JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 25, Number 3, July 2012, Pages 617–635
S 0894-0347(2012)00733-4
Article electronically published on March 15, 2012

NONNEGATIVE POLYNOMIALS AND SUMS OF SQUARES

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1. Introduction

A real polynomial in n variables is called nonnegative if it is greater than or equal to 0 at all points in $\mathbb{R}^n$. It is a central question in real algebraic geometry whether a nonnegative polynomial can be written in a way that makes its nonnegativity apparent, i.e. as a sum of squares of polynomials (or more general objects). Algorithms to obtain such representations, when they are known, have many applications in polynomial optimization [9], [10], [11].

The investigation of the relation between nonnegativity and sums of squares began in the seminal paper of Hilbert from 1888. Hilbert showed that every nonnegative polynomial is a sum of squares of polynomials only in the following three cases: univariate polynomials, quadratic polynomials, and bivariate polynomials of degree 4. In all other cases Hilbert showed the existence of nonnegative polynomials that are not sums of squares. Hilbert's proof used the fact that polynomials of degree d satisfy linear relations, known as the Cayley-Bacharach relations, which are not satisfied by polynomials of full degree $2d$ [14], [15].

Hilbert then showed that every bivariate nonnegative polynomial is a sum of squares of rational functions and Hilbert's 17th problem asked whether this is true in general. In the 1920's Artin and Schreier solved Hilbert's 17th problem in the affirmative. However, there is no known algorithm to obtain this representation. In particular we may need to use numerators and denominators of very large degree, thus representing a simple object (the polynomial) as a sum of squares of significantly more complex objects [3].

It should be noted that Hilbert did not provide an explicit nonnegative polynomial that is not a sum of squares of polynomials. He only proved its existence. The first explicit example appeared only eighty years later and is due to Motzkin. Since then many explicit examples of nonnegative polynomials that are not sums of squares have appeared [14]. For some low-dimensional, symmetric families there are also descriptions of the exact differences between nonnegative polynomials and sums of squares [5]. However even in the smallest cases where nonnegative polynomials are different from sums of squares, three variables of degree 4 and two variables of degree 6, we have not had a complete understanding of what makes nonnegative polynomials different from sums of squares.

We show that, in these cases, all linear inequalities that separate nonnegative polynomials from sums of squares come from the Cayley-Bacharach relations. The Cayley-Bacharach relations were already used by Hilbert in the original proof of

Received by the editors December 11, 2010 and, in revised form, August 12, 2011, and December 17, 2011.

2010 *Mathematics Subject Classification.* Primary 14N05, 14P99; Secondary 52A20.

the existence of nonnegative polynomials that are not sums of squares. We show that, in fact, these relations are the fundamental reason underlying the existence of any such polynomial, and we provide explicit structure for the linear inequalities separating nonnegative polynomials from sums of squares. The algebra and geometry involved in these two cases is quite similar, and we give a complete unified geometric description of the differences between nonnegative polynomials and sums of squares.

1.1. **Main results.** By analogy with quadratic forms we will refer to nonnegative polynomials as *positive semidefinite*, or *psd* for short, and sums of squares will be called *sos*. Any psd polynomial can be made homogeneous by adding an extra variable and it will remain nonnegative. The same holds for sums of squares. We will therefore work with homogeneous polynomials (forms).

Our goal is to investigate the cases of forms in three variables of degree 6, known as ternary sextics, and forms in four variables of degree 4, known as quaternary quartics. We will denote these cases as $(3, 6)$ and $(4, 4)$, respectively.

Let $H_{n,d}$ be the vector space of real forms in n variables of degree d. Nonnegative forms and sums of squares both form full-dimensional closed convex cones in $H_{n,2d}$, which we call $P_{n,2d}$ and $\Sigma_{n,2d}$, respectively:

$$P_{n,2d} = \{p \in H_{n,2d} \ \ | \ \ p(x) \geq 0 \ \text{ for all } \ x \in \mathbb{R}^n\},$$

and

$$\Sigma_{n,2d} = \left\{p \in H_{n,2d} \ \middle| \ p(x) = \sum q_i^2 \ \text{ for some } \ q_i \in H_{n,d}\right\}.$$

It is clear that $\Sigma_{n,2d} \subseteq P_{n,2d}$ and by Hilbert's theorem this inclusion is actually strict in the cases $(3, 6)$ and $(4, 4)$.

The defining linear inequalities for the psd cone $P_{n,2d}$ are easy to describe. They are given by

$$f(v) \geq 0 \qquad \text{for all } v \in \mathbb{R}^n.$$

By homogeneity of forms it suffices to only consider points v in the unit sphere $\mathbb{S}^{n-1}$. We remark that with this characterization and an appropriate choice of the inner product it is not hard to show that the dual cone to $P_{n,2d}$ is the conic hull of the real Veronese variety of degree $2d$ and thus the dual cone of $P_{n,2d}$ is essentially equivalent to the Veronese orbitope [16].

The above inequalities are clearly satisfied by all sums of squares but when the sos cone is strictly smaller, it must satisfy additional linear inequalities. We prove the following characterization for $(3, 6)$:

Theorem 1.1. *Suppose that $p \in P_{3,6}$ and p is not sos. Then there exist two real cubics $q_1, q_2 \in H_{3,3}$ intersecting in nine (possibly complex) projective points $\gamma_1, \ldots, \gamma_9$ such that the values of p on γ_i certify that p is not a sum of squares. More precisely, let $z_1, \ldots, z_9$ be affine representatives of γ_i. Then there exists a real linear functional $\ell : H_{3,6} \to \mathbb{R}$ given by*

$$\ell(f) = \sum \mu_i f(z_i),$$

for some $\mu_i \in \mathbb{C}$ such that $\ell(r) \geq 0$ for all $r \in \Sigma_{3,6}$ and $\ell(p) < 0$. Furthermore at most two of the points γ_i are complex.

We also prove a similar theorem for the case $(4,4)$:

Theorem 1.2. *Suppose that $p \in P_{4,4}$ and p is not sos. Then there exist three real quadrics $q_1, q_2, q_3 \in H_{4,2}$ intersecting in eight (possibly complex) projective points $\gamma_1, \ldots, \gamma_8$ such that the values of p on γ_i certify that p is not a sum of squares. More precisely, let $z_1, \ldots, z_8$ be affine representatives of γ_i. Then there exists a real linear functional $\ell : H_{4,4} \to \mathbb{R}$ given by*

$$\ell(f) = \sum \mu_i f(z_i),$$

for some $\mu_i \in \mathbb{C}$ such that $\ell(r) \geq 0$ for all $r \in \Sigma_{4,4}$ and $\ell(p) < 0$. Furthermore at most two of the points γ_i are complex.

These theorems are proved at the end of Section 5. The cases $(3,6)$ and $(4,4)$ are quite similar, and we provide a unified presentation of the proofs. The main ingredient in the proofs is the Cayley-Bacharach theorem [6], which shows that the values of forms in $H_{3,3}$ (resp. $H_{4,2}$) on the points z_i defined above are linearly related and this relation is unique. It was already observed by Hilbert in his original proof that the Cayley-Bacharach relations can be used to construct nonnegative polynomials that are not sums of squares. A modern exposition of Hilbert's construction along with generalizations is given by Reznick in [15]. We show that the Cayley-Bacharach relations are more than just a way of constructing examples and in fact they are the fundamental reason that prevents sums of squares from filling out the entire psd cone. We note that for the cases where $P_{n,2d} = \Sigma_{n,2d}$ the Cayley-Bacharach relations do not exist and it is possible to prove the equality of the psd and sos cones based on the nonexistence of the relations.

Complex zeroes of real forms come in conjugate pairs. In Section 4.1 we show how to exclude the cases of the intersection containing more than one conjugate pair of complex zeroes. We also show how to explicitly derive the inequalities ℓ, given the Cayley-Bacharach relation. This is done for a fully real intersection case in Section 6 and in Section 7 for the case of one conjugate pair of complex zeroes.

We also obtain the following interesting corollaries:

Corollary 1.3. *Suppose that $p \in \Sigma_{3,6}$ lies on the boundary of the cone of sums of squares and p is a strictly positive form. Then p is a sum of three squares and cannot be written as a sum of two squares.*

Also, for the case $(4,4)$ we have the following:

Corollary 1.4. *Suppose that $p \in \Sigma_{4,4}$ lies on the boundary of the cone of sums of squares and p is a strictly positive form. Then p is a sum of four squares and cannot be written as a sum of three squares.*

Corollaries 1.3 and 1.4 were used as a starting point to investigate the algebraic boundary of the cones $\Sigma_{3,6}$ and $\Sigma_{4,4}$ in [2]. Here we briefly note that sextics that are sums of three squares of cubics and quartics that are sums of four squares of quadratics form hypersurfaces in $H_{3,6}$ and $H_{4,4}$. One of the main results of [2] is establishing the degree of these hypersurfaces with a connection with K3 surfaces.

In Section 3 we examine in detail the case of an arbitrary completely real transverse intersection of two cubics for the case $(3,6)$ and three quadratics for the case $(4,4)$. We provide a complete description of the differences between attainable values of psd forms and sos forms on the intersection points z_i. Let $E : H_{n,2d} \to \mathbb{R}^m$

be the evaluation map, sending $p \in H_{n,2d}$ to its values on z_i:

$$E(p) = (p(z_1), \dots, p(z_m)).$$

Here $m = 9$ for the case $(3,6)$ and $m = 8$ for the case $(4,4)$. Let $\mathbb{R}^m_+$ be the nonnegative orthant of $\mathbb{R}^m$, and let $\mathbb{R}^m_{++}$ denote the (open) strictly positive orthant. Let P' and Sq' be the images of $P_{n,2d}$ and $\Sigma_{n,2d}$ under E. We show that in the cases $(3,6)$ and $(4,4)$ with z_i coming from any completely real transverse intersection of two cubics or three quadratics the image P' of $P_{n,2d}$ contains the positive orthant $\mathbb{R}^m_{++}$. In other words any combination of strictly positive values on the points z_i is realizable by psd forms. However the Cayley-Bacharach relation forces restrictions on values of sos forms. We show the following (Theorem 3.4):

Theorem 1.5. *We can choose affine representatives $z_1, \dots, z_m$ for the projective points γ_i so that the image Sq' of the sos cone $\Sigma_{n,2d}$ under E is given by*

$$Sq' = \left\{ (x_1, \dots, x_m) \in \mathbb{R}^m_+ \quad \middle| \quad \sum_{i=1}^m \sqrt{x_i} \geq 2\sqrt{x_k} \quad \textit{for all} \quad k \right\}.$$

If we intersect the images P' and Sq' with the hyperplane

$$L = \left\{ x \in \mathbb{R}^m \quad \middle| \quad \sum_{i=1}^m x_i = 1 \right\},$$

then $P' \cap L$ is essentially just a simplex since $\mathbb{R}^m_{++} \subset P'$, while Sq' is a simplex with cut-off corners.

The proofs for the main theorems are obtained by analyzing the dual cone $\Sigma^*_{n,2d}$. Let K be a convex cone in a real vector space V. Its dual cone K^* is defined as the set of all linear functionals in the dual space V^* that are nonnegative on K:

$$K^* = \{\ell \in V^* \quad | \quad \ell(x) \geq 0 \text{ for all } x \in K\}.$$

Let us consider the dual space $H^*_{n,2d}$ of linear functionals on $H_{n,2d}$. To every linear functional $\ell \in H^*_{n,2d}$ we can associate a quadratic form Q_ℓ defined on $H_{n,d}$ by setting

$$Q_\ell(f) = \ell(f^2) \text{ for all } f \in H_{n,d}.$$

We classify the extreme rays of the dual cone $\Sigma^*_{n,2d}$ which provides us with the description of all linear inequalities that define the sos cone. We prove the following theorems, which we think are interesting in themselves:

Theorem 1.6. *Suppose that ℓ spans an extreme ray of $\Sigma^*_{3,6}$. Then* $\operatorname{rank} Q_\ell = 1$ *or* $\operatorname{rank} Q_\ell = 7$.

Also, for the case $(4,4)$ we have the following:

Theorem 1.7. *Suppose that ℓ spans an extreme ray of $\Sigma^*_{4,4}$. Then* $\operatorname{rank} Q_\ell = 1$ *or* $\operatorname{rank} Q_\ell = 6$.

We remark that in real analysis the functionals $\ell \in H^*_{n,2d}$ are represented by their values on the monomial basis and are called *truncated moment sequences*. The matrix of the quadratic form Q_ℓ, when written with respect to the monomial basis of $H_{n,d}$, has several names: it is called *the moment matrix*, or *generalized Hankel matrix*, in real analysis, and it is called the *symmetric catalecticant matrix* in algebraic geometry. We prefer to keep a basis-free approach, but our results have interesting consequences when stated in terms of moment terminology.

2. Dual cones

Let $S_{n,d}$ be the vector space of real quadratic forms on $H_{n,d}$. We can view the dual space $H^*_{n,2d}$ as a subspace of $S_{n,d}$ by identifying the linear functional $\ell \in H^*_{n,2d}$ with its quadratic form Q_ℓ defined by $Q_\ell(f) = \ell(f^2)$. If we choose the basis of monomials for $H_{n,2d}$, then $H^*_{n,2d}$ is identified with the subspace of generalized Hankel matrices in $S_{n,d}$ [13]. However, it is advantageous in our approach to not work with a fixed basis.

Let $S^+_{n,d}$ be the cone of positive semidefinite forms in $S_{n,d}$:

$$S^+_{n,d} = \{Q \in S_{n,d} \mid Q(f) \geq 0 \quad \text{for all} \quad f \in H_{n,d}\}.$$

The following lemma is a well-known connection between $\Sigma^*_{n,2d}$ and $S^+_{n,d}$ that allows sums of squares problems to be solved by semidefinite programming. Viewed with the monomial basis, it says that $\Sigma^*_{n,2d}$ is the intersection of $S^+_{n,d}$ with the subspace of generalized Hankel matrices; thus $\Sigma^*_{n,2d}$ is the Hankel spectrahedron.

Lemma 2.1. *The cone $\Sigma^*_{n,2d}$ is the section of the cone of psd matrices $S^+_{n,d}$ with the subspace $H^*_{n,2d}$:*

$$\Sigma^*_{n,2d} = S^+_{n,d} \cap H^*_{n,2d}.$$

Proof. Suppose that $\ell \in \Sigma^*_{n,2d}$. Then $\ell(f^2) \geq 0$ for all $f \in H_{n,d}$. By definition of Q_ℓ we see that it must be psd. Thus $\Sigma^*_{n,2d} \subseteq S^+_{n,d} \cap H^*_{n,2d}$.

Now suppose that $Q_\ell \in S^+_{n,d} \cap H^*_{n,2d}$. Then it follows that $\ell(f^2) \geq 0$ for all $f \in H_{n,d}$ and thus $\ell \in \Sigma^*_{n,2d}$. Thus $S^+_{n,d} \cap H^*_{n,2d} \subseteq \Sigma^*_{n,2d}$ and the lemma follows. $\square$

We now need a general lemma about extreme rays of sections of the cone of positive semidefinite forms. Let S be the vector space of quadratic forms on a real vector space V. Let S^+ be the cone of psd forms in S. The following lemma is from [12] (Corollary 4), we provide a proof for completeness:

Lemma 2.2. *Let L be a linear subspace of S and let K be the section of S^+ with L:*

$$K = S^+ \cap L.$$

Suppose that a quadratic form Q spans an extreme ray of K. Then the kernel of Q is maximal for all quadratic forms in L: if $P \in L$ and $\ker Q \subseteq \ker P$, then $P = \lambda Q$ for some $\lambda \in \mathbb{R}$.

Proof. Suppose not, so that there exists an extreme ray Q of K and a quadratic form $P \in L$ such that $\ker Q \subseteq \ker P$ and $P \neq \lambda Q$. Since $\ker Q \subseteq \ker P$, it follows that all eigenvectors of both Q and P corresponding to nonzero eigenvalues lie in the orthogonal complement $(\ker Q)^\perp$ of $\ker Q$. Furthermore, Q is positive definite on $(\ker Q)^\perp$.

It follows that Q and P can be simultaneously diagonalized to matrices Q' and P' with the additional property that whenever the diagonal entry Q'_{ii} is 0, the corresponding entry P'_{ii} is also 0. Therefore, for sufficiently small $\epsilon \in \mathbb{R}$ we have that $Q + \epsilon P$ and $Q - \epsilon P$ are positive semidefinite and therefore $Q + \epsilon P, Q - \epsilon P \in K$. Thus Q is not an extreme ray of K, which is a contradiction. $\square$

Combining Lemma 2.1 and Lemma 2.2, we obtain the following corollary, which will be a critical tool for describing the extreme rays of $\Sigma^*_{n,2d}$:

Corollary 2.3. *Suppose that Q spans an extreme ray of $\Sigma^*_{n,2d}$. Then either* $\operatorname{rank} Q = 1$ *or the forms in the kernel of Q have no common projective zeroes, real or complex.*

Proof. Let $W \subset H_{n,d}$ be the kernel of Q and suppose that the forms in W have a common real zero $v \neq 0$. Let $\ell \in H^*_{n,2d}$ be the linear functional given by evaluation at v: $\ell(f) = f(v)$ for all $f \in H_{n,2d}$. Then Q_ℓ is a rank 1 positive semidefinite quadratic form and $\ker Q \subseteq \ker Q_\ell$. By Lemma 2.2 it follows that $Q = \lambda Q_\ell$ and thus Q has rank 1.

Now suppose that the forms in W have a common nonreal zero $z \neq 0$. Let $\ell \in H^*_{n,2d}$ be the linear functional given by taking the real part of the value at z: $\ell(f) = \operatorname{Re} f(z)$ for all $f \in H_{n,2d}$. It is easy to check that the kernel of Q_ℓ includes all forms that vanish at z and therefore $W \subseteq \ker Q_\ell$. Therefore by applying Lemma 2.2 we again see that $Q = \lambda Q_\ell$. However, we claim that Q_ℓ is not a psd form.

The quadratic form Q_ℓ is given by $Q_\ell(f) = \operatorname{Re} f^2(z)$ for $f \in H_{n,d}$. However, there exists $f \in H_{n,d}$ such that $f(z)$ is purely imaginary and therefore $Q_\ell(f) < 0$. The corollary now follows. □

We note that if we can find a nonzero psd quadratic form Q_ℓ such that the forms in its kernel W_ℓ have no common real zeroes, then ℓ will indeed provide a linear inequality that holds for all sos forms but fails for some psd forms. Since Q_ℓ is psd, we know that $\ell \in \Sigma^*_{n,2d}$ and we need to construct a nonnegative $f \in H_{n,2d}$ such that $\ell(f) < 0$. Since forms in W_ℓ have no common real zeroes, we can find $f_i \in W_\ell$ such that $q = \sum_i f_i^2$ is strictly positive. We have $Q_\ell(f_i) = \ell(f_i^2) = 0$ for all i. Therefore $\ell(q) = 0$ and q is strictly positive on the unit sphere. For sufficiently small $\epsilon > 0$ we know that $f = q - \epsilon(x_1^2 + \ldots + x_n^2)^d$ is nonnegative. On the other hand we have $\ell(f) = -\epsilon\ell((x_1^2 + \ldots + x_n^2)^d) < 0$.

We will also need the following classification of all rank 1 forms in $H^*_{n,2d}$. For $v \in \mathbb{R}^n$ let ℓ_v be the linear functional in $H^*_{n,2d}$ given by evaluation at v:

$$\ell_v(f) = f(v) \text{ for } f \in H_{n,2d},$$

and let Q_v be the quadratic form associated to ℓ_v: $Q_v(f) = f^2(v)$. In this case we say that Q_v (or ℓ_v) corresponds to point evaluation. Note that the inequalities $\ell_v \geq 0$ are the defining inequalities of $P^*_{n,2d}$. The following lemma shows that all rank 1 forms in $H^*_{n,2d}$ correspond to point evaluations. Since we are interested in the inequalities that are valid on $\Sigma_{n,2d}$ but not valid on $P_{n,2d}$, it allows us to disregard rank 1 forms $Q \in H^*_{n,2d}$.

Lemma 2.4. *Suppose that Q is a rank 1 form in $H^*_{n,2d}$. Then $Q = \lambda Q_v$ for some $v \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.*

Proof. Let Q be a rank 1 form in $H^*_{n,2d}$. Then $Q(f) = \lambda s^2(f)$ for some linear functional $s \in H^*_{n,d}$. Therefore it suffices to show that if $Q = s^2(f)$ for some $s \in H^*_{n,d}$, then $Q = Q_v$ for some $v \in \mathbb{R}^n$.

Since $Q \in H^*_{n,d}$, we know that Q is defined by $Q(f) = \ell(f^2)$ for a linear functional $\ell \in H^*_{n,2d}$ and therefore $\ell(f^2) = s^2(f)$ for all $f \in H_{n,d}$. We have $Q(f+g) = \ell((f+g)^2) = \ell(f^2) + 2\ell(fg) + \ell(g^2) = (s(f)+s(g))^2 = s^2(f) + 2s(f)s(g) + s^2(g)$ and it follows that $\ell(fg) = s(f)s(g)$ for all $f, g \in H_{n,d}$.

Let x^α denote the monomial $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. If we take monomials x^α, x^β, x^γ, x^δ in $H_{n,d}$ such that $x^\alpha x^\beta = x^\gamma x^\delta$, then we must have $s(x^\alpha)s(x^\beta) = s(x^\gamma)s(x^\delta)$.

Suppose that $s(x_i^d) = 0$ for all i. Then we see that $s(x_i^{d-1}x_j)^2 = s(x_i^d)s(x_i^{d-2}x_j^2) = 0$ and continuing in similar fashion we have $s(x^\alpha) = 0$ for all monomials. Then ℓ is the zero functional and Q does not have rank 1, a contradiction.

We may assume without loss of generality that $s(x_1^d) \neq 0$. Since we are interested in $\ell(f^2) = s^2(f)$, we can work with $-s$, if necessary, and thus we may assume that $s(x_1^d) > 0$. Let $s_i = s(x_1^{d-1}x_i)$ for $1 \leq i \leq n$. We will express $s(x^\alpha)$ in terms of s_i for all $x^\alpha \in H_{n,d}$. Since $(x_1^d)(x_1^{d-2}x_ix_j) = (x_1^{d-1}x_i)(x_1^{d-1}x_j)$, we have $s(x_1^{d-2}x_ix_j) = s_is_j/s_1$. Continuing in this fashion, we find that

$$s(x_1^{\alpha_1}\cdots x_n^{\alpha_n}) = \frac{s_2^{\alpha_2}\cdots s_n^{\alpha_n}}{s_1^{d-1-\alpha_1}}.$$

Now let $v \in \mathbb{R}^n$ be the vector

$$v = (s_1^{1/d}, s_1^{-(d-1)/d}s_2, \ldots, s_1^{-(d-1)/d}s_n).$$

Let s_v be the linear operator on $H_{n,d}$ defined by evaluating a form at v: $s_v(f) = f(v)$. Then we have $s_v(x_1^{d-1}x_i) = s_i$ and

$$s_v(x_1^{\alpha_1}\cdots x_n^{\alpha_n}) = s_2^{\alpha_2}\cdots s_n^{\alpha_n}s_1^{\alpha_1/d-(d-1)(d-\alpha_1)/d} = \frac{s_2^{\alpha_2}\cdots s_n^{\alpha_n}}{s_1^{d-1-\alpha_1}}.$$

Since s agrees with s_v on monomials, it follows that $s = s_v$ and thus $\ell(f^2) = s^2(f) = f(v)^2 = f^2(v)$. Therefore ℓ indeed corresponds to point evaluation and we are done. $\square$

2.1. **Kernels of extreme rays.** Let Q_ℓ span an extreme ray of $\Sigma_{n,2d}^*$ that does not correspond to point evaluation. Let W_ℓ be the kernel of Q_ℓ and let $J(\ell)$ be the ideal generated by W_ℓ. By Corollary 2.3 and Lemma 2.4 we know that the forms in W_ℓ have no common projective zeroes real or complex, i.e. $\mathcal{V}_{\mathbb{C}}(W_\ell) = \emptyset$. We now investigate the kernel W_ℓ further.

Forms $p_1, \ldots, p_n \in H_{n,d}$ are said to form a *sequence of parameters* if they have no common projective complex zeroes:

$$\mathcal{V}_{\mathbb{C}}(p_1, \ldots, p_n) = \emptyset.$$

It follows that we can find a sequence of parameters $p_1, \ldots, p_n \in W_\ell$. Let I be the ideal generated by $p_1, \ldots, p_n$. We will need the following theorem (special case of [6, Theorem CB8]):

Theorem 2.5. *Suppose that $p_1, \ldots, p_n \in H_{n,d}$ are a sequence of parameters and let I be the ideal generated by $p_1, \ldots, p_n$ in $\mathbb{C}[x_1, \ldots, x_n]$. Then I is a Gorenstein ideal with socle of degree $n(d-1)$.*

We also prove a simple but very useful characterization of kernels of the forms $Q_\ell \in H_{n,2d}^*$:

Lemma 2.6. *Let Q_ℓ be a quadratic form in $H_{n,2d}^*$. Then $p \in W_\ell$ if and only if $\ell(pq) = 0$ for all $q \in H_{n,d}$.*

Proof. In order to investigate W_ℓ, we need to define the associated bilinear form B_ℓ:

$$B_\ell(p, q) = \frac{Q_\ell(p+q) - Q_\ell(p) - Q_\ell(q)}{2} \quad \text{for } p, q \in H_{n,d}.$$

By definition of Q_ℓ we have $Q_\ell(p) = \ell(p^2)$. Therefore it follows that

$$B_\ell(p,q) = \ell(pq). \tag{2.1}$$

A form $p \in H_{n,d}$ is in the kernel of Q_ℓ if and only if $B_\ell(p,q) = 0$ for all $q \in H_{n,d}$. Using (2.1), the lemma follows. □

We are now in a position to prove Theorems 1.6 and 1.7, which we restate in a unified way:

Theorem 2.7. *Suppose that ℓ is an extreme ray of $\Sigma^*_{n,2d}$ in the cases $(3,6)$ and $(4,4)$ and that ℓ does not correspond to point evaluation. Then the rank of Q_ℓ is equal to $\dim H_{n,d} - n$.*

Proof. Let $p_1,\dots,p_n$ be a sequence of parameters in W_ℓ and let I be the ideal generated by $p_1,\dots,p_n$. We claim that $W_\ell = I_d$, or in other words, linear combinations of $p_1,\dots,p_n$ generate W_ℓ. We note that this claim implies the desired theorem, since it shows that the kernel of Q_ℓ has dimension exactly n.

By Theorem 2.5 we know that the socle of I has degree $n(d-1) = 2d$ in the cases $(3,6)$ and $(4,4)$. Suppose that W_ℓ is strictly larger than I_d. The ideal I is Gorenstein with socle of degree $2d$, and hence $J(\ell)_{2d}$ is strictly larger than I_{2d}, which means that $J(\ell)_{2d} = H_{n,2d}$.

It follows from Lemma 2.6 that

$$\ell(f) = 0 \qquad \text{for all} \qquad f \in J(\ell)_{2d}.$$

Therefore ℓ is the zero linear functional, which is a contradiction. □

Given an extreme ray Q_ℓ of $\Sigma^*_{n,2d}$ that does not correspond to point evaluation, we can pass to its kernel W_ℓ, and in the cases $(3,6)$ and $(4,4)$ the kernel W_ℓ has dimension exactly n and further $\mathcal{V}_{\mathbb{C}}(W_\ell) = \emptyset$. It follows from Theorem 2.5 that an n-dimensional subspace W with $\mathcal{V}_{\mathbb{C}}(W) = \emptyset$ uniquely determines (up to a constant multiple) the linear functional ℓ such that the kernel of Q_ℓ is W. The linear functional ℓ is the unique linear functional vanishing on the degree $2d$ part $\langle W \rangle_{2d}$ of the ideal generated by W. This correspondence is a special case of the *global residue map* [4, §1.6].

Therefore instead of directly studying the extreme rays ℓ of $\Sigma^*_{n,2d}$, we can look instead for n-dimensional subspaces W of $H_{n,d}$, with $\mathcal{V}_{\mathbb{C}}(W) = \emptyset$, whose corresponding linear functionals are extreme rays of $\Sigma^*_{n,2d}$. The linear functionals $\ell \in \Sigma^*_{n,2d}$ have the defining property of being nonnegative on squares. In order to see when a subspace W of $H_{n,d}$ gives rise to an extreme ray of $\Sigma^*_{n,2d}$, we need to get a handle on the linear functional $\ell \in H^*_{n,2d}$ that W defines. We do this by passing to point evaluations. We need the following general lemma, which allows us to extract a transverse 0-dimensional intersection from forms in W.

Lemma 2.8. *Suppose that $p_1,\dots,p_n \in H_{n,d}$ are a sequence of parameters. Then there exist $f_1,\dots,f_{n-1}$ in the real linear span of p_i such that the forms $f_1,\dots,f_{n-1}$ intersect transversely in d^{n-1} (possibly complex) points.*

Proof. Let W be the linear span of $p_1,\dots,p_n$ with complex coefficients. We begin by showing that there exist linear combinations $f_1,\dots,f_{n-1} \in W$ such that $f_1,\dots f_{n-1}$ intersect transversely in $\mathbb{CP}^{n-1}$.

By Bertini's theorem a general form in W is smooth. Let f_1 be such a form. Let V_1 be the smooth variety defined by f_1 and let W_1 be a subspace of W complementary to f_1. Then W_1 defines a linear system of divisors on V_1 and by Bertini's theorem the intersection of V_1 with a general element of W_1 is a smooth variety of dimension $n-2$. Let f_2 be such an element of W_1. Now we can let V_2 be the smooth variety defined by f_1 and f_2, let W_2 be the complementary subspace to f_1 and f_2, and repeatedly apply Bertini's theorem until we get a 0-dimensional smooth intersection. Hence the forms $f_1, \dots, f_{n-1}$ that we constructed intersect transversely.

Now we argue that there exist *real* linear combinations $f_1, \dots, f_{n-1}$ which intersect transversely. Suppose not and let $f_i = \sum_{j=1}^{n} \alpha_{ij} p_j$. Then for all $\alpha_{ij} \in \mathbb{R}$ the forms f_i do not intersect transversely. This is an algebraic condition on the coefficients α_{ij}, given by vanishing of some polynomials in the variables α_{ij}. However, if a polynomial vanishes on all real points, then it must be identically zero. Therefore, no complex linear combinations of p_i intersect transversely, which is a contradiction. $\square$

Now suppose that we have an n-dimensional subspace W of $H_{n,d}$ with $\mathcal{V}_{\mathbb{C}}(W) = \emptyset$ and we locate $f_1, \dots, f_{n-1} \in W$ that intersect transversely. Let $s = d^{n-1}$ and let Γ be the complex projective variety defined by $f_1, \dots, f_{n-1}$:

$$\Gamma = \mathcal{V}_{\mathbb{C}}(f_1, \dots, f_{n-1}) = \{\gamma_1, \dots, \gamma_s\} \subset \mathbb{CP}^{n-1}.$$

Let $S = \{z_1, \dots, z_s\}$ be a set of affine representatives for projective points γ_i. The functional ℓ determined by W is the unique linear functional vanishing on $\langle W \rangle_{2d}$. In particular ℓ vanishes on $\langle f_1, \dots, f_{n-1} \rangle_{2d}$. Since the forms $f_1, \dots, f_{n-1}$ intersect transversely, the ideal generated by f_i is radical [8]. It follows therefore that ℓ can be expressed as a linear combination of evaluations at points v_i:

$$\ell = \sum_{i=1}^{s} \mu_i \ell_{v_i}; \qquad \ell(p) = \sum_{i=1}^{s} \mu_i p(v_i), \quad p \in H_{n,2d}.$$

The coefficients μ_i are determined uniquely from any form f_n so that $f_1, \dots, f_n$ form a basis of W. In order to see how this occurs, we need to introduce Cayley-Bacharach relations.

2.2. Cayley-Bacharach relations.

We now recall the Cayley-Bacharach theorem as applicable to ternary cubics and quaternary quadrics [6, Theorem CB6]:

Lemma 2.9. *For the cases* $(3,6)$ *and* $(4,4)$ *let* $f_1, \dots, f_{n-1} \in H_{n,d}$ *be forms intersecting transversely in* $s = d^{n-1}$ *complex projective points* $\gamma_1, \dots, \gamma_s$. *Let* $z_1, \dots, z_s$ *be affine representatives of the projective points* γ_i. *Then there is a unique linear relation on the values of any form in* $H_{n,d}$ *on* z_i*:*

$$u_1 p(z_1) + \ldots + u_s p(z_s) = 0 \quad \textit{for all} \quad p \in H_{n,d},$$

with nonzero $u_i \in \mathbb{C}$.

As we will see later, evaluation on transverse intersections will be enough to distinguish nonnegative forms from sums of squares. Before we proceed with that, we would like to show explicitly the geometry of values on transverse intersections when all of the intersection points are real.

3. Cones of point evaluations

Since the geometry of the cases $(3,6)$ and $(4,4)$ is very similar, we will give a unified presentation. For these cases let $f_1, \dots, f_{n-1}$ be forms in $H_{n,d}$ intersecting transversely in $s = d^{n-1}$ real projective points $\gamma_1, \dots, \gamma_s$. Let $v_1, \dots, v_s \in \mathbb{R}^n$ be arbitrary nonzero affine representatives for $\gamma_1, \dots, \gamma_s$ with v_i corresponding to γ_i. Then by Lemma 2.9 there exists a unique linear relation

$$u_1 p(v_1) + \ldots + u_s p(v_s) = 0 \quad \text{for all} \quad p \in H_{n,d}$$

and since we have all points $v_i \in \mathbb{R}^n$ coming from the intersection of real forms, all the coefficients u_i must be real.

We first look at general real 0-dimensional intersections. Suppose that $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ is a set of real projective points that can be given as the complete set of common real zeroes of some forms $f_1, \dots, f_k \in H_{n,d}$:

$$\Gamma = \mathcal{V}_{\mathbb{R}}(f_1, \dots, f_k).$$

Let $S = \{v_1, \dots, v_m\} \subset \mathbb{R}^n$ be a set of affine representatives for γ_i. Let E be the evaluation map that sends $p \in H_{n,2d}$ to its values on the points v_i:

$$E : H_{n,2d} \longrightarrow \mathbb{R}^m, \qquad E(p) = (p(v_1), \dots, p(v_m)).$$

Note that E is defined on forms of degree $2d$. Let P' and Sq' be the images of $P_{n,2d}$ and $\Sigma_{n,2d}$ under E, respectively, and let H' be the image of $H_{n,2d}$. We observe that H' does not have to be all of $\mathbb{R}^m$, since the values of forms in $H_{n,2d}$ on points v_i may be linearly dependent. Since we are evaluating nonnegative forms, it follows that P' lies inside the intersection of H' and $\mathbb{R}^m_+$:

$$P' \subseteq H' \cap \mathbb{R}^m_+.$$

The following theorem shows that this inclusion is almost an equality.

Theorem 3.1. *Let $\mathbb{R}^m_{++}$ be the positive orthant of $\mathbb{R}^m$. The intersection of H' with the positive orthant is contained in P':*

$$H' \cap \mathbb{R}^m_{++} \subset P'.$$

Proof. Let $s = (s_1, \dots, s_m)$ be a point in the intersection of H' and $\mathbb{R}^m_{++}$. Since $s \in H$, there exists a form $p \in H_{n,2d}$ such that $p(v_i) = s_i$. Let $g = f_1^2 + \ldots + f_k^2$. We claim that for large enough $\lambda \in \mathbb{R}$ the form $\bar{p} = p + \lambda g$ will be nonnegative, and since each f_i is zero on S, we will also have $E(\bar{p}) = s$.

By homogeneity of $\bar{p}$ it suffices to show that it is nonnegative on the unit sphere $\mathbb{S}^{n-1}$. Furthermore, we may assume that the evaluation points v_i lie on the unit sphere. Since we are dealing with forms, evaluation on the points outside of the unit sphere amounts to rescaling of the values on $\mathbb{S}^{n-1}$.

Let $B_\epsilon(S)$ be the open epsilon neighborhood of S in the unit sphere $\mathbb{S}^{n-1}$. Since $p(v_i) > 0$ for all i, it follows that for sufficiently small ϵ the form p is strictly positive on $B_\epsilon(S)$:

$$p(x) > 0 \quad \text{for all} \quad x \in B_\epsilon(S).$$

The complement of $B_\epsilon(S)$ in $\mathbb{S}^{n-1}$ is compact, and therefore we can let m_1 be the minimum of g and m_2 be the minimum of p on $\mathbb{S}^{n-1} \setminus B_\epsilon(S)$. If $m_2 \geq 0$, then p itself was nonnegative and we are done. Therefore, we may assume $m_2 < 0$. We also note that since g vanishes on S only, it follows that m_1 is strictly positive.

Now let $\lambda \geq -\frac{m_2}{m_1}$. The form $\bar{p} = p + \lambda g$ is positive on $B_\epsilon(S)$. By construction of $B_\epsilon(S)$ we also see that the minimum of $\bar{p}$ on the complement of $B_\epsilon(S)$ is at least 0. Therefore $\bar{p}$ is nonnegative on the unit sphere $\mathbb{S}^{n-1}$ and we are done. □

We obtain the following corollary for the cases $(3, 6)$ and $(4, 4)$:

Corollary 3.2. *Suppose that Γ comes from the transverse intersection of two ternary cubics (resp. three quaternary quadrics). Then $\mathbb{R}^9_{++} \subset P'$ (resp. $\mathbb{R}^8_{++} \subset P'$).*

Proof. We need to show that in our two cases the cone P' is full dimensional. This happens if and only if the values on the points v_i are linearly independent for forms in $H_{3,6}$ (resp. $H_{4,4}$). This is an easy special case of the Cayley-Bacharach theorem [6, Theorem CB6]. □

We now show how the presence of a Cayley-Bacharach relation impacts the values attainable by sums of squares. Suppose now that the points $v_1, \ldots, v_m \in \mathbb{R}^n$ are such that there exists a unique Cayley-Bacharach relation satisfied by all forms $p \in H_{n,d}$: $u_1 p(v_1) + \ldots + u_m p(v_m) = 0$ with nonzero coefficients $u_i \in \mathbb{R}$.

We first describe Sq' if the affine representatives are chosen so that the coefficients in the Cayley-Bacharach relation have absolute value 1. Let $w_i = |u_i|^{1/d} v_i$. Then $p(w_i) = |u_i| p(v_i)$ for all $f \in H_{n,d}$. Thus we see that the values of forms in $H_{n,d}$ on w_i satisfy a unique relation $a_1 f(w_1) + \ldots + a_m f(w_m) = 0$ with $a_i = \pm 1$. Now redefine the map E using this particular set of affine representatives w_i and let Sq' be the image of $\Sigma_{n,2d}$ under E.

Let T_m be the subset of the nonnegative orthant $\mathbb{R}^m_+$ defined by the following m inequalities:

$$T_m = \left\{ (x_1, \ldots, x_m) \in \mathbb{R}^m_+ \quad \middle| \quad \sum_{i=1}^{m} \sqrt{x_i} \geq 2\sqrt{x_k} \quad \text{for all} \quad k \right\}.$$

Lemma 3.3. *The set T_m is a closed convex cone. Moreover, T_m is the convex hull of the points $x = (x_1, \ldots, x_m) \in \mathbb{R}^m_+$ where $\sum_{i=1}^m \sqrt{x_i} = 2\sqrt{x_k}$ for some k.*

Proof. T_m is defined as a subset of $\mathbb{R}^m$ by the following $2m$ inequalities: $x_k \geq 0$ and $\sqrt{x_1} + \ldots + \sqrt{x_m} \geq 2\sqrt{x_k}$ for all k. Therefore it is clear that T_m is a closed set.

For $x = (x_1, \ldots, x_m) \in \mathbb{R}^m_+$ let $||x||_{1/2}$ denote the $L^{1/2}$ norm of x:

$$||x||_{1/2} = (\sqrt{x_1} + \ldots + \sqrt{x_m})^2.$$

We can restate inequalities of T_m as $x_k \geq 0$ and $||x||_{1/2} \geq 4x_k$ for all k. Now suppose that $x, y \in T_m$ and let $z = \lambda x + (1 - \lambda)y$ for some $0 \leq \lambda \leq 1$. It is clear that $z_k \geq 0$ for all k. It is known by the Minkowski inequality ([7], p. 30) that the $L^{1/2}$ norm is a concave function: $||\lambda x + (1-\lambda)y||_{1/2} \geq \lambda\, ||x||_{1/2} + (1-\lambda)\, ||y||_{1/2}$. Therefore

$$||z||_{1/2} \geq \lambda\, ||x||_{1/2} + (1-\lambda)\, ||y||_{1/2} \geq 4\lambda x_k + 4(1-\lambda) y_k = 4z_k.$$

Therefore T_m is a convex cone.

To show that T_m is the convex hull of the points where $||x||_{1/2} = 4x_k$ for some k, we proceed by induction. The base case $m = 2$ is simple since T_2 is just a ray spanned by the point $(1, 1)$. For the induction step we observe that any convex set is the convex hull of its boundary. For any point in the boundary of T_m one of the defining $2m$ inequalities must be sharp. If a point x is in the boundary of T_m and

$x_i \neq 0$ for all i, then the inequalities $x_i \geq 0$ are not sharp at x and therefore the inequality $||x||_{1/2} \geq 4x_k$ must be sharp for some k and we are done.

If $x_i = 0$ for some i, then the point x lies in the set T_{m-1} in the subspace spanned by the $m-1$ standard basis vectors excluding e_i and we are done by induction. □

Theorem 3.4. *With the choices of affine representatives w_i, so that the coefficients in the unique Cayley-Bacharach relation are of absolute value 1, we have $Sq' = T_m$.*

Proof. By slight abuse of notation we will also use E as the evaluation map at w_i for forms in $H_{n,d}$. Let $a = (a_1, \dots, a_m)$ be the vector of coefficients in the Cayley-Bacharach relation $a_1 p(w_1) + \ldots + a_m p(w_m) = 0$ with $a_i = \pm 1$ and $f \in H_{n,d}$.

By uniqueness of the Cayley-Bacharach relation we know that $L = E(H_{n,d})$ is the hyperplane in $\mathbb{R}^m$ perpendicular to a. To show that $Sq' \subseteq T_m$, it suffices to show that $E(q^2) \in T_m$ for any $q \in H_{n,d}$. Let $s = E(q)$ and $t = E(q^2)$. We know that $E(q^2) = (t_1, \dots, t_m) = (s_1^2, \dots, s_m^2)$. By the Cayley-Bacharach relation we have $a_1 s_1 + \ldots + a_m s_m = 0$ with $a_i = \pm 1$. Without loss of generality, we may assume that s_1 has the maximal absolute value among the s_i. Multiplying the Cayley-Bacharach relation by -1, if necessary, we can make $a_1 = -1$. Then we have $s_1 = a_2 s_2 + \ldots + a_m s_m$. We can now write $\sqrt{t_1} = \pm\sqrt{t_2} \pm \sqrt{t_3} \pm \ldots \pm \sqrt{t_m}$ with the exact signs depending on a_i and signs of s_i. Therefore we see that $2\sqrt{t_1} \leq \sqrt{t_1} + \ldots + \sqrt{t_m}$. Since s_1 had the largest absolute value among the s_i, it follows that $Sq' \subseteq T_m$.

To show the reverse inclusion $T_m \subseteq Sq'$, we use Lemma 3.3. It suffices to show that all points in $x \in T_m$ with $2\sqrt{x_k} = \sqrt{x_1} + \ldots + \sqrt{x_m}$ for some k are also in Sq'. Without loss of generality we may assume that $k = 1$ and we have $\sqrt{x_1} = \sqrt{x_2} + \ldots + \sqrt{x_m}$. Let $y = (y_1, \dots, y_m)$ with $y_1 = -\sqrt{x_1}/a_1$ and $y_i = \sqrt{x_i}/a_i$ for $2 \leq i \leq m$. It follows that $a_1 y_1 + \ldots + a_m y_m = 0$. Therefore $y \in E(H_{n,d})$ and $y = E(q)$ for some quadratic form q. Then $E(q^2) = x$ and we are done. □

Note that this already proves Hilbert's theorem that there exist nonnegative polynomials that are not sums of squares for the cases $(3, 6)$ and $(4, 4)$. In fact Hilbert's proof, by different methods, established that the standard basis vectors are not in Sq', while we provide a complete description of Sq'.

We now describe what happens if we do not rescale the affine representatives and the coefficients in the Cayley-Bacharach relation are arbitrary real numbers. Suppose that the unique Cayley-Bacharach relation satisfied by all forms $f \in H_{n,d}$ is given by $u_1 f(v_1) + \ldots + u_m f(v_m) = 0$ with nonzero coefficients u_i. Let E be the evaluation map at the points v_i and let Sq' be the image of $\Sigma_{n,2d}$ under E.

Corollary 3.5. *The cone Sq' is the subset of $\mathbb{R}^m_+$ satisfying the following m inequalities:*

$$|u_1|\sqrt{x_1} + \ldots + |u_m|\sqrt{x_m} \geq 2|u_k|\sqrt{x_k},$$

for all $1 \leq k \leq m$.

Proof. Let $a \in \mathbb{R}^m$ be a vector with $a_i = u_i/|u_i|$ and let D be the diagonal $m \times m$ matrix with $D_{ii} = |u_i|$. Let L_a be the hyperplane of vectors in $\mathbb{R}^m$ perpendicular to a and let L_u be the hyperplane of vectors perpendicular to u. The linear transformation $\bar{D}$ sending $x \in \mathbb{R}^m$ to Dx sends L_u to L_a.

Since the Cayley-Bacharach relation is unique, it follows that Sq' is the convex hull of the points $(y_1^2, \dots, y_m^2)$ with $y = (y_1, \dots, y_m) \in L_u$. We have shown in

Theorem 3.4 that the convex hull of squares from L_a is T_m. Since $\bar{D}$ sends L_u to L_a, it follows that $\bar{D}^2$ sends Sq' to T_m.

By Lemma 3.3 we know that T_m is the set of $x \in \mathbb{R}^m_+$ satisfying inequalities $\sqrt{x_1} + \ldots + \sqrt{x_m} \geq 2\sqrt{x_k}$ for all $1 \leq k \leq m$. Now suppose that $x = D^2 y$ with $x \in T_m$ and $y \in Sq'$. Then $x_i = |u_i|^2 y_i$ and it follows that y satisfies $|u_1|\sqrt{x_1} + \ldots + |u_m|\sqrt{x_m} \geq 2|u_k|\sqrt{x_k}$ for all $1 \leq k \leq m$. Since D^2 is an invertible linear transformation ($u_i \neq 0$), it follows that all y satisfying these inequalities are in Sq'. □

4. Structure of extreme rays of $\Sigma^*_{3,6}$ and $\Sigma^*_{4,4}$

We now return to the study of extreme rays of $\Sigma^*_{n,2d}$ for the cases $(3, 6)$ and $(4, 4)$. Let W be an n-dimensional linear subspace of $H_{n,d}$ with $\mathcal{V}_{\mathbb{C}}(W) = \emptyset$. Let $f_1, \ldots, f_{n-1} \in W$ be forms intersecting transversely in $s = d^{n-1}$ points $\gamma_1, \ldots, \gamma_s$. Let $z_1, \ldots, z_s$ be affine representatives for points γ_i. By Lemma 2.9 there is a unique linear relation for values of forms in $H_{n,d}$ on the points z_i: $u_1 f(z_1) + \cdots + u_s f(z_s) = 0$, for all $f \in H_{n,d}$. The unique (up to a constant multiple) linear functional ℓ vanishing on $\langle W \rangle_{2d}$ can be written as a linear combination of point evaluations on the points z_i:

$$\ell = \sum_{i=1}^{s} \mu_i \ell_{z_i}, \quad \text{for some} \quad \mu_i \in \mathbb{C}.$$

Let f_n be a form in W such that $f_1, \ldots, f_n$ form a basis of W. Note that the values $f_n(z_i)$ are the same regardless of which $f_n \in W$ we choose. We now explain how to determine the coefficients μ_i from the knowledge of the Cayley-Bacharach coefficients u_i and the values $f_n(z_i)$.

Lemma 4.1.

$$\mu_i = \frac{u_i}{f_n(z_i)}; \qquad \ell = \sum_{i=1}^{s} \frac{u_i}{f_n(z_i)} \ell_{z_i} \quad i = 1, \ldots, s.$$

Proof. Let ℓ be defined as above. We need to show that ℓ vanishes on all forms in $\langle W \rangle_{2d}$. We observe that for any form $q \in H_{n,d}$ we have

$$\ell(f_1 q) = \cdots = \ell(f_{n-1} q) = 0$$

since ℓ is defined by values at common zeroes of $f_1, \ldots, f_{n-1}$. Also

$$\ell(f_n q) = \sum_{i=1}^{s} \frac{u_i}{f_n(z_i)} f_n(z_i) q(z_i) = \sum_{i=1}^{s} u_i q(z_i) = 0,$$

by the Cayley-Bacharach relation. □

Since the forms $f_1, \ldots, f_{n-1}$ are real, the set $\Gamma = \{\gamma_1, \ldots, \gamma_s\}$ is invariant under conjugation. Hence we can choose affine representatives z_i so that the set $S = \{z_1, \ldots, z_s\}$ is invariant under conjugation. By uniqueness of the Cayley-Bacharach relation it follows that if $z_i = \bar{z}_j$, then $u_i = \bar{u}_j$. We now show that if the functional ℓ is nonnegative on squares, then we can restrict the number of complex points z_i, forcing most of the intersection points to be real.

4.1. **Number of complex points.** Suppose that S is a finite set of points in $\mathbb{C}^n$ that is invariant under conjugation: $\bar{S} = S$. Let S be given by $S = \{r_1, \dots, r_k, z_1, \dots, z_m, \bar{z}_1, \dots, \bar{z}_m\}$ with $r_i \in \mathbb{R}^n$ and $z_i, \bar{z}_i \in \mathbb{C}^n$, $z_i \neq \bar{z}_i$. Let $\ell : H_{n,2d} \to \mathbb{R}$ be a linear functional given as a combination of evaluations on S:

$$\ell(p) = \sum_{i=1}^{k} \lambda_i p(r_i) + \sum_{i=1}^{m} \left(\mu_i p(z_i) + \bar{\mu}_i p(\bar{z}_i)\right), \quad p \in H_{n,2d},$$

with $\lambda_i \in \mathbb{R}$, $\mu_i \in \mathbb{C}$, and $\lambda_i, \mu_i \neq 0$.

Let $E_{\mathbb{R}} : H_{n,d} \to \mathbb{R}^{k+2m}$ be the real evaluation projection of forms in $H_{n,d}$ given by

$$E_{\mathbb{R}}(p) = \left(p(r_1), \dots, p(r_k), \operatorname{Re} p(z_1), \operatorname{Im} p(z_1), \dots, \operatorname{Re} p(z_m), \operatorname{Im} p(z_m)\right), \quad p \in H_{n,d}.$$

Let c be the dimension of the image of $E_{\mathbb{R}}$:

$$c = \dim E_{\mathbb{R}}(H_{n,d}).$$

Lemma 4.2. *Suppose that the quadratic form Q_ℓ is positive semidefinite. Then $c \leq k + m$.*

Proof. The quadratic form $Q_\ell : H_{n,d} \to \mathbb{R}$ is defined by

$$Q_\ell(q) = \sum_{i=1}^{k} \lambda_i q^2(r_i) + \sum_{i=1}^{m} \left(\mu_i q^2(z_i) + \bar{\mu}_i q^2(\bar{z}_i)\right), \quad q \in H_{n,d}.$$

Let $\bar{Q}_\ell$ be the quadratic form on $\mathbb{C}^{k+2m}$ given by

$$\sum_{i=1}^{k} \lambda_i x_i^2 + \sum_{i=1}^{m} \mu_i \left(x_{2i-1} + \sqrt{-1} x_{2i}\right)^2 + \bar{\mu}_i \left(x_{2i-1} - \sqrt{-1} x_{2i}\right)^2.$$

By its definition, the form Q_ℓ is a composition on $E_{\mathbb{R}}$ and $\bar{Q}_\ell$:

$$Q_\ell = \bar{Q}_\ell \circ E_{\mathbb{R}}.$$

Each of the two variable blocks $\mu_i \left(x_{2i-1} + \sqrt{-1} x_{2i}\right)^2 + \bar{\mu}_i \left(x_{2i-1} - \sqrt{-1} x_{2i}\right)^2$ has one positive and one negative eigenvalue, since $\mu_i \neq 0$. Therefore the form $\bar{Q}_\ell$ has at least m negative eigenvalues, and thus Q_ℓ is strictly negative on a subspace of dimension at least m.

Recall that the form Q_ℓ is positive semidefinite, which implies that $\bar{Q}_\ell$ is psd on the image of $E_{\mathbb{R}}$. Thus the image of $E_{\mathbb{R}}$ has codimension at least m and the lemma follows. $\square$

We can restate the lemma as follows: note that $|S| = k + 2m$ and $|S| - c$ is the number of linearly independent relations that evaluation on S imposes on forms in $H_{n,d}$. Hence we get the following corollary:

Corollary 4.3. *Suppose that Q_ℓ is positive semidefinite. Then the number of complex conjugate pairs in S is at most equal to the number of linearly independent relations on S for forms of degree d.*

Applying this to transversal intersections in the cases $(3, 6)$ and $(4, 4)$ we get

Corollary 4.4. *Suppose that ℓ is an extreme ray of $\Sigma^*_{n,2d}$ that does not correspond to point evaluation, and let $f_1, \dots, f_{n-1}$ be forms in the kernel W_ℓ of Q_ℓ intersecting transversely in $s = d^{n-1}$ points, $\Gamma = \{\gamma_1, \dots, \gamma_s\}$. Then the set Γ includes at most 1 complex conjugate pair and the rest of the points in Γ are real.*

5. Proofs of main theorems

We now prove Theorem 1.1 and Theorem 1.2 in a unified manner.

Proof of Theorems 1.1 *and* 1.2. Suppose that $p \in P_{n,2d}$ and p is not sos. Then there exists an extreme ray ℓ of $\Sigma^*_{n,2d}$ such that $\ell(p) < 0$ and $\ell(q) \geq 0$ for all $q \in \Sigma_{n,2d}$. Since $\ell(p) < 0$ and p is nonnegative, it follows that ℓ does not correspond to point evaluation. Let Q_ℓ be the quadratic from associated with ℓ and let W_ℓ be the kernel of Q_ℓ. Then by Theorem 2.7 we have $\dim W_\ell = n$ and by Lemma 2.8 we can find $f_1, \ldots, f_{n-1} \in W_\ell$ intersecting transversely in $s = d^{n-1}$ projective points $\gamma_1, \ldots, \gamma_s$. By Corollary 4.4 we know that at most two of the γ_i are complex and by Lemma 4.1 the linear functional ℓ has the desired form. □

Proof of Corollaries 1.3 *and* 1.4. Let p be a strictly positive form on the boundary of $\Sigma_{n,2d}$. Then there exists an extreme ray ℓ of the dual cone $\Sigma^*_{n,2d}$, such that $\ell(p) = 0$. Now suppose that $p = \sum f_i^2$ for some $f_i \in H_{n,d}$. It follows that $Q_\ell(f_i) = 0$ for all i and since Q_ℓ is a positive semidefinite quadratic form, we see that all the f_i lie in the kernel W_ℓ of Q_ℓ. By Theorem 2.7 we know that $\dim W_\ell = n$ and therefore p is a sum of squares of forms coming from an n-dimensional subspace of $H_{n,2d}$. It follows that p is a sum of at most n squares.

Now suppose that p is a sum of $n-1$ or fewer squares, $p = f_1^2 + \cdots + f_{n-1}^2$ with some f_i possibly zero. Since p is strictly positive, we know that the forms f_i have no common real zeroes. Therefore we found $n-1$ forms $f_i \in W_\ell$ that have no common real zeroes (if p were a sum of fewer than $n-1$ squares, then we could add arbitrary f_i to get their number up to $n-1$). By the proof of Lemma 2.8 we know that $n-1$ generic forms in W_ℓ intersect transversely, and hence we can find forms $f_i' \in W_\ell$ in a neighborhood of f_i such that the f_i' intersect transversely in d^{n-1} complex points. This is a contradiction by Corollary 4.4. □

We now examine the two cases of Corollary 4.4: the case of fully real intersection and the case of one complex conjugate pair. In each of these cases there exist psd forms Q_ℓ corresponding to extreme rays of $\Sigma^*_{n,2d}$. We provide explicit equations of these extreme rays, based on the Cayley-Bacharach relation, thus giving us a complete description of the extreme rays of $\Sigma^*_{n,2d}$.

6. Fully real intersection

We have already examined the difference between attainable values on fully real intersection for psd and sos forms in Section 3. Now we describe the dual picture of all the linear inequalities that come from fully real intersections, which hold on $\Sigma_{n,2d}$ but fail on $P_{n,2d}$.

Suppose that for the cases $(3,6)$ and $(4,4)$ a linear functional $\ell \in H^*_{n,2d}$ spans an extreme ray of Σ^*_{n2d} that does not correspond to point evaluation. Let W_ℓ be the kernel of Q_ℓ and suppose that $f_1, \ldots, f_{n-1} \in W_\ell$ intersect transversely in $s = d^{n-1}$ real projective points $\gamma_1, \ldots, \gamma_s$. Let $v_1, \ldots, v_s$ be affine representatives for $\gamma_1, \ldots, \gamma_s$ and let $u_1 p(v_1) + \ldots + u_s p(v_s) = 0$ with $u_i \in \mathbb{R}$ be the unique Cayley-Bacharach relation on the points v_i.

Theorem 6.1. *The form Q_ℓ can be uniquely written as*

$$Q_\ell(f) = a_1 f(v_1)^2 + \ldots + a_s f(v_s)^2 \quad \textit{for} \quad f \in H_{n,d},$$

with a single negative coefficient a_k, the rest of the a_i positive, and

$$\sum_{i=1}^{s} \frac{u_i^2}{a_i} = 0.$$

Furthermore any such form is extreme in $\Sigma^*_{n,2d}$

The key to the unified description in these cases is the uniqueness of the Cayley-Bacharach relation, which holds for both $(3,6)$ and $(4,4)$.

Let $E : H_{n,d} \to \mathbb{R}^s$ be the evaluation map that sends $f \in H_{n,d}$ to its values at the points v_i:

$$E(f) = (f(v_1), \dots, f(v_s)).$$

Let L be the image of $H_{n,d}$ under E. Since the forms in $H_{n,d}$ satisfy a unique relation, it follows that L is the following hyperplane:

$$L = \{x \in \mathbb{R}^s \quad | \quad u_1x_1 + \ldots + u_sx_s = 0\}.$$

We would like to classify all positive semidefinite quadratic forms Q_ℓ on $H_{n,d}$ with

$$Q_\ell = a_1 f^2(v_1) + \ldots + a_s f^2(v_s)$$

and coefficients $a_i \in \mathbb{R}$. By Lemma 4.1 the extreme rays ℓ of $\Sigma^*_{n,2d}$ are guaranteed to have this form with points v_i coming from the transverse intersection of two cubics or three quadratics. In terms of the evaluations map we would like to find all quadratic forms $Q : \mathbb{R}^s \to \mathbb{R}$ given by $Q = a_1x_1^2 + \ldots + a_sx_s^2$ that are positive semidefinite on the hyperplane L.

Let S_L be the cone of diagonal quadratic forms $Q = a_1x_1^2 + \ldots + a_sx_s^2$ that are positive semidefinite on the hyperplane L. Theorem 6.1 follows immediately from the following proposition:

Proposition 6.2. *Suppose that Q spans an extreme ray of S_L. Then either $Q = a_ix_i^2$ for some i and $a_i > 0$ or Q has the form specified in Theorem* 6.1.

Proof. Let $Q = a_1x_1^2 + \ldots + a_sx_s^2$ span an extreme ray of S_L. If all coefficients a_i are nonnegative, then since Q spans an extreme ray, it follows that $Q = a_ix_i^2$ for some i and $a_i > 0$.

Suppose now that one of the coefficients a_i is zero. Without loss of generality we may assume $a_s = 0$. Then we claim that $Q = a_ix_i^2$ for some $i < s$ and $a_i > 0$.

First we show that all other coefficients must be nonnegative. Suppose that $a_s = 0$ and $a_1 < 0$. From the equation of L we can write $x_1 = -(u_2x_2 + \ldots + u_sx_s)/u_1$. Therefore the form

$$Q = a_1\frac{(u_2x_2 + \ldots + u_sx_s)^2}{u_1^2} + a_2x_2^2 + \ldots + a_{s-1}x_{s-1}^2$$

is positive semidefinite for all values of $x_2, \dots, x_s$. However, the coefficient of x_s^2 is strictly negative, which is a contradiction. Therefore we can write $Q = a_1x_1^2 + \ldots + a_{s-1}x_{s-1}^2$ with $a_i \geq 0$. Since Q spans an extreme ray, it follows that $Q = a_ix_i^2$ for some $i < s$ and $a_i > 0$.

Next we claim that if one of the a_i is negative, then the rest are strictly positive. Suppose that $a_1 < 0$ and $a_2 \leq 0$. Then again write $x_1 = -(u_2x_2 + \ldots + u_sx_s)/u_1$ and $Q = a_1\frac{(u_2x_2+\ldots+u_sx_s)^2}{u_1^2} + a_2x_2^2 + \ldots + a_sx_s^2$. Now the coefficient of x_2^2 is strictly negative, which is a contradiction.

Now we have only one case left: one a_i is negative and the rest are strictly positive. Suppose that $a_s < 0$. Write $x_s = -(u_1x_1 + \ldots + u_{s-1}x_{s-1})/u_s$ and

$$Q = a_1x_1^2 + \ldots + a_{s-1}x_{s-1}^2 + a_s\frac{(u_1x_1 + \ldots + u_{s-1}x_{s-1})^2}{u_s^2}.$$

Let us maximize $\frac{(u_1x_1+\ldots+u_{s-1}x_{s-1})^2}{u_s^2}$ subject to $a_1x_1^2 + \ldots + a_{s-1}x_{s-1}^2 = 1$. Applying Lagrange multipliers, we see that $x_i = \lambda u_i/a_i$ for some λ and all $i \leq s-1$. Now we find the value of a_s that makes $Q(u_1/a_1, \ldots, u_{s-1}/a_{s-1}) = 0$. We see that this happens for

$$a_s^\star = \frac{-u_s^2}{\frac{u_1^2}{a_1} + \ldots + \frac{u_{s-1}^2}{a_{s-1}}}.$$

It is clear that any $a_s \geq a_s^\star$ will result in a psd form Q. However, if $a_s > a_s^\star$, then the form Q is positive definite on L and therefore it does not lie on the boundary of S_L and does not span an extreme ray.

With $a_s = a_s^\star$ the kernel of Q is spanned by the vector $v = (u_1/a_1, \ldots, u_s/a_s)$. We see that (up to a constant multiple) Q is the only form in S_L with kernel that includes v. Therefore Q is extreme in S_L. □

7. One complex pair

We now examine the last case of intersection with one complex conjugate pair of zeroes. Suppose that ℓ spans an extreme ray of $\Sigma_{n,2d}^*$ that does not correspond to point evaluation. Let W_ℓ be the kernel of Q_ℓ and suppose that $f_1, \ldots, f_{n-1} \in W_\ell$ intersect transversely in $s = d^{n-1}$ projective points $\gamma_1, \ldots, \gamma_s$ with a single complex conjugate pair and the rest of the γ_i real. Let $v_1, \ldots, v_{s-2}$ be affine representatives for the real γ_i and let $z, \bar{z}$ be affine representatives for the complex roots chosen such that

$$u_1f(v_1) + \ldots + u_{s-2}f(v_{s-2}) + f(z) + f(\bar{z}) = 0,$$

with $u_i \in \mathbb{R}$, is the unique Cayley-Bacharach relation on the points $v_i, z, \bar{z}$.

Theorem 7.1. *The form Q_ℓ can be uniquely written as*

$$Q_\ell(f) = a_1f(v_1)^2 + \ldots + a_{s-2}f(v_{s-2})^2 + 4m\,(\operatorname{Re} z)^2 - 4t\,(\operatorname{Im} z)^2 \qquad \textit{for} \quad f \in H_{n,d},$$

with all the $a_i > 0$ and m and t satisfying

$$\frac{2m}{m^2+t^2} + \sum_{i=1}^{s-2}\frac{u_i^2}{a_i} = 0.$$

Furthermore any such form is extreme in $\Sigma_{n,2d}^*$.

Again we give a unified presentation based on the uniqueness of the Cayley-Bacharach relation. We construct the real evaluation map $E_{\mathbb{R}} : H_{n,d} \to \mathbb{R}^s$ as follows:

$$E_{\mathbb{R}}(f) = (f(v_1), \ldots, f(v_{s-2}), 2\operatorname{Re} f(z), 2\operatorname{Im} f(z)).$$

Let L be the image of $H_{n,d}$ under E. Then L is the following hyperplane:

$$L = \{x \in \mathbb{R}^s \quad | \quad u_1x_1 + \ldots + u_{s-2}x_{s-2} + x_{s-1} = 0\}.$$

Note that L does not depend on $x_s = 2\operatorname{Im} f(z)$.

We would like to classify all positive semidefinite quadratic forms Q_ℓ on $H_{n,d}$ with

$$Q_\ell = a_1 f^2(v_1) + \ldots + a_{s-2} f^2(v_{s-2}) + b f^2(z) + \bar{b} f^2(\bar{z})$$

and coefficients $a_i \in \mathbb{R}$ and $b \in \mathbb{C}$. By Lemma 4.1 the extreme rays ℓ of $\Sigma^*_{n,2d}$ are guaranteed to have this form with points v_i and z coming from the transverse intersection of two cubics or three quadratics.

Let $b = m + t\sqrt{-1}$. In terms of the evaluation map we would like to find all quadratic forms $Q : \mathbb{R}^s \to \mathbb{R}$ given by

$$Q = a_1 x_1^2 + \ldots + a_{s-2} x_{s-2}^2 + \frac{m}{2}\left(x_{s-1}^2 - x_s^2\right) - t x_{s-1} x_s$$

that are positive semidefinite on the hyperplane L. Let S_L be the convex cone of all such quadratic forms.

Proposition 7.2. *Suppose that Q spans an extreme ray of S_L. Then either $Q = a_i x_i^2$ for some i and $a_i > 0$ or Q has the form specified by Theorem* 7.1. *Conversely, all such forms span extreme rays of S_L.*

Proof. Let $Q = a_1 x_1^2 + \ldots + a_{s-2} x_{s-2}^2 + \frac{m}{2}\left(x_{s-1}^2 - x_s^2\right) - t x_{s-1} x_s$ span an extreme ray of S_L. If $m = 0$, then in order for Q to be psd on L we must have $t = 0$ and then all the coefficients a_i are nonnegative and Q is a nonnegative combination of point evaluations. Since Q is extreme in S_L, it follows that $Q = a_i x_i^2$ for some i and $a_i > 0$.

If $m \neq 0$, then it must be strictly negative since x_s^2 is not constrained by L and its coefficient is $-m/2$. We can complete the square in Q and write

$$Q = a_1 x_1^2 + \ldots + a_{s-2} x_{s-2}^2 + \frac{m^2 + t^2}{2m} x_{s-1}^2 - \frac{1}{2m}(t x_{s-1} + m x_s)^2.$$

Since the term $-\frac{1}{2m}(t x_{s-1} + m x_s)^2$ is always nonnegative and x_s is unconstrained, we can always make it equal to zero by taking $x_s = -t x_{s-1}/m$. Therefore Q is psd if and only if $Q' = a_1 x_1^2 + \ldots + a_{s-2} x_{s-2}^2 + \frac{m^2+t^2}{m} x_{s-1}^2$ is psd on $L' = \{x \in \mathbb{R}^{s-1} \mid u_1 x_1 + \ldots + u_{s-2} x_{s-2} + x_{s-1} = 0\}$. We are in exactly the same situation as the case of fully real intersection and since the coefficient of x_{s-1} is guaranteed to be negative, we know from Proposition 6.2 that all the a_i are positive and

$$\frac{m^2 + t^2}{2m} = \frac{-1}{\frac{u_1^2}{a_1} + \ldots + \frac{u_{s-2}^2}{a_{s-2}}}.$$

The resulting quadratic form will have a unique projective zero in L at

$$v = \left(\frac{u_1}{a_1}, \ldots, \frac{u_{s-2}}{a_{s-2}}, \frac{2m}{m^2 + t^2}, \frac{-2t}{m^2 + t^2}\right).$$

It is easy to verify that up to a constant multiple there is a unique form Q in S_L with v in the kernel, which guarantees that Q is extreme and completes the proof. $\square$

We would like to close the paper with a conjecture. Let W_ℓ be the kernel of an extreme ray Q_ℓ of $\Sigma^*_{n,2d}$, in the cases $(3, 6)$ and $(4, 4)$ that does not correspond to point evaluation. It follows from Corollary 4.4 that *any* transversal intersection of $n - 1$ forms in the kernel W_ℓ has at most one complex pair of zeroes. We have examined extreme rays that can be defined on such transverse intersections in Sections 6 and 7. However, we conjecture that the case of one complex pair of zeroes is not truly necessary, as we can always find a fully real intersection inside W_ℓ:

Conjecture 7.3. *Suppose that for the cases* $(3,6)$ *and* $(4,4)$, W *is an* n*-dimensional subspace of* $H_{n,d}$ *such that* $\mathcal{V}_{\mathbb{C}}(W)=\emptyset$ *and any collection of forms* $f_1,\dots,f_{n-1}$ *intersecting transversely in* W *has at most* 1 *complex pair of zeroes. Then there exist forms* $f_1,\dots,f_{n-1}\in W$ *intersecting transversely in only real points.*

Acknowledgments

The author would like to thank the anonymous referees whose suggestions greatly helped in improving the exposition and clarity of the paper. The author is also thankful for the hospitality of the Institute for Pure and Applied Mathematics at UCLA, where the paper was written.

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