unless $j \preceq i$.

Applying this to the space $\mathscr{T}_{\leq d}^{m|n}$ for fixed $d \in \mathbb{Z}$, the basis $\{M_f\}_{f \in \mathbb{Z}_{\leq d}^{m|n}}$ and the antilinear involution $\pi_{\leq d} \circ - : \mathscr{T}_{\leq d}^{m|n} \to \mathscr{T}_{\leq d}^{m|n}$, we deduce:

Lemma 2.16. *There exist unique bases $\{T_f^{(d)}\}_{f \in \mathbb{Z}_{\leq d}^{m|n}}$, $\{L_f^{(d)}\}_{f \in \mathbb{Z}_{\leq d}^{m|n}}$ for $\mathscr{T}_{\leq d}^{m|n}$ such that*

(i) *$\pi_{\leq d}(\overline{T_f^{(d)}}) = T_f^{(d)}$ and $\pi_{\leq d}(\overline{L_f^{(d)}}) = L_f^{(d)}$;*
(ii) *$T_f^{(d)} \in M_f + \sum_{g \in \mathbb{Z}_{\leq d}^{m|n}} q\mathbb{Z}[q] M_g$ and $L_f^{(d)} \in M_f + \sum_{g \in \mathbb{Z}_{\leq d}^{m|n}} q^{-1}\mathbb{Z}[q^{-1}] M_g$.*

Moreover, the coefficient of M_g in $T_f^{(d)}$ (resp. $L_f^{(d)}$) is zero unless $g \preceq f$.

Passing to the completion gives us bases for $\widehat{\mathscr{T}}^{m|n}$:

Theorem 2.17. *There exist unique topological bases $\{T_f\}_{f\in\mathbb{Z}^{m|n}}$, $\{L_f\}_{f\in\mathbb{Z}^{m|n}}$ for $\widehat{\mathscr{T}}^{m|n}$ such that*

(i) $\overline{T_f} = T_f$ *and* $\overline{L_f} = L_f$*;*
(ii) $T_f \in M_f + \widehat{\sum}_{g\in\mathbb{Z}^{m|n}} q\mathbb{Z}[q]M_g$ *and* $L_f \in M_f + \widehat{\sum}_{g\in\mathbb{Z}^{m|n}} q^{-1}\mathbb{Z}[q^{-1}]M_g$.

Moreover,

(iii) $T_f = M_f + (*)$ *and* $L_f = M_f + (**)$*, where* $(*)$ *and* $(**)$ *are (possibly infinite) linear combinations of* M_g*'s for* $g \prec f$*;*
(iv) $\omega(T_f) = T_{\omega(f)}$ *and* $\omega(L_f) = L_{\omega(f)}$.

Proof. Take $e \geq d$ and $f \in \mathbb{Z}^{m|n}_{\leq d}$. Consider the elements $T_f^{(e)} \in \mathscr{T}^{m|n}_{\leq e}$ and $T_f^{(d)} \in \mathscr{T}^{m|n}_{\leq d}$ given by Lemma 2.16. We know that if $\pi_{\leq d}(u) = 0$ then $\pi_{\leq d}(\overline{u}) = 0$. Hence we have that $\pi_{\leq d}(\overline{\pi_{\leq d}(u) - u}) = 0$ for all $u \in \widehat{\mathscr{T}}^{m|n}$. Applying this to $u = T_f^{(e)}$ we deduce that $\pi_{\leq d}(\overline{\pi_{\leq d}(T_f^{(e)})}) = \pi_{\leq d}(T_f^{(e)})$. Hence by the uniqueness in Lemma 2.16, we have that $\pi_{\leq d}(T_f^{(e)}) = T_f^{(d)}$. Similarly, $\pi_{\leq d}(L_f^{(e)}) = L_f^{(d)}$. Hence, for all $f \in \mathbb{Z}^{m|n}$ there exist unique elements $T_f, L_f \in \widehat{\mathscr{T}}^{m|n}$ such that

$$\pi_{\leq d}(T_f) = T_f^{(d)}, \qquad \pi_{\leq d}(L_f) = L_f^{(d)}$$

for all $d \in \mathbb{Z}$ and all $f \in \mathbb{Z}^{m|n}_{\leq d}$. Using the lemma for each $d \in \mathbb{Z}$, one now easily checks that these satisfy (i)–(iii), while (iv) follows from Theorem 2.14(v). $\square$

We call the basis $\{T_f\}_{f\in\mathbb{Z}^{m|n}}$ the *canonical basis* of $\widehat{\mathscr{T}}^{m|n}$, and $\{L_f\}_{f\in\mathbb{Z}^{m|n}}$ is the *dual canonical basis*. Let us introduce notation for the coefficients: let

$$T_f = \sum_{g\in\mathbb{Z}^{m|n}} t_{g,f}(q)M_g, \qquad L_f = \sum_{g\in\mathbb{Z}^{m|n}} l_{g,f}(q)M_g \tag{2.18}$$

for polynomials $t_{g,f}(q) \in \mathbb{Z}[q]$ and $l_{g,f}(q) \in \mathbb{Z}[q^{-1}]$. We know that $t_{g,f}(q) = l_{g,f}(q) = 0$ unless $g \preceq f$, and that $t_{f,f}(q) = l_{f,f}(q) = 1$.

Example 2.19. If $m = n = 1$, the bases $\{T_f\}_{f\in\mathbb{Z}^{m|n}}$ and $\{L_f\}_{f\in\mathbb{Z}^{m|n}}$ are

$$\begin{aligned}
&\{w_a \otimes v_b \mid a, b \in \mathbb{Z}, a \neq b\} \cup \{w_a \otimes v_a + q w_{a+1} \otimes v_{a+1} \mid a \in \mathbb{Z}\},\\
&\{w_a \otimes v_b \mid a, b \in \mathbb{Z}, a \neq b\} \cup \{w_a \otimes v_a - q^{-1} w_{a+1} \otimes v_{a+1}\\
&\qquad\qquad + q^{-2} w_{a+2} \otimes v_{a+2} - \cdots \mid a \in \mathbb{Z}\},
\end{aligned}$$

respectively.

§2-i. **Duality.** We wish next to explain the relationship between the bases $\{T_f\}_{f\in\mathbb{Z}^{m|n}}$ and $\{L_f\}_{f\in\mathbb{Z}^{m|n}}$. Recall the definitions (2.7) and (2.8). Define a new bilinear form $\langle .,. \rangle_{\mathscr{T}}$ on $\widehat{\mathscr{T}}^{m|n}$ by

$$\langle u, v \rangle_{\mathscr{T}} = (u, \sigma(\overline{v}))_{\mathscr{T}} \tag{2.20}$$

for $u, v \in \widehat{\mathscr{T}}^{m|n}$. Note that this makes sense, even though the expression $\sigma(\overline{v})$ may not make sense in its own right. Indeed, it is clear that on expanding u and $\overline{v}$ in

terms of the basis $\{M_f\}$, there are only finitely many f such that M_f is involved in u and M_{-f} is involved in $\overline{v}$. So we can interpret $(u, \sigma(\overline{v}))_{\mathscr{T}}$ as

$$\sum_{f \in \mathbb{Z}^{m|n}} (u, M_f)_{\mathscr{T}} \overline{(M_{-f}, \overline{v})_{\mathscr{T}}},$$

all but finitely many terms in the sum being zero.

Lemma 2.21. *$\langle XuH, v\rangle_{\mathscr{T}} = \langle u, \sigma(X)v\tau(H)\rangle_{\mathscr{T}}$ for all $X \in \mathscr{U}, H \in \mathscr{H}_{m|n}$ and $u, v \in \widehat{\mathscr{T}}^{m|n}$.*

Proof. According to Lemma 2.9, we have that

$$\begin{aligned}\langle XuH, v\rangle_{\mathscr{T}} &= (XuH, \sigma(\overline{v}))_{\mathscr{T}} = (u, \tau(X)\sigma(\overline{v})\tau(H))_{\mathscr{T}} \\ &= (u, \tau(\overline{\sigma\overline{(\sigma(X))}})\sigma(\overline{v})\tau(H))_{\mathscr{T}} = (u, \sigma(\overline{\sigma(X)}\overline{v}\overline{\tau(H)}))_{\mathscr{T}} \\ &= (u, \sigma(\overline{\sigma(X)v\tau(H)}))_{\mathscr{T}} = \langle u, \sigma(X)v\tau(H)\rangle_{\mathscr{T}}.\end{aligned}$$

□

Lemma 2.22. *The bilinear form $\langle ., .\rangle_{\mathscr{T}}$ is symmetric.*

Proof. Let us first show that $\langle u, M_f\rangle_{\mathscr{T}} = \langle M_f, u\rangle_{\mathscr{T}}$ for all $u \in \widehat{\mathscr{T}}^{m|n}$ and typical antidominant $f \in \mathbb{Z}^{m|n}$. We need to show that $(u, M_{-f})_{\mathscr{T}} = \overline{(\overline{u}, M_{-f})_{\mathscr{T}}}$, for which it suffices to consider the special case $u = M_g$. Then, $(M_g, M_{-f})_{\mathscr{T}} = \delta_{g,-f}$. Consider $(\overline{M_g}, M_{-f})_{\mathscr{T}}$. By Theorem 2.14(iv), it is zero unless $\mathrm{wt}(g) = \mathrm{wt}(-f)$. So since $-f$ is typical, g must be too. By Theorem 2.14(iv),

$$\overline{M_g} = M_g + (\text{a linear combination of } M_h\text{'s with } h \prec g).$$

Since f is antidominant, we deduce that $(\overline{M_g}, M_{-f})_{\mathscr{T}} = (M_g, M_{-f})_{\mathscr{T}} = \delta_{g,-f}$.

Now we show that $\langle u, M_f\rangle_{\mathscr{T}} = \langle M_f, u\rangle_{\mathscr{T}}$ for all $u \in \widehat{\mathscr{T}}^{m|n}$ and all typical $f \in \mathbb{Z}^{m|n}$. Let $x \in S_{m|n}$ be of minimal length such that $f \cdot x$ is antidominant. Then, by the previous paragraph and Lemma 2.21, we have that

$$\begin{aligned}\langle u, M_f\rangle_{\mathscr{T}} &= \langle u, M_{f\cdot x}T_{x^{-1}}\rangle_{\mathscr{T}} = \langle uT_x, M_{f\cdot x}\rangle_{\mathscr{T}} \\ &= \langle M_{f\cdot x}, uT_x\rangle_{\mathscr{T}} = \langle M_{f\cdot x}T_{x^{-1}}, u\rangle_{\mathscr{T}} = \langle M_f, u\rangle_{\mathscr{T}},\end{aligned}$$

as required.

Now let us consider the general case. In view of Theorem 2.12, we may assume that $v = XM_g$ for typical g and $X = F_{a_1}^{(r_1)} \cdots F_{a_s}^{(r_s)} \in \mathscr{U}$. Then, by the previous paragraph and Lemma 2.21, we have that

$$\begin{aligned}\langle u, v\rangle_{\mathscr{T}} &= \langle u, XM_g\rangle_{\mathscr{T}} = \langle \sigma(X)u, M_g\rangle_{\mathscr{T}} \\ &= \langle M_g, \sigma(X)u\rangle_{\mathscr{T}} = \langle XM_g, u\rangle_{\mathscr{T}} = \langle v, u\rangle_{\mathscr{T}}.\end{aligned}$$

This completes the proof. □

The following theorem characterizes $\{L_f\}_{f \in \mathbb{Z}^{m|n}}$ uniquely as the dual basis to $\{T_{-g}\}_{g \in \mathbb{Z}^{m|n}}$ under the bilinear form $\langle ., .\rangle_{\mathscr{T}}$.

Theorem 2.23. *For $f, g \in \mathbb{Z}^{m|n}$, $\langle L_f, T_{-g}\rangle_{\mathscr{T}} = \delta_{f,g}$.*

Proof. Consider $\langle L_f, T_{-g}\rangle_{\mathscr{T}} = (L_f, \sigma(T_{-g}))_{\mathscr{T}}$. We observe that L_f is equal to M_f plus a $q^{-1}\mathbb{Z}[q^{-1}]$-linear combination of M_h's with $h \prec f$. Also $\sigma(T_{-g})$ equals M_g plus a $q^{-1}\mathbb{Z}[q^{-1}]$-linear combination of M_h's with $h \succ g$. Hence, $\langle L_f, T_{-g}\rangle_{\mathscr{T}}$ is zero unless $f \succeq g$, it is 1 if $f = g$ and it is in $q^{-1}\mathbb{Z}[q^{-1}]$ if $f \succ g$.

On the other hand, by Lemma 2.22, $\langle L_f, T_{-g}\rangle_{\mathscr{T}} = (\sigma(L_f), T_{-g})_{\mathscr{T}}$. Hence, arguing as in the previous paragraph, $\langle L_f, T_{-g}\rangle_{\mathscr{T}}$ is zero unless $f \succeq g$, it is 1 if $f = g$ and it is in $q\mathbb{Z}[q]$ if $f \succ g$. Since $q\mathbb{Z}[q] \cap q^{-1}\mathbb{Z}[q^{-1}] = \{0\}$, this completes the proof. □

Corollary 2.24. *For $f \in \mathbb{Z}^{m|n}$,*

$$M_f = \sum_{g \in \mathbb{Z}^{m|n}} t_{-f,-g}(q^{-1}) L_g = \sum_{g \in \mathbb{Z}^{m|n}} l_{-f,-g}(q^{-1}) T_g.$$

Proof. By the theorem, we can write $M_f = \sum_{g \in \mathbb{Z}^{m|n}} \langle M_f, T_{-g}\rangle_{\mathscr{T}} L_g$. Now a computation from the definition (2.20) of the form $\langle .,. \rangle_{\mathscr{T}}$ gives that $\langle M_f, T_{-g}\rangle_{\mathscr{T}} = t_{-f,-g}(q^{-1})$. The second equality is proved similarly. □

§2-j. **An algorithm.** The goal in this subsection is to explain an algorithm to compute $T_f^{(d)}$ (cf. Lemma 2.16) for each $d \in \mathbb{Z}$ and $f \in \mathbb{Z}^{m|n}_{\leq d}$. Assuming a certain positivity conjecture which ensures that the $T_f^{(d)}$ converge to T_f in finitely many steps, the algorithm can be modified to actually compute the canonical basis elements T_f themselves.

The algorithm proceeds by induction on the degree of atypicality $\#f$ of f. To begin with, we describe the base of the induction by explaining how to compute T_f (hence all $T_f^{(d)}$) for *typical* f. If f is typical and antidominant, then we have that $T_f = M_f$ and we are done. Otherwise, we can find $i \in I(m-1|n-1)$ such that $f \cdot s_i \prec f$. We may assume by induction that $T_{f \cdot s_i}$ is already known, and consider the bar invariant element $T_{f \cdot s_i}(H_i + q)$, which we view as a first approximation to T_f. It equals M_f plus a sum of terms $p_{g,f}(q) M_g$ for polynomials $p_{g,f}(q) \in \mathbb{Z}[q]$ and g with $g \prec f$ (there being only finitely many such g's since f is typical). For each such g with $p_{g,f}(0) \neq 0$, we make a correction by subtracting $p_{g,f}(0) T_g$ from our first approximation. The result is a bar invariant expression that equals M_f plus a $q\mathbb{Z}[q]$-linear combination of M_g's. This must be T_f by the uniqueness in Theorem 2.17.

We have just described the usual algorithm to compute the parabolic Kazhdan-Lusztig polynomials associated to the Hecke algebra $\mathscr{H}_{m|n}$ [KL], [Deo]. To make this precise, let us recall the definition of the latter, following [So1]. Let $f \in \mathbb{Z}^{m|n}$ be antidominant. Then, $S_f := \mathrm{stab}_{S_{m|n}}(f)$ is a parabolic subgroup of $S_{m|n}$. Let $\mathscr{H}_f$ be the corresponding parabolic subalgebra of $\mathscr{H}_{m|n}$. Let $\mathbf{1}_{\mathscr{H}_f}$ denote the one dimensional right $\mathscr{H}_f$-module on basis 1 with action $1H_i = q^{-1}1$ for each $H_i \in \mathscr{H}_f$. We consider the induced module $\mathbf{1}_{\mathscr{H}_f} \otimes_{\mathscr{H}_f} \mathscr{H}_{m|n}$. This has a basis given by the elements $M_x^{(f)} := 1 \otimes H_x$ as x runs over the set D_f of minimal length $S_f \backslash S_{m|n}$-coset representatives. The bar involution on $\mathbf{1}_{\mathscr{H}_f} \otimes_{\mathscr{H}_f} \mathscr{H}_{m|n}$ is the antilinear map defined by $\overline{1 \otimes H_x} := 1 \otimes \overline{H_x}$. The Kazhdan-Lusztig basis of $\mathbf{1}_{\mathscr{H}_f} \otimes_{\mathscr{H}_f} \mathscr{H}_{m|n}$ is the unique bar invariant basis $\{C_x^{(f)}\}_{x \in D_f}$ such that

$$C_x^{(f)} \in M_x^{(f)} + \sum_{y \in D_f} q\mathbb{Z}[q] M_y^{(f)}.$$

The corresponding parabolic Kazhdan-Lusztig polynomials are defined from

$$C_x^{(f)} = \sum_{y \in D_f} m_{y,x}^{(f)}(q) M_y^{(f)}.$$

The relationship to our situation is as follows:

Lemma 2.25. *Suppose that f is typical and antidominant. Then, for $x \in D_f$ and $g \in \mathbb{Z}^{m|n}$,*

$$t_{g,f\cdot x}(q) = \begin{cases} m_{y,x}^{(f)}(q) & \text{if } g = f \cdot y \text{ for some } y \in D_f, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Note that $M_f H_i = q^{-1} M_f$ for all $H_i \in \mathscr{H}_f$. So we get from Frobenius reciprocity a unique $\mathscr{H}_{m|n}$-module homomorphism $\theta : \mathbf{1}_{\mathscr{H}_f} \otimes_{\mathscr{H}_f} \mathscr{H}_{m|n} \to \mathscr{T}^{m|n}$ under which $M_x^{(f)}$ maps to $M_{f\cdot x}$. By (B2) from the proof of Theorem 2.14, $\overline{M_{f\cdot x}} = M_f \overline{H_x}$. So $\theta(\overline{u}) = \overline{\theta(u)}$ for each $u \in \mathbf{1}_{\mathscr{H}_f} \otimes_{\mathscr{H}_f} \mathscr{H}_{m|n}$. Therefore, $\theta(C_x^{(f)})$ is bar invariant, and it equals $M_{f\cdot x}$ plus a $q\mathbb{Z}[q]$-linear combination of other M_g's. Hence, $\theta(C_x^{(f)}) = T_{f\cdot x}$ by the uniqueness. This shows that for each $x \in D_f$,

$$T_{f\cdot x} = \sum_{y \in D_f} m_{y,x}^{(f)}(q) M_{f\cdot y}.$$

The lemma follows. □

Remark 2.26. In the case $m = 0$, this lemma shows that in type A the parabolic Kazhdan-Lusztig polynomials coincide with the coefficients of the canonical basis of $\mathscr{V}^{\otimes n}$. This is a well-known observation; see for example [FKK].

Now we describe the algorithm to compute $T_f^{(d)}$ for atypical f. We assume therefore that we are given $d \in \mathbb{Z}$ and $f \in \mathbb{Z}^{m|n}_{\leq d}$ with $\#f > 0$, and that we have already constructed an algorithm to compute $T_g^{(e)}$ for each $e \in \mathbb{Z}$ and g with $\#g < \#f$. Let us write $f_- = (f(-m), \dots, f(-1))$ and $f_+ = (f(1), \dots, f(n))$. Define a_1 to be the greatest integer that appears in both the tuples f_- and f_+. Now we iterate a certain *bumping procedure*:

Let $n_{1,1}$ be the number of entries equal to a_1 appearing in the tuple f_+, and label all such entries. If there are no entries equal to (a_1+1) appearing to the right of labeled a_1's, move on to the next paragraph. Otherwise, let $n_{1,2}$ be the number of entries equal to (a_1+1) appearing to the right of labeled a_1's, and label all such (a_1+1)'s. Next, if there are no (a_1+2)'s to the right of labeled (a_1+1)'s, move on to the next paragraph. Otherwise let $n_{1,3}$ be the number of (a_1+2)'s to the right of labeled (a_1+1)'s, and label all such (a_1+2)'s. Continue in this way.

When the process just described terminates, we are left with a sequence $n_{1,1}$, $n_{1,2}, \dots, n_{1,k_1}$ for some $k_1 \geq 1$, where there are $n_{1,i}$ labeled (a_1+i-1)'s in the tuple f_+. We define $X_1 := E_{a_1+k_1-1}^{(n_{1,k_1})} \cdots E_{a_1+1}^{(n_{1,2})} E_{a_1}^{(n_{1,1})}$ and $a_2 := a_1 + k_1$. If there are no entries equal to a_2 in the tuple f_-, the bumping procedure is finished. Otherwise, we need to repeat the bumping procedure but applied to f_- instead, as follows. Let $n_{2,1}$ be the number of entries equal to a_2 appearing in the tuple f_-, and label all such entries. If there are no entries equal to (a_2+1) appearing to the left of labeled a_2's, move on to the next paragraph. Otherwise, let $n_{2,2}$ be the number of entries equal to (a_2+1) appearing to the left of labeled a_2's, and label all such (a_2+1)'s. Continue in this way until the process terminates.

We are left with a sequence $n_{2,1}, n_{2,2}, \dots, n_{2,k_1}$ for some $k_2 \geq 1$, where there are $n_{2,i}$ labeled (a_2+i-1)'s in the tuple f_-. Let $X_2 := F_{a_2+k_2-1}^{(n_{2,k_2})} \cdots F_{a_2+1}^{(n_{2,2})} F_{a_2}^{(n_{2,1})}$ and $a_3 := a_2 + k_2$. This time if there are no entries equal to a_3 in the tuple f_+, the bumping procedure is finished. Otherwise, we repeat the whole process once more from the beginning, but using a_3 in place of a_1, to construct $X_3, a_4, X_4, \dots$ and so on.

When the bumping procedure finally ends, we are left with a sequence of monomials $X_1, \dots, X_N$ and integers $a_1 < a_2 < \cdots < a_{N+1}$. Increase all labeled entries in the tuples f_-, f_+ by 1 and let $h \in \mathbb{Z}^{m|n}$ be the corresponding function. Note that $\#h < \#f$. So by induction, we can compute $T_h^{(e)}$, where $e = \max(d, a_{N+1})$. Now consider the bar invariant element $\pi_{\leq d}(X_N \cdots X_1 T_h^{(e)}) \in \mathscr{T}_{\leq d}^{m|n}$, which is our first approximation to $T_f^{(d)}$. By Lemmas 2.10 and 2.11, it equals M_f plus a finite linear combination of terms $p_{g,f}(q) M_g$ for polynomials $p_{g,f}(q) \in \mathbb{Z}[q, q^{-1}]$ and $g \in \mathbb{Z}_{\leq d}^{m|n}$ with $g \prec f$. Now we make corrections to the first approximation. Let $g \prec f$ be maximal such that $p_{g,f}(q) \notin q\mathbb{Z}[q]$. Let $p'_{g,f}(q)$ be the unique bar invariant element of $\mathbb{Z}[q, q^{-1}]$ such that $p'_{g,f}(q) \equiv p_{g,f}(q) \pmod{q\mathbb{Z}[q]}$. Proceeding by induction on the ordering on $\mathbb{Z}_{\leq d}^{m|n}$, we may assume that $T_g^{(d)}$ is already known. Subtract $p'_{g,f}(q) T_g^{(d)}$ from the first approximation, to obtain a second approximation to $T_f^{(d)}$. Repeating the correction procedure, we reduce in finitely many steps to a bar invariant expression that equals M_f plus a $q\mathbb{Z}[q]$-linear combination of M_g's. This must be $T_f^{(d)}$ by the uniqueness. We are done.

Example 2.27. We explain how to compute $T_{(0,4,1|0,2,3)}^{(4)}$ using the algorithm. The bumping procedure proceeds as follows:

$$\begin{aligned}(0,4,1|0,2,3) &\xrightarrow{a_1=0} (0,4,1|\underline{0},2,3) \xrightarrow{a_2=1} (0,4,\underline{1}|\underline{0},2,3)\\ &\xrightarrow{a_3=2} (0,4,\underline{1}|\underline{0},\underline{2},\underline{3}) \xrightarrow{a_4=4} (0,\underline{4},\underline{1}|\underline{0},\underline{2},\underline{3}).\end{aligned}$$

Now, $(0,5,2|1,3,4)$ is typical, so we can compute $\pi_{\leq 4}(F_4 E_3 E_2 F_1 E_0 T_{(0,5,2|1,3,4))}^{(5)})$ using the Kazhdan-Lusztig algorithm. It turns out that this equals $M_{(0,4,1|0,2,3)} + M_{(1,4,1|1,2,3)}$ plus a $q\mathbb{Z}[q]$-linear combination of lower terms. Now one computes $T_{(1,4,1|1,2,3)}^{(4)}$ needed for the correction procedure by repeating the algorithm (which is rather lengthy). Finally one obtains

$$\begin{aligned}T_{(0,4,1|0,2,3)}^{(4)} = M_{(0,4,1|0,2,3)} &+ qM_{(1,4,0|0,2,3)} + qM_{(4,0,1|0,2,3)}\\ &+ q^2 M_{(1,4,1|1,2,3)} + q^2 M_{(4,1,0|0,2,3)} + q^3 M_{(4,1,1|1,2,3)}.\end{aligned}$$

Note that there is no reason why we chose to start the bumping procedure with f_+ in describing the algorithm. One could also start the bumping procedure with f_-, increasing all entries in f_- equal to a_1 by 1, and so on. In practice, one should always choose to start with the side for which the resulting word $X_N \cdots X_1 \in \mathscr{U}$ is as short as possible. In the present example, it is better to start the bumping procedure with f_-, since then there is only one step:

$$(0,4,1|0,2,3) \xrightarrow{a_1=0} (\underline{0},4,1|0,2,3).$$

Thus, we need to compute $\pi_{\leq 4}(F_0 T^{(4)}_{(1,4,1|0,2,3)})$ instead, which is much quicker as only one generator of $\mathscr{U}$ needs to be applied. It turns out that this equals $T^{(4)}_{(0,4,1|1,2,3)}$ directly (indeed it already equals $T_{(0,4,1|1,2,3)}$), with no corrections needed.

Computer calculations using the above algorithm support the following positivity conjecture:

Conjecture 2.28. *Let $f \in \mathbb{Z}^{m|n}$.*

(i) *The coefficients $t_{g,f}(q)$ of T_f when expanded in the basis $\{M_g\}_{g\in\mathbb{Z}^{m|n}}$ belong to $\mathbb{N}[q]$.*

(ii) *The coefficients $l_{g,f}(q)$ of L_f when expanded in the basis $\{M_g\}_{g\in\mathbb{Z}^{m|n}}$ belong to $\mathbb{N}[-q^{-1}]$.*

(iii) *For each $a \in \mathbb{Z}$, $f \in \mathbb{Z}^{m|n}$ and $r \geq 1$, the coefficients of $E_a^{(r)} T_f$ and $F_a^{(r)} T_f$ when expanded in the basis $\{T_g\}_{g\in\mathbb{Z}^{m|n}}$ belong to $\mathbb{N}[q, q^{-1}]$.*

(iv) *For each $a \in \mathbb{Z}$, $f \in \mathbb{Z}^{m|n}$ and $r \geq 1$, the coefficients of $E_a^{(r)} L_f$ and $F_a^{(r)} L_f$ when expanded in the basis $\{L_g\}_{g\in\mathbb{Z}^{m|n}}$ belong to $\mathbb{N}[q, q^{-1}]$.*

If this positivity conjecture is true, it follows in particular that each T_f belongs to $\mathscr{T}^{m|n}$ rather than the completion $\widehat{\mathscr{T}}^{m|n}$, i.e. each T_f is a finite linear combination of M_g's. To see this, we modify the above algorithm to obtain an algorithm that computes T_f itself (not just the $T_f^{(d)}$'s) in finitely many steps, as follows. To start with, one follows the bumping procedure to obtain h and the elements $X_1, \ldots, X_N \in \mathscr{U}$ exactly as above. Since $\#h < \#f$, we may assume that T_h is known inductively and is a finite sum of M_g's. Consider $X_N \cdots X_1 T_h \in \mathscr{T}^{m|n}$, and choose e to be minimal so that $X_N \cdots X_1 T_h \in \mathscr{T}^{m|n}_{\leq e}$. In view of Conjecture 2.28(iii), $X_N \cdots X_1 T_h$ equals T_f plus a $\mathbb{N}[q, q^{-1}]$-linear combination of T_g's. So by Conjecture 2.28(i), we must have that $T_f \in \mathscr{T}^{m|n}_{\leq e}$, hence $T_f = T_f^{(e)}$. Now follow the above algorithm to compute $T_f^{(e)}$.

§2-k. **Crystal structures.** Finally in this section, we review some results of Kashiwara; see e.g. [K2] for the basic language used here. Let $\mathscr{A}$ be the subring of $\mathbb{Q}(q)$ consisting of rational functions having no pole at $q = 0$. Evaluation at $q = 0$ induces an isomorphism $\mathscr{A}/q\mathscr{A} \to \mathbb{Q}$.

Let $\mathscr{V}_{\mathscr{A}}$ be the $\mathscr{A}$-lattice in $\mathscr{V}$ spanned by the v_a's, and let $\mathscr{W}_{\mathscr{A}}$ be the $\mathscr{A}$-lattice in $\mathscr{W}$ spanned by the w_a's. Then, $\mathscr{V}_{\mathscr{A}}$ together with the basis of the $\mathbb{Q}$-vector space $\mathscr{V}_{\mathscr{A}}/q\mathscr{V}_{\mathscr{A}}$ given by the images of the v_a's is a lower crystal basis for $\mathscr{V}$ at $q = 0$ in the sense of [K2, 4.1]. Similarly, $\mathscr{W}_{\mathscr{A}}$ together with the basis for $\mathscr{W}_{\mathscr{A}}/q\mathscr{W}_{\mathscr{A}}$ given by the images of the w_a's is a lower crystal basis for $\mathscr{W}$ at $q = 0$. Let $\mathscr{T}^{m|n}_{\mathscr{A}} = \mathscr{W}_{\mathscr{A}}^{\otimes m} \otimes_{\mathscr{A}} \mathscr{V}_{\mathscr{A}}^{\otimes n}$ be the $\mathscr{A}$-lattice in $\mathscr{T}^{m|n}$ spanned by the M_f's. Then, by [K2, Theorem 4.1], $\mathscr{T}^{m|n}_{\mathscr{A}}$ together with the basis for $\mathscr{T}^{m|n}_{\mathscr{A}}/q\mathscr{T}^{m|n}_{\mathscr{A}}$ given by the images of the M_f for $f \in \mathbb{Z}^{m|n}$ is a lower crystal basis for $\mathscr{T}^{m|n}$ at $q = 0$. Moreover, we can easily describe the associated crystal graph using Kashiwara's tensor product rule.

To do this, let us identify the set $\{M_f + q\mathscr{T}^{m|n}_{\mathscr{A}}\}_{f\in\mathbb{Z}^{m|n}}$ underlying the crystal basis with the set $\mathbb{Z}^{m|n}$ in the obvious way. Then, Kashiwara's crystal operators induce maps $\tilde{E}'_a, \tilde{F}'_a : \mathbb{Z}^{m|n} \to \mathbb{Z}^{m|n} \sqcup \{\varnothing\}$. (We are using $\tilde{E}'_a, \tilde{F}'_a$ because $\tilde{E}_a, \tilde{F}_a$ are used for something else later on.) Fix $a \in \mathbb{Z}$ and $f \in \mathbb{Z}^{m|n}$. The *a-signature*

$(\sigma_{-m},\dots,\sigma_{-1},\sigma_1,\dots,\sigma_n)$ of f is defined by

$$\sigma_i = \begin{cases} + & \text{if } i>0 \text{ and } f(i)=a, \text{ or if } i<0 \text{ and } f(i)=a+1,\\ - & \text{if } i>0 \text{ and } f(i)=a+1, \text{ or if } i<0 \text{ and } f(i)=a,\\ 0 & \text{otherwise.}\end{cases} \tag{2.29}$$

From this, we form the *reduced a-signature* by successively replacing subsequences of the form $+-$ (possibly separated by 0's) in the signature with 00 until no $-$ appears to the right of a $+$. Recall the definition of $d_j \in \mathbb{Z}^{m|n}$ from §2-a. We define

$$\tilde{E}'_a(f) = \begin{cases} \varnothing & \text{if there are no } -\text{'s in the reduced } a\text{-signature},\\ f-d_j & \text{if the rightmost } - \text{ is in position } j\in I(m|n),\end{cases}$$

and

$$\tilde{F}'_a(f) = \begin{cases} \varnothing & \text{if there are no } +\text{'s in the reduced } a\text{-signature},\\ f+d_j & \text{if the leftmost } + \text{ is in position } j\in I(m|n).\end{cases}$$

Also let

$$\begin{aligned}\varepsilon'_a(f) &= \max\{r\geq 0 \mid (\tilde{E}'_a)^r(f)\neq 0\}\\ &= \text{the total number of } -\text{'s in the reduced } a\text{-signature},\\ \varphi'_a(f) &= \max\{r\geq 0 \mid (\tilde{F}'_a)^r(f)\neq 0\}\\ &= \text{the total number of } +\text{'s in the reduced } a\text{-signature}.\end{aligned}$$

Then, the datum $(\mathbb{Z}^{m|n},\tilde{E}'_a,\tilde{F}'_a,\varepsilon'_a,\varphi'_a,\mathrm{wt})$ is the crystal associated to the module $\mathscr{T}^{m|n}$.

Example 2.30. Consider the function $f=(3,6,2,0,2,1|3,2,1)\in\mathbb{Z}^{6|3}$. The 2-signature is $(+,0,-,0,-,0|-,+,0)$. Cancelling off $+-$ pairs, we deduce that the reduced 2-signature is $(0,0,0,0,-,0|-,+,0)$. Hence, the 2-string through f in the crystal graph is

$$\begin{aligned}(3,6,2,0,3,1|2,2,1) \xrightarrow{\tilde{F}'_2} (3,6,2,0,2,1|2,2,1) &\xrightarrow{\tilde{F}'_2} (3,6,2,0,2,1|3,2,1)\\ &\xrightarrow{\tilde{F}'_2} (3,6,2,0,2,1|3,3,1).\end{aligned}$$

Theorem 2.31. *Let $f\in\mathbb{Z}^{m|n}$ and $a\in\mathbb{Z}$.*

(i) *$E_aT_f = [\varphi'_a(f)+1]T_{\tilde{E}'_a(f)} + \widehat{\sum}_{g\in\mathbb{Z}^{m|n}} u^a_{g,f}T_g$, where the coefficient $u^a_{g,f}$ belongs to $q^{2-\varphi'_a(g)}\mathbb{Z}[q]$ and is zero unless $\varepsilon'_b(g)\geq\varepsilon'_b(f)$ for all $b\in\mathbb{Z}$.*

(ii) *$F_aT_f = [\varepsilon'_a(f)+1]T_{\tilde{F}'_a(f)} + \widehat{\sum}_{g\in\mathbb{Z}^{m|n}} v^a_{g,f}T_g$, where the coefficient $v^a_{g,f}$ belongs to $q^{2-\varepsilon'_a(g)}\mathbb{Z}[q]$ and is zero unless $\varphi'_b(g)\geq\varphi'_b(f)$ for all $b\in\mathbb{Z}$.*

(In (i), resp. (ii), the first term on the right-hand side should be omitted if $\tilde{E}'_a(f)$, resp. $\tilde{F}'_a(f)$, equals $\varnothing$.)

Proof. Fix $d\in\mathbb{Z}$, and consider $\mathscr{T}=\mathscr{T}^{m|n}_{\leq d}$, which is an integrable module in the sense of [K1, 1.3] with respect to the subalgebra of $\mathscr{U}$ generated by all $E_a,F_a,K_b^{\pm1}$ for $a<d$, $b\leq d$. Let $\mathscr{T}_0$ (resp. $\mathscr{T}_{\mathbb{Q}}$) be the $\mathscr{A}$- (resp. $\mathbb{Q}[q,q^{-1}]$-)lattice in $\mathscr{T}$ spanned by the basis elements $\{M_f\}_{f\in\mathbb{Z}^{m|n}_{\leq d}}$. Let $\mathscr{T}_\infty=\pi_{\leq d}(\overline{\mathscr{T}_0})$, an $\overline{\mathscr{A}}$-lattice in $\mathscr{T}$. The canonical map $\mathscr{T}_{\mathbb{Q}}\cap\mathscr{T}_0\cap\mathscr{T}_\infty\to\mathscr{T}_0/q\mathscr{T}_0$ is an isomorphism; this follows at once from Lemma 2.16 since that shows that all three lattices are generated by the elements $\{T^{(d)}_f\}_{f\in\mathbb{Z}^{m|n}_{\leq d}}$. The preimage of the crystal basis element $M_f+q\mathscr{T}_0$

is $T_f^{(d)}$, for each $f \in \mathbb{Z}^{m|n}_{\leq d}$. In the language of Kashiwara [K1], this shows that $(\mathscr{T}_{\mathbb{Q}}, \mathscr{T}_0, \mathscr{T}_\infty)$ is a balanced triple, and that $\{T_f^{(d)}\}_{f \in \mathbb{Z}^{m|n}_{\leq d}}$ is a lower global crystal basis for $\mathscr{T}$ at $q = 0$.

Now we get from [K1, Proposition 5.3.1] (which is about upper global crystal bases) and an argument involving duality [K1, §3.2], that

$$E_a T_f^{(d)} = [\varphi'_a(f) + 1] T^{(d)}_{\tilde{E}'_a(f)} + \sum_{g \in \mathbb{Z}^{m|n}_{\leq d}} u^a_{g,f} T_g^{(d)},$$

where the coefficient $u^a_{g,f}$ belongs to $q^{2-\varphi'_a(g)}\mathbb{Z}[q]$ and is zero unless $\varepsilon'_b(g) \geq \varepsilon'_b(f)$ for all $b < d$. Taking the limit as $d \to \infty$, we get (i). The proof of (ii) is similar. □

We will also meet certain dual crystal operators on $\mathbb{Z}^{m|n}$. Define

$$\tilde{E}^*_a(f) := -\tilde{F}'_{-1-a}(-f), \qquad \tilde{F}^*_a(f) := -\tilde{E}'_{-1-a}(-f), \tag{2.32}$$

$$\varepsilon^*_a(f) := \varphi'_{-1-a}(-f), \qquad \varphi^*_a(f) := \varepsilon'_{-1-a}(-f). \tag{2.33}$$

These can be described explicitly in a similar way to the above: for fixed $a \in \mathbb{Z}$ and $f \in \mathbb{Z}^{m|n}$, let $(\sigma_{-m}, \ldots, \sigma_{-1}, \sigma_1, \ldots, \sigma_n)$ be the a-signature as defined in (2.29). Form the *dual reduced a-signature* by successively replacing sequences of the form $-+$ (possibly separated by 0's) with 00 until no $-$ appears to the left of a $+$. Then:

$$\tilde{E}^*_a(f) = \begin{cases} 0 & \text{if there are no } -\text{'s in the dual reduced } a\text{-signature,} \\ f - d_j & \text{if the leftmost } - \text{ is in position } j \in I(m|n), \end{cases}$$

and

$$\tilde{F}^*_a(f) = \begin{cases} 0 & \text{if there are no } +\text{'s in the dual reduced } a\text{-signature,} \\ f + d_j & \text{if the rightmost } + \text{ is in position } j \in I(m|n). \end{cases}$$

Also

$$\varepsilon^*_a(f) = \text{the total number of } -\text{'s in the dual reduced } a\text{-signature,}$$
$$\varphi^*_a(f) = \text{the total number of } +\text{'s in the dual reduced } a\text{-signature.}$$

In this way, we obtain the dual crystal structure $(\mathbb{Z}^{m|n}, \tilde{E}^*_a, \tilde{F}^*_a, \varepsilon^*_a, \varphi^*_a, \mathrm{wt})$ on the underlying set $\mathbb{Z}^{m|n}$.

Theorem 2.34. *Let $f \in \mathbb{Z}^{m|n}$ and $a \in \mathbb{Z}$.*

(i) $E_a L_f = [\varepsilon^*_a(f)] L_{\tilde{E}^*_a(f)} + \widehat{\sum}_{g \in \mathbb{Z}^{m|n}} w^a_{g,f} L_g$, *where the coefficient $w^a_{g,f}$ belongs to $q^{2-\varepsilon^*_a(f)}\mathbb{Z}[q]$ and is zero unless $\varphi^*_b(g) \leq \varphi^*_b(f)$ for all $b \in \mathbb{Z}$.*

(ii) $F_a L_f = [\varphi^*_a(f)] L_{\tilde{F}^*_a(f)} + \widehat{\sum}_{g \in \mathbb{Z}^{m|n}} x^a_{g,f} L_g$, *where the coefficient $x^a_{g,f}$ belongs to $q^{2-\varphi^*_a(f)}\mathbb{Z}[q]$ and is zero unless $\varepsilon^*_b(g) \leq \varepsilon^*_b(f)$ for all $b \in \mathbb{Z}$.*

Proof. Dualize Theorem 2.31 using Theorem 2.23 and Lemma 2.21. □

3. Exterior algebra

Now we descend from the tensor space $\mathscr{T}^{m|n}$ to $\mathscr{E}^{m|n}$. We continue with the same notation as in section 2.

§3-a. **The space $\mathscr{E}^{m|n}$.** Let w_0 denote the longest element of $S_{m|n}$. Let

$$H_0 := \sum_{x \in S_{m|n}} (-q)^{\ell(x) - \ell(w_0)} H_x \in \mathscr{H}_{m|n}. \tag{3.1}$$

The first lemma summarizes some elementary properties.

Lemma 3.2. *The following properties hold:*

(i) $H_i H_0 = -qH_0 = H_0 H_i$ *for any* $i \in I(m-1|n-1)$*;*
(ii) $\overline{H_0} = H_0$*;*
(iii) $H_0^2 = -[m]![n]!H_0$*;*
(iv) $H_0 = \tau(H_0)$*;*
(v) *the map* $\omega : \mathscr{H}_{m|n} \to \mathscr{H}_{n|m}$ *maps* $H_0 \in \mathscr{H}_{m|n}$ *to* $H_0 \in \mathscr{H}_{n|m}$*.*

Proof. Part (i) is an easy exercise. For (ii), use [So1, Proposition 2.9] and apply the map *dia* there. For (iii), one gets at once using (i) that

$$H_0^2 = \sum_{x \in S_{m|n}} (-q)^{\ell(x)-\ell(w_0)} H_x H_0 = \sum_{x \in S_{m|n}} (-q)^{2\ell(x)-\ell(w_0)} H_0.$$

Now use the well-known formula for the Poincaré polynomial of $S_{m|n}$ to rewrite the sum. Finally, (iv) and (v) are obvious. □

Let $\mathscr{E}^{m|n} := \mathscr{T}^{m|n} H_0$, a $\mathscr{U}$-submodule of $\mathscr{T}^{m|n}$. Note that $\mathscr{E}^{m|n}$ is the q-analogue of the exterior power $\bigwedge^m \mathscr{W} \otimes \bigwedge^n \mathscr{V}$. Form the completion $\widehat{\mathscr{E}}^{m|n} = \widehat{\mathscr{T}}^{m|n} H_0$ as in §2-d. By Lemmas 2.9(i) and 3.2(v), the restriction of the map $\omega : \widehat{\mathscr{T}}^{m|n} \to \widehat{\mathscr{T}}^{n|m}$ is an isomorphism $\omega : \widehat{\mathscr{E}}^{m|n} \to \widehat{\mathscr{E}}^{n|m}$.

We will call $f \in \mathbb{Z}^{m|n}$ *dominant* if $f(-m) < \cdots < f(-1),\ f(1) > \cdots > f(n)$. We warn the reader that the inequality signs here are strict, unlike in the earlier definition of antidominant! Let $\mathbb{Z}_+^{m|n}$ denote the set of all dominant $f \in \mathbb{Z}^{m|n}$. For $f \in \mathbb{Z}_+^{m|n}$, let

$$K_f := M_{f \cdot w_0} H_0 \in \mathscr{E}^{m|n}. \tag{3.3}$$

The following lemma implies that the $\{K_f\}_{f \in \mathbb{Z}_+^{m|n}}$ form a basis for $\mathscr{E}^{m|n}$.

Lemma 3.4. *Suppose* $f \in \mathbb{Z}^{m|n}$ *and let* $x \in S_{m|n}$ *be the unique element of minimal length such that* $f \cdot x$ *is antidominant. Then,*

$$M_f H_0 = \begin{cases} (-q)^{\ell(x)} K_{f \cdot x w_0} & \text{if } f \cdot x w_0 \text{ is dominant,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have that $M_f = M_{f \cdot x} H_{x^{-1}}$. So applying Lemma 3.2(i), $M_f H_0 = M_{f \cdot x} H_{x^{-1}} H_0 = (-q)^{\ell(x)} M_{f \cdot x} H_0$. Finally, note that if $f \cdot x w_0$ is not dominant, then $M_{f \cdot x} H_0 = 0$. □

§3-b. **Canonical bases.** Since $\overline{H_0} = H_0$ by Lemma 3.2(ii), the bar involution on $\widehat{\mathscr{T}}^{m|n}$ leaves $\widehat{\mathscr{E}}^{m|n}$ invariant. Moreover, for dominant f, $\overline{K_f} = \overline{M_{f \cdot w_0}} H_0$. So using Lemma 3.4 and the explicit description of $\overline{M_{f \cdot w_0}}$ given by (B1) and (B3) in the proof of Theorem 2.14, we see that $\overline{K_f} = K_f + (*)$, where $(*)$ is a (possibly infinite) $\mathbb{Z}[q, q^{-1}]$-linear combination of K_g's for $g \in \mathbb{Z}_+^{m|n}$ with $g \prec f$. Moreover, for typical dominant f, we have that $\overline{K_f} = K_f$. As in Theorem 2.14, these properties uniquely characterize the bar involution on $\widehat{\mathscr{E}}^{m|n}$:

Theorem 3.5. *There exists a unique continuous, antilinear map* $- : \widehat{\mathscr{E}}^{m|n} \to \widehat{\mathscr{E}}^{m|n}$ *such that*

(i) $\overline{K_f} = K_f$ *for all typical* $f \in \mathbb{Z}_+^{m|n}$*;*

(ii) $\overline{Xu} = \overline{X}\,\overline{u}$ *for all* $X \in \mathscr{U}$ *and* $u \in \widehat{\mathscr{E}}^{m|n}$.

Moreover,

(iii) *bar is an involution;*
(iv) $\overline{K_f} = K_f + (*)$ *where* $(*)$ *is a (possibly infinite)* $\mathbb{Z}[q, q^{-1}]$*-linear combination of* K_g*'s for dominant* $g \prec f$*;*
(v) $\overline{\omega(u)} = \omega(\overline{u})$ *for all* $u \in \widehat{\mathscr{E}}^{m|n}$.

Proof. We have already proved the existence above. For uniqueness, note on applying H_0 to the conclusion of Theorem 2.12 that we can write each K_f as a possibly infinite linear combination of $X_g K_g$'s for $X_g \in \mathscr{U}$ and typical $g \in \mathbb{Z}_+^{m|n}$. Hence, as in Corollary 2.13, the only continuous $\mathscr{U}$-endomorphism of $\widehat{\mathscr{E}}^{m|n}$ that fixes K_f for all typical $f \in \mathbb{Z}_+^{m|n}$ is the identity map. Now using this one gets uniqueness by exactly the same argument as in the proof of Theorem 2.14. □

Now applying the general principles used in the proof of Theorem 2.17, we deduce:

Theorem 3.6. *There exist unique topological bases* $\{U_f\}_{f \in \mathbb{Z}_+^{m|n}}$, $\{L_f\}_{f \in \mathbb{Z}_+^{m|n}}$ *for* $\widehat{\mathscr{E}}^{m|n}$ *such that*

(i) $\overline{U_f} = U_f$ *and* $\overline{L_f} = L_f$*;*
(ii) $U_f \in K_f + \widehat{\sum}_{g \in \mathbb{Z}_+^{m|n}} q\mathbb{Z}[q] K_g$ *and* $L_f \in K_f + \widehat{\sum}_{g \in \mathbb{Z}_+^{m|n}} q^{-1}\mathbb{Z}[q^{-1}] K_g$.

Moreover,

(iii) $U_f = K_f + (*)$ *and* $L_f = K_f + (**)$*, where* $(*)$ *and* $(**)$ *are (possibly infinite) linear combinations of* K_g*'s for dominant* $g \prec f$*;*
(iv) $\omega(U_f) = U_{\omega(f)}$ *and* $\omega(L_f) = L_{\omega(f)}$.

We use the following notation for the coefficients:

$$L_f = \sum_{g \in \mathbb{Z}_+^{m|n}} l_{g,f}(q) K_g, \qquad U_f = \sum_{g \in \mathbb{Z}_+^{m|n}} u_{g,f}(q) K_g, \tag{3.7}$$

for polynomials $l_{g,f}(q) \in \mathbb{Z}[q^{-1}], u_{g,f}(q) \in \mathbb{Z}[q]$. We know that $l_{g,f}(q) = u_{g,f}(q) = 0$ unless $g \preceq f$, and $l_{f,f}(q) = u_{f,f}(q) = 1$.

Note that $K_f = M_f + (*)$, where $(*)$ is a $q^{-1}\mathbb{Z}[q^{-1}]$-linear combination of M_g's. So the element L_f defined in Theorem 3.6 is bar invariant and equals M_f plus a $q^{-1}\mathbb{Z}[q^{-1}]$-linear combination of M_g's. So by the uniqueness in Theorem 2.17, the elements L_f and the polynomials $l_{g,f}(q)$ defined here are *the same* as the ones defined in §2-h, for dominant g, f. Thus our notation is consistent with the earlier notation. The relationship between the elements U_f here and the T_f's from before is given by:

Lemma 3.8. *For* $f \in \mathbb{Z}_+^{m|n}$, $U_f = T_{f \cdot w_0} H_0$.

Proof. Note that $T_{f \cdot w_0} H_0$ is a bar invariant element of $\widehat{\mathscr{E}}^{m|n}$. Recall that $T_{f \cdot w_0}$ equals $M_{f \cdot w_0}$ plus a $q\mathbb{Z}[q]$-linear combination of M_g's. So applying Lemma 3.4, $T_{f \cdot w_0} H_0$ equals K_f plus a $q\mathbb{Z}[q]$-linear combination of K_g's. Hence $T_{f \cdot w_0} H_0 = U_f$ by the uniqueness in Theorem 3.6. □

§3-c. **Duality.** Recall the antilinear involution $\sigma : \mathscr{T}^{m|n} \to \mathscr{T}^{m|n}$ defined in (2.8). In view of Lemmas 2.9(iii) and 3.2(ii), this leaves the subspace $\mathscr{E}^{m|n}$ invariant. Indeed, by Lemma 3.4, we have that

$$\sigma(K_f) = (-q)^{\ell(w_0)} K_{-f\cdot w_0} \tag{3.9}$$

for each $f \in \mathbb{Z}_+^{m|n}$. Let $(.,.)_{\mathscr{E}}$ be the bilinear form on $\mathscr{E}^{m|n}$ defined so that the elements $\{K_f\}_{f\in\mathbb{Z}_+^{m|n}}$ are orthonormal. Note by Lemmas 2.9(ii) and 3.2(iii) that

$$\begin{aligned}(K_f, K_g)_{\mathscr{T}} &= (M_{f\cdot w_0}H_0, M_{g\cdot w_0}H_0)_{\mathscr{T}} = -[m]![n]!(M_{f\cdot w_0}H_0, M_{g\cdot w_0})_{\mathscr{T}}\\ &= -(-q)^{-\ell(w_0)}[m]![n]!\delta_{f,g}.\end{aligned}$$

Hence,

$$(u,v)_{\mathscr{E}} = -\frac{(-q)^{\ell(w_0)}}{[m]![n]!}(u,v)_{\mathscr{T}} \tag{3.10}$$

for all $u, v \in \mathscr{E}^{m|n}$. Finally, define a bilinear form $\langle .,.\rangle_{\mathscr{E}}$ on $\widehat{\mathscr{E}}^{m|n}$ by setting

$$\langle u, v\rangle_{\mathscr{E}} := (-q)^{-\ell(w_0)}(u, \sigma(\overline{v}))_{\mathscr{E}} \tag{3.11}$$

for all $u, v \in \widehat{\mathscr{E}}^{m|n}$. Comparing with the definition of the form $\langle .,.\rangle_{\mathscr{T}}$ from (2.20) and using (3.10), one sees immediately that

$$\langle u, v\rangle_{\mathscr{E}} = -\frac{1}{[m]![n]!}\langle u, v\rangle_{\mathscr{T}} \tag{3.12}$$

for all $u, v \in \widehat{\mathscr{E}}^{m|n}$. Hence in particular we get from Lemma 2.22 that the form $\langle .,.\rangle_{\mathscr{E}}$ is symmetric.

Theorem 3.13. *For $f, g \in \mathbb{Z}_+^{m|n}$, $\langle L_f, U_{-g\cdot w_0}\rangle_{\mathscr{E}} = \delta_{f,g}$.*

Proof. Since $L_f \in \widehat{\mathscr{E}}^{m|n}$, we have by Lemma 3.2(iii) that $L_f H_0 = -[m]![n]!L_f$. So applying Theorem 2.23 and Lemmas 3.8 and 2.21, we have that

$$\begin{aligned}\langle L_f, U_{-g\cdot w_0}\rangle_{\mathscr{E}} &= -\frac{1}{[m]![n]!}\langle L_f, T_{-g}H_0\rangle_{\mathscr{T}} = -\frac{1}{[m]![n]!}\langle L_f H_0, T_{-g}\rangle_{\mathscr{T}}\\ &= \langle L_f, T_{-g}\rangle_{\mathscr{T}} = \delta_{f,g}.\end{aligned}$$

□

By the theorem and the argument used to prove Corollary 2.24, we get:

Corollary 3.14. *For $f \in \mathbb{Z}_+^{m|n}$,*

$$K_f = \sum_{g\in\mathbb{Z}_+^{m|n}} u_{-f\cdot w_0,-g\cdot w_0}(q^{-1})L_g = \sum_{g\in\mathbb{Z}_+^{m|n}} l_{-f\cdot w_0,-g\cdot w_0}(q^{-1})U_g.$$

§3-d. **Crystal structures.** Next we describe the crystal structure on $\widehat{\mathscr{E}}^{m|n}$, following the same language as in §2-k. Recalling (3.3), let

$$\tilde{E}_a(f) := (\tilde{E}'_a(f\cdot w_0))\cdot w_0, \qquad \tilde{F}_a(f) := (\tilde{F}'_a(f\cdot w_0))\cdot w_0, \tag{3.15}$$

$$\varepsilon_a(f) := \varepsilon'_a(f\cdot w_0), \qquad \varphi_a(f) := \varphi'_a(f\cdot w_0) \tag{3.16}$$

for $f \in \mathbb{Z}_+^{m|n}$. Then, $(\mathbb{Z}_+^{m|n}, \tilde{E}_a, \tilde{F}_a, \varepsilon_a, \varphi_a, \mathrm{wt})$ is the crystal associated to the $\mathscr{U}$-module $\mathscr{E}^{m|n}$. Actually, the crystal structure is so simple in this case, that we can list all the possibilities explicitly. There are ten possible configurations for edges of

color a in the crystal graph, listed below. Here, $\dots$ denotes entries different from $a, a+1$.

(1) $(\dots, a, a+1, \dots | \dots, a+1, a, \dots)$;
(2) $(\dots, a, a+1, \dots | \dots, a, \dots) \xrightarrow{\tilde{F}_a} (\dots, a, a+1, \dots | \dots, a+1, \dots)$;
(3) $(\dots, a+1, \dots | \dots, a+1, a, \dots) \xrightarrow{\tilde{F}_a} (\dots, a, \dots | \dots, a+1, a, \dots)$;
(4) $(\dots, a+1, \dots | \dots, a, \dots) \xrightarrow{\tilde{F}_a} (\dots, a, \dots | \dots, a, \dots)$
$\xrightarrow{\tilde{F}_a} (\dots, a, \dots | \dots, a+1, \dots)$;
(5) $(\dots, a+1, \dots | \dots, a+1, \dots)$;
(6) $(\dots, a, a+1, \dots | \dots)$;
(7) $(\dots | \dots, a+1, a, \dots)$;
(8) $(\dots | \dots, a, \dots) \xrightarrow{\tilde{F}_a} (\dots | \dots, a+1, \dots)$;
(9) $(\dots, a+1, \dots | \dots) \xrightarrow{\tilde{F}_a} (\dots, a, \dots | \dots)$;
(10) $(\dots | \dots)$.

We also have the *dual crystal* $(\mathbb{Z}_+^{m|n}, \tilde{E}_a^*, \tilde{F}_a^*, \varepsilon_a^*, \varphi_a^*, \mathrm{wt})$, where $\tilde{E}_a^*, \tilde{F}_a^*, \varepsilon_a^*$ and φ_a^* are the restrictions of the functions from §2-k to $\mathbb{Z}_+^{m|n}$. Combining (2.32) and (3.15), we have that

$$\tilde{E}_a^*(-f \cdot w_0) = -\tilde{F}_{-1-a}(f) \cdot w_0, \qquad \tilde{F}_a^*(-f \cdot w_0) = -\tilde{E}_{-1-a}(f) \cdot w_0. \tag{3.17}$$

Again, there are ten possible configurations for the edges in the corresponding dual crystal graph, all of which are exactly the same as (1)–(10) above (replacing $\tilde{F}_a$ with $\tilde{F}_a^*$) with the exception of (4) and (5) which change to

(4*) $(\dots, a+1, \dots | \dots, a, \dots) \xrightarrow{\tilde{F}_a^*} (\dots, a+1, \dots | \dots, a+1, \dots)$
$\xrightarrow{\tilde{F}_a^*} (\dots, a, \dots | \dots, a+1, \dots)$;
(5*) $(\dots, a, \dots | \dots, a, \dots)$.

Remark 3.18. In (3.31) and Lemma 3.32(v) below we will define mutually inverse bijections $\mathtt{L}, \mathtt{R} : \mathbb{Z}_+^{m|n} \to \mathbb{Z}_+^{m|n}$. By considering all the above cases (1)–(10) one by one, it is not hard to check that $\mathtt{L}$ satisfies, indeed is characterized uniquely by, the following properties

(1) if $f \in \mathbb{Z}_+^{m|n}$ is typical then $\mathtt{L}(f) = f$;
(2) for every $a \in \mathbb{Z}$ and $f \in \mathbb{Z}_+^{m|n}$, $\tilde{E}_a^* \mathtt{L}(f) = \mathtt{L}(\tilde{E}_a f)$ and $\tilde{F}_a^* \mathtt{L}(f) = \mathtt{L}(\tilde{F}_a f)$;
(3) for every $f \in \mathbb{Z}_+^{m|n}$, $\mathrm{wt}(\mathtt{L}(f)) = \mathrm{wt}(f)$.

Hence, $\mathtt{L} : (\mathbb{Z}_+^{m|n}, \tilde{E}_a, \tilde{F}_a, \varepsilon_a, \varphi_a, \mathrm{wt}) \to (\mathbb{Z}_+^{m|n}, \tilde{E}_a^*, \tilde{F}_a^*, \varepsilon_a^*, \varphi_a^*, \mathrm{wt})$ is an isomorphism of crystals with inverse $\mathtt{R}$.

The crucial observation to be made from the above description of the crystal graph is that *all a-strings are of length at most* 2. The following lemma is a consequence of this particularly simple structure.

Lemma 3.19. *Let $f \in \mathbb{Z}_+^{m|n}$ and $a \in \mathbb{Z}$.*

(i) *If $\varepsilon_a(f) > 0$, then $E_a U_f = [\varphi_a(f) + 1] U_{\tilde{E}_a(f)}$.*
(ii) *If $\varphi_a(f) > 0$, then $F_a U_f = [\varepsilon_a(f) + 1] U_{\tilde{F}_a(f)}$.*

Proof. We prove (i), (ii) being similar. Dualizing Theorem 2.34(i) using Theorem 3.13 and Lemma 2.21 (or by considering the effect of the Kashiwara operators directly and arguing as in the proof of Theorem 2.31) gives us that $E_a U_f = [\varphi_a(f)+1]U_{\tilde{E}_a(f)} + \widehat{\sum}_{g\in\mathbb{Z}_+^{m|n}} y_{g,f}^a U_g$, where $y_{g,f}^a$ belongs to $q^{2-\varphi_a(g)}\mathbb{Z}[q]$ and is zero unless $\varepsilon_b(g) \geq \varepsilon_b(f)$ for all $b \in \mathbb{Z}$. Suppose that $y_{g,f}^a \neq 0$ for some g. By assumption, $\varepsilon_a(g) \geq \varepsilon_a(f) \geq 1$, so $\varphi_a(g) \leq 1$ since all a-strings are of length ≤ 2. Therefore, $0 \neq y_{g,f}^a \in q\mathbb{Z}[q]$. But $y_{g,f}^a$ is bar invariant, so this is a contradiction. □

§3-e. **Two algorithms.** In this subsection, we describe algorithms to compute the canonical basis $\{U_f\}$ and the coefficients $\{l_{g,f}(q)\}$ of the dual canonical basis of $\mathscr{E}^{m|n}$ explicitly. The first algorithm computes U_f, and is similar to the algorithm for computing the T_f's explained in §2-j — but it is *much* simpler since no corrections are needed thanks to Lemma 3.19.

Procedure 3.20. Suppose we are given $f \in \mathbb{Z}_+^{m|n}$ with $\#f > 0$. Compute $h \in \mathbb{Z}_+^{m|n}$ and operators $X_a, Y_a \in \{E_a, F_a\}_{a\in\mathbb{Z}}$ by following the instructions below starting at step (0).

(0) Choose the largest $i \in \{-m, \dots, -1\}$ such that $f(i) = f(j)$ for some $j \in \{1, \dots, n\}$. Go to step (1).
(1) If $i < -1$ and $f(i+1) = f(i)+1$, replace i by $(i+1)$ and repeat step (1). Otherwise, go to step (2).
(2) If $f(i)+1 = f(j)$ for some (necessarily unique) $j \in \{1, \dots, n\}$ go to step (1)′. Otherwise, set $X_a = F_{f(i)}, Y_a = E_{f(i)}$ and $h = f - d_i$. Stop.
(1)′ If $j > 1$ and $f(j-1) = f(j)+1$, replace j by $(j-1)$ and repeat step (1)′. Otherwise, go to step (2)′.
(2)′ If $f(j)+1 = f(i)$ for some (necessarily unique) $i \in \{-m, \dots, -1\}$ go to step (1). Otherwise, set $X_a = E_{f(j)}, Y_a = F_{f(j)}$ and $h = f + d_j$. Stop.

The following lemma follows immediately from the nature of the above procedure and Lemma 3.19.

Lemma 3.21. *Take $f \in \mathbb{Z}_+^{m|n}$ with $\#f > 0$. Define h and operators $X_a, Y_a \in \{E_a, F_a\}_{a\in\mathbb{Z}}$ according to Procedure 3.20. Then, one of the following holds:*

(i) *$\#h = \#f$. In this case, the a-string through f is $h \xrightarrow{\tilde{X}_a} f$, of length 1. Moreover, $X_a U_h = U_f$, $Y_a U_f = U_h$ and $X_a K_h = K_f$.*
(ii) *$\#h = \#f - 1$. In this case, the a-string through f is $h \xrightarrow{\tilde{X}_a} f \xrightarrow{\tilde{X}_a} g$, of length 2. Moreover $X_a U_h = U_f$, $Y_a U_f = [2]U_h$ and $X_a K_h = K_f + qK_{\tilde{X}_a^*(h)}$.*

Case (ii) (when the atypicality gets strictly smaller) must occur after at most $(m+n-1)$ repetitions of the procedure. Hence after finitely many recursions, the procedure reduces f to a typical weight.

Lemma 3.21 implies the following algorithm for computing U_f. If $f \in \mathbb{Z}_+^{m|n}$ is typical then $U_f = K_f$, since such f's are minimal in the ordering $\preceq$ in $\mathbb{Z}_+^{m|n}$. Otherwise, apply Procedure 3.20 to get $h \in \mathbb{Z}_+^{m|n}$ and $X_a \in \{E_a, F_a\}_{a\in\mathbb{Z}}$. Since the procedure always reduces f to a typical weight in finitely many steps, we may assume U_h is known recursively. Then $U_f = X_a U_h$.

Example 3.22. Applying the algorithm repeatedly, we get that

$$\begin{aligned} U_{(0,1,3,4|2,1,0)} &= F_4F_3E_2F_1F_5F_4E_3F_2E_1F_0K_{(1,3,5,6|4,2,0)} \\ &= K_{(0,1,3,4|2,1,0)} + qK_{(1,3,4,6|6,2,1)} + qK_{(0,3,4,5|5,2,0)} + q^2K_{(3,4,5,6|6,5,2)}. \end{aligned}$$

In the next subsection, we will apply the above algorithm to derive a closed formula for U_f. We turn now to describing the second algorithm, which computes the polynomials $l_{g,f}(q)$. It will not be needed until §4-f below. First we state a variation on Procedure 3.20.

Procedure 3.23. Suppose we are given $g \in \mathbb{Z}_+^{m|n}$ with $\#g > 0$. Compute $h \in \mathbb{Z}_+^{m|n}$ and operators $X_a, Y_a \in \{E_a, F_a\}_{a\in\mathbb{Z}}$ by following the instructions below starting at step (0).

(0) Choose the smallest $i \in \{-m, \dots, -1\}$ such that $g(i) = g(j)$ for some $j \in \{1, \dots, n\}$. Go to step (1).
(1) If $i > -m$ and $g(i-1) = g(i) - 1$, replace i by $(i-1)$ and repeat step (1). Otherwise, go to step (2).
(2) If $g(i) - 1 = g(j)$ for some (necessarily unique) $j \in \{1, \dots, n\}$ go to step $(1)'$. Otherwise, set $h = g + d_i$, $X_a = E_{h(i)}$ and $Y_a = F_{h(i)}$. Stop.
$(1)'$ If $j < n$ and $g(j+1) = g(j) - 1$, replace j by $(j+1)$ and repeat step $(1)'$. Otherwise, go to step $(2)'$.
$(2)'$ If $g(j) - 1 = g(i)$ for some (necessarily unique) $i \in \{-m, \dots, -1\}$ go to step (1). Otherwise, set $h = g - d_j$, $X_a = F_{h(j)}$ and $Y_a = E_{h(j)}$. Stop.

Lemma 3.24. *Suppose $g, f \in \mathbb{Z}_+^{m|n}$ with $\#g > 0$. Define h and operators $X_a, Y_a \in \{E_a, F_a\}_{a\in\mathbb{Z}}$ according to Procedure 3.23. Then,*

$$l_{g,f}(-q^{-1}) = \begin{cases} l_{h,\tilde{Y}_a^*(f)}(-q^{-1}) & \text{if } \#h = \#g, \\ l_{h,\tilde{Y}_a^*(f)}(-q^{-1}) + ql_{\tilde{X}_a(h),f}(-q^{-1}) & \text{if } \#h = \#g - 1, \end{cases}$$

interpreting $l_{h,\tilde{Y}_a^*(f)}(-q^{-1})$ as 0 if $\tilde{Y}_a^*(f) = \varnothing$.

Proof. Let $f, g \in \mathbb{Z}_+^{m|n}$ with $\#f > 0$. Apply Procedure 3.20 to construct h and operators $X_a, Y_a \in \{E_a, F_a\}_{a\in\mathbb{Z}}$. Apply the operator X_a to both sides of the equation

$$K_h = \sum_{k\in\mathbb{Z}_+^{m|n}} l_{-h\cdot w_0,-k\cdot w_0}(q^{-1})U_k$$

from Corollary 3.14. In the case that $\#h = \#f$, we know by Lemma 3.21 that h, hence also all $k \in \mathbb{Z}_+^{m|n}$ with the same weight as h, is at one end of an a-string of length 1. So by Lemma 3.19, $X_aU_k = U_{\tilde{X}_a(k)}$ for all k with $l_{-h\cdot w_0,-k\cdot w_0}(q^{-1}) \neq 0$. Hence,

$$K_f = X_aK_h = \sum_{k\in\mathbb{Z}_+^{m|n}} l_{-h\cdot w_0,-k\cdot w_0}(q^{-1})U_{\tilde{X}_a(k)}.$$

On the other hand, if $\#h = \#f - 1$, then h, hence also all k of the same weight as h, lies at one end of an a-string of length 2, so $X_aU_k = U_{\tilde{X}_a(k)}$. We also know from Lemma 3.21 that $X_aK_h = K_f + qK_{\tilde{X}_a^*(h)}$. So in this case,

$$K_f = \sum_{k\in\mathbb{Z}_+^{m|n}} l_{-h\cdot w_0,-k\cdot w_0}(q^{-1})U_{\tilde{X}_a(k)} - q\sum_{h\in\mathbb{Z}_+^{m|n}} l_{-\tilde{X}_a^*(h)\cdot w_0,-g\cdot w_0}(q^{-1})U_g.$$

Now compute the coefficient of U_g in the above expressions for K_f using Corollary 3.14 again, to get

$$l_{-f\cdot w_0,-g\cdot w_0}(q^{-1}) = \begin{cases} l_{-h\cdot w_0,-\tilde{Y}_a(g)\cdot w_0}(q^{-1}) & \text{if } \#h = \#f, \\ l_{-h\cdot w_0,-\tilde{Y}_a(g)\cdot w_0}(q^{-1}) & \text{if } \#h = \#f - 1, \\ \qquad -ql_{-\tilde{X}_a^*(h)\cdot w_0,-g\cdot w_0}(q^{-1}) & \end{cases}$$

interpreting $l_{-h\cdot w_0,-\tilde{Y}_a(g)\cdot w_0}(q)$ as 0 if $\tilde{Y}_a(g) = \varnothing$. The lemma follows from this, replacing f by $-g\cdot w_0$ and g by $-f\cdot w_0$ and using (3.17), since Procedure 3.23 is just Procedure 3.20 twisted by the involution $f \mapsto -f\cdot w_0$. □

Now to compute $l_{g,f}(-q^{-1})$, we have that $l_{g,f}(-q^{-1}) = \delta_{g,f}$ if g is typical, and it is 0 if $g \not\preceq f$. Otherwise, if $\#g > 0$ and $g \preceq f$, apply Procedure 3.23 and Lemma 3.24 to write $l_{g,f}(-q^{-1})$ in terms of $l_{h,\tilde{Y}_a^*(f)}(-q^{-1})$ and (in case $\#h = \#g-1$) $l_{\tilde{X}_a(h),f}(-q^{-1})$, and repeat. This process terminates in finitely many steps, because h is closer to being typical than g in the sense of Procedure 3.23, and $\tilde{X}_a(h)$ is closer than g to failing the condition $g \preceq f$. Note that this algorithm shows in particular that $l_{g,f}(-q^{-1}) \in \mathbb{N}[q]$, as also follows from the explicit description given in Corollary 3.39(ii) below.

§3-f. **Combinatorial description of canonical bases.** We now introduce some combinatorics to enable us to write down closed formulae for the canonical basis and dual canonical basis elements. The material in this subsection was inspired originally by [JZ]. Suppose $f \in \mathbb{Z}^{m|n}$ is conjugate under the action of $S_{m|n}$ to an element of $\mathbb{Z}_+^{m|n}$. We will denote this "dominant conjugate" of f by f^+. For $-m \le i < 0 < j \le n$ with $f(i) = f(j)$, let

$$\mathtt{L}_{i,j}(f) := f - a(d_i - d_j), \tag{3.25}$$

where a is the smallest positive integer such that $f - a(d_i - d_j)$ and all $\mathtt{L}_{k,l}(f) - a(d_i - d_j)$ for $i < k < 0 < l < j$ with $f(k) = f(l)$ are conjugate to elements of $\mathbb{Z}_+^{m|n}$. Similarly, let

$$\mathtt{R}_{i,j}(f) := f + b(d_i - d_j), \tag{3.26}$$

where b is the smallest positive integer such that $f + b(d_i - d_j)$ and all $\mathtt{R}_{k,l}(f) + b(d_i - d_j)$ for $-m \le k < i, j < l \le n$ with $f(k) = f(l)$ are conjugate to elements of $\mathbb{Z}_+^{m|n}$.

Now take $f \in \mathbb{Z}_+^{m|n}$. Let $r = \#f$ and $-m \le i_1 < \cdots < i_r < 0 < j_r < \cdots < j_1 \le n$ be the unique integers with $f(i_s) = f(j_s)$ for each $s = 1, \dots, r$. For a tuple $\theta = (\theta_1, \dots, \theta_r) \in \mathbb{N}^r$, $|\theta|$ denotes $\theta_1 + \cdots + \theta_r$. Let

$$\mathtt{L}_\theta(f) = \left(\mathtt{L}_{i_r,j_r}^{\theta_r} \circ \mathtt{L}_{i_{r-1},j_{r-1}}^{\theta_{r-1}} \circ \cdots \circ \mathtt{L}_{i_1,j_1}^{\theta_1}(f)\right)^+, \tag{3.27}$$

$$\mathtt{L}'_\theta(f) = \left(\mathtt{L}_{i_1,j_1}^{\theta_1} \circ \mathtt{L}_{i_2,j_2}^{\theta_2} \circ \cdots \circ \mathtt{L}_{i_r,j_r}^{\theta_r}(f)\right)^+, \tag{3.28}$$

$$\mathtt{R}_\theta(f) = \left(\mathtt{R}_{i_1,j_1}^{\theta_1} \circ \mathtt{R}_{i_2,j_2}^{\theta_2} \circ \cdots \circ \mathtt{R}_{i_r,j_r}^{\theta_r}(f)\right)^+, \tag{3.29}$$

$$\mathtt{R}'_\theta(f) = \left(\mathtt{R}_{i_r,j_r}^{\theta_r} \circ \mathtt{R}_{i_{r-1},j_{r-1}}^{\theta_{r-1}} \circ \cdots \circ \mathtt{R}_{i_1,j_1}^{\theta_1}(f)\right)^+. \tag{3.30}$$

Note that $\mathtt{L}'_\theta(f), \mathtt{L}_\theta(f) \preceq f \preceq \mathtt{R}_\theta(f), \mathtt{R}'_\theta(f)$. The operators $\mathtt{L}_\theta$ and $\mathtt{R}_\theta$ will only ever be used for θ belonging to the set $\{0,1\}^r$. In the special case that $\theta_1 = \cdots = \theta_r = 1$, we let

$$\mathtt{L}(f) := \mathtt{L}_\theta(f), \qquad \mathtt{R}(f) := \mathtt{R}_\theta(f). \tag{3.31}$$

The following combinatorial lemma lists some elementary properties of the lowering and raising operators, which follow immediately from the definition.

Lemma 3.32. *Let $f \in \mathbb{Z}_+^{m|n}$ and $r = \#f$.*

(i) *Suppose $\theta \in \mathbb{N}^r$ and let $\varphi = (\theta_r, \dots, \theta_1)$. Then, $\mathtt{R}_\theta(-f \cdot w_0) = -\mathtt{L}_\varphi(f) \cdot w_0$ and $\mathtt{R}'_\theta(-f \cdot w_0) = -\mathtt{L}'_\varphi(f) \cdot w_0$.*

(ii) *The sets $\{\mathtt{L}_\theta(f)\}_{\theta \in \{0,1\}^r}$ and $\{\mathtt{R}_\theta(f)\}_{\theta \in \{0,1\}^r}$ contain 2^r distinct elements.*

(iii) *Suppose $\theta \in \{0,1\}^r$ and let $\varphi = (1-\theta_r, \dots, 1-\theta_1)$. Then, $\mathtt{L}_\theta(\mathtt{R}(f)) = \mathtt{R}_\varphi(f)$ and $\mathtt{R}_\theta(\mathtt{L}(f)) = \mathtt{L}_\varphi(f)$. In particular, taking $\theta_1 = \cdots = \theta_r = 1$, the maps $\mathtt{L}, \mathtt{R} : \mathbb{Z}_+^{m|n} \to \mathbb{Z}_+^{m|n}$ are mutually inverse bijections.*

Example 3.33. Take $f = (0,1,3,4|2,1,0)$ as in Example 3.22, so $\#f = 2$. Then we have that $\mathtt{L}_{(0,0)}(f) = (0,1,3,4|2,1,0)$, $\mathtt{L}_{(1,0)}(f) = (1,3,4,6|6,2,1)$, $\mathtt{L}_{(0,1)}(f) = (0,3,4,5|5,2,0)$ and $\mathtt{L}_{(1,1)}(f) = (3,4,5,6|6,5,2)$. Observe that these are exactly the K_g's appearing in the expression for U_f computed in Example 3.22.

The main theorem of the subsection is the following.

Theorem 3.34. *For $f \in \mathbb{Z}_+^{m|n}$ and $r = \#f$,*

(i) $U_f = \displaystyle\sum_{\theta \in \{0,1\}^r} q^{|\theta|} K_{\mathtt{L}_\theta(f)}$;

(ii) $K_f = \displaystyle\sum_{\theta \in \mathbb{N}^r} (-q)^{|\theta|} U_{\mathtt{L}'_\theta(f)}$.

Proof. (i) If f is typical, then $U_f = K_f$ and there is nothing to prove. So suppose that $\#f > 0$ and define h and $X_a, Y_a \in \{E_a, F_a\}_{a \in \mathbb{Z}}$ according to Procedure 3.20. We may assume by induction that the result has already been established for h. Recalling Lemma 3.21, we need to consider two cases. In the first case $\#h = \#f$, we know that $U_h = \sum_{\theta \in \{0,1\}^r} q^{|\theta|} K_{\mathtt{L}_\theta(h)}$. Applying X_a to both sides, noting that $X_a U_h = U_f$ and that $X_a K_{\mathtt{L}_\theta(h)} = K_{\mathtt{L}_\theta(f)}$ for each θ, gives the desired conclusion. In the second case $\#h = \#f - 1$. This time, we know that $U_h = \sum_{\theta \in \{0,1\}^{r-1}} q^{|\theta|} K_{\mathtt{L}_\theta(h)}$. For each θ, $\mathtt{L}_\theta(h)$ here has the form $(\dots, a+1, \dots | \dots, a, \dots)$, so

$$X_a K_{\mathtt{L}_\theta(h)} = K_{(\dots,a,\dots|\dots,a,\dots)} + qK_{(\dots,a+1,\dots|\dots,a+1,\dots)} = K_{\mathtt{L}_{\theta\cup 0}(f)} + qK_{\mathtt{L}_{\theta\cup 1}(f)},$$

where $\theta \cup x$ denotes $(\theta_1, \dots, \theta_{r-1}, x) \in \{0,1\}^r$. So again we see on applying X_a to both sides that $U_f = X_a U_h = \sum_{\theta \in \{0,1\}^r} q^{|\theta|} K_{\mathtt{L}_\theta(f)}$.

(ii) To deduce this from (i), we will work in the free $\mathbb{Z}[q,q^{-1}]$-module $\mathcal{M}^{m|n}$ on basis $\{[f]\}_{f \in \mathbb{Z}_+^{m|n} \cdot S_{m|n}}$, completed to a topological $\mathbb{Z}[q,q^{-1}]$-module $\widehat{\mathcal{M}}^{m|n}$ exactly as in §2-d so that expressions of the form $[f]$+(a possibly infinite linear combination of $[g]'s$ with $g \prec f$) make sense. We define continuous linear maps $U, K : \widehat{\mathcal{M}}^{m|n} \to \widehat{\mathscr{E}}^{m|n}$ by letting $U([f]) = U_{f^+}, K([f]) = K_{f^+}$. These maps have the right inverses $U^{-1}, K^{-1} : \widehat{\mathscr{E}}^{m|n} \to \widehat{\mathcal{M}}^{m|n}$ with $U^{-1}(U_f) = [f], K^{-1}(K_f) = [f]$ for each $f \in \mathbb{Z}_+^{m|n}$. Finally, define continuous linear operators $\lambda_{i,j} : \widehat{\mathcal{M}}^{m|n} \to \widehat{\mathcal{M}}^{m|n}$ for each $-m \leq$

$i < 0 < j \leq n$ by

$$\lambda_{i,j}([f]) = \begin{cases} [\mathtt{L}_{i,j}(f)] & \text{if } f(i) = f(j), \\ 0 & \text{if } f(i) \neq f(j), \end{cases}$$

for each $f \in \mathbb{Z}_+^{m|n} \cdot S_{m|n}$. Now consider the maps

$$P := K \circ \left(\prod_{-m \leq i < 0 < j \leq n}^{\rightarrow} (1 + q\lambda_{i,j}) \right) \circ U^{-1} : \widehat{\mathscr{E}}^{m|n} \to \widehat{\mathscr{E}}^{m|n},$$

$$Q := U \circ \left(\prod_{-m \leq i < 0 < j \leq n}^{\leftarrow} \frac{1}{1 + q\lambda_{i,j}} \right) \circ K^{-1} : \widehat{\mathscr{E}}^{m|n} \to \widehat{\mathscr{E}}^{m|n},$$

where $\prod^{\rightarrow}$ is taken in some ordering with i's decreasing and j's increasing from left to right and $\prod^{\leftarrow}$ is taken in the opposite ordering, and $\frac{1}{1+q\lambda_{i,j}}$ denotes $(1 - q\lambda_{i,j} + q^2\lambda_{i,j} - \cdots)$. By (i) and the definition of the operator $\mathtt{L}_{i,j}$, the map P sends U_f to U_f, so $P = \mathrm{id}$. On the other hand the result we are trying to prove is equivalent to the statement that Q sends K_f to K_f. Therefore we will be done if we can show that $P \circ Q = \mathrm{id}$, i.e. that for every $f \in \mathbb{Z}_+^{m|n}$,

$$K \circ \left(\prod_{-m \leq i < 0 < j \leq n}^{\rightarrow} (1 + q\lambda_{i,j}) \right) \circ (U^{-1} \circ U) \circ \left(\prod_{-m \leq i < 0 < j \leq n}^{\leftarrow} \frac{1}{1 + q\lambda_{i,j}} \right) ([f]) = K_f.$$

This is obvious if we can show that the inside map $(U^{-1} \circ U) : [g] \mapsto [g^+]$ on the left-hand side can be omitted. For this, we check that

(3.35)
$$K \circ \left(\prod_{-m \leq i < 0 < j \leq n}^{\rightarrow} (1 + q\lambda_{i,j}) \right) ([g^+]) = K \circ \left(\prod_{-m \leq i < 0 < j \leq n}^{\rightarrow} (1 + q\lambda_{i,j}) \right) ([g])$$

for every $g \in \mathbb{Z}^{m|n}$ such that $[g]$ is involved in $\left(\prod^{\leftarrow} \frac{1}{1+q\lambda_{i,j}} \right) ([f])$ with non-zero coefficient. Suppose we have such a g. The crucial observation is that whenever there exist $-m \leq i' < i < 0 < j < j' \leq n$ with $g(j) = g(i) < g(i') = g(j')$, one can find c with $g(i) < c < g(i')$ that does not arise in the tuple g. Given this it is not hard to see that (3.35) holds. □

Corollary 3.36. *For $f \in \mathbb{Z}_+^{m|n}$,*

(i) *$K_f = \sum_g q^{-|\theta_g|} L_g$ where the sum is over all $g \in \mathbb{Z}_+^{m|n}$ such that $\mathtt{R}_{\theta_g}(g) = f$ for some (unique) $\theta_g \in \{0,1\}^{\#g}$;*

(ii) *$L_f = \sum_{g,\theta} (-q)^{-|\theta|} K_g$ where the sum is over all $g \in \mathbb{Z}_+^{m|n}$ and $\theta \in \mathbb{N}^{\#g}$ such that $\mathtt{R}'_\theta(g) = f$.*

Proof. (i) Recall from Corollary 3.14 that the coefficient of L_g in K_f is equal to $u_{-f \cdot w_0, -g \cdot w_0}(q^{-1})$. By Theorem 3.34(i) and Lemma 3.32(ii), $u_{-f \cdot w_0, -g \cdot w_0}(q^{-1}) = q^{-|\theta|}$ if $-f \cdot w_0 = \mathtt{L}_\theta(-g \cdot w_0)$ for some (necessarily unique) $\theta \in \{0,1\}^{\#g}$, and is zero otherwise. Equivalently, invoking Lemma 3.32(i), $u_{-f \cdot w_0, -g \cdot w_0}(q^{-1}) = q^{-|\theta|}$ if $f = \mathtt{R}_\theta(g)$ for some θ, and is zero otherwise.

(ii) By Corollary 3.14 again, $l_{g,f}(q^{-1})$ is equal to the coefficient of $U_{-f\cdot w_0}$ in $K_{-g\cdot w_0}$. By Theorem 3.34(ii), this equals $\sum_\theta (-q)^{|\theta|}$ where the sum is over all $\theta \in \mathbb{N}^{\#g}$ with $\mathtt{L}'_\theta(-g\cdot w_0) = -f\cdot w_0$, equivalently, $\mathtt{R}'_\theta(g) = f$. □

Example 3.37. Suppose $f = (-m, \dots, -2, -1| -1, -2, \dots, -n)$, so $r = \#f = \min(m, n)$. We observe that any $g \preceq f$ can be represented as $\mathtt{L}'_\theta(f)$ for a unique element $\theta \in \mathbb{N}^r$ with $\theta_1 \le \cdots \le \theta_r$. Moreover, this θ is also the unique element of $\mathbb{N}^r$ with the property that $f = \mathtt{R}'_\theta(g)$. We deduce from Corollary 3.36(ii) that

$$L_f = \sum_{\theta=(\theta_1\le\cdots\le\theta_r)\in\mathbb{N}^r} (-q)^{-|\theta|} K_{\mathtt{L}'_\theta(f)}. \tag{3.38}$$

Recalling the definitions from (3.7), we can restate Theorem 3.34(i) and Corollary 3.36(ii) as follows:

Corollary 3.39. *For $g, f \in \mathbb{Z}_+^{m|n}$,*

(i) *$u_{g,f}(q) = q^{|\theta|}$ if $g = \mathtt{L}_\theta(f)$ for some $\theta \in \{0,1\}^{\#f}$, $u_{g,f}(q) = 0$ otherwise;*
(ii) *$l_{g,f}(-q^{-1}) = \sum_\theta q^{|\theta|}$ summing over all $\theta \in \mathbb{N}^{\#g}$ with $\mathtt{R}'_\theta(g) = f$.*

Example 3.40. Using Corollary 3.39(ii) and arguing by induction on n, one gets that $l_{g,f}(-q^{-1}) = q^2(1+q^2)^{n-1}$, for $f = (0, 2, \dots, 2n-2|2n-2, \dots, 2, 0)$ and $g = (2, 4, \dots, 2n|2n, \dots, 4, 2)$.

Corollary 3.41. *For $f, g \in \mathbb{Z}_+^{m|n}$, $u_{-g\cdot w_0, -f\cdot w_0}(q) = q^{\#f} u_{g,\mathtt{R}(f)}(q^{-1})$.*

Proof. Let $r = \#f$. By Corollary 3.39(i) and Lemma 3.32(i), we know that $u_{-g\cdot w_0,-f\cdot w_0}(q) = q^{|\theta|}$ if $g = \mathtt{R}_\theta(f)$ for some $\theta \in \{0,1\}^r$ and is zero otherwise. Similarly, by Lemma 3.32(iii), $u_{g,\mathtt{R}(f)}(q)$ is $q^{|\theta|} = q^{r-|\varphi|}$ if $g = \mathtt{L}_\theta(\mathtt{R}(f)) = \mathtt{R}_\varphi(f)$ for some $\theta \in \{0,1\}^r$ and $\varphi = (1-\theta_r, \dots, 1-\theta_1)$ and is zero otherwise. □

§3-g. **Length function.** We now consider some further properties of the polynomials $l_{g,f}(q)$.

Lemma 3.42. *Let $g, f \in \mathbb{Z}_+^{m|n}$ with $g \preceq f$ and set $r = \#g = \#f$. There exists a unique $\theta = \theta(g, f) \in \mathbb{N}^r$ such that*

(i) *$f = \mathtt{R}'_\theta(g)$;*
(ii) *if $f = \mathtt{R}'_\varphi(g)$ for some $\theta \ne \varphi \in \mathbb{N}^r$ then $|\varphi| < |\theta|$ and $|\varphi| \equiv |\theta| \pmod 2$.*

Given in addition $h \in \mathbb{Z}_+^{m|n}$ with $h \preceq g \preceq f$, $\theta(h, f) = \theta(h, g) + \theta(g, f)$.

Proof. We just explain how to construct θ, and leave the rest of the proof to the reader. Define $-m \le i_1 < \cdots < i_r < 0 < j_r < \cdots < j_1 \le n$ such that $g(i_s) = g(j_s)$ for each $s = 1, \dots, r$, and $-m \le i'_1 < \cdots < i'_r < 0 < j'_r < \cdots < j'_1 \le n$ such that $f(i'_s) = f(j'_s)$ for each $s = 1, \dots, r$. For $0 \le s \le r$, let

$$g_s = g + \sum_{t=1}^{s} (g(i_s) - f(i'_s))(d_{i_s} - d_{j_s}),$$

so $g_0 = g$ and $(g_r)^+ = f$. Now for each $s = 1, \dots, r$, let θ_s be the unique non-negative integer such that $\mathtt{R}^{\theta_s}_{i_s,j_s}(g_{s-1}) = g_s$, recalling (3.26), and take $\theta = \theta(g, f)$ to be the tuple $(\theta_1, \dots, \theta_r)$. □

Using Lemma 3.42, we can introduce a length function on $\mathbb{Z}_+^{m|n}$. Suppose to start with that $g \preceq f$. Let $\ell(g,f) = |\theta(g,f)|$, where $\theta(g,f)$ is the tuple defined in the lemma. Notice that if $h \preceq g \preceq f$, then

$$\ell(h,g) + \ell(g,f) = \ell(h,f), \tag{3.43}$$

as follows from the stronger fact that $\theta(h,g) + \theta(g,f) = \theta(h,f)$ established by Lemma 3.42. This allows us to extend the notion of length to arbitrary $g, f \in \mathbb{Z}_+^{m|n}$ with $\mathrm{wt}(g) = \mathrm{wt}(f)$: pick $h \in \mathbb{Z}_+^{m|n}$ with $h \preceq g$ and $h \preceq f$ and set $\ell(g,f) = \ell(h,f) - \ell(h,g)$. To check that this is well defined, suppose $h' \in \mathbb{Z}_+^{m|n}$ also satisfies $h' \preceq g$ and $h' \preceq f$. Choose another $k \in \mathbb{Z}_+^{m|n}$ with $k \preceq h$ and $k \preceq h'$. Then using (3.43),

$$\begin{aligned}\ell(h,f) - \ell(h,g) &= (\ell(k,h) + \ell(h,f)) - (\ell(k,h) + \ell(h,g)) \\ &= \ell(k,f) - \ell(k,g) = \ell(h',f) - \ell(h',g),\end{aligned}$$

as required. So we have now defined $\ell(g,f)$, the *length of f relative to g*, for arbitrary $g, f \in \mathbb{Z}_+^{m|n}$ with $\mathrm{wt}(g) = \mathrm{wt}(f)$. It is immediate from the definition that (3.43) holds for all $h, g, f \in \mathbb{Z}_+^{m|n}$ with $\mathrm{wt}(h) = \mathrm{wt}(g) = \mathrm{wt}(f)$.

Finally we can somewhat arbitrarily introduce an absolute notion of length. For each weight γ of $\mathscr{E}^{m|n}$, we fix a choice of "origin" $o_\gamma \in \mathbb{Z}_+^{m|n}$ with $\mathrm{wt}(o_\gamma) = \gamma$. Then for any $f \in \mathbb{Z}_+^{m|n}$, we define

$$\ell(f) := \ell(o_\gamma, f) \tag{3.44}$$

where $\gamma = \mathrm{wt}(f)$. The important thing is that if $\mathrm{wt}(g) = \mathrm{wt}(f)$, then $\ell(g,f) = \ell(f) - \ell(g)$, so we can recover the length of f relative to g from the absolute lengths of f and g. In this notation, Lemma 3.42(ii) and Corollary 3.39(ii) combine to show:

Corollary 3.45. *For $g, f \in \mathbb{Z}_+^{m|n}$ with $g \preceq f$, the polynomial $l_{g,f}(-q^{-1})$ belongs to $q^{\ell(f)-\ell(g)}\mathbb{N}[q^{-2}]$, and the coefficient of $q^{\ell(f)-\ell(g)}$ is 1.*

4. Representations of $\mathfrak{gl}(m|n)$

We now relate the combinatorics developed in sections 2 and 3 to two natural categories $\mathcal{O}_{m|n}$ and $\mathcal{F}_{m|n}$ of representations of $\mathfrak{gl}(m|n)$. For basic notions regarding Lie superalgebras, we follow [Ka1]. We denote the parity of a vector v in a vector superspace by $\bar{v} \in \mathbb{Z}_2$. For a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ and $\mathfrak{g}$-supermodules M, N, the space $\mathrm{Hom}_{\mathfrak{g}}(M,N)$ has a canonical $\mathbb{Z}_2$-grading, and the category of all $\mathfrak{g}$-supermodules is a superadditive category in the sense of [M, Chapter 3, §2.7]. We will use the notation $M \simeq N$ as opposed to the usual $M \cong N$ to indicate that there is an *even* isomorphism between M and N. Also Π denotes the parity change functor.

§4-a. **Two categories.** From now on, we let $\mathfrak{g}$ denote the Lie superalgebra $\mathfrak{gl}(m|n)$. So $\mathfrak{g}$ consists of $(m+n)\times(m+n)$ matrices over $\mathbb{C}$, where we label rows and columns of such matrices by the ordered index set $I(m|n) = \{-m, \dots, -1, 1, \dots, n\}$ as in the introduction. For $i \in I(m|n)$, let $\bar{i} = \bar{0}$ if $i > 0$ and $\bar{1}$ if $i < 0$. Then, the parity of the ij-matrix unit $e_{i,j} \in \mathfrak{g}$ is $\bar{i} + \bar{j}$, and the superbracket satisfies

$$[e_{i,j}, e_{k,l}] = \delta_{j,k} e_{i,l} - (-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{l})} \delta_{i,l} e_{k,j}. \tag{4.1}$$

Note that the subalgebra $\mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}$ is isomorphic to $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$. We will need some other important subalgebras: let $\mathfrak{h}$ denote the standard Cartan subalgebra of $\mathfrak{g}$ consisting of all diagonal matrices, let $\mathfrak{b}$ be the standard Borel subalgebra of all upper triangular matrices, and let $\mathfrak{p} = \mathfrak{g}_{\bar{0}} + \mathfrak{b}$.

For $\lambda \in \mathfrak{h}^*$ and a $\mathfrak{g}$-supermodule M, we define the λ-weight space M_λ of M with respect to $\mathfrak{h}$ as usual: $M_\lambda = \{m \in M \mid hm = \lambda(h)m \text{ for all } h \in \mathfrak{h}\}$. Given a $\mathfrak{g}$-supermodule M such that $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$, we can consider the graded dual $M^\star := \bigoplus_{\lambda \in \mathfrak{h}^*} \mathrm{Hom}_{\mathbb{C}}(M_\lambda, \mathbb{C})$ with the usual $\mathbb{Z}_2$-grading and $\mathfrak{g}$-action. Twisting the $\mathfrak{g}$-action on $M^\star$ with the automorphism $X \mapsto -X^{st}$, where $st : \mathfrak{g} \to \mathfrak{g}$ is the *supertranspose* $e_{i,j} \mapsto (-1)^{\bar{i}(\bar{i}+\bar{j})} e_{j,i}$, we obtain a new $\mathfrak{g}$-supermodule denoted M^τ. If all the weight spaces are finite dimensional, then there are natural isomorphisms $(M^\star)^\star \simeq M$ and $(M^\tau)^\tau \simeq M$.

Let $\{\delta_i\}_{i \in I(m|n)}$ be the basis for $\mathfrak{h}^*$ dual to the basis $\{e_{i,i}\}_{i \in I(m|n)}$ for $\mathfrak{h}$. Define a symmetric bilinear form $(.|.)$ on $\mathfrak{h}^*$ by declaring that $(\delta_i|\delta_j) = (-1)^{\bar{i}} \delta_{i,j}$. The Weyl group W associated to the reductive Lie algebra $\mathfrak{g}_{\bar{0}}$ can be identified with the symmetric group $S_{m|n}$ from §2-a. It acts linearly on $\mathfrak{h}^*$ so that $x\delta_i = \delta_{xi}$ for $x \in W, i \in I(m|n)$. As before, we write w_0 for the longest element of W. We will also need the *dot action* of W on $X(m|n)$ defined by $x \cdot \lambda := x(\lambda + \rho) - \rho$, where

$$\rho = -\sum_{i \in I(m|n)} i\delta_i. \tag{4.2}$$

The *root system* of $\mathfrak{g}$ is the set $R = \{\delta_i - \delta_j \mid i, j \in I(m|n), i \neq j\}$. We write $R = R_{\bar{0}} \cup R_{\bar{1}}$, where $R_{\bar{0}}$ consists of all *even roots* $\delta_i - \delta_j$ with $\bar{i} = \bar{j}$, and $R_{\bar{1}}$ consists of the remaining *odd roots*. Corresponding to the Borel subalgebra $\mathfrak{b}$, we have the standard choice of *positive roots* $R^+ = R^+_{\bar{0}} \cup R^+_{\bar{1}} = \{\delta_i - \delta_j | i, j \in I(m|n), i < j\}$. The *dominance ordering* on $\mathfrak{h}^*$ is defined by $\lambda \leq \mu$ if $(\mu - \lambda)$ is an $\mathbb{N}$-linear combination of positive roots.

From now on, we will restrict our attention to the *integral weights*, i.e. the weights belonging to the subset $X(m|n)$ of $\mathfrak{h}^*$ consisting of all $\mathbb{Z}$-linear combinations of $\{\delta_i\}_{i \in I(m|n)}$. For $\lambda = \sum_{i \in I(m|n)} \lambda_i \delta_i \in X(m|n)$, we define its *parity*

$$\bar{\lambda} := \lambda_{-m} + \cdots + \lambda_{-2} + \lambda_{-1} \in \mathbb{Z}_2. \tag{4.3}$$

Let $X^+(m|n)$ be the set of all *dominant integral weights*, namely, the $\lambda = \sum_{i \in I(m|n)} \lambda_i \delta_i \in X(m|n)$ such that $\lambda_{-m} \geq \cdots \geq \lambda_{-1}$ and $\lambda_1 \geq \cdots \geq \lambda_n$. Define a bijection

$$X(m|n) \to \mathbb{Z}^{m|n}, \quad \lambda \mapsto f_\lambda, \tag{4.4}$$

where $f_\lambda \in \mathbb{Z}^{m|n}$ is the function defined by $f_\lambda(i) = (\lambda + \rho|\delta_i)$ for $i \in I(m|n)$. Under this bijection, $X^+(m|n)$ maps onto $\mathbb{Z}^{m|n}_+$; see §3-a. Also $f_{x \cdot \lambda} = f_\lambda \cdot x^{-1}$ for each $x \in W = S_{m|n}$, i.e. the dot action of W on $X(m|n)$ corresponds to the action of $S_{m|n}$ on $\mathbb{Z}^{m|n}$ introduced in §2-a. Now we lift all the remaining combinatorial definitions involving $\mathbb{Z}^{m|n}$ directly to $X(m|n)$. For instance, recalling (2.3), we define the degree of atypicality $\#\lambda$ of $\lambda \in X(m|n)$ by $\#\lambda := \#f_\lambda$; this is the same notion as in [S3, (1.1)]. Similarly, let $\mathrm{wt}(\lambda) := \mathrm{wt}(f_\lambda)$, an element of the weight lattice P (see (2.2)), and write $\lambda \preceq \mu$ if $f_\lambda \preceq f_\mu$; see §2-b. This ordering on $X(m|n)$ plays the role of the *Bruhat ordering*; see e.g. Theorem 4.31(ii) below. It should not be confused with the dominance ordering $\leq$: we have that $\lambda \preceq \mu \Rightarrow \lambda \leq \mu$ but not conversely.

We are ready to introduce two categories of representations of $\mathfrak{g}$. All the results summarized in the remainder of this subsection are taken from [B2, section 7], where they are deduced from a general framework for representations of graded Lie superalgebras similar to that of Soergel [So2].

The first category is denoted $\mathcal{O}_{m|n}$, and is the (integral weight) analogue of the [BGG] category $\mathcal{O}$ for a semisimple Lie algebra. By definition, $\mathcal{O}_{m|n}$ is the category of all finitely generated $\mathfrak{g}$-supermodules M that are locally finite dimensional over $\mathfrak{b}$ and satisfy

$$M = \bigoplus_{\lambda \in X(m|n)} M_\lambda. \tag{4.5}$$

An object $P \in \mathcal{O}_{m|n}$ is *projective* if every (not necessarily even) morphism from P to a quotient of an object $M \in \mathcal{O}_{m|n}$ lifts to a morphism from P to M. By [B2, Lemma 7.3], the category $\mathcal{O}_{m|n}$ has enough projectives, i.e. every object is a quotient of a projective object. Moreover, $\mathcal{O}_{m|n}$ is finite, i.e. every object has a composition series. For each $\lambda \in X(m|n)$, we have the *Verma module*

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda, \tag{4.6}$$

where $\mathbb{C}_\lambda$ is the one dimensional $\mathfrak{b}$-module of weight λ *concentrated in degree* $\bar{\lambda}$. The significance of the choice of parity here will be explained in §4-e below. As usual, $M(\lambda)$ has a unique irreducible quotient denoted $L(\lambda)$, and $\{L(\lambda)\}_{\lambda \in X(m|n)}$ is a complete set of pairwise non-isomorphic irreducibles in $\mathcal{O}_{m|n}$.

We say that an object $M \in \mathcal{O}_{m|n}$ has a *Verma flag* if it has a filtration $0 = M_0 < \cdots < M_r = M$ such that each M_i/M_{i-1} is $\cong M(\lambda_i)$ for some $\lambda_i \in X(m|n)$. If M has a Verma flag and $\mu \in X(m|n)$, we let

$$(M : M(\mu)) = \dim \mathrm{Hom}_{\mathcal{O}_{m|n}}(M, M(\mu)^\tau). \tag{4.7}$$

By [B2, (6.1)], this computes the number of subquotients of a Verma flag of M that are $\cong M(\mu)$. There is an obvious refinement of these multiplicities: for $p \in \mathbb{Z}_2$,

$$(M : M(\mu))_p := \dim \mathrm{Hom}_{\mathcal{O}_{m|n}}(M, M(\mu)^\tau)_p \tag{4.8}$$

counts the number of subquotients of a Verma flag of M that are $\simeq \Pi^p M(\mu)$.

By [B2, Theorem 6.3], there is for each $\lambda \in X(m|n)$ a unique (up to even isomorphism) indecomposable module $T(\lambda) \in \mathcal{O}_{m|n}$ satisfying the following properties:

(T1) $T(\lambda)$ has a Verma flag starting with $M(\lambda)$ at the bottom;
(T2) $\mathrm{Ext}^1_{\mathcal{O}_{m|n}}(M(\mu), T(\lambda)) = 0$ for all $\mu \in X(m|n)$.

Moreover, by [B2, (7.4)], the multiplicity of $M(\mu)$ in a Verma flag of $T(\lambda)$ is equal to the composition multiplicity of $L(-\lambda - 2\rho)$ in $M(-\mu - 2\rho)$, i.e.

$$(T(\lambda) : M(\mu)) = [M(-\mu - 2\rho) : L(-\lambda - 2\rho)], \tag{4.9}$$

for $\lambda, \mu \in X(m|n)$. In particular, $(T(\lambda) : M(\lambda)) = 1$ and $(T(\lambda) : M(\mu)) = 0$ unless $\mu \leq \lambda$. Consequently, we call $T(\lambda)$ the *infinite dimensional tilting module of highest weight* λ. Finally, note that for every $\lambda \in X(m|n)$,

$$f_{-\lambda-2\rho} = -f_\lambda, \tag{4.10}$$

so the involution $\lambda \mapsto -\lambda - 2\rho$ on $X(m|n)$ appearing in the formula (4.9) corresponds to the involution $f \mapsto -f$ on $\mathbb{Z}^{m|n}$ in ρ-shifted notation.

The second category we shall consider is the category $\mathcal{F}_{m|n}$ of all finite dimensional $\mathfrak{g}$-supermodules satisfying (4.5). Again, this is finite and has enough projectives. As is well known, the irreducible finite dimensional $\mathfrak{g}_{\bar{0}}$-supermodules with integral highest weights are parametrized by the set $X^+(m|n)$. For $\lambda \in X^+(m|n)$, let us write $L'(\lambda)$ for the corresponding irreducible highest weight representation of $\mathfrak{g}_{\bar{0}}$ *concentrated in degree* $\bar{\lambda}$. Then, for each $\lambda \in X^+(m|n)$, we have the *Kac module*

$$K(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L'(\lambda), \tag{4.11}$$

where we are viewing $L'(\lambda)$ here as a $\mathfrak{p}$-supermodule with elements of $\mathfrak{p}_{\bar{0}} = \mathfrak{g}_{\bar{0}}$ acting as given and elements of $\mathfrak{p}_{\bar{1}}$ acting trivially. For $\lambda \in X^+(m|n)$, the irreducible module $L(\lambda)$ defined earlier can also be realized as the unique irreducible quotient of $K(\lambda)$, and $\{L(\lambda)\}_{\lambda \in X^+(m|n)}$ is a complete set of pairwise non-isomorphic irreducibles in $\mathcal{F}_{m|n}$.

When working in $\mathcal{F}_{m|n}$, we will talk about *Kac flags* in place of Verma flags. If M has a Kac flag, the number of subquotients of a Kac flag of M that are $\cong K(\mu)$ is denoted $(M : K(\mu))$, and can be computed by

$$(M : K(\mu)) = \dim \operatorname{Hom}_{\mathcal{F}_{m|n}}(M, K(\mu)^\tau). \tag{4.12}$$

Like in (4.8), there is a refinement denoted $(M : K(\mu))_p$ for $p \in \mathbb{Z}_2$, counting the number of subquotients of a Kac flag of M that are $\simeq \Pi^p K(\mu)$. By [B2, (7.6)] or [Z, Proposition 2.5], the projective cover $P(\lambda)$ of $L(\lambda)$ in the category $\mathcal{F}_{m|n}$ has a Kac flag with $K(\lambda)$ appearing at the top, satisfying the BGG reciprocity

$$(P(\lambda) : K(\mu)) = [K(\mu) : L(\lambda)]. \tag{4.13}$$

There are also indecomposable tilting modules in category $\mathcal{F}_{m|n}$, denoted $U(\lambda)$ for $\lambda \in X^+(m|n)$. Here, by [B2, Theorem 6.3], $U(\lambda) \in \mathcal{F}_{m|n}$ is the unique (up to even isomorphism) indecomposable object such that

(U1) $U(\lambda)$ has a Kac flag starting with $K(\lambda)$ at the bottom;
(U2) $\operatorname{Ext}^1_{\mathcal{F}_{m|n}}(K(\mu), U(\lambda)) = 0$ for all $\mu \in X^+(m|n)$.

Let $\beta = n(\delta_{-m} + \cdots + \delta_{-1}) - m(\delta_1 + \cdots + \delta_n)$ be the sum of the positive odd roots. Then, by [B2, (7.7)–(7.8)] and parity considerations, we have that

$$K(\lambda)^\star \simeq K(\beta - w_0\lambda), \tag{4.14}$$

$$U(\lambda)^\star \simeq P(\beta - w_0\lambda). \tag{4.15}$$

Note that (4.13), (4.14) and (4.15) together imply

$$(U(\lambda) : K(\mu)) = [K(\beta - w_0\mu) : L(\beta - w_0\lambda)], \tag{4.16}$$

for $\lambda, \mu \in X^+(m|n)$. In particular, $(U(\lambda) : K(\lambda)) = 1$ and $(U(\lambda) : K(\mu)) = 0$ unless $\mu \leq \lambda$. Accordingly, we will call $U(\lambda)$ the *finite dimensional tilting module of highest weight* λ. We remark finally that

$$f_{\beta - w_0\lambda} = -f_\lambda \cdot w_0 - (m+n+1)\mathbf{1}, \tag{4.17}$$

where $\mathbf{1} \in \mathbb{Z}^{m|n}$ is the constant function $i \mapsto 1$. Thus, up to a constant shift which can usually be ignored, the involution $\lambda \mapsto \beta - w_0\lambda$ on $X^+(m|n)$ appearing in the formula (4.16) corresponds to the involution $f \mapsto -f \cdot w_0$ on $\mathbb{Z}_+^{m|n}$.

§4-b. **Translation functors.** We need some basic facts about central characters. Let Z be the (even) center of $U(\mathfrak{g})$. The fixed choices of $\mathfrak{h} \subset \mathfrak{b}$ determine a Harish-Chandra homomorphism $\varphi : Z \to U(\mathfrak{h})$; see [Dix, 7.4.3]. Each $\lambda \in \mathfrak{h}^*$ yields

a central character χ_λ defined by $\chi_\lambda(z) = \lambda(\varphi(z))$. To parametrize the *integral central characters*, i.e. the χ_λ for $\lambda \in X(m|n)$, we use the following consequence of results of Sergeev [Sg2], [Sg3]; see [S3, Corollary 1.9]:

Lemma 4.18. *Given $\lambda, \mu \in X(m|n)$, we have that $\chi_\lambda = \chi_\mu$ if and only if* $\mathrm{wt}(\lambda) = \mathrm{wt}(\mu)$ *(where* $\mathrm{wt}(\lambda) = \mathrm{wt}(f_\lambda)$*; see* (2.2)*).*

For each central character χ, let $\mathcal{O}_\chi$ denote the full subcategory of $\mathcal{O}_{m|n}$ consisting of the modules all of whose composition factors have central character χ. We have the *block decomposition*

$$\mathcal{O}_{m|n} = \bigoplus_{\chi} \mathcal{O}_\chi$$

as χ runs over all integral central characters. Lemma 4.18 shows that we can parametrize the integral characters χ instead by the weights $\gamma \in P$ arising non-trivially in the tensor space $\mathscr{T}^{m|n}$ of §2-d. Let us introduce some notation to do this formally. Suppose that $\gamma \in P$. Let $\mathcal{O}_\gamma = \{0\}$ if γ is not a weight of $\mathscr{T}^{m|n}$; else let $\mathcal{O}_\gamma = \mathcal{O}_{\chi_\lambda}$, where $\lambda \in X(m|n)$ is such that $\mathrm{wt}(\lambda) = \gamma$. Then, we can rewrite the above block decomposition as

$$\mathcal{O}_{m|n} = \bigoplus_{\gamma \in P} \mathcal{O}_\gamma, \tag{4.19}$$

where $\mathcal{O}_\gamma$ is non-zero if and only if γ is a weight of $\mathscr{T}^{m|n}$. We let $\mathrm{pr}_\gamma : \mathcal{O}_{m|n} \to \mathcal{O}_\gamma$ be the natural projection functor. In an entirely similar way, we have the block decomposition of $\mathcal{F}_{m|n}$,

$$\mathcal{F}_{m|n} = \bigoplus_{\gamma \in P} \mathcal{F}_\gamma, \tag{4.20}$$

where this time $\mathcal{F}_{m|n}$ is non-zero if and only if γ is a weight of $\mathscr{E}^{m|n}$; see §3-a.

Let V be the natural $\mathfrak{g}$-supermodule. So, V is the vector superspace on basis $\{v_i\}_{i \in I(m|n)}$, where $\bar{v}_i := \bar{i}$, and the action of the matrix unit $e_{i,j} \in \mathfrak{g}$ is given by $e_{i,j} v_k = \delta_{j,k} v_i$. For $r \geq 0$, let $S^r V$ be the rth supersymmetric power of V, a finite dimensional irreducible representation of $\mathfrak{g}$. Let $S^r V^\star = S^r(V^\star) \simeq (S^r V)^\star$. For $a \in \mathbb{Z}$ and $r \geq 0$, we define additive functors $F_a^{(r)}, E_a^{(r)} : \mathcal{O}_{m|n} \to \mathcal{O}_{m|n}$ as follows. It suffices by additivity to define them on objects belonging to $\mathcal{O}_\gamma$ for each $\gamma \in P$. So if $M \in \mathcal{O}_\gamma$, we let

$$F_a^{(r)} M := \mathrm{pr}_{\gamma - r(\varepsilon_a - \varepsilon_{a+1})}(M \otimes S^r V), \tag{4.21}$$

$$E_a^{(r)} M := \mathrm{pr}_{\gamma + r(\varepsilon_a - \varepsilon_{a+1})}(M \otimes S^r V^\star). \tag{4.22}$$

On a morphism $\theta : M \to N$, $F_a^{(r)}\theta$ and $E_a^{(r)}\theta$ are defined simply to be the restrictions of the natural maps $\theta \otimes \mathrm{id}$. Clearly, the restrictions of $F_a^{(r)}$ and $E_a^{(r)}$ to $\mathcal{F}_{m|n}$ give functors $F_a^{(r)}, E_a^{(r)} : \mathcal{F}_{m|n} \to \mathcal{F}_{m|n}$ too. The first well-known lemma gives the elementary properties.

Lemma 4.23. *On either category $\mathcal{O}_{m|n}$ or $\mathcal{F}_{m|n}$, $F_a^{(r)}$ and $E_a^{(r)}$ are exact functors, they commute with the τ-duality, and are both left and right adjoint to each other.*

The next lemma is also quite standard, though we have included a proof since we wish to keep track of parity information.

Lemma 4.24. *Let $\nu_1, \dots, \nu_N$ be the set of weights of $S^r V$ ordered so that $\nu_i > \nu_j \Rightarrow i < j$. Let $\lambda \in X(m|n)$.*

(i) *$M(\lambda) \otimes S^r V$ has a multiplicity-free Verma flag with subquotients $\simeq M(\lambda+\nu_1), \dots, M(\lambda+\nu_N)$ in order from bottom to top.*
(ii) *$M(\lambda) \otimes S^r V^\star$ has a multiplicity-free Verma flag with subquotients $\simeq M(\lambda-\nu_1), \dots, M(\lambda-\nu_N)$ in order from top to bottom.*

Proof. We prove (i), (ii) being entirely similar. By the tensor identity,

$$M(\lambda) \otimes S^r V = (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda) \otimes S^r V \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\mathbb{C}_\lambda \otimes S^r V).$$

So it suffices by exactness of the functor $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})}?$ to show that $M := \mathbb{C}_\lambda \otimes S^r V$ has a filtration $0 = M_0 < M_1 < \cdots < M_N = M$ as a $\mathfrak{b}$-module with $M_i/M_{i-1} \simeq \mathbb{C}_{\lambda+\nu_i}$. Let $x_1, \dots, x_N$ be a basis for $S^r V$, where x_i is of weight ν_i. Then $1 \otimes x_i \in \mathbb{C}_\lambda \otimes S^r V$ is of weight $\lambda + \nu_i$ and degree $\bar{\lambda} + \bar{x}_i = \bar{\lambda} + \bar{\nu}_i = \overline{\lambda + \nu_i}$ (recall (4.3)). So taking M_i to be the subspace spanned by $1 \otimes x_1, \dots, 1 \otimes x_i$ gives the required filtration. □

Corollary 4.25. *Let $\lambda \in X(m|n)$ and $a \in \mathbb{Z}$. Let $(\sigma_{-m}, \dots, \sigma_{-1}, \sigma_1, \dots, \sigma_n)$ be the a-signature of f_λ; see* (2.29).

(i) *$F_a^{(r)} M(\lambda)$ has a multiplicity-free Verma flag with subquotients $\simeq M(\lambda + \delta_{i_1} + \cdots + \delta_{i_r})$ for all distinct $i_1, \dots, i_r \in I(m|n)$ such that $\sigma_{i_1} = \cdots = \sigma_{i_r} = +$.*
(ii) *$E_a^{(r)} M(\lambda)$ has a multiplicity-free Verma flag with subquotients $\simeq M(\lambda - \delta_{j_1} - \cdots - \delta_{j_r})$ for all distinct $j_1, \dots, j_r \in I(m|n)$ such that $\sigma_{j_1} = \cdots = \sigma_{j_r} = -$.*

In both (i) and (ii), the Verma flag can be chosen so that subquotients appear in order refining dominance, most dominant at the bottom.

Proof. The Verma module $M(\lambda)$ has central character χ_λ and so belongs to $\mathcal{O}_{\mathrm{wt}(\lambda)}$ by Lemma 4.18. Applying the exact functor $\mathrm{pr}_{\mathrm{wt}(\lambda) - r(\varepsilon_a - \varepsilon_{a+1})}$ to the filtration in Lemma 4.24(i), we deduce that $F_a^{(r)} M(\lambda)$ has a Verma flag with subquotients being the $M(\lambda + \nu_i)$ such that $\mathrm{wt}(\lambda + \nu_i) = \mathrm{wt}(\lambda) - r(\varepsilon_a - \varepsilon_{a+1})$. This implies that $\nu_i = \delta_{i_1} + \cdots + \delta_{i_s}$ for distinct $i_1, \dots, i_s \in I(m|n)$ such that $\sigma_{i_1} = \cdots = \sigma_{i_s} = +$, giving (i). Part (ii) is similar. □

There is an analogous statement in the finite dimensional setting.

Corollary 4.26. *Let $\lambda \in X^+(m|n)$ and $a \in \mathbb{Z}$. Let $(\sigma_{-m}, \dots, \sigma_{-1}, \sigma_1, \dots, \sigma_n)$ be the a-signature of f_λ; see* (2.29).

(i) *$F_a^{(r)} K(\lambda)$ has a multiplicity-free Kac flag with subquotients $\simeq K(\lambda + \delta_{i_1} + \cdots + \delta_{i_r})$ for all distinct $i_1, \dots, i_r \in I(m|n)$ such that $\lambda + \delta_{i_1} + \cdots + \delta_{i_r} \in X^+(m|n)$ and $\sigma_{i_1} = \cdots = \sigma_{i_r} = +$.*
(ii) *$E_a^{(r)} K(\lambda)$ has a multiplicity-free Kac flag with subquotients $\simeq K(\lambda - \delta_{j_1} - \cdots - \delta_{j_r})$ for all distinct $j_1, \dots, j_r \in I(m|n)$ such that $\lambda - \delta_{j_1} - \cdots - \delta_{j_r} \in X^+(m|n)$ and $\sigma_{j_1} = \cdots = \sigma_{j_r} = -$.*

In both (i) and (ii), the Kac flag can be chosen so that subquotients appear in order refining dominance, most dominant at the bottom.

Proof. We prove (i). By universal properties, $K(\lambda)$ is the largest finite dimensional quotient of $M(\lambda)$. So since $F_a^{(r)}$ is exact, $F_a^{(r)} K(\lambda)$ is a quotient of $F_a^{(r)} M(\lambda)$ and Corollary 4.25 implies that $F_a^{(r)} K(\lambda)$ has a filtration with subquotients being finite dimensional quotients of $M(\lambda + \delta_{i_1} + \cdots + \delta_{i_r})$ for all distinct $i_1, \dots, i_r \in I(m|n)$

such that $\sigma_{i_1} = \cdots = \sigma_{i_r} = +$. But such a quotient is zero unless $\lambda+\delta_{i_1}+\cdots+\delta_{i_r} \in X^+(m|n)$. Hence, $F_a^{(r)}K(\lambda)$ has a filtration with subquotients being quotients of the Kac modules $K(\lambda+\delta_{i_1}+\cdots+\delta_{i_r})$ for all distinct $i_1,\dots,i_r \in I(m|n)$ such that $\lambda+\delta_{i_1}+\cdots+\delta_{i_r} \in X^+(m|n)$ and $\sigma_{i_1} = \cdots = \sigma_{i_r} = +$. Finally the fact that each factor is actually isomorphic to the corresponding Kac module, rather than a proper quotient, follows by a character calculation using the Kac character formula for $K(\mu)$, the Pieri formulae [Mac, (5.16), (5.17)] and Lemma 4.18. □

Corollary 4.27. *Let $a \in \mathbb{Z}$ and $r \geq 1$.*

(i) *For each $\lambda \in X(m|n)$, each indecomposable summand of $F_a^{(r)}T(\lambda)$ or of $E_a^{(r)}T(\lambda)$ is $\simeq T(\mu)$ for $\mu \in X(m|n)$.*

(ii) *For each $\lambda \in X^+(m|n)$, each indecomposable summand of $F_a^{(r)}U(\lambda)$ or of $E_a^{(r)}U(\lambda)$ is $\simeq U(\mu)$ for $\mu \in X^+(m|n)$.*

Proof. We prove (i) for $E_a^{(r)}$, the other cases being similar. Let T be an indecomposable summand of $E_a^{(r)}T(\lambda)$. We need to show that it has a Verma flag with subquotients $\simeq M(\nu)$ for various $\nu \in X(m|n)$, and that $\mathrm{Ext}^1_{\mathcal{O}_{m|n}}(M(\mu),T)=0$ for all $\mu \in X(m|n)$. The first statement is immediate since $E_a^{(r)}T(\lambda)$ has such a Verma flag by Corollary 4.25, and summands of modules with a Verma flag also have a Verma flag; see [B2, Corollary 4.3]. For the second statement, Lemma 4.23 and a standard argument (see e.g. [J2, I.4.4]) show that $\mathrm{Ext}^1_{\mathcal{O}_{m|n}}(M(\mu),E_a^{(r)}T(\lambda)) \simeq \mathrm{Ext}^1_{\mathcal{O}_{m|n}}(F_a^{(r)}M(\mu),T(\lambda))$. To see that the right-hand side is zero, note that $F_a^{(r)}M(\mu)$ has a Verma flag by Corollary 4.25. By induction on length using the long exact sequence and the defining property (T2) of $T(\lambda)$, $\mathrm{Ext}^1_{\mathcal{O}_{m|n}}(M,T(\lambda))=0$ for every $M \in \mathcal{O}_{m|n}$ with a Verma flag. □

Let $\mathcal{O}^\Delta_{m|n}$ be the full subcategory of $\mathcal{O}_{m|n}$ consisting of all modules possessing a Verma flag. Let $K(\mathcal{O}^\Delta_{m|n})$ denote the Grothendieck group of the superadditive category $\mathcal{O}^\Delta_{m|n}$ in the sense of [BK, §2-c]. Note that $K(\mathcal{O}^\Delta_{m|n})$ is a free $\mathbb{Z}$-module on basis $\{[M(\lambda)]\}_{\lambda\in X(m|n)}$. Similarly, let $\mathcal{F}^\Delta_{m|n}$ be the full subcategory of $\mathcal{F}_{m|n}$ consisting of all modules possessing a Kac flag, and let $K(\mathcal{F}^\Delta_{m|n})$ denote its Grothendieck group. Thus, $K(\mathcal{F}^\Delta_{m|n})$ is a free $\mathbb{Z}$-module on basis $\{[K(\lambda)]\}_{\lambda\in X^+(m|n)}$. In view of Corollaries 4.25 and 4.26, the functors $F_a^{(r)}$ and $E_a^{(r)}$ map objects in $\mathcal{O}^\Delta_{m|n}$ (resp. $\mathcal{F}^\Delta_{m|n}$) to objects in $\mathcal{O}^\Delta_{m|n}$ (resp. $\mathcal{F}^\Delta_{m|n}$). Moreover, they preserve short exact sequences in $\mathcal{O}^\Delta_{m|n}$ (resp. $\mathcal{F}^\Delta_{m|n}$). Hence they induce $\mathbb{Z}$-linear operators on $K(\mathcal{O}^\Delta_{m|n})$ and on $K(\mathcal{F}^\Delta_{m|n})$.

Now we make the connection to the modules $\mathscr{T}^{m|n}$ and $\mathscr{E}^{m|n}$ from sections 2 and 3 of the article. Actually we need to specialize these modules at $q=1$. So let $\mathscr{T}^{m|n}_{\mathbb{Z}[q,q^{-1}]}$ be the $\mathbb{Z}[q,q^{-1}]$-lattice in $\mathscr{T}^{m|n}$ spanned by $\{M_f\}_{f\in\mathbb{Z}^{m|n}}$, in the notation of §2-d. Let $\mathscr{E}^{m|n}_{\mathbb{Z}[q,q^{-1}]}$ be the $\mathbb{Z}[q,q^{-1}]$-lattice in $\mathscr{E}^{m|n}$ spanned by $\{K_f\}_{f\in\mathbb{Z}^{m|n}_+}$, in the notation of §3-a. Viewing $\mathbb{Z}$ as a $\mathbb{Z}[q,q^{-1}]$-module so that q acts as 1, we define

$$\mathscr{T}^{m|n}_{\mathbb{Z}} := \mathbb{Z} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathscr{T}^{m|n}_{\mathbb{Z}[q,q^{-1}]},$$
$$\mathscr{E}^{m|n}_{\mathbb{Z}} := \mathbb{Z} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathscr{E}^{m|n}_{\mathbb{Z}[q,q^{-1}]}.$$

We write $M_f(1)$ (resp. $K_f(1)$) for the basis element $1 \otimes M_f$ of $\mathscr{T}_{\mathbb{Z}}^{m|n}$ (resp. $1 \otimes K_f$ of $\mathscr{E}_{\mathbb{Z}}^{m|n}$). Similarly, we define $T_f(1) = 1 \otimes T_f$ and $U_f(1) = 1 \otimes U_f$ (in the case of $T_f(1)$, recall that as a consequence of Conjecture 2.28 we expect it is a finite sum of $K_g(1)$'s and so belongs to $\mathscr{T}_{\mathbb{Z}}^{m|n}$, but without this we mean here to work in the completion $\widehat{\mathscr{T}}_{\mathbb{Z}}^{m|n}$ constructed as in §2-d).

Note that the generators $E_a^{(r)}$ and $F_a^{(r)}$ of $\mathscr{U} = U_q(\mathfrak{gl}_\infty)$ specialize at $q = 1$ to the usual divided powers $E_a^r/r!$ and $F_a^r/r!$ in the Chevalley generators of the Lie algebra $\mathfrak{gl}_\infty$, so we can view $\mathscr{T}_{\mathbb{Z}}^{m|n}$ (resp. $\mathscr{E}_{\mathbb{Z}}^{m|n}$) as modules over the Kostant $\mathbb{Z}$-form $\mathscr{U}_{\mathbb{Z}}$ for the universal enveloping algebra $U(\mathfrak{gl}_\infty)$.

Theorem 4.28. *Identify $K(\mathcal{O}_{m|n}^{\Delta})$ with $\mathscr{T}_{\mathbb{Z}}^{m|n}$ via the $\mathbb{Z}$-module isomorphism*

$$i : K(\mathcal{O}_{m|n}^{\Delta}) \to \mathscr{T}_{\mathbb{Z}}^{m|n}, \quad [M(\lambda)] \mapsto M_{f_\lambda}(1).$$

Then, the representation theoretically defined operators $F_a^{(r)}, E_a^{(r)}$ act in the same way as the Chevalley generators $F_a^{(r)}, E_a^{(r)}$ of $\mathscr{U}_{\mathbb{Z}}$.

Proof. Corollary 4.25 shows that the operators induced by the functors $F_a^{(r)}, E_a^{(r)}$ act on $[M(\lambda)] \in K(\mathcal{O}_{m|n}^{\Delta})$ in exactly the same way as $F_a^{(r)}, E_a^{(r)} \in \mathscr{U}_{\mathbb{Z}}$ act on $M_{f_\lambda}(1) \in \mathscr{T}_{\mathbb{Z}}^{m|n}$. □

An entirely similar argument, using Corollary 4.26 instead, gives the analogous theorem for category $\mathcal{F}_{m|n}^{\Delta}$:

Theorem 4.29. *Identify $K(\mathcal{F}_{m|n}^{\Delta})$ with $\mathscr{E}_{\mathbb{Z}}^{m|n}$ via the $\mathbb{Z}$-module isomorphism*

$$j : K(\mathcal{F}_{m|n}^{\Delta}) \to \mathscr{E}_{\mathbb{Z}}^{m|n}, \quad [K(\lambda)] \mapsto K_{f_\lambda}(1).$$

Then, the representation theoretically defined operators $F_a^{(r)}, E_a^{(r)}$ act in the same way as the Chevalley generators $F_a^{(r)}, E_a^{(r)}$ of $\mathscr{U}_{\mathbb{Z}}$.

§4-c. **Tilting modules in category $\boldsymbol{\mathcal{O}_{m|n}}$.** We proceed to prove some results and formulate some conjectures about the infinite dimensional tilting modules $T(\lambda)$. For $\lambda \in X(m|n)$, write

$$M'(\lambda) := U(\mathfrak{g}_{\bar{0}}) \otimes_{\mathfrak{b}_{\bar{0}}} \mathbb{C}_\lambda$$

for the purely even Verma module for $\mathfrak{g}_{\bar{0}}$ concentrated in degree $\bar{\lambda}$, and $L'(\lambda)$ for its unique irreducible quotient. We will need the following result of Kac [Ka3, Proposition 2.9]. Actually in *loc. cit.*, Kac is only concerned with finite dimensional representations, but the same argument works for the general case stated here.

Lemma 4.30. *If $\lambda \in X(m|n)$ is typical, then $L(\lambda) \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L'(\lambda)$.*

Recall the definition of the polynomials $t_{g,f}(q)$ and $l_{g,f}(q)$ from (2.18). We use the bijection (4.4) to shift notation, letting $t_{\mu,\lambda}(q) := t_{f_\mu,f_\lambda}(q)$ and $l_{\mu,\lambda}(q) := l_{f_\mu,f_\lambda}(q)$. The first part of the following theorem is a reformulation of the Kazhdan-Lusztig conjecture [KL] for $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$, proved in [BB], [BrK].

Theorem 4.31. *Let $\lambda \in X(m|n)$.*

(i) *If λ is typical then $(T(\lambda) : M(\mu)) = t_{\mu,\lambda}(1)$ for each $\mu \in X(m|n)$.*
(ii) *For arbitrary λ, each subquotient of a Verma flag of $T(\lambda)$ is $\simeq M(\mu)$ for $\mu \preceq \lambda$.*

Proof. (i) For the proof, we will assume instead that $\lambda \in X(m|n)$ is typical with $\lambda + \rho \in X^+(m|n)$. Let W_λ be the stabilizer in $W \cong S_{m|n}$ of λ under the dot action, and let D^λ be the set of all maximal length W/W_λ-coset representatives. Let w_λ be the longest element of W_λ. By the Kazhdan-Lusztig conjecture for the Lie algebra $\mathfrak{g}_{\bar{0}}$ proved in [BB], [BrK], combined with the translation principle [J1] (see also [BGS, Theorem 3.11.4]), we have that

$$[M'(x \cdot \lambda) : L'(y \cdot \lambda)] = P_{x,y}(1)$$

for arbitrary $x, y \in D^\lambda$. Here, $P_{x,y}(1)$ denotes the usual Kazhdan-Lusztig polynomial associated to $x, y \in W$ evaluated at 1; see [KL].

We claim that $P_{x,y}(1) = t_{-x\cdot\lambda-2\rho,-y\cdot\lambda-2\rho}(1)$ for all $x, y \in D^\lambda$. To see this, let $f := -f_\lambda$, which is antidominant in the sense of §2-a. Define S_f, D_f as in §2-j. We will use the fact that the map $D^\lambda \to D_f, x \mapsto w_\lambda x^{-1}$ is a bijection. Observe using (4.10) that $f_{-y\cdot\lambda-2\rho} = f \cdot w_\lambda y^{-1}$. So by Lemma 2.25, $t_{-x\cdot\lambda-2\rho,-y\cdot\lambda-2\rho}(1) = t_{f\cdot w_\lambda x^{-1}, f\cdot w_\lambda y^{-1}}(1) = m^{(f)}_{w_\lambda x^{-1}, w_\lambda y^{-1}}(1)$. Noting that $m^{(f)}_{w_\lambda x^{-1}, w_\lambda y^{-1}}(1)$ is the same as the element with the same name in [So1], [So1, Remark 2.6] and [So1, Proposition 3.4], show that $m^{(f)}_{w_\lambda x^{-1}, w_\lambda y^{-1}}(1) = P_{x^{-1},y^{-1}}(1) = P_{x,y}(1)$. This proves the claim.

Now $M(x \cdot \lambda) \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M'(x \cdot \lambda)$ by associativity of tensor product, while Lemma 4.30 shows that $L(y\cdot\lambda) \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L'(y\cdot\lambda)$. So, as the functor $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} ?$ is exact, we deduce from the previous two paragraphs that

$$[M(x \cdot \lambda) : L(y \cdot \lambda)] = t_{-x\cdot\lambda-2\rho,-y\cdot\lambda-2\rho}(1).$$

Note moreover that this argument shows that every subquotient of $M(x \cdot \lambda)$ that is $\cong L(y \cdot \lambda)$ is actually $\simeq L(y \cdot \lambda)$. Finally applying (4.16) gives that

$$(T(-y \cdot \lambda - 2\rho) : M(-x \cdot \lambda - 2\rho)) = t_{-x\cdot\lambda-2\rho,-y\cdot\lambda-2\rho}(1).$$

Part (i) of the theorem follows easily from this and central character considerations. Moreover, by an obvious refinement of (4.16) keeping track of parity information too, we see that every subquotient of a Verma flag of $T(-y \cdot \lambda - 2\rho)$ that is $\cong M(-x \cdot \lambda - 2\rho)$ is actually $\simeq M(-x \cdot \lambda - 2\rho)$.

(ii) We proceed by induction on $\#\lambda$. The case that λ is typical follows from (i). So suppose that $\#\lambda > 0$ and the theorem has been proved for all μ with $\#\mu < \#\lambda$. Let $i : K(\mathcal{O}^\Delta_{m|n}) \to \mathscr{T}^{m|n}_{\mathbb{Z}}$ be the map defined in Theorem 4.28. Apply the algorithm explained in §2-j to $f = f_\lambda$ to construct $h = f_\nu$ for $\nu \in X(m|n)$ with $\#\nu < \#\lambda$ and a sequence $X_1, \dots, X_N$ of monomials in $E_a^{(r)}$ and $F_a^{(r)}$. Let $M := X_N \cdots X_1 T(\nu)$. Note by Corollary 4.25 that M has a Verma flag, and each subquotient of a Verma flag of M is $\simeq M(\mu)$ for some $\mu \in X(m|n)$. By the induction hypothesis, $i([T(\nu)])$ equals $M_h(1)$ plus a linear combination of $M_g(1)$'s with $g \prec h$. By Lemmas 2.10 and 2.11, we deduce that $X_N \cdots X_1 i([T(\nu)])$ equals $M_f(1)$ plus a linear combination of $M_g(1)$'s with $g \prec f$. So by Theorem 4.28,

$$[M] = [M(\lambda)] + (\text{a linear combination of } [M(\mu)]\text{'s with } \mu \prec \lambda).$$

By Corollary 4.27(i), $T(\lambda)$ is a summand of M, and the result follows. □

Motivated by the theorem, we formulate the following conjecture.

Conjecture 4.32. *Let $i : K(\mathcal{O}^\Delta_{m|n}) \to \mathscr{T}^{m|n}_{\mathbb{Z}}$ be the map defined in Theorem 4.28. Then, $i([T(\lambda)]) = T_{f_\lambda}(1)$ for each $\lambda \in X(m|n)$.*

In view of (4.9) this conjecture is equivalent to either of the statements

$$(T(\lambda) : M(\mu)) = t_{\mu,\lambda}(1), \tag{4.33}$$

$$[M(\lambda) : L(\mu)] = t_{-\lambda-2\rho,-\mu-2\rho}(1) \tag{4.34}$$

for all $\lambda, \mu \in X(m|n)$. By Corollary 2.24 and (4.10), the unitriangular matrices $(l_{\mu,\lambda}(1))_{\mu,\lambda\in X(m|n)}$ and $(t_{-\lambda-2\rho,-\mu-2\rho}(1))_{\mu,\lambda\in X^+(m|n)}$ are inverse to each other, so inverting (4.34) also gives that

$$\mathrm{ch}\, L(\lambda) = \sum_{\lambda\in X(m|n)} l_{\mu,\lambda}(1)\mathrm{ch}\, M(\mu). \tag{4.35}$$

Although the summation is infinite here, it involves only finitely many non-zero contributions to the dimensions of each fixed weight space of $L(\lambda)$; thus it can be viewed as a conjectural character formula for irreducibles in $\mathcal{O}_{m|n}$.

Further evidence for Conjecture 4.32 is given by the main theorem in the next subsection. We finally mention one other result which is in keeping with the conjecture; compare in particular with Theorem 2.34. Recall the definition of the dual crystal operators $\tilde{E}_a^*, \tilde{F}_a^*, \varepsilon_a^*$ and φ_a^* from §2-k. Again, we lift these directly to $X(m|n)$ via the bijection (4.4).

Theorem 4.36 (Kujawa)**.** *Let $\lambda \in X(m|n)$ and $a \in \mathbb{Z}$.*

(i) *$F_a L(\lambda) \neq 0$ if and only if $\varphi_a^*(\lambda) \neq 0$, in which case it is a τ-self-dual indecomposable module with irreducible socle and cosocle $\simeq L(\tilde{F}_a^*(\lambda))$. Moreover, $F_a L(\lambda)$ is irreducible if and only if $\varphi_a^*(\lambda) = 1$.*

(ii) *$E_a L(\lambda) \neq 0$ if and only if $\varepsilon_a^*(\lambda) \neq 0$, in which case it is a τ-self-dual indecomposable module with irreducible socle and cosocle $\simeq L(\tilde{E}_a^*(\lambda))$. Moreover, $E_a L(\lambda)$ is irreducible if and only if $\varepsilon_a^*(\lambda) = 1$.*

Theorem 4.36 is a result of Jon Kujawa that will form part of his PhD thesis [Ku]. The proof, which will hopefully appear elsewhere, is similar to the proof given in [B1] of Kleshchev's modular branching rules from [Kv]. It involves some explicit calculations with certain lowering operators in $U(\mathfrak{g})$.

§4-d. **Tilting modules in category $\mathcal{F}_{m|n}$.** Now we study the finite dimensional tilting modules $U(\lambda)$. Lift the crystal operators $\tilde{E}_a, \tilde{F}_a, \varepsilon_a, \varphi_a$ from §3-d to $X^+(m|n)$ through the bijection (4.4), as well as the mutually inverse bijections $\mathtt{L}$ and $\mathtt{R}$ from (3.31).

Theorem 4.37. *Let $j : K(\mathcal{F}^{\Delta}_{m|n}) \to \mathscr{E}^{m|n}_{\mathbb{Z}}$ be the map defined in Theorem 4.29. Then, $j([U(\lambda)]) = U_{f_\lambda}(1)$ for each $\lambda \in X^+(m|n)$. Moreover:*

(i) *Each subquotient of a Kac flag of $U(\lambda)$ is $\simeq K(\mu)$ for $\mathtt{L}(\lambda) \preceq \mu \preceq \lambda$;*

(ii) *$U(\lambda) \simeq P(\mathtt{L}(\lambda))$;*

(iii) *$U(\lambda) \simeq U(\lambda)^\tau$.*

Proof. If $\#\lambda = 0$ then Lemma 4.30 implies that $U(\lambda) = P(\lambda) = K(\lambda) = L(\lambda)$ and the theorem follows in this case. Now suppose that $\#\lambda > 0$. Let $f = f_\lambda$ and define $h = f_\nu$ for $\nu \in X^+(m|n)$ and operators $X_a \in \{E_a, F_a\}_{a\in\mathbb{Z}}$ according to Procedure 3.20. We may assume by induction that the theorem has been proved for ν.

Consider $X_a U(\nu)$. Theorem 4.29, Lemma 3.21 and the induction hypothesis show that $j([X_a U(\nu)]) = X_a U_{f_\nu}(1) = U_{f_\lambda}(1)$. So we get from the explicit description of

$U_{f_\lambda}(1)$ in Theorem 3.34(i) that $[X_aU(\nu)] = [K(\lambda)] + (*) + [K(\mathtt{L}(\lambda))]$, where $(*)$ is a sum of $[K(\mu)]$'s for $\mathtt{L}(\lambda) \prec \mu \prec \lambda$. Using Corollary 4.26, we deduce from this and the induction hypothesis that $X_aU(\nu)$ has a Kac flag with subquotients $\simeq K(\lambda)$, $K(\mathtt{L}(\lambda))$ and all other subquotients $\simeq K(\mu)$ for $\mathtt{L}(\lambda) \prec \mu \prec \lambda$. So $X_aU(\nu)$ must have a summand that is $\simeq U(\lambda)$, recalling Lemma 4.27(ii). Also $U(\nu) \simeq U(\nu)^\tau$ is projective by the induction hypothesis, hence $X_aU(\nu) \simeq (X_aU(\nu))^\tau$ is projective by Lemma 4.23. Thus, $X_aU(\nu)$ must have a summand that is $\simeq P(\mathtt{L}(\lambda))$. To complete the proof, it just remains to show that $X_aU(\nu)$ is indecomposable. For this, we give two different arguments, the first based on Theorem 4.36 and the second using instead a fundamental fact proved by Serganova in [S3].

Method one. Suppose the space

$$\mathrm{Hom}_{\mathcal{F}_{m|n}}(X_aU(\nu), L(\mu)) \simeq \mathrm{Hom}_{\mathcal{F}_{m|n}}(U(\nu), Y_aL(\mu))$$

is non-zero for some $\mu \in X^+(m|n)$. By the choice of a in Procedure 3.20, λ is not at the end of an a-string of length 2 in the crystal graph. Since we must have that $\mathrm{wt}(\mu) = \mathrm{wt}(\lambda)$ by Lemma 4.18, it follows that μ is also not at the end of an a-string of length 2. Theorem 4.36 now implies immediately that $Y_aL(\mu) \simeq L(\tilde{Y}_a^*(\mu))$. By the induction hypothesis, $U(\nu)$ is the projective cover of $L(\mathtt{L}(\nu))$, so we deduce from the non-vanishing of the right-hand hom space above that $\tilde{Y}_a^*(\mu) = \mathtt{L}(\nu)$. Hence, $\mu = \tilde{X}_a^*(\mathtt{L}(\nu)) = \mathtt{L}(\tilde{X}_a(\nu)) = \mathtt{L}(\lambda)$, using Remark 3.18(2) for the penultimate equality. We have now shown that $\mathrm{cosoc}_{\mathfrak{g}}(X_aU(\nu)) \simeq L(\mathtt{L}(\lambda))$, so it is indecomposable.

Method two. Suppose $X_aU(\nu)$ is decomposable. Then, by what we have shown already, we can write $X_aU(\nu) = T_1 \oplus T_2$, where $T_1 \simeq U(\lambda)$ and $T_2 \neq 0$ is a direct sum of indecomposable tilting modules. Note that $Y_aT_i \neq 0$ for each i; indeed we have by adjointness that

$$\mathrm{Hom}_{\mathcal{F}_{m|n}}(U(\nu), Y_aT_i) \simeq \mathrm{Hom}_{\mathcal{F}_{m|n}}(X_aU(\nu), T_i) \neq 0.$$

Recalling Lemma 3.21, we now consider two cases. First, suppose that ν is at the end of an a-string of length 1. Then, we have that $Y_aU_{f_\lambda}(1) = U_{f_\nu}(1)$, i.e. $[Y_aX_aU(\nu)] = [U(\nu)]$. Since $Y_aX_aU(\nu)$ is a direct sum of indecomposable tilting modules, we deduce that $U(\nu) \cong Y_aX_aU(\nu) \cong Y_aT_1 \oplus Y_aT_2$, a contradiction since $U(\nu)$ is indecomposable. Otherwise, we have that ν is at the end of an a-string of length 2, and $[Y_aX_aU(\nu)] = 2[U(\nu)]$. Hence this time we must have that $Y_aT_1 \cong Y_aT_2 \cong U(\nu)$. In particular, we get that $[Y_aU(\lambda) : L(\nu)] = 1$. We now show that $[Y_aU(\lambda) : L(\nu)] \geq 2$, to get the desired contradiction.

Let $\mu = \tilde{X}_a^*(\nu)$, so $\mu = \lambda - \alpha$ for some $\alpha \in R_{\bar{1}}^+$ with $(\lambda + \rho|\alpha) = 0$. By [S3, Theorem 5.5] and (4.16), we have that $[K(\lambda) : L(\mu)] \geq 1$ and that $(U(\lambda) : K(\mu)) = [K(\beta - w_0\mu) : L(\beta - w_0\lambda)] \geq 1$. Hence, $[U(\lambda) : L(\mu)] \geq 2$, since it has a Kac flag involving both $K(\lambda)$ and $K(\mu)$, each of which have $L(\mu)$ as a composition factor. Now $X_aK(\nu)$ has a two-step filtration with $K(\lambda)$ at the bottom and $K(\mu)$ at the top, hence

$$\mathrm{Hom}_{\mathcal{F}_{m|n}}(K(\nu), Y_aL(\mu)) \simeq \mathrm{Hom}_{\mathcal{F}_{m|n}}(X_aK(\nu), L(\mu)) \neq 0.$$

This shows that $[Y_aL(\mu) : L(\nu)] \geq 1$. Finally applying the exact functor Y_a to $U(\lambda)$ and combining our two facts $[U(\lambda) : L(\mu)] \geq 2$ and $[Y_aL(\mu) : L(\nu)] \geq 1$ gives that $[Y_aU(\lambda) : L(\nu)] \geq 2$ as required. $\square$

Now recall the definition of the polynomials $u_{g,f}(q)$ and $l_{g,f}(q)$ from (3.7). As usual we shift notation, writing $u_{\mu,\lambda}(q) := u_{f_\mu,f_\lambda}(q)$ and $l_{\mu,\lambda}(q) := l_{f_\mu,f_\lambda}(q)$. Combining the theorem with (4.16), we get that

$$(U(\lambda) : K(\mu)) = u_{\mu,\lambda}(1), \tag{4.38}$$

$$[K(\lambda) : L(\mu)] = u_{\beta-w_0\lambda,\beta-w_0\mu}(1). \tag{4.39}$$

The Main Theorem stated in the introduction follows immediately from the second of these formulae and Corollary 3.36(i), since in view of (4.17) and Corollary 3.14 that gives an explicit formula for $u_{\beta-w_0\lambda,\beta-w_0\mu}(1)$. In particular, $[K(\lambda) : L(\mu)] \leq 1$ for all $\lambda, \mu \in X^+(m|n)$, as was conjectured in [HKJ, Conjecture 7.2], and $L(\mu)$ appears as a composition factor in exactly $2^{\#\mu}$ different Kac modules $K(\lambda)$, as was conjectured in [HKJ, Corollary 7.3].

By Corollary 3.14 and (4.17), the unitriangular matrix $(l_{\mu,\lambda}(1))_{\mu,\lambda\in X^+(m|n)}$ is the inverse of $(u_{\beta-w_0\lambda,\beta-w_0\mu}(1))_{\mu,\lambda\in X^+(m|n)}$. So on inverting (4.39), we also get that

$$\mathrm{ch}\, L(\lambda) = \sum_{\mu\in X^+(m|n)} l_{\mu,\lambda}(1)\mathrm{ch}\, K(\mu). \tag{4.40}$$

This can be viewed as a character formula for the finite dimensional irreducible $\mathfrak{gl}(m|n)$-supermodules with integral highest weight. The explicit description of the coefficients $l_{\mu,\lambda}(1)$ given by Corollary 3.39(ii) seems to be quite different from the explicit description given by Serganova [S3, Theorem 2.3], and I have been unable to prove combinatorially that they are equivalent.

To conclude the subsection, let us record one more consequence of Theorem 4.37.

Corollary 4.41. *For $\lambda \in X^+(m|n)$, $L(\lambda)^\star \simeq L(\beta - w_0\mathtt{R}(\lambda))$.*

Proof. By (4.15) and Theorem 4.37, $P(\beta - w_0\mathtt{R}(\lambda)) \simeq U(\mathtt{R}(\lambda))^\star \simeq P(\lambda)^\star$ and it is self-dual under the duality τ. Hence $L(\beta - w_0\mathtt{R}(\lambda)) \simeq \mathrm{soc}_{\mathfrak{g}} P(\beta - w_0\mathtt{R}(\lambda)) \simeq \mathrm{soc}_{\mathfrak{g}} P(\lambda)^\star \simeq (\mathrm{cosoc}_{\mathfrak{g}} P(\lambda))^\star \simeq L(\lambda)^\star$. □

Remark 4.42. A different description of the highest weight of $L(\lambda)^\star$ can be derived using Serganova's odd reflections; see [S1], [PS1, Lemma 0.3] and [BKu, Theorem 4.5]. In view of (4.39) and (4.14), Corollary 4.41 implies (indeed is equivalent to) the equality $u_{\beta-w_0\mu,\beta-w_0\lambda}(1) = u_{\mu,\mathtt{R}(\lambda)}(1)$; see Corollary 3.41 for a stronger statement.

§4-e. **Highest weight categories.** Let $\mathcal{F}^{\bar{0}}_{m|n}$ (resp. $\mathcal{F}^{\bar{1}}_{m|n}$) be the full subcategory of $\mathcal{F}_{m|n}$ consisting of the modules all of whose composition factors are $\simeq L(\lambda)$ (resp. $\simeq \Pi L(\lambda)$) for $\lambda \in X^+(m|n)$. Obviously, the parity change functor Π defines an isomorphism between $\mathcal{F}^{\bar{0}}_{m|n}$ and $\mathcal{F}^{\bar{1}}_{m|n}$. Since each $\mathrm{End}_{\mathcal{F}_{m|n}}(L(\lambda))$ is concentrated in degree $\bar{0}$, each $\mathrm{Hom}_{\mathcal{F}_{m|n}}(M, N)$ for $M, N \in \mathcal{F}^{\bar{0}}_{m|n}$ is also concentrated in degree $\bar{0}$, hence $\mathcal{F}^{\bar{0}}_{m|n}$ is an abelian category.

Lemma 4.43. *For $\lambda \in X^+(m|n)$, each of the objects $U(\lambda), P(\lambda), K(\lambda)$ and $L(\lambda)$ belongs to $\mathcal{F}^{\bar{0}}_{m|n}$. Moreover, the dualities $\star$ and τ and the functors $F_a^{(r)}$ and $E_a^{(r)}$ map objects in $\mathcal{F}^{\bar{0}}_{m|n}$ to objects in $\mathcal{F}^{\bar{0}}_{m|n}$.*

Proof. By Theorem 4.37(i), each subquotient of a Kac flag of $U(\lambda)$ is $\simeq K(\mu)$ for some $\mu \in X^+(m|n)$. We deduce using (4.14)–(4.15) that each subquotient of a Kac flag of $P(\lambda)$ is $\simeq K(\mu)$ for some $\mu \in X^+(m|n)$. By the obvious refinement

of (4.13) keeping track of parities, it follows that each composition factor of $K(\mu)$ is $\simeq L(\lambda)$ for some $\lambda \in X^+(m|n)$. Combining these statements shows that all of $U(\lambda), P(\lambda), K(\lambda)$ and $L(\lambda)$ belong to $\mathcal{F}^{\bar{0}}_{m|n}$. For the remaining statement, we obviously have that $L(\lambda)^\tau \simeq L(\lambda)$, hence τ leaves $\mathcal{F}^{\bar{0}}_{m|n}$ invariant. The same thing for $\star$ follows from Corollary 4.41. Finally, Corollary 4.26 shows that the exact functors $F_a^{(r)}$ and $E_a^{(r)}$ send $K(\lambda)$ to an object in $\mathcal{F}^{\bar{0}}_{m|n}$, and $L(\lambda)$ is a quotient of $K(\lambda)$ so they must also send $L(\lambda)$ to an object in $\mathcal{F}^{\bar{0}}_{m|n}$. □

Corollary 4.44. *For any $M, N \in \mathcal{F}^{\bar{0}}_{m|n}$ and $i \geq 0$, the space $\mathrm{Ext}^i_{\mathcal{F}_{m|n}}(M, N)$ is concentrated in degree $\bar{0}$.*

Proof. We have already noted this is the case if $i = 0$. To get the general case from this, note by the lemma that every composition factor of every term of the obvious minimal projective resolution of M belongs to $\mathcal{F}^{\bar{0}}_{m|n}$. □

It follows easily from the corollary that every object $M \in \mathcal{F}_{m|n}$ decomposes uniquely as $M = M^{\bar{0}} \oplus M^{\bar{1}}$ with $M^p \in \mathcal{F}^p_{m|n}$ for each $p \in \mathbb{Z}_2$. We deduce that there is a decomposition $\mathcal{F}_{m|n} = \mathcal{F}^{\bar{0}}_{m|n} \oplus \Pi\mathcal{F}^{\bar{0}}_{m|n}$ allowing us to reconstruct the superadditive category $\mathcal{F}_{m|n}$ from the additive category $\mathcal{F}^{\bar{0}}_{m|n}$. For example, for $M, N \in \mathcal{F}_{m|n}$ and $i \geq 0$, we have that

$$\mathrm{Ext}^i_{\mathcal{F}_{m|n}}(M, N)_{\bar{0}} = \mathrm{Ext}^i_{\mathcal{F}^{\bar{0}}_{m|n}}(M^{\bar{0}}, N^{\bar{0}}) \oplus \mathrm{Ext}^i_{\mathcal{F}^{\bar{0}}_{m|n}}(\Pi M^{\bar{1}}, \Pi N^{\bar{1}}), \tag{4.45}$$

$$\mathrm{Ext}^i_{\mathcal{F}_{m|n}}(M, N)_{\bar{1}} = \mathrm{Ext}^i_{\mathcal{F}^{\bar{0}}_{m|n}}(M^{\bar{0}}, \Pi N^{\bar{1}}) \oplus \mathrm{Ext}^i_{\mathcal{F}^{\bar{0}}_{m|n}}(\Pi M^{\bar{1}}, N^{\bar{0}}). \tag{4.46}$$

At this point, we refer the reader to [CPS1], [CPS2] for the definition of a *highest weight category with duality.*

Theorem 4.47. *The category $\mathcal{F}^{\bar{0}}_{m|n}$ is a highest weight category with weight poset $(X^+(m|n), \preceq)$ and duality τ. For $\lambda \in X^+(m|n)$, $U(\lambda), P(\lambda), K(\lambda)$ and $L(\lambda)$ are the indecomposable tilting, projective, standard and irreducible modules parametrized by λ, respectively.*

Proof. We have seen in Theorem 4.37 that $(P(\lambda) : K(\mu)) \neq 0 \Rightarrow \lambda \preceq \mu$. Given this and (4.13) it is a routine matter to check that $\mathcal{F}^{\bar{0}}_{m|n}$ satisfies the axioms for a highest weight category with duality. □

Remark 4.48. In an entirely similar fashion, we define $\mathcal{O}^{\bar{0}}_{m|n}$ to be the full subcategory of $\mathcal{O}_{m|n}$ consisting of the objects M all of whose composition factors are $\simeq L(\lambda)$ for $\lambda \in X(m|n)$. Using Theorem 4.31(ii) and the refined versions of (4.9) and BGG reciprocity [B2, (6.6)] keeping track of parity, one can prove analogues of all the results in this subsection for $\mathcal{O}^{\bar{0}}_{m|n}$: there is a decomposition $\mathcal{O}_{m|n} = \mathcal{O}^{\bar{0}}_{m|n} \oplus \Pi\mathcal{O}^{\bar{0}}_{m|n}$, and $\mathcal{O}^{\bar{0}}_{m|n}$ is a highest weight category with weight poset $(X(m|n), \preceq)$ and duality τ.

§4-f. **Kazhdan-Lusztig polynomials.** In this subsection, we explain the true significance of the polynomials $l_{\mu,\lambda}(q) = l_{f_\mu, f_\lambda}(q)$ for $\mu, \lambda \in X^+(m|n)$.

Lemma 4.49. *Let $\mu \in X^+(m|n)$. Then, $K(\mu)$ has a projective resolution $\cdots \to P_1(\mu) \to P_0(\mu) \to K(\mu) \to 0$ in $\mathcal{F}^{\bar{0}}_{m|n}$ such that for every $\lambda \in X^+(m|n)$,*

$$\sum_{i \geq 0} \dim \mathrm{Hom}_{\mathcal{F}_{m|n}}(P_i(\mu), L(\lambda)) q^i = l_{\mu,\lambda}(-q^{-1}).$$

Proof. We first explain how to construct for fixed $d \geq 0$ an exact sequence $P_d(\mu) \to \cdots \to P_0(\mu) \to K(\mu) \to 0$ with each $P_i(\mu)$ projective. In case $\#\mu = 0$, $K(\mu)$ is already projective, so we can simply take $P_0(\mu) = K(\mu)$ and $P_i(\mu) = 0$ for $i > 0$. Now suppose $\#\mu > 0$. Let $g = f_\mu$ and apply Procedure 3.23 to construct $h = f_\nu$ and operators $X_a, Y_a \in \{E_a, F_a\}_{a \in \mathbb{Z}}$. Since Procedure 3.23 reduces μ to a typical weight in finitely many steps, we may assume inductively that we have already constructed an exact sequence

$$P_d(\nu) \longrightarrow \cdots \longrightarrow P_0(\nu) \longrightarrow K(\nu) \longrightarrow 0. \tag{4.50}$$

Now we consider two cases. Suppose first that $\#\nu = \#\mu$. Then $X_a K(\nu) \simeq K(\mu)$, so applying X_a to (4.50) gives us the desired sequence with $P_i(\mu) = X_a P_i(\nu)$. In the second case, $\#\nu = \#\mu - 1$, and $X_a K(\nu)$ has a two step filtration with $K(\mu)$ at the top and $K(\tilde{X}_a(\nu))$ at the bottom. Applying X_a to (4.50) gives us an exact sequence $X_a P_d(\nu) \to \cdots \to X_a P_0(\nu) \to X_a K(\nu) \to 0$. By induction on d, we may assume in addition that we have already constructed an exact sequence $P_{d-1}(\tilde{X}_a(\nu)) \to \cdots \to P_0(\tilde{X}_a(\nu)) \to K(\tilde{X}_a(\nu)) \to 0$. Applying the comparison theorem [W, 2.2.6] to the embedding $i : K(\tilde{X}_a(\nu)) \hookrightarrow X_a K(\nu)$, we get vertical maps making the diagram commute:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_1(\tilde{X}_a(\nu)) & \longrightarrow & P_0(\tilde{X}_a(\nu)) & \longrightarrow & K(\tilde{X}_a(\nu)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle i} & & \\
\cdots & \longrightarrow & X_a P_1(\nu) & \longrightarrow & X_a P_0(\nu) & \longrightarrow & X_a K(\nu) & \longrightarrow & 0
\end{array}$$

The total complex of this double complex is exact by the acyclic assembly lemma [W, 2.7.3]. Factoring out $K(\tilde{X}_a(\mu))$ yields the required exact sequence

$$\cdots \longrightarrow X_a P_1(\nu) \oplus P_0(\tilde{X}_a(\nu)) \longrightarrow X_a P_0(\nu) \longrightarrow K(\mu) \longrightarrow 0.$$

This time, $P_i(\mu) = X_a P_i(\nu) \oplus P_{i-1}(\tilde{X}_a(\nu))$.

Replacing d by $(d+1)$, the same procedure constructs an exact sequence $P_{d+1}(\mu) \to P_d(\mu) \to \cdots \to P_0(\mu) \to K(\mu) \to 0$, where we can always ensure that the first d terms are *the same* as the ones constructed before. Now letting $d \to \infty$ we get a projective resolution of $K(\mu)$. We note moreover by the construction that whenever $P(\lambda)$ is a summand of $P_i(\mu)$ for some $i \geq 0$, i.e. $\mathrm{Hom}_{\mathcal{F}_{m|n}}(P_i(\mu), L(\lambda)) \neq 0$, we must have that $\mu \preceq \lambda$.

Finally let $p_{\mu,\lambda}(q) = \sum_{i \geq 0} \dim \mathrm{Hom}_{\mathcal{F}_{m|n}}(P_i(\mu), L(\lambda)) q^i$. To complete the proof, we need to show that $p_{\mu,\lambda}(q) = l_{\mu,\lambda}(-q^{-1})$ for each $\mu, \lambda \in X^+(m|n)$. For this, we show that the polynomials $p_{\mu,\lambda}(q)$ satisfy the same relations as the polynomials $l_{\mu,\lambda}(-q^{-1})$ in Lemma 3.24. Once this is established, the algorithm explained at the end of §3-e to compute $l_{\mu,\lambda}(-q^{-1})$ also computes $p_{\mu,\lambda}(q)$, hence $p_{\mu,\lambda}(q) = l_{\mu,\lambda}(-q^{-1})$. So take $\lambda, \mu \in X^+(m|n)$ with $\#\mu > 0$, where we may assume that $\mathrm{wt}(\lambda) = \mathrm{wt}(\mu)$, since otherwise $p_{\mu,\lambda}(q) = 0$ and the conclusion holds trivially. Apply Procedure 3.23 to $g = f_\mu$ to get $h = f_\nu$ and operators X_a, Y_a, and consider the two cases $\#\nu = \#\mu$ or $\#\nu = \#\mu - 1$, just like above. Let us just explain the argument in the second case, the first case being easier. Since $\mathrm{wt}(\lambda) = \mathrm{wt}(\mu)$ and μ

is not at the end of an a-string of length 2 in the dual crystal graph, Theorem 4.36 shows that $Y_a L(\lambda)$ equals $L(\tilde{Y}_a^*(\lambda))$, interpreted as 0 if $\tilde{Y}_a^*(\lambda) = \varnothing$. So we get that

$$\sum_{i\geq 0} \dim \mathrm{Hom}_{\mathcal{F}_{m|n}}(X_a P_i(\nu), L(\lambda))q^i = \sum_{i\geq 0} \dim \mathrm{Hom}_{\mathcal{F}_{m|n}}(P_i(\nu), Y_a L(\lambda))q^i$$
$$= \sum_{i\geq 0} \dim \mathrm{Hom}_{\mathcal{F}_{m|n}}(P_i(\nu), L(\tilde{Y}_a^*(\lambda)))q^i = p_{\nu,\tilde{Y}_a^*(\lambda)}(q),$$

interpreted as 0 in case $\tilde{Y}_a^*(\lambda) = \varnothing$. We noted above that $P_i(\mu) = X_a P_i(\nu) \oplus P_{i-1}(\tilde{X}_a(\nu))$, hence we get that $p_{\mu,\lambda}(q) = p_{\nu,\tilde{Y}_a^*(\lambda)}(q) + q p_{\tilde{X}_a(\nu),\lambda}(q)$, which is what we wanted in this case; cf. Lemma 3.24. □

Now choose a length function on $\mathbb{Z}_+^{m|n}$ as explained in §3-g, and lift it to $X^+(m|n)$ by setting $\ell(\lambda) := \ell(f_\lambda)$.

Theorem 4.51. *For $\mu, \lambda \in X^+(m|n)$, the superspace $\mathrm{Ext}^i_{\mathcal{F}_{m|n}}(K(\mu), L(\lambda))$ is concentrated in degree $\bar{0}$, and*

$$\sum_{i\geq 0} \dim \mathrm{Ext}^i_{\mathcal{F}_{m|n}}(K(\mu), L(\lambda))q^i = l_{\mu,\lambda}(-q^{-1}).$$

Hence, $\mathrm{Ext}^\bullet_{\mathcal{F}_{m|n}}(K(\mu), L(\lambda)) \neq 0$ if and only if $\mu \preceq \lambda$, in which case

(i) $\mathrm{Ext}^i_{\mathcal{F}_{m|n}}(K(\mu), L(\lambda)) \neq 0 \Rightarrow i \leq \ell(\lambda) - \ell(\mu), i \equiv \ell(\lambda) - \ell(\mu) \pmod 2$*;*
(ii) $\mathrm{Ext}^{\ell(\lambda)-\ell(\mu)}_{\mathcal{F}_{m|n}}(K(\mu), L(\lambda))$ *is exactly one dimensional;*
(iii) $\mathrm{Ext}^1_{\mathcal{F}_{m|n}}(K(\mu), L(\lambda))$ *is at most one dimensional.*

Proof. Apply the functor $\mathrm{Hom}_{\mathcal{F}_{m|n}}(?, L(\lambda))$ to the projective resolution constructed in Lemma 4.49 and use Corollaries 3.45, 3.39(ii) and 4.44. □

By [Z, Theorem 7.6], Theorem 4.51 shows that the polynomials $l_{\mu,\lambda}(-q^{-1})$ defined here agree with the Kazhdan-Lusztig polynomials $K_{\lambda,\mu}(q)$ defined by Serganova in [S2], [S3]. (It also proves [JZ, Conjecture 4.4], and answers a question raised at the end of [Z].) Thus we have a cohomological interpretation of the polynomial $l_{\mu,\lambda}(-q^{-1})$, analogous to Vogan's interpretation [V, Conjecture 3.4] of Kazhdan-Lusztig polynomials in category $\mathcal{O}$ for a semisimple Lie algebra. The even-odd vanishing established in Theorem 4.51(i) is especially important: in the language of [CPS3], it shows that the highest weight category $\mathcal{F}^{\bar{0}}_{m|n}$ has a "Kazhdan-Lusztig theory". Applying [CPS4, Corollary 3.9] (and Corollary 4.44 again), we obtain:

Corollary 4.52. *For $\mu, \lambda \in X^+(m|n)$, the superspace $\mathrm{Ext}^i_{\mathcal{F}_{m|n}}(L(\mu), L(\lambda))$ is concentrated in degree $\bar{0}$, and*

$$\sum_{i\geq 0} \dim \mathrm{Ext}^i_{\mathcal{F}_{m|n}}(L(\mu), L(\lambda))q^i = \sum_{\nu\in X^+(m|n)} l_{\nu,\mu}(-q^{-1}) l_{\nu,\lambda}(-q^{-1}).$$

In particular, $\mathrm{Ext}^i_{\mathcal{F}_{m|n}}(L(\mu), L(\lambda)) \neq 0 \Rightarrow i \equiv \ell(\lambda) - \ell(\mu) \pmod 2$.

Example 4.53. Take $\mu = \lambda = 0$ and let $r = \min(m, n)$. In this case, the polynomials $l_{\nu,\mu}(-q^{-1})$ are computed explicitly in Example 3.37. Combining this with

Corollary 4.52, one deduces that $\dim \mathrm{Ext}^{2i}_{\mathcal{F}_{m|n}}(\mathbb{C},\mathbb{C})$ equals the number of partitions of i with at most r non-zero parts. Hence:

$$\sum_{i\geq 0} \dim \mathrm{Ext}^{i}_{\mathcal{F}_{m|n}}(\mathbb{C},\mathbb{C})q^i = \frac{1}{(1-q^2)(1-q^4)\cdots(1-q^{2r})}. \tag{4.54}$$

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