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SQUARE FUNCTION/NON-TANGENTIAL MAXIMAL FUNCTION ESTIMATES AND THE DIRICHLET PROBLEM FOR NON-SYMMETRIC ELLIPTIC OPERATORS

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1. Introduction and statements of results

We consider a divergence form elliptic operator

$$L := -\operatorname{div} A(x)\nabla,$$

defined in $\mathbb{R}^{n+1}$, where A is $(n+1)\times(n+1)$, real, L^∞, t-independent, possibly non-symmetric, and satisfies the uniform ellipticity condition

$$\lambda|\xi|^2 \leq \langle A(x)\xi,\xi\rangle\text{`} <:= \sum_{i,j=1}^{n+1} A_{ij}(x)\xi_j\xi_i, \quad \|A\|_{L^\infty(\mathbb{R}^n)} \leq \lambda^{-1}, \tag{1.1}$$

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for some $\lambda > 0$, and for all $\xi \in \mathbb{R}^{n+1}$, $x \in \mathbb{R}^n$. As usual, the divergence form equation is interpreted in the weak sense i.e., we say that $Lu = 0$ in a domain Ω if $u \in W^{1,2}_{loc}(\Omega)$ and

$$\int A\nabla u \cdot \nabla \Psi = 0,$$

for all $\Psi \in C_0^\infty(\Omega)$. For us, Ω will be a Lipschitz graph domain

$$\Omega_\psi := \{(x,t) \in \mathbb{R}^{n+1} : t > \psi(x)\}, \tag{1.2}$$

where $\psi : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function, or more specifically (but without loss of generality), Ω will be the half-space $\mathbb{R}^{n+1}_+ := \{(x,t) \in \mathbb{R}^n \times (0,\infty)\}$.

The purpose of this paper is two-fold.

First, we shall establish global and local L^p bounds for the square function

$$S^\alpha(u)(x) := \left(\iint_{|x-y|<\alpha t} |\nabla u(y,t)|^2 \frac{dydt}{t^{n-1}} \right)^{1/2}, \tag{1.3}$$

in terms of the non-tangential maximal function

$$N^\alpha_*(u)(x) := \sup_{(y,t):|x-y|<\alpha t} |u(y,t)| \tag{1.4}$$

(for the sake of brevity we shall refer to such bounds as "$S < N$" estimates), and vice versa (we designate these as "$N < S$" estimates)[1]. As regards the latter, we recall that global $N < S$ bounds were already known [AA]; our new contribution here is to prove a local version. On the other hand, our $S < N$ estimates are completely new, for all $n \geq 2$ (the case $n = 1$ appeared previously in [KKPT]).

Second, having established (local) S/N estimates, we then use these, along with the method of "ϵ-approximability", to obtain absolute continuity of L-harmonic measure ω with respect to "surface" measure dx, on the boundary of $\mathbb{R}^{n+1}_+$. In fact, we prove a stronger, scale-invariant version of absolute continuity, namely that ω belongs to the class A_∞. Let us recall that the latter notion is defined as follows. In the sequel, Q will denote a cube in $\mathbb{R}^n$.

Definition 1.5. (A_∞, $A_\infty(Q_0)$). A non-negative Borel measure ω defined on $\mathbb{R}^n$ (resp., on a fixed cube Q_0) is said to belong to the class A_∞ (resp. $A_\infty(Q_0)$) if there are positive constants C and θ such that for every cube Q (resp. every cube $Q \subseteq Q_0$), and every Borel set $F \subset Q$, we have

$$\omega(F) \leq C \left(\frac{|F|}{|Q|} \right)^\theta \omega(Q). \tag{1.6}$$

It is well known (see [CF]) that the A_∞ property is equivalent to the condition that ω is absolutely continuous with respect to the Lebesgue measure, and that there is an exponent $q > 1$ such that the Radon-Nykodym derivative $k := d\omega/dx$ satisfies the "reverse Hölder" estimate

$$\left(\fint_Q k(x)^q dx \right)^{1/q} \leq C \fint_Q k(x)\, dx,$$

uniformly for every cube Q (resp. every $Q \subseteq Q_0$).

[1]We note that in the sequel, when the value of the aperture α is unimportant, or is clear in context, we shall often simply write S and N_* in lieu of S^α and N^α_*. It is well known that L^p norms for $N^\alpha_* f$ are equivalent for any choice of α, and similarly for $S^\alpha f$ (see [FS], [CMS].)

It is also well known (see [Ke, Theorem 1.7.3]) that the fact that harmonic measure belongs to the class A_∞ is equivalent to the solvability of the following L^p Dirichlet problem, for some $p < \infty$ (in fact for p dual to the exponent q in the reverse Hölder inequality):

$$(D_p) \qquad \begin{cases} Lu = 0 \text{ in } \mathbb{R}^{n+1}_+ \\ \lim_{t\to 0} u(\cdot,t) = f \text{ in } L^p(\mathbb{R}^n) \text{ and n.t.} \\ \|N_*(u)\|_{L^p(\mathbb{R}^n)} < \infty. \end{cases}$$

Here, the notation "$u \to f$ n.t." means that $\lim_{(y,t)\to(x,0)} u(y,t) = f(x)$, for a.e. $x \in \mathbb{R}^n$, where the limit runs over $(y,t) \in \Gamma(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < t\}$.

We also remark that we obtain, as another immediate corollary of the A_∞ property of harmonic measure, that the layer potentials associated to the operator L, as well as its complex perturbations, enjoy L^2 estimates ([H], [AAAHK]).

The corresponding problems have been studied for a long time in the case of real symmetric matrices (see, e.g., [JK], [KP]). The non-symmetric case, however, was not achievable via previously devised methods (we shall comment more on this point below), while being important for several independent reasons. First, the well-posed results for equations with real non-symmetric coefficients and associated estimates on solutions provide the first step towards understanding the operators with complex coefficients, in the non-Hermitian case. The latter is absolutely necessary to establish analyticity of the solution as a function of the coefficients even when the coefficients are real, one of Kato's long-time goals [Ka], which would then allow one to study also hyperbolic problems (see [AHLT] for the "block matrix" case). From the analytic point of view, this can be viewed as a far-reaching extension of the Kato square root problem [HLMc], [AHLMcT] and Kato's program. Furthermore, the equations with complex coefficients offer the simplest model of elliptic systems retaining their major difficulties, with multiple entry points to the theory of elasticity and other applications.

In a different direction, one has to mention that many problems which arise in the homogenization theory have non-symmetric coefficients [BLP]. The estimates established in the present paper provide a foundation to study the uniform bounds, independent of the scaling parameter in the homogenization theory, in the absence of symmetry. For instance, in the study of homogenization for equations on Lipschitz domains [KS1], the corresponding estimates in the symmetric case are needed to treat "small scales" and their proof (in the symmetric case) inspires the proof of the "large scales" estimates. Note that the homogenization estimates in [KS1], [KS2], established for Lipschitz domains, were necessary to prove the corresponding "Schauder-type estimates" for the Neumann problem in smooth domains [KLS], still in the presence of symmetry. Again, the extension of all this to the non-symmetric case would be very desirable [BLP] and the present paper should be a first step in this direction.

We now state our results precisely. In the sequel, our ambient space will always be $\mathbb{R}^{n+1}$, with $n \geq 2$.

Theorem 1.7. *Let L be an elliptic operator as above, defined in $\mathbb{R}^{n+1}$, with t-independent coefficients, and suppose that $Lu = 0$ in $\mathbb{R}^{n+1}_+$. Then*

$$\|S(u)\|_{L^p(\mathbb{R}^n)} \lesssim \|N_*(u)\|_{L^p(\mathbb{R}^n)}, \qquad 0 < p < \infty, \tag{1.8}$$

where the implicit constant depends upon p, n, ellipticity, and the apertures of the cones defining S and N_*.

The previous theorem has the following immediate local corollary. Given a cube $Q \subset \mathbb{R}^n$, let

$$T_Q := Q \times \big(0, \ell(Q)\big) \subset \mathbb{R}^{n+1}_+ \tag{1.9}$$

denote the standard Carleson box above Q, where here and in the sequel, $\ell(Q)$ is the side length of Q.

Corollary 1.10. *Under the same hypotheses as in Theorem* 1.7*, for a bounded solution u, we have the Carleson measure estimate*

$$\sup_Q \frac{1}{|Q|} \iint_{T_Q} |\nabla u(x,t)|^2 t dt dx \leq C \, \|u\|_{L^\infty(\Omega)}, \tag{1.11}$$

where C depends only upon dimension and ellipticity.

Sketch of proof of Corollary 1.10. The corollary may be deduced from the theorem by a variant of the argument in [FS]: we divide the boundary data into a "local" part plus a "far-away" part (which we further sub-divide in a dyadic annular fashion), and then use Theorem 1.7 to handle the local part, and Hölder continuity at the boundary to obtain a summable decay for the dyadic terms in the far-away part. The treatment of the local part requires, in addition, the use of a decay estimate for solutions with boundary data vanishing outside a cube (cf. Lemma 4.9 below). We have omitted the details. Alternatively, (1.11) may be gleaned directly from local estimates established in our proof of Theorem 1.7 (cf. Section 3 below, where we shall make note of the local estimates in question during the course of the proof). □

We recall that the converse direction to Theorem 1.7, at least in the case $p = 2$, has recently been obtained by Auscher and Axelsson, and appears in [AA, Theorem 2.4, part (i)], as follows:

$$\|N_*(u)\|_{L^2(\mathbb{R}^n)} \lesssim \|S(u)\|_{L^2(\mathbb{R}^n)}. \tag{1.12}$$

In fact, the result of [AA] is considerably more general in that (1.12) holds in the case of complex coefficients and even strongly elliptical systems. Furthermore, the hypothesis of t-independence may be relaxed to a sort of scale-invariant square Dini smoothness in the t-variable, averaged in x. We refer the reader to [AA] for details. We remark that it is still an (apparently difficult) open problem to extend Theorem 1.7 (that is, the $S < N$ direction) to the case of complex coefficients, even when assuming t-independence as we do here.

With (1.12), the global estimate of [AA], in hand, we shall deduce a local version. Given a cube $Q \subset \mathbb{R}^n$, let θQ denote the concentric cube of side length $\theta\, \ell(Q)$, and let

$$R_Q := Q \times \big(0, \ell(Q)/2\big), \tag{1.13}$$

be the "short" Carleson box above Q.

Theorem 1.14. *Let L be a t-independent elliptic operator as above, and suppose that $u \in L^\infty$ is a solution of $Lu = 0$ in $\mathbb{R}^{n+1}_+$. Then, for each cube $Q \subset \mathbb{R}^n$, and*

each $0 < \theta < 1$, there is a set $K_Q = K_Q(\theta) \subset\subset R_Q$, with $\operatorname{dist}(K_Q, \partial R_Q) \approx \ell(Q)$ (depending upon θ), such that

$$\fint_{\theta Q} |u(x)|^2\, dx \leq C_\theta \left(\frac{1}{|Q|} \iint_{R_Q} |\nabla u(x,t)|^2 t dt dx + \sup_{K_Q} |u|^2 \right), \tag{1.15}$$

where the constant C_θ depends also on dimension and ellipticity.

Remark 1.16. We note that our proof of Theorem 1.14 (see Section 4 below) will actually show something stronger, namely, that (1.15) holds with the left-hand side replaced by $\fint_{\theta Q} N_{*,Q}(u)^2\, dx$, where $N_{*,Q}$ is a truncated non-tangential maximal operator, defined with respect to cones that have been truncated at height $\approx \ell(Q)$.

We note that Theorem 1.7, the global $N < S$ bound (1.12), and Theorem 1.14, imply generalizations of themselves. These respective generalizations may be summarized as follows:

Corollary 1.17. *Let L be as above, let Ω_ψ be a Lipschitz graph domain (cf.* (1.2)*), and suppose that $Lu = 0$ in Ω_ψ. Then, for every $p \in (0, \infty)$ we have*

$$\int_{\partial\Omega_\psi} S_\psi(u)^p d\sigma \lesssim \int_{\partial\Omega_\psi} N_{*,\psi}(u)^p d\sigma, \tag{1.18}$$

and

$$\int_{\partial\Omega_\psi} N_{*,\psi}(u)^p d\sigma \lesssim \int_{\partial\Omega_\psi} S_\psi(u)^p d\sigma, \tag{1.19}$$

where the implicit constants depend upon n, p, ellipticity, and $\|\nabla\psi\|_\infty$. Moreover, for $0 < \theta < 1$, if $0 \leq \psi(x) \leq \ell(Q)/8$ in Q, and if $u \in L^\infty$ is a solution of $Lu = 0$ in Ω_ψ, then there is a set $K_Q^\psi = K_Q^\psi(\theta) \subset\subset T_Q \cap \Omega_\psi$, with $\operatorname{dist}\left(K_Q^\psi, \partial(T_Q \cap \Omega_\psi)\right) \approx \ell(Q)$ (depending on θ and $\|\nabla\psi\|_\infty$), such that

(1.20)

$$\fint_{\theta Q} |u(x,\psi(x))|^2 dx \leq C_\theta \left(\frac{1}{|Q|} \iint_{T_Q \cap \Omega_\psi} |\nabla u(x,t)|^2 \left(t - \psi(x)\right) dt dx + \sup_{K_Q^\psi} |u|^2 \right),$$

where C_θ depends also upon n, ellipticity, and the Lipschitz constant of ψ.

Here, $d\sigma = d\sigma(x) := \sqrt{1 + |\nabla\psi(x)|^2} dx \approx dx$ denotes the standard surface measure on the Lipschitz graph $\partial\Omega_\psi$. The square function $S_\psi(u)$ and non-tangential maximal function $N_{*,\psi}(u)$ are defined on Ω_ψ as follows:

$$S_\psi(u)(x) := \left(\iint_{\Gamma(x)} |\nabla u(Y)|^2 \frac{dY}{\delta(Y)^{n-1}} \right)^{1/2}, \tag{1.21}$$

$$N_{*,\psi}(u)(x) := \sup_{\Gamma(x)} |u(Y)|, \tag{1.22}$$

where $\delta(Y) := \operatorname{dist}(Y, \partial\Omega_\psi)$, and where $\Gamma(x) \subset \Omega_\psi$ is a vertical cone with vertex at $x \in \partial\Omega_\psi$ of sufficiently narrow aperture (depending upon the Lipschitz constant of ψ) that $\delta(Y) \approx |Y - x|$, $\forall Y \in \Gamma(x)$.

Sketch of proof of Corollary 1.17. Since Theorem 1.7, Theorem 1.14, and (1.12) hold (or will be shown to hold) for the entire class of t-independent divergence form operators as described above, one may reduce matters to the case that $\psi \equiv 0$

(i.e., the case that $\Omega_\psi = \mathbb{R}^{n+1}_+$), by "pulling back" under the mapping $(x,t) \to (x, t+\psi(x))$, which preserves the class of t-independent elliptic operators under consideration, and maps $\Omega_\psi \to \mathbb{R}^{n+1}_+$, and $\partial\Omega_\psi \to \partial\mathbb{R}^{n+1}_+$, bijectively. In the case of (1.19), the pullback mechanism and (1.12) yield directly only the case $p = 2$; however, since we also establish local "$N < S$" estimates (see Remark 1.16), we may obtain the full range of p in (1.19) by a well-known "good-lambda" argument. We have omitted the details, which are standard. □

Using the local estimates (1.11) and (1.20), we shall deduce the following theorem. Given a cube $Q \in \mathbb{R}^n$, we let x_Q denote the center of Q, and let $X_Q := (x_Q, \ell(Q))$ be the "Corkscrew point" relative to Q. For $X \in \mathbb{R}^{n+1}_+$, and an elliptic operator L as above, we let ω^X denote the L-harmonic measure at X.

Theorem 1.23. *Let L be a divergence form elliptic operator as above, with t-independent coefficients. Then, there is a $p < \infty$ such that the Dirichlet problem D_p is well-posed; equivalently, for each cube $Q \subset \mathbb{R}^n$, the L-harmonic measure $\omega^{X_Q} \in A_\infty(Q)$, with constants that are uniform in Q.*

The proof of Theorem 1.23 will be deduced from (1.11) and (1.20) via the method of "ϵ-approximability". We defer, until Section 5, a detailed discussion of this notion, but we mention at this point that it was introduced by Garnett [G], who showed that the property is enjoyed by bounded harmonic functions in the plane. An alternative proof of Garnett's result was provided by Varopoulos [V]. A third proof, which extended to bounded harmonic functions in all dimensions, was found by Dahlberg [D], who made the connection with square function estimates on bounded Lipschitz domains. In [KKPT], it was observed by the second and fourth named authors of this paper, jointly with Koch and Toro, that Dahlberg's argument may be carried over to bounded solutions of general divergence form elliptic operators, in the presence of square function estimates on bounded Lipschitz domains. Moreover, these authors showed that ϵ-approximability, in turn, implies that harmonic measure belongs to A_∞ with respect to surface measures on the boundary. In the present paper, we invoke the latter result of [KKPT] "off-the -shelf": the essence of the proof of our Theorem 1.23 is to show that our solutions are ϵ-approximable. Having done this (in Section 5), we then immediately obtain the conclusion of Theorem 1.23 by [KKPT, Theorem 2.3]. We remark that our approach here, although relying upon ideas from the proofs in both [G] and [D], does *not*, in contrast to the proofs of ϵ-approximability in [D] and [KKPT], require S/N estimates on Lipschitz sub-domains of arbitrary orientation, but rather only local S/N estimates on Lipschitz graph domains Ω_ψ as in (1.2), for which the fixed vertical (i.e., t) direction is transverse to $\partial\Omega_\psi$. This refinement of the ϵ-approximability method is significant for us because it is not clear how (or whether) one could exploit the t-independence of our coefficients to obtain S/N estimates on Lipschitz domains with other orientations (i.e., for which the t-direction may fail to be transverse to the boundary).

Finally, we note that by [H] and [AAAHK], Theorem 1.23 has as an immediate corollary that the layer potentials associated to any t-independent operator L as above, and to its complex perturbations, are L^2 bounded. More precisely, let $\mathcal{E}_L(x,t,y,s)$ be the fundamental solution for L, and define the single layer potential

operator by

$$S_t^L f(x) := \int_{\mathbb{R}^n} \mathcal{E}_L(x,t,y,0)\, f(y)\, dy, \quad t \in \mathbb{R} \tag{1.24}$$

Corollary 1.25. [2] *Let $L = -\operatorname{div} A(x)\nabla$ be a t-independent divergence form elliptic operator, where A is real, or more generally, where A has complex entries and there is a real, elliptic, t-independent matrix $A'(x)$ such that $\|A - A'\|_{L^\infty(\mathbb{R}^n)} < \varepsilon_0$. If ε_0 is small enough, depending only upon dimension and ellipticity, then*

$$\sup_{t>0} \int_{\mathbb{R}^n} |\nabla_{x,t} S_t^L f(x)|^2\, dx + \iint_{\mathbb{R}^{n+1}_+} |\nabla_{x,t}\partial_t S_t^L f(x)|^2\, \frac{dxdt}{t} \leq C \int_{\mathbb{R}^n} |f(x)|^2\, dx,$$

where C depends upon n, ellipticity, and $\|A - A'\|_{L^\infty(\mathbb{R}^n)}$.

The case that A has real entries follows immediately from Theorem 1.23 and [H, Theorem 3.1 and its proof]. In turn, the perturbation result follows from the proof of [AAAHK, Theorem 1.12], plus the global $N < S$ bound of [AA] (that is, (1.12) above). We have omitted the details.

1.1. **Historical comments, and remarks on the proofs of the theorems.** The earlier mentioned solution of the Kato problem [AHLMcT] has stimulated a recent surge of interest on boundary problems for the operators with complex coefficients. In particular, in [AA] the authors proved the perturbation results which allow one to guarantee the L^2 solvability of boundary problems for the t-dependent matrices with coefficients satisfying the small Carleson measure condition, *if the corresponding solvability results for matrices with "frozen", t-independent coefficients hold true.* The analogous results in L^p were obtained in [HMaMo]. There is, however, a notorious lack of knowledge of the full scope of well-posedness in the t-independent case. Prior to the present paper, one could rely only on the results in either the symmetric, constant coefficient, or block matrix cases, or on the real non-symmetric case *in ambient dimension* $n + 1 = 2$. Our results ultimately allow one to establish solvability for boundary problems arising via small Carleson measure perturbations of the real non-symmetric coefficients in all dimensions [HMaMo], [HKMP2]. Moreover, the well-posedness for complex t-independent matrices which are L^∞ perturbations of the real, not necessarily symmetric case, now also becomes accessible via [AAAHK], [HMiMo], and [HKMP2].

It turns out that both the methods of attack and the scope of results pertaining to symmetric and non-symmetric equations are necessarily different. In the symmetric case, solvability of the Dirichlet problem D_2 was proved in [JK], by means of a so-called "Rellich identity" obtained via integration by parts. In turn, given the solvability result, S/N bounds follow the main theorem in [DJK] (thus, for symmetric matrices, the logic of our proof strategy in the present paper, in which we establish S/N bounds first and then deduce solvability, was reversed). Such an approach uses symmetry in two vital ways. The integration by parts argument used to prove the Rellich identity relies heavily on self-adjointness, and thus is inappli-

[2] Between the submission and publication of the present paper, the particular result of Corollary 1.25 has been proved in a greater generality of t-independent operators with complex bounded measurable coefficients in [R], by different methods.

cable to the non-symmetric case treated here. Furthermore, self-adjointness plays another role: in the case of real symmetric coefficients, one obtains L^2 solvability of the Dirichlet problem (equivalently, that the Poisson kernel satisfies a reverse Hölder inequality with exponent $q = 2$); whereas, in the case of non-symmetric coefficients, by the counter-examples of [KKPT], one cannot make precise the exponent p for which one has solvability of D_p (equivalently, one cannot specify the reverse Hölder exponent q enjoyed by the Poisson kernel), in particular, D_2, along with the L^2 Rellich estimate, may actually fail. Thus, for non-symmetric operators, the conclusion that $\omega \in A_\infty$ is the best possible. The counter-examples in [KKPT] arise from a certain transmission problem in dimension 2. The results proved here should allow one to obtain positive results for the corresponding transmission problem in higher dimensions as well.

Our main results, Theorems 1.7, 1.14 and 1.23, are extensions to $\mathbb{R}^{n+1}_+$, $n \geq 2$, of analogous results of [KKPT], which were valid in the plane (i.e., $n = 1$). The proof of Theorem 1.14 will follow that of its antecedent, Theorem 3.18 of [KKPT], very closely, with some minor changes required by the higher dimensional setting. As noted above, the proof of Theorem 1.23 is based on the "ϵ-approximability" arguments of [G], [D] and [KKPT], in which S/N estimates on Lipschitz sub-domains is used to obtain a certain approximability property of solutions, and in turn, to deduce solvability of D_p for some finite p. In this paper, we present a non-trivial refinement of the method, which requires us to establish (local) comparability of S and N only on Lipschitz graph domains, for which the t-direction is transverse to the boundary.

The $S < N$ estimates proved in [KKPT] relied on the fact that in the plane, a 2×2 t-independent matrix can be triangularized by "pushing forward" to an appropriate Lipschitz graph domain Ω_1. In turn, one can prove square function estimates for operators with upper triangular coefficient matrices, by a standard integration by parts argument, since for such operators the function $v(x,t) \equiv t$ is an adjoint null solution. Having triangularized the matrix, this integration by parts may be carried out in the half-plane $\mathbb{R}^2_+$, and even in Lipschitz graph domains, after "pulling back" to the half-space with the Dahlberg-Kenig-Stein change of variable.

In higher dimensions, this approach fails, but the proof of Theorem 1.7 exploits a more general principle in the same spirit, namely, that by pushing forward to the domain above the graph of an appropriate $W^{1,2+\varepsilon}$ function φ, which arises in a (local) L-adapted Hodge decomposition of the coefficient vector $\mathbf{c} := (A_{n+1,j})_{1\leq j\leq n}$, one may put the coefficient matrix into a better form, in which the vector $\mathbf{c}$ is replaced by a divergence-free vector. In turn, this observation may be combined with an L-adapted variant of the Dahlberg-Kenig-Stein pullback mapping, along with the solution of the Kato problem [HLMc], [AHLMcT], to carry out a refined version of the classical integration by parts argument. Of course, some care must be taken with the push forward/pullback mapping based on φ, since the latter is merely $W^{1,2+\varepsilon}$, and not Lipschitz.

The latter argument, establishing $S < N$ bounds, is a completely novel development and the technical core of this paper. In particular, the introduction of an adapted Hodge decomposition to modify the matrix of coefficients, to the best of the authors' knowledge, does not appear earlier in the literature on square function estimates, and is one of the fundamental novelties of our approach. Beyond the basic idea of employing suitable "push forward" and "pull-back" from a Lipschitz

domain pioneered in [KKPT], it bears little resemblance with previously developed techniques. The only common thread with the aforementioned results for complex coefficients (e.g., [AA]) is a profound use of the Kato square root estimate [AHLMcT], albeit in a decisively different way.

1.2. **Notation.** In the sequel, we shall use the notational convention that a generic constant C, as well as the constants implicit in the expressions $a \lesssim b, a \approx b, a \gtrsim b$, shall be allowed to depend on dimension, ellipticity, the aperture of the cones used in the definition of S and N_* (with one exception, to be noted shortly), and, when working in Lipschitz graph domains, the Lipschitz constant, unless there is an explicit qualification to the contrary. As regards the constants depending on the aperture of the cones, in "Step 2" of the proof of Theorem 1.7, we shall consider non-tangential maximal functions taken with respect to a narrow aperture η, and we shall indicate explicitly any dependence on η of the norms of these maximal functions (thus, if no dependence on η is indicated, there is none, or we have reached a stage of the argument where such dependence is irrelevant; cf. (2.20)–(2.21) and Subsection 3.2 below.) We shall sometimes write $X = (x,t)$ to denote points in $\mathbb{R}^{n+1}$, and we let $B(X_0, r) := \{X \in \mathbb{R}^{n+1} : |X - X_0| < r\}$ denote the standard Euclidean ball in $\mathbb{R}^{n+1}$. We shall denote cubes in $\mathbb{R}^n$ and in $\mathbb{R}^{n+1}$, respectively, by $Q \subset \mathbb{R}^n$ and $I \subset \mathbb{R}^{n+1}$.

2. Proof of Theorem 1.7: preliminaries for "$S < N$"

Let $A(x)$ be an $(n+1)\times(n+1)$, real, elliptic, L^∞, t-independent and possibly non-symmetric matrix, as in the introduction. We represent the matrix A schematically as follows:

$$A = \left[\begin{array}{c|c} A_\parallel & \mathbf{b} \\ \hline \mathbf{c} & d \end{array}\right], \tag{2.1}$$

where $d := A_{n+1,n+1}$, $\mathbf{b} := (A_{i,n+1})_{1\leq i\leq n}, \mathbf{c} := (A_{n+1,j})_{1\leq j\leq n}$, and $A_\parallel$ denotes the $n\times n$ submatrix of A with entries $(A_\parallel)_{i,j} := A_{i,j}, 1 \leq j \leq n$. Given any matrix $B = (B_{i,j})$ (no matter its dimensions), we let $B^* = (B_{j,i})$ denotes its adjoint (i.e. transpose, since our coefficients are real). Thus,

$$A^* = \left[\begin{array}{c|c} A_\parallel^* & \mathbf{c} \\ \hline \mathbf{b} & d \end{array}\right]. \tag{2.2}$$

Eventually, we shall establish "good-lambda" estimates for square functions of solutions of the equation $Lu = 0$, and thus, as usual, we shall work locally, on a given cube $Q \subset \mathbb{R}^n$. Since our coefficients clearly belong to L^p_{loc} for any finite p, having fixed a cube Q, we can make a $W^{1,2+\varepsilon}$ Hodge decomposition with sufficiently small $\varepsilon > 0$ (see, e.g., [AT]), and write

$$\mathbf{c}1_{5Q} = -A_\parallel^*\nabla\varphi + \mathbf{h}, \qquad \mathbf{b}1_{5Q} = A_\parallel\nabla\tilde{\varphi} + \widetilde{\mathbf{h}}, \tag{2.3}$$

where $\varphi, \tilde{\varphi} \in W^{1,2+\varepsilon}_0(5Q)$, and $\mathbf{h}, \widetilde{\mathbf{h}}$ are divergence free and supported in $5Q$, and where

$$\fint_{5Q} \Big(|\nabla\varphi(x)| + |\mathbf{h}(x)|\Big)^{2+\varepsilon} dx \leq C \fint_{5Q} |\mathbf{c}(x)|^{2+\varepsilon} dx \leq C \tag{2.4}$$

$$\fint_{5Q} \Big(|\nabla\tilde{\varphi}(x)| + |\widetilde{\mathbf{h}}(x)|\Big)^{2+\varepsilon} dx \leq C \fint_{5Q} |\mathbf{b}(x)|^{2+\varepsilon} dx \leq C. \tag{2.5}$$

We define an n-dimensional divergence form operator

$$L_{\|} := -\operatorname{div}_x(A_{\|}\nabla_x),$$

and let $\mathcal{P}_t := e^{-t^2 L_{\|}}$ and $\mathcal{P}_t^* := e^{-t^2 L_{\|}^*}$ denote, respectively, the heat semigroup associated to $L_{\|}$ and to its adjoint $L_{\|}^*$, but endowed with "elliptic" homogeneity (thus, t has been squared).

In the sequel, we shall want to consider the pullback of L under the mapping

$$\rho(x,t) := \big(x, \tau(x,t)\big) := \big(x, t - \varphi(x) + \mathcal{P}^*_{\eta t}\varphi(x)\big)\,, \tag{2.6}$$

where $\eta > 0$ is a small but fixed number to be chosen, and φ is as in (2.3), and has been extended to all of $\mathbb{R}^n$ by setting $\varphi \equiv 0$ in $\mathbb{R}^n \backslash 5Q$. A computation shows that if u is a solution of $Lu = 0$ in $\mathbb{R}^{n+1}_+$, then $u_1 := u \circ \rho$ is a solution of $L_1 u_1 = 0$ (at least formally), where $L_1 := -\operatorname{div}(A_1\nabla)$, and, for J and $\mathbf{p}$ to be defined momentarily,

$$A_1 := \left[\begin{array}{c|c} J\,A_{\|} & \mathbf{b} + A_{\|}\nabla_x\varphi - A_{\|}\nabla_x\mathcal{P}^*_{\eta t}\varphi \\ \hline \mathbf{h} - A^*_{\|}\nabla_x\mathcal{P}^*_{\eta t}\varphi & \frac{\langle A\,\mathbf{p},\mathbf{p}\rangle}{J} \end{array}\right]. \tag{2.7}$$

Here, $\mathbf{h}$ is the divergence free vector in the Hodge decomposition (2.3), and we define J and $\mathbf{p}$ as follows:

$$J(x,t) := 1 + \partial_t \mathcal{P}^*_{\eta t}\varphi(x), \tag{2.8}$$

is the Jacobian of the change of variable $t \to \tau(x,t)$, with $x \in \mathbb{R}^n$ fixed, and

$$\mathbf{p}(x,t) := (\nabla_x \tau(x,t), -1) = (\nabla_x \mathcal{P}^*_{\eta t}\varphi(x) - \nabla_x\varphi(x), -1). \tag{2.9}$$

Let us make precise our statement that $L_1 u_1 = 0$. In fact, in the sequel, we shall consider u_1 in a certain sawtooth domain Ω_0 in which the mapping $(x,t) \to \rho(x,t)$ is 1-1, with range contained in $\mathbb{R}^{n+1}_+$, and in which $J(x,t) \approx 1$ (uniformly). The fact that $L_1 u_1 = 0$ in the sawtooth region then follows from the pointwise identity

$$A\big((\nabla u)\circ\rho\big)\cdot\big((\nabla v)\circ\rho\big)\,J = A_1\nabla u_1 \cdot \nabla v_1, \tag{2.10}$$

for $v \in W^{1,2}(\Omega_0)$, where $v_1 := v \circ \rho$.

We conclude these preliminaries with some estimate for square functions and non-tangential maximal functions built from the "ellipticized" heat semigroup operators $\mathcal{P}_t$ and $\mathcal{P}_t^*$. By the solution of the Kato problem [HLMc], [AHLMcT], we have for every $\alpha > 0$ that

$$\begin{aligned} &\int_{\mathbb{R}^n} \iint_{|x-y|<\alpha t} |t\,\mathcal{P}_t \operatorname{div}_x \mathbf{f}(y)|^2 \frac{dydt}{t^{n+1}} dx \\ &\quad\approx \iint_{\mathbb{R}^{n+1}_+} |t\,\mathcal{P}_t \operatorname{div}_x \mathbf{f}(x)|^2 \frac{dxdt}{t} \leq C\,\|\mathbf{f}\|^2_{L^2(\mathbb{R}^n)}, \end{aligned} \tag{2.11}$$

where the implicit constants depend upon the aperture α (but in fact are uniform for all $\alpha \leq 1$. Also, by standard semigroup theory (more precisely, that $\mathcal{P}_t = e^{-t^2 L_{\|}/2} e^{-t^2 L\|/2}$, and that $t\nabla_{x,t} e^{-t^2 L_{\|}/2}$ is bounded on $L^2(\mathbb{R}^n)$, uniformly in t; cf.

[Ka]), the latter bounds imply that

$$\int_{\mathbb{R}^n} \iint_{|x-y|<\alpha t} |t^2 \nabla_{x,t}\, \mathcal{P}_t \operatorname{div}_x \mathbf{f}(y)|^2 \frac{dydt}{t^{n+1}} dx \approx \iint_{\mathbb{R}^{n+1}_+} |t^2 \nabla_{x,t}\, \mathcal{P}_t \operatorname{div}_x \mathbf{f}(x)|^2 \frac{dxdt}{t} \leq C \|\mathbf{f}\|^2_{L^2(\mathbb{R}^n)}. \tag{2.12}$$

Of course, analogous bounds hold for $\mathcal{P}_t^*$. By a well-known argument of Fefferman and Stein [FS], the bounds in (2.11)–(2.12) imply corresponding Carleson measure estimates when $\mathbf{f} \in L^\infty(\mathbb{R}^n)$, and thus by tent space interpolation [CMS] we obtain that

$$\|\mathcal{A}_1^\alpha \mathbf{f}\|_{L^p(\mathbb{R}^n)} + \|\mathcal{A}_2^\alpha \mathbf{f}\|_{L^p(\mathbb{R}^n)} \leq C_{\alpha,p} \|\mathbf{f}\|_{L^p(\mathbb{R}^n)}, \tag{2.13}$$

for every $p \in [2, \infty)$, where

$$\mathcal{A}_1^\alpha \mathbf{f}(x) := \left(\iint_{|x-y|<\alpha t} |t\, \mathcal{P}_t \operatorname{div}_x \mathbf{f}(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \tag{2.14}$$

$$\mathcal{A}_2^\alpha \mathbf{f}(x) := \left(\iint_{|x-y|<\alpha t} |t^2 \nabla_{x,t}\, \mathcal{P}_t \operatorname{div}_x \mathbf{f}(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}. \tag{2.15}$$

Trivially, (2.11)–(2.12) also entail L^2 bounds for the vertical square functions

$$\mathcal{G}_1 \mathbf{f}(x) := \left(\int_0^\infty |t\, \mathcal{P}_t \operatorname{div}_x \mathbf{f}(x)|^2 \frac{dt}{t} \right)^{1/2} \tag{2.16}$$

$$\mathcal{G}_2 \mathbf{f}(x) := \left(\int_0^\infty |t^2 \nabla_{x,t}\, \mathcal{P}_t \operatorname{div}_x \mathbf{f}(x)|^2 \frac{dt}{t} \right)^{1/2}. \tag{2.17}$$

The L^2 bounds for these vertical square functions may also be extended to L^p:

$$\|\mathcal{G}_1 \mathbf{f}\|_{L^p(\mathbb{R}^n)} + \|\mathcal{G}_2 \mathbf{f}\|_{L^p(\mathbb{R}^n)} \leq C_p \|\mathbf{f}\|_{L^p(\mathbb{R}^n)}, \tag{2.18}$$

for every $p \in [2, 2+\varepsilon_0)$, with $\varepsilon_0 > 0$ chosen small enough depending on dimension and ellipticity. For $\mathcal{G}_1$, the latter fact is a routine consequence of local Hölder regularity in x, of the kernel of $\mathcal{P}_t$, and in fact the L^p bounds hold more generally for $2 \leq p < \infty$; for $\mathcal{G}_2$, the L^p estimates in the range $2 < p < 2 + \varepsilon_0$ are essentially due to Auscher [A], and in that case the upper endpoint $2 + \varepsilon_0$ is the best possible.

Clearly, (2.13) and (2.18) hold also for the analogous operators corresponding to $\mathcal{P}_t^*$.

Finally, we note that for $2 \leq p < \infty$,

$$\|N_*^\alpha(\partial_t \mathcal{P}_t f)\|_p \leq C_{\alpha,p} \|\nabla_x f\|_p \tag{2.19}$$

$$\|\eta^{-1} N_*^\eta(\partial_t \mathcal{P}_{\eta t} f)\|_p \leq C_p \|\nabla_x f\|_p \tag{2.20}$$

$$\|\widetilde{N}_*^\eta(\nabla_x \mathcal{P}_{\eta t} f)\|_p \leq C_p \|\nabla_x f\|_p \tag{2.21}$$

and similarly for $\mathcal{P}_t^*$, where we shall define $\widetilde{N}_*^\eta$ momentarily. Indeed, since the kernel of the operator $t\partial_t \mathcal{P}_t$ enjoys pointwise Gaussian bounds and kills constants, we have

$$|\partial_t \mathcal{P}_t f(y)| = |\partial_t \mathcal{P}_t (f - f_{x,t})(y)| \leq C_\alpha M(\nabla_x f)(x),$$

whenever $|x-y|<\alpha t$, where $f_{x,t}:=\fint_{|x-z|<t} f(z)dz$, and where in the last step we have used a dyadic annular decomposition, the decay of the kernel, a telescoping identity, and the L^1 Poincare inequality. The bound (2.19) now follows immediately. A slightly more careful version of the same argument, in which we replace $f_{x,t}$ by $f_{x,\eta t}$, yields (2.20), since the kernel of $t\partial_t \mathcal{P}_{\eta t}$, call it $k_{\eta t}(x,y)$ enjoys the Gaussian estimate

$$|k_{\eta t}(x,y)| \lesssim (\eta t)^{-n} \exp\left(-\frac{|x-y|^2}{\eta^2 t^2}\right).$$

Here, our interest is in the case that η is fairly small, so it is important that we have specified that the aperture of the cone in (2.20) is equal to η (it would of course also be fine to allow any aperture $\alpha \lesssim \eta$). To prove (2.21), in which

$$\widetilde{N}_*^{\eta}(v)(x) := \sup_{(y,t):|x-y|<\eta t} \left(\fint_{|y-z|<\eta t} |v(z,t)|^2 dz\right)^{1/2}, \tag{2.22}$$

we may argue as in [KP], using a variant of Caccioppoli's inequality to obtain a bound in terms of $N_*^{2\eta}(\partial_t \mathcal{P}_{\eta t} f)$, $\sup_{t>0}|\partial_t \mathcal{P}_t f|$, and a tangential gradient on the boundary. We omit the details.

3. Proof of Theorem 1.7: main arguments for "$S<N$"

In this section, we present the main arguments in the proof of Theorem 1.7, in three steps. We first show that $S(u)$ is controlled, in L^p norm for sufficiently large p, by a vertical square function involving only the t-derivative of u (plus $N_*(u)$). We then show that this vertical square function is controlled by $N_*(u)$, again in L^p norm for large p. Finally, we shall remove the restriction on p. We will sometimes vary the apertures of our cones, in the definitions of $S(u)$ and $N_*(u)$, from one of these steps to the next; but, as we have already noted, this is harmless as all choices of aperture yield equivalent L^p norms ([FS], [CMS].) Within each step, we shall always maintain a consistent choice of aperture.

3.1. Step 1: $S(u)$ is controlled by a vertical square function of $\partial_t u$.

Set

$$g(u)(x) := \left(\int_0^\infty |\partial_t u(x,t)|^2\, t dt\right)^{1/2}. \tag{3.1}$$

Our goal at this stage is to establish the following "good-λ" inequality, for arbitrary positive λ, and for all sufficiently small γ:

$$\left|\left\{x\in Q : S(u)(x)>3\lambda, \left(M\left(g(u)^2+N_*(u)^2\right)(x)\right)^{1/2}\le \gamma\lambda\right\}\right| \le C\gamma^2|Q|, \tag{3.2}$$

whenever Q is a Whitney cube for the open set $\{S(u)>\lambda\}$. Here, and in the sequel, M denotes the non-centered Hardy-Littlewod maximal operator, taken with respect to averages on the cubes. As is well known, (3.2) implies the global L^p bound:

$$\|S(u)\|_{L^p(\mathbb{R}^n)} \le C_p\left(\|g(u)\|_{L^p(\mathbb{R}^n)} + \|N_*(u)\|_{L^p(\mathbb{R}^n)}\right), \qquad 2<p<\infty. \tag{3.3}$$

For the sake of specificity, let us fix the aperture of the cones defining $S(u)$ to be 1, and that of the cones defining $N^*(u)$ to be $\gg 1$.

We now fix a cube Q in the Whitney decomposition of $\{S(u) > \lambda\}$, and we introduce a truncated square function

$$S_Q(u)(x) := \left(\iint_{|x-y|<t<\ell(Q)} |\nabla u(y,t)|^2 \frac{dydt}{t^{n-1}} \right)^{1/2}.$$

To prove (3.2), we may suppose that there is at least one point in Q, call it x_*, for which

$$\left(M\left(g(u)^2 + N_*(u)^2\right)(x_*)\right)^{1/2} \leq \gamma\lambda. \tag{3.4}$$

Then, by the arguments of [DJK] (which are now standard), using interior estimates for solutions, properties of Whitney cubes, and the fact that the cones defining $N_*(u)$ have apertures much larger than do those defining $S(u)$, the set on the left hand side of (3.2) is contained in $\{x \in Q : S_Q(u)(x) > \lambda\}$, provided γ is chosen small enough depending on dimension and ellipticity. We omit the details, which may be found in [DJK]. By Tchebychev's inequality, and then Fubini's Theorem, we therefore have that the left hand side of (3.2) is bounded by

$$\begin{aligned} \left|\left\{x \in Q : S_Q(u)(x) > \lambda\right\}\right| &\leq \frac{1}{\lambda^2} \int_Q S_Q(u)^2(x)\, dx \\ &\lesssim \frac{1}{\lambda^2} \int_{3Q} \int_0^{\ell(Q)} |\nabla u(y,t)|^2\, tdtdy =: \frac{1}{\lambda^2}\, I. \end{aligned} \tag{3.5}$$

We claim that

$$I \lesssim |Q|\, M\left(g(u)^2 + N_*(u)^2\right)(x_*), \tag{3.6}$$

whence (3.2) follows from (3.4).

Let us now verify the claim. Set $\Phi_Q(t) \equiv \Phi(t/\ell(Q))$, where $\Phi \in C^\infty(\mathbb{R})$, with $0 \leq \Phi \leq 1$, $\Phi(t) \equiv 1$ if $t \leq 1$, and $\Phi(t) \equiv 0$ if $t \geq 2$. Integrating by parts in t, we then have that

$$\begin{aligned} I &\leq \int_{3Q} \int_0^{2\ell(Q)} |\nabla u(y,t)|^2\, \Phi_Q(t)\, tdtdy \approx \int_{3Q} \int_0^{2\ell(Q)} \partial_t \left(|\nabla u(y,t)|^2\, \Phi_Q(t) \right) t^2 dtdy \\ &\lesssim \int_{3Q} \int_0^{2\ell(Q)} \left\langle \nabla \partial_t u(y,t), \nabla u(y,t) \right\rangle \Phi_Q(t)\, t^2 dtdy + \int_{3Q} \fint_{\ell(Q)}^{2\ell(Q)} |\nabla u(y,t)|^2\, t^2 dtdy \\ &=: I' + I''. \end{aligned}$$

By Caccioppoli's inequality, $I'' \lesssim |Q|\, M\left(N_*(u)^2\right)(x_*)$. Moreover, by Cauchy's inequality, we have that

$$I' \lesssim \epsilon \int_{3Q} \int_0^{2\ell(Q)} |\nabla u(y,t)|^2\, \Phi_Q(t)\, tdtdy + \frac{1}{\epsilon} \int_{3Q} \int_0^{2\ell(Q)} |\nabla \partial_t u(y,t)|^2\, t^3 dtdy.$$

Fixing ϵ small enough, depending only upon allowable parameters, we may hide the first of these terms (to do this rigorously, we would smoothly truncate the t-integral away from 0, to guarantee that I is finite; the truncation results in additional error terms which may be shown, via Caccioppoli's inequality, to be controlled by $|Q|M(N_*(u)^2)(x_*)$; we omitted the routine details). Covering the region $3Q \times (0, 2\ell(Q))$ by Whitney boxes (of the decomposition of the open set $\mathbb{R}^{n+1}_+$), and using

Caccioppoli's inequality (as we may, since by t-independence $\partial_t u$ is a solution), we find that the last term is bounded by a constant time

$$\int_{4Q}\int_0^{3\ell(Q)} |\partial_t u(y,t)|^2\, t dt dy \lesssim |Q|\, M\left(g(u)^2\right)(x_*).$$

Collecting estimates, we obtain (3.6) as claimed. This concludes Step 1.

To conclude this subsection, let us note that in the context of the Carleson measure estimate of Corollary 1.10, the preceding argument shows that the left hand side of (1.11) may be replaced by a similar expression, but with ∇u replaced by $\partial_t u$, modulo errors in the order of $\|u\|_\infty$. Thus, to establish Corollary 1.10, it suffices to verify:

$$\sup_Q \frac{1}{|Q|}\iint_{T_Q} |\partial_t u(x,t)|^2 t dt dx \leq C\,\|u\|_{L^\infty(\Omega)}.$$

We further note that since $\partial_t u$ is a solution, it satisfies the De Giorgi/Nash local Hölder continuity estimates. Consequently, by [AHLT, Lemma 2.14], it is enough to show that there is a uniform constant c, and for each cube Q, a set $F \subset Q$, with $|F| \geq c|Q|$, for which

$$\frac{1}{|Q|}\int_F\int_0^{\ell(Q)} |\partial_t u(x,t)|^2 t dt dx \leq C\,\|u\|_{L^\infty(\Omega)}, \tag{3.7}$$

3.2. **Step 2: a "good-λ" inequality for the vertical square function.** We turn now to the heart of the proof of Theorem 1.7, namely, to establish a "good-λ" inequality for the vertical square function (3.1) in terms of $N_*(u)$. Throughout this subsection, we may assume that our solution u is continuous up the boundary of $\mathbb{R}^{n+1}_+$; indeed, having established the desired bounds for continuous u, we may apply those bounds to $u_\delta(x,t) := u(x,t+\delta)$ with $\delta > 0$ which is a solution of the same equation, by t-independence of the coefficients. In turn, these bounds are preserved in the limit, as $\delta \to 0$, by a monotone convergence argument. We omitted the routine details.

For a given $\lambda > 0$, suppose that Q is a Whitney cube for the open set

$$E_\lambda := \{x \in \mathbb{R}^n : M\left(g(u)\right)(x) > \lambda\}.$$

We now fix $\varepsilon > 0$ so that $2+\varepsilon$ is an exponent for which the Hodge decomposition holds for $L_\parallel$ and $L^*_\parallel$ (cf. (2.3)–(2.5).) Let $\varphi, \tilde{\varphi} \in W_0^{1,2+\varepsilon}(5Q)$ be as in (2.3), and for a small $\eta > 0$ to be chosen, set

$$\Lambda_1 := \eta^{-1} N^\eta_*(\partial_t \mathcal{P}^*_{\eta t}\varphi) + N_*(\partial_t \mathcal{P}^*_t \varphi) + \widetilde{N}^\eta_*(\nabla \mathcal{P}^*_{\eta t}\varphi) + \left(M(|\nabla_x \varphi|^2)\right)^{1/2} \tag{3.8}$$

$$\Lambda_2 := \eta^{-1} N^\eta_*(\partial_t \mathcal{P}_{\eta t}\tilde{\varphi}) + N_*(\partial_t \mathcal{P}_t \tilde{\varphi}) + \widetilde{N}^\eta_*(\nabla \mathcal{P}_{\eta t}\tilde{\varphi}) + \left(M(|\nabla_x \tilde{\varphi}|^2)\right)^{1/2}, \tag{3.9}$$

where the non-tangential maximal operator N_* in the second terms on the two right hand sides are defined with respect to cones of aperture 1. We define a certain "maximal differentiation operator"

$$D_{*,p} f(x) := \sup_{r>0}\left(\fint_{|x-y|<r} \left(\frac{|f(x)-f(y)|}{|x-y|}\right)^p dy\right)^{1/p}, \tag{3.10}$$

which obeys the estimate

$$\|D_{*,p_1} f\|_p \leq C_{p,p_1,n}\,\|\nabla f\|_p, \quad 1 \leq p_1 < p < \infty. \tag{3.11}$$

Indeed, by a classical "Morrey type" inequality (see, e.g., [GT, Lemma 7.16]), we have

$$\frac{|f(x)-f(y)|}{|x-y|} \lesssim M(\nabla f)(x) + M(\nabla f)(y),$$

whence it follows that

$$D_{*,p_1} f(x) \lesssim M(\nabla f)(x) + \big(M\big(M(\nabla f)\big)^{p_1}(x)\big)^{1/p_1}.$$

The latter bound clearly implies (3.11).

We then fix $p_1 \in (1,2)$ and define

$$F := \{x \in Q : \Lambda_1(x) + \Lambda_2(x) + D_{*,p_1}\varphi(x) + D_{*,p_1}\tilde{\varphi}(x) \leq \kappa_0\}, \tag{3.12}$$

and note that by (2.19)–(2.21), (3.11), and Tchebychev's inequality, we have

$$|Q\backslash F| \lesssim \kappa_0^{-2-\varepsilon}\,|Q|, \tag{3.13}$$

uniformly in η.

Set $p_0 := 2(2+\varepsilon)/\varepsilon$. Our goal is to prove that for some aperture, sufficiently large α,

(3.14)

$$\Big|\Big\{x \in Q : g(u)(x) > 3\lambda, \big(M\big(N_*^\alpha(u)^{p_0}\big)(x)\big)^{1/p_0} \leq \gamma\lambda\Big\}\Big| \leq C\left(C_{\kappa_0,\eta}\,\gamma^2 + \kappa_0^{-2-\varepsilon}\right)|Q|,$$

for all $\gamma > 0$ sufficiently small, for all κ_0 sufficiently large, and for η chosen small enough depending on κ_0. Here, γ is at our disposal, and (3.13) holds uniformly in η, so we may choose first κ_0, then η, and finally γ, to obtain a bound on the RHS of (3.14) which is a small portion of $|Q|$, whence the standard good-lambda arguments may be carried out to show that

$$\|g(u)\|_p \leq C_p\,\|N_*^\alpha(u)\|_p, \qquad \forall p_0 < p < \infty. \tag{3.15}$$

Let us note at this point that the latter bound, together with (3.3), yield that

$$\|S(u)\|_p \leq C_p\,\|N_*(u)\|_p, \qquad \forall p_0 < p < \infty. \tag{3.16}$$

By (3.13), it is enough to prove the following modified version of (3.14):

$$\Big|\Big\{x \in F : g(u)(x) > 3\lambda, \big(M\big(N_*^\alpha(u)^{p_0}\big)(x)\big)^{1/p_0} \leq \gamma\lambda\Big\}\Big| \leq C_{\eta,\kappa_0}\,\gamma^2\,|Q|, \tag{3.17}$$

As usual, we may assume that there is a point in Q, call it x_*, such that

$$N_*^\alpha(u)(x_*) \leq \big(M\big(N_*^\alpha(u)^{p_0}\big)(x_*)\big)^{1/p_0} \leq \gamma\lambda, \tag{3.18}$$

otherwise there is nothing to prove. Let us note that

$$g(u) \leq \left(\int_0^{\ell(Q)} |\partial_t u|^2\, t dt\right)^{1/2} + \left(\int_{\ell(Q)}^\infty |\partial_t u|^2\, t dt\right)^{1/2} =: g_1(u) + g_2(u).$$

We claim that

$$g_2(u)(x) \leq (1 + C\gamma)\lambda, \qquad \forall x \in Q. \tag{3.19}$$

Indeed, we have that

$$g_2(u)(x) \leq g(u)(x_Q) + \left(\int_{\ell(Q)}^\infty |\partial_t u(x,t) - \partial_t u(x_Q,t)|^2\, t dt\right)^{1/2},$$

where we may choose $x_Q \in \mathbb{R}^n \backslash E_\lambda$, with $\operatorname{dist}(x_Q, Q) \approx \ell(Q)$, since Q is a Whitney cube for E_λ. Then $g_2(u)(x_Q) \leq \lambda$, by definition of E_λ. Moreover, since our coefficients are t-independent, we may apply standard De Giorgi/Nash/Moser interior estimates to obtain that

$$\left(\int_{\ell(Q)}^{\infty} |\partial_t u(x,t) - \partial_t u(x_Q,t)|^2 \, tdt\right)^{1/2} \lesssim \left(\int_{\ell(Q)}^{\infty} \left(\frac{\ell(Q)}{t}\right)^{2\beta} \frac{dt}{t}\right)^{1/2} N_*^\alpha(u)(x_0) \lesssim \gamma\lambda,$$

by (3.18), where $\beta > 0$ is the De Giorgi/Nash exponent, and where we have taken the aperture α to be sufficiently large. This proves the claim.

Taking γ sufficiently small, we may therefore suppose that $g_2(u) < 2\lambda$ in Q, so that the LHS of (3.17) is bounded by

$$\left|\left\{x \in F : g_1(u)(x) > \lambda\right\}\right| \leq \frac{1}{\lambda^2} \int_F \int_0^{\ell(Q)} |\partial_t u|^2 t dt dx \lesssim \frac{1}{\lambda^2} \int_F \int_0^{\ell(Q)} A(x)\nabla u(x,t)\cdot\nabla u(x,t)\, t dt dx, \tag{3.20}$$

where in the last step, we have crudely dominated $|\partial_t u|$ by $|\nabla u|$ and then used ellipticity. We note at this point that in the context of Corollary 1.10, the integral in the middle term is precisely that which appears in (3.7). In the remainder of this subsection, we shall prove that

$$\int_F \int_0^{\ell(Q)} A(x)\nabla u(x,t)\cdot\nabla u(x,t)\, t dt dx \leq C_{\eta,\kappa_0}\, |Q| \left(\fint_{2Q} N_*^\alpha(u)^{p_0}\right)^{2/p_0}. \tag{3.21}$$

Clearly, this estimate yields both our desired "good-lambda" inequality, as well as the bound (3.7).

We turn to the proof of (3.21). By the change of variable $t \to t - \varphi(x) + \mathcal{P}^*_{\eta t}\varphi(x)$ (that this change of variable is "legal" follows from (3.23) and (3.24) below), we have

$$\int_F \int_0^{\ell(Q)} A\nabla u\cdot\nabla u\, t dt dx \lesssim \int_F \int_0^{2\ell(Q)} A_1\nabla u_1\cdot\nabla u_1\, t dt dx, \tag{3.22}$$

where $u_1(x,t) := u(x, t-\varphi(x)+\mathcal{P}^*_{\eta t}\varphi(x))$, and where A_1 and u_1 are as in Section 2 above. Here, we have chosen $\eta \ll \kappa_0^{-2}$, so that

$$|(I - \mathcal{P}^*_{\eta t})\varphi(x)| = \left|\int_0^{\eta t} \partial_s \mathcal{P}^*_s \varphi(x) ds\right| \leq \eta t \kappa_0 \ll \eta^{1/2} t \ll t/8, \quad \forall x \in F. \tag{3.23}$$

We note at this point that the analogue of (3.23) holds also for $(I - \mathcal{P}_{\eta t})\tilde{\varphi}$, and moreover, by (3.8)–(3.12), we have

$$\max\left(|\partial_t \mathcal{P}_{\eta t}\tilde{\varphi}(x)|, |\partial_t \mathcal{P}^*_{\eta t}\varphi(x)|\right) \leq \eta\kappa_0 \ll \eta^{1/2}, \qquad \forall (x,t) \in \Omega_0, \tag{3.24}$$

where Ω_0 is the sawtooth domain

$$\Omega_0 := \bigcup_{x\in F} \Gamma_0(x), \tag{3.25}$$

and $\Gamma_0(x)$ denotes the cone with vertex at x and aperture η. Thus, if $(x,t) \in \Omega_0$, then $|x - x_0| < \eta t$ for some $x_0 \in F$, so that, setting $\varphi_{x_0,\eta t} := \fint_{|x_0-y|<2\eta t} \varphi(y)dy$,

we have

$$|\mathcal{P}^*_{\eta t}\left(\varphi - \varphi_{x_0,\eta t}\right)(x)| \lesssim \eta t M(\nabla\varphi)(x_0) \lesssim \eta t \kappa_0 \ll \eta^{1/2} t, \quad \forall (x,t) \in \Omega_0, \tag{3.26}$$

by a telescoping argument and Poincaré's inequality, and by the Gaussian bounds for $\mathcal{P}^*_{\eta t}$.

We now define a smooth cut-off adapted to Ω_0, or to be more precise, to a slightly smaller sawtooth domain $\Omega_1 := \cup_{x\in F}\Gamma_1(x)$, where $\Gamma_1(x)$ has aperture $\eta/8$. Let $\delta(x) := \operatorname{dist}(x,F)$, and let $\Phi \in C^\infty(\mathbb{R})$, with $\Phi(r) \equiv 1$ if $r \leq 1/16$, and $\Phi(r) \equiv 0$, if $r > 1/8$. We then set

$$\Psi(x,t) := \Phi\left(\frac{\delta(x)}{\eta t}\right)\, \Phi\left(\frac{t}{32\,\ell(Q)}\right). \tag{3.27}$$

Let us record some observations concerning the cut-off Ψ, and certain related sawtooth regions. To begin, we note that

$$\Psi(x,t) \equiv 1, \qquad \forall (x,t) \in F \times (0, 2\,\ell(Q)), \tag{3.28}$$

and also, since η is small, that

$$\operatorname{supp}(\Psi) \subset \Omega_{1,Q} := \Omega_1 \cap \big(2Q \times (0, 4\ell(Q))\big).$$

Next, we claim that

$$|(I - \mathcal{P}^*_{\eta t})\varphi(x)| \ll \eta^{1/2} t, \qquad \forall (x,t) \in \Omega_{0,Q} := \Omega_0 \cap \big(2Q \times (0, 4\ell(Q))\big), \tag{3.29}$$

and that an analogous bound holds for $(I - \mathcal{P}_{\eta t})\tilde{\varphi}$. To verify the claim, we first observe that for $(x,t) \in \Omega_0$, there is a point $x_0 \in F$ such that

$$x \in \Delta := \Delta(x_0, \eta t) := \{x : |x - x_0| < \eta t\}.$$

Let us further observe that $2\Delta \subset 5Q$, since $t \leq 4\ell(Q)$, and η is small. Next, we note that by (2.3), φ is a $W^{1,2}$ weak solution of the inhomogeneous PDE

$$L^*_{\|}\varphi = \operatorname{div}(\mathbf{c}),$$

in the domain $5Q$, and the same is true with φ replaced by $\varphi - c$, for any constant c. Thus, by Moser-type interior estimates, and the definition of F (cf. (3.12)) we have that

$$\begin{aligned} \sup_{\Delta} |\varphi - \varphi(x_0)| &\lesssim \left(\fint_{2\Delta} |\varphi(z) - \varphi(x_0)|^{p_1}\, dz\right)^{1/p_1} + \eta t\, \|\mathbf{c}\|_\infty \\ &\lesssim \eta t \left(D_{*,p_1}\varphi(x_0) + \|\mathbf{c}\|_\infty\right) \lesssim \eta t\, (\kappa_0 + \|\mathbf{c}\|_\infty) \ll \eta^{1/2} t, \end{aligned} \tag{3.30}$$

where the implicit constants depends only upon p_1, ellipticity and dimension (see, e.g., [GT, Theorem 8.17, p. 194]). Consequently, for **every** $y \in \Delta$, we then have

$$\begin{aligned} |(I - \mathcal{P}^*_{\eta t})\varphi(y)| &\leq |\varphi(y) - \varphi(x_0)| + |(I - \mathcal{P}^*_{\eta t})\varphi(x_0)| \\ &\quad + |\mathcal{P}^*_{\eta t}\left(\varphi - \varphi_{x_0,\eta t}\right)(x_0)| + |\mathcal{P}^*_{\eta t}\left(\varphi - \varphi_{x_0,\eta t}\right)(y)| \ll \eta^{1/2} t, \end{aligned} \tag{3.31}$$

where we have used (3.23) and (3.26), along with (3.30). In particular, since $x \in \Delta$, we obtain (3.29), as claimed. The corresponding bound for $(I - \mathcal{P}_{\eta t})\tilde{\varphi}$ follows by an identical argument.

Moreover, for $(x,t) \in \Omega_0$, by (3.24) we have

$$J(x,t) = \partial_t \left(t - \varphi(x) + \mathcal{P}^*_{\eta t}\varphi(x)\right) \approx 1 \tag{3.32}$$

$$\widetilde{J}(x,t) = \partial_t \left(t - \tilde{\varphi}(x) + \mathcal{P}_{\eta t}\tilde{\varphi}(x)\right) \approx 1. \tag{3.33}$$

We then have that the mapping $\rho(x,t) := (x, \tau(x,t)) := (x, t + \mathcal{P}^*_{\eta t}\varphi(x) - \varphi(x))$ is 1-1 on $\mathrm{supp}(\Psi)$, with

$$7t/8 < \tau(x,t) < 9t/8, \qquad \forall (x,t) \in \mathrm{supp}(\Psi). \tag{3.34}$$

Consequently, if $\Omega_\beta := \cup_{x\in F}\Gamma_\beta(x)$ is the sawtooth domain with respect to F, with cones of aperture β, we have that

$$\Omega_{8\beta/9} \subset \rho(\Omega_\beta) \subset \Omega_{8\beta/7}, \qquad \forall \beta \leq \eta. \tag{3.35}$$

Let us note also that

$$|\nabla_{x,t}\Psi(x,t)| \lesssim \frac{1}{\eta t} 1_{E_1}(x,t) + \frac{1}{\ell(Q)} 1_{E_2}(x,t), \tag{3.36}$$

where

$$\begin{aligned} E_1 &:= \{(x,t) \in 2Q \times (0, 4\,\ell(Q)) : \eta t/16 \leq \delta(x) \leq \eta t/8\} \\ E_2 &:= 2Q \times \big(2\,\ell(Q), 4\,\ell(Q)\big) \end{aligned} \tag{3.37}$$

By (3.28), we have that the RHS of (3.22) is bounded by

$$\begin{aligned} &\iint_{\mathbb{R}^{n+1}_+} A_1 \nabla u_1 \cdot \nabla u_1 \, \Psi^2 \, t \, dtdx \\ &= -\frac{1}{2} \iint_{\mathbb{R}^{n+1}_+} L_1(u_1^2)\, \Psi^2 \, t dtdx \\ &= -\frac{1}{2} \iint_{\mathbb{R}^{n+1}_+} u_1^2 \, L_1^*(t)\, \Psi^2 dtdx - \frac{1}{2} \iint_{\mathbb{R}^{n+1}_+} A_1 \nabla(u_1^2) \cdot \nabla(\Psi^2) t dtdx \\ &\quad + \frac{1}{2} \iint_{\mathbb{R}^{n+1}_+} (u_1)^2 \, e_{n+1} \cdot A_1 \nabla(\Psi^2)\, dxdt + \frac{1}{2} \int_F u^2 \, A_{n+1,n+1}\, dx \\ &=: \mathcal{S} + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{B}, \end{aligned} \tag{3.38}$$

where $e_{n+1} := (0, \dots 0, 1)$, and where in the boundary term $\mathcal{B}$ we have used that $(A_1^*)_{n+1,n+1}(x,0) = A_{n+1,n+1}(x)$, that $u_1(x,0) = u(x,0)$ on F (cf. (3.23)), and that $\Psi(x,0) = 1_F(x)$. We note that

$$|\mathcal{B}| \leq C\,|Q| \fint_Q N_*^\alpha(u)^2 \leq C(\gamma\lambda)^2 |Q|, \tag{3.39}$$

by Hölder's inequality and (3.18). Let us now consider the "error terms" $\mathcal{E}_1$ and $\mathcal{E}_2$. For a small constant σ to be chosen later, we have that

$$\begin{aligned} |\mathcal{E}_1| &\leq \sigma \iint_{\mathbb{R}^{n+1}_+} A_1 \nabla u_1 \cdot \nabla u_1 \, \Psi^2 \, t \, dtdx \\ &\quad + \frac{1}{\sigma} \iint_{\mathbb{R}^{n+1}_+} u_1^2 \, A_1 \nabla\Psi \cdot \nabla\Psi \, t \, dtdx =: \mathcal{E}_1' + \mathcal{E}_1''. \end{aligned} \tag{3.40}$$

Choosing σ small enough, we shall eventually hide $\mathcal{E}_1'$, along with several copies of it that will arise later, on the LHS of (3.38). By (3.36), and the definition of A_1 (2.7), writing $\mathbf{h} = \mathbf{c} 1_{5Q} + A_\parallel^* \nabla\varphi$ (cf. (2.3)), and using (3.32) and the fact that the original coefficient matrix is bounded, we find that

$$\mathcal{E}_1'' \leq \mathcal{E}_{11}'' + \mathcal{E}_{12}'',$$

where

$$\mathcal{E}''_{11} = \frac{C_\eta}{\sigma} \iint_{E_1} u_1^2 \left[1 + |\nabla_x (I - \mathcal{P}^*_{\eta t})\varphi(x)|^2\right] \frac{dxdt}{t},$$

and where $\mathcal{E}''_{12}$ is a similar integral over the region E_2. We shall treat only $\mathcal{E}''_{11}$, as the term $\mathcal{E}''_{12}$ is easier.

To this end, we write

$$\mathcal{E}''_{11} = \frac{C_\eta}{\sigma} \sum_k \sum_{Q' \in \mathbb{D}^\eta_k} \int_{Q'} \int_{2^{-k}}^{2^{-k+1}} u_1^2 \left(1 + |\nabla_x (I - \mathcal{P}^*_{\eta t})\varphi(x)|^2\right) 1_{E_1} \frac{dxdt}{t}, \tag{3.41}$$

where $\mathbb{D}^\eta_k$ denotes the grid of dyadic cubes such that

$$\frac{1}{64}\eta 2^{-k} \leq \operatorname{diam} Q' < \frac{1}{32}\eta 2^{-k}, \qquad Q' \in \mathbb{D}^\eta_k. \tag{3.42}$$

Consider now any fixed k and $Q' \in \mathbb{D}^\eta_k$, for which the double integral in (3.41) is non-zero, thus, for which there is a point

$$(x_1, t_1) \in E_1 \cap \left(Q' \times \left[2^{-k}, 2^{-k+1}\right]\right). \tag{3.43}$$

We now fix such a point (x_1, t_1). By definition of E_1,

$$\frac{\eta t_1}{16} \leq \delta(x_1) \leq \frac{\eta t_1}{8}. \tag{3.44}$$

In particular, there is a point $x_0 \in F$ such that $|x_1 - x_0| < (\eta t_1)/8$. Note that

$$Q' \subset \Delta' := \Delta(x_0, \eta 2^{-k}) := \{z : |x_0 - z| < \eta 2^{-k}\}, \tag{3.45}$$

by (3.42). Consequently,

$$Q' \times [2^{-k}, 2^{-k+1}] \subset \Omega_{0,Q} \tag{3.46}$$

(we recall that $\Omega_{0,Q}$ is defined in (3.29)). Furthermore, since δ is Lipschitz with norm 1, using (3.42) and (3.44), we obtain that there is a uniform constant C such that

$$Q' \times \left[2^{-k}, 2^{-k+1}\right] \subset \widetilde{E}_1 := \left\{(y,s) \in 2Q \times (0, 4\ell(Q)) : \frac{\eta s}{C} \leq \delta(y) \leq C\eta s\right\}. \tag{3.47}$$

It then follows that

$$|Q'| \lesssim \int_{Q'} \int_{2^{-k}}^{2^{-k+1}} 1_{\widetilde{E}_1}(y,s) \frac{ds}{s} dy. \tag{3.48}$$

Now, by (2.22), (3.8), and (3.12), we have that for every $t \in [2^{-k}, 2^{-k+1}]$,

$$\begin{aligned} \fint_{Q'} |\nabla_x (I - \mathcal{P}^*_{\eta t})\varphi(x)|^2 \, dx &\lesssim \fint_{\Delta'} |\nabla_x \mathcal{P}^*_{\eta t}\varphi(x)|^2 \, dx + \fint_{\Delta'} |\nabla_x \varphi(x)|^2 \, dx \\ &\lesssim \left(\widetilde{N}^\eta_*(\nabla \mathcal{P}^*_{\eta t}\varphi)\right)^2(x_0) + M(|\nabla_x \varphi|^2)(x_0) \lesssim \kappa_0^2. \end{aligned} \tag{3.49}$$

Moreover, by (3.29), (3.46), and the definition of u_1, for α large enough we have

$$\sup |u_1(x,t)| \leq \operatorname{essinf}_{y \in Q'} N^\alpha_*(u)(y), \tag{3.50}$$

where the supremum runs over all $(x,t)\in Q'\times(2^{-k},2^{-k+1})$. Thus,

$$\begin{aligned}(3.51)\qquad &\int_{Q'}\int_{2^{-k}}^{2^{-k+1}} u_1^2\left(1+|\nabla_x(I-\mathcal{P}^*_{\eta t})\varphi(x)|^2\right)1_{E_1}\,\frac{dxdt}{t}\\ &\leq \int_{2^{-k}}^{2^{-k+1}} \operatorname{essinf}_{Q'}\left(N^\alpha_*(u)^2\right)\fint_{Q'}\left(1+|\nabla_x(I-\mathcal{P}^*_{\eta t})\varphi(x)|^2\right)dx\;|Q'|\,\frac{dt}{t}\\ &\lesssim \left(1+\kappa_0^2\right)\int_{Q'} N^\alpha_*(u)^2(y)\int_{2^{-k}}^{2^{-k+1}} 1_{\widetilde{E}_1}(y,s)\,\frac{ds}{s}dy,\end{aligned}$$

where we have used (3.48) and (3.49). Returning to (3.41), we then have

$$\begin{aligned}\mathcal{E}''_{11}&\leq C_{\eta,\kappa_0,\sigma}\sum_k\sum_{Q'\in\mathbb{D}^\eta_k}\int_{Q'} N^\alpha_*(u)^2(y)\int_{2^{-k}}^{2^{-k+1}} 1_{\widetilde{E}_1}(y,s)\,\frac{ds}{s}\,dy\\ &\leq C_{\eta,\kappa_0,\sigma}\int_{2Q} N^\alpha_*(u)^2(y)\int_{\delta(y)/(C\eta)}^{C\delta(y)/\eta}\frac{ds}{s}\,dy\leq C_{\eta,\kappa_0,\sigma}(\gamma\lambda)^2|Q|,\end{aligned}$$

where in the last step we have used (3.18).

The term $\mathcal{E}_2$ in (3.38) satisfies the same bounds as $\mathcal{E}''_1$. It therefore remains to treat the main term $\mathcal{S}$. To this end, we first observe that

$$L_1^*(t)=\operatorname{div}_x A^*_\parallel\nabla_x\mathcal{P}^*_{\eta t}\varphi-\partial_t\left(\frac{1}{J}\langle A\,\mathbf{p},\mathbf{p}\rangle\right)=:-L^*_\parallel\mathcal{P}^*_{\eta t}\varphi-\partial_t\left(\frac{1}{J}\langle A\,\mathbf{p},\mathbf{p}\rangle\right),$$

since $\operatorname{div}_x\mathbf{h}=0$. We then have that

$$\begin{aligned}(3.52)\qquad \mathcal{S}&=\frac{1}{2}\iint_{\mathbb{R}^{n+1}_+}u_1^2\left(L^*_\parallel\mathcal{P}^*_{\eta t}\varphi\right)\Psi^2dtdx\\ &\quad+\frac{1}{2}\iint_{\mathbb{R}^{n+1}_+}u_1^2\,\partial_t\left(\frac{1}{J}\langle A\,\mathbf{p},\mathbf{p}\rangle\right)\Psi^2dtdx=:\mathcal{S}_1+\mathcal{S}_2.\end{aligned}$$

We treat $\mathcal{S}_1$ first. We note that by definition of $\mathcal{P}^*_{\eta t}$, we have

$$(3.53)\qquad \partial_t\mathcal{P}^*_{\eta t}=-2\eta^2tL^*_\parallel\mathcal{P}^*_{\eta t}=-2\eta^2t\,\mathcal{P}^*_{\eta t}\,L^*_\parallel.$$

Integrating by parts in t, we then obtain

$$\begin{aligned}(3.54)\qquad \mathcal{S}_1&=-\frac{1}{2}\iint_{\mathbb{R}^{n+1}_+}u_1^2\,\partial_t\left(L^*_\parallel\mathcal{P}^*_{\eta t}\varphi\right)\Psi^2\,t\,dtdx\\ &\quad+C_\eta\iint_{\mathbb{R}^{n+1}_+}(u_1\,\partial_tu_1)\,\partial_t\mathcal{P}^*_{\eta t}\varphi\,\Psi^2\,dtdx\\ &\quad+C_\eta\iint_{\mathbb{R}^{n+1}_+}u_1^2\,\partial_t\mathcal{P}^*_{\eta t}\varphi\,(\Psi\,\partial_t\Psi)\,dtdx=:\mathcal{S}'_1+\mathcal{S}''_1+\mathcal{S}'''_1.\end{aligned}$$

The term $\mathcal{S}'''_1$ may be handled like $\mathcal{E}''_1$ and $\mathcal{E}_2$ above, except that the present term is somewhat easier, since $\partial_t\mathcal{P}^*_{\eta t}\varphi$ is bounded in the support of Ψ (cf. (3.8) and (3.12).)

Next, using (3.53), and that the original matrix $A \in L^\infty$, we have

$$
\begin{aligned}
(3.55)\qquad |\mathcal{S}_1'| &\leq \left| \iint_{\mathbb{R}^{n+1}_+} u_1 \, \nabla_x u_1 \cdot A_\parallel^* \nabla_x \partial_t \mathcal{P}^*_{\eta t} \varphi \, \Psi^2 \, t \, dt dx \right| \\
&\quad + \left| \iint_{\mathbb{R}^{n+1}_+} u_1^2 \, \left(A_\parallel^* \nabla_x \partial_t \mathcal{P}^*_{\eta t} \varphi \cdot \nabla_x \Psi \right) \, \Psi \, t \, dt dx \right| =: J + K \\
&\lesssim \sigma \iint_{\mathbb{R}^{n+1}_+} |\nabla_x u_1|^2 \, \Psi^2 \, t \, dt dx \\
&\quad + \left(\frac{1}{\sigma} + 1 \right) \iint_{\mathbb{R}^{n+1}_+} u_1^2 \, \left| \eta^2 \nabla_x \mathcal{P}^*_{\eta t} \, L_\parallel^* \varphi \right|^2 \, \Psi^2 \, t^3 dt dx \\
&\quad + \iint_{\mathbb{R}^{n+1}_+} u_1^2 \, |\nabla_x \Psi|^2 \, t \, dt dx := \mathcal{S}_{11}' + \mathcal{S}_{12}' + \mathcal{S}_{13}',
\end{aligned}
$$

where once again σ is a small number at our disposal. The term $\mathcal{S}_{13}'$ is a slightly simpler version of $\mathcal{E}_1''$, and may be handled by a similar argument.

Next, we consider $\mathcal{S}_{12}'$. By (3.29), and the definition of u_1, we have that

$$
(3.56)\qquad |u_1(x,t)| \leq \sup_{s>0} |u(x,s)| \leq N_*^\alpha(u)(x), \qquad \forall (x,t) \in \Omega_{0,Q}.
$$

Consequently,

(3.57)

$$
\begin{aligned}
\mathcal{S}_{12}' &\leq C_\sigma \int_{2Q} N_*^\alpha(u)^2(x) \, \left(\widetilde{\mathcal{G}}_2(A_\parallel^* \nabla \varphi)(x) \right)^2 \, dx \\
&\leq C_\sigma \left(\int_{2Q} N_*^\alpha(u)^{2(2+\varepsilon)/\varepsilon} \, dx \right)^{\varepsilon/(2+\varepsilon)} \left(\int_{\mathbb{R}^n} \left(\widetilde{\mathcal{G}}_2(A_\parallel^* \nabla \varphi) \right)^{2+\varepsilon} \, dx \right)^{2/(2+\varepsilon)} \\
&\leq C_\sigma (\gamma \lambda)^2 |Q|,
\end{aligned}
$$

where $\widetilde{G}_2$ is the $\mathcal{P}_t^*$ analogue of the vertical square function defined in (2.17), and where we have used (2.18), (2.4), and (3.18) (with $p_0 := 2(2+\varepsilon)/\varepsilon$.)

We would like to handle $\mathcal{S}_{11}'$ by simply hiding it on the LHS of (3.38), with σ chosen small enough, but there is a slightly delicate issue of ellipticity that we must address in order to do this. Before doing so, let us observe that

$$
|\mathcal{S}_1''| \leq \sigma \iint_{\mathbb{R}^{n+1}_+} |\partial_t u_1|^2 \, \Psi^2 \, t dt dx + C_{\eta,\sigma} \iint_{\mathbb{R}^{n+1}_+} |u_1|^2 \, |\partial_t \mathcal{P}^*_{\eta t} \varphi|^2 \, \Psi^2 \, \frac{dx dt}{t} =: \mathcal{S}_{11}'' + \mathcal{S}_{12}''.
$$

The term $\mathcal{S}_{12}''$ may be handled exactly like $\mathcal{S}_{12}'$ above, but with the $\mathcal{P}_t^*$ analogue of (2.16) in place of (2.17), and we obtain the bound

$$
\mathcal{S}_{12}'' \leq C_{\eta,\sigma} (\gamma \lambda)^2 |Q|.
$$

The term $\mathcal{S}_{11}''$ is of the same nature as $\mathcal{S}_{11}'$, and we shall treat them together. In fact,

$$
(3.58)\qquad \mathcal{S}_{11}' + \mathcal{S}_{11}'' = \sigma \iint_{\mathbb{R}^{n+1}_+} |\nabla u_1|^2 \, \Psi^2 \, t dt dx,
$$

where, unless otherwise specified, $\nabla := \nabla_{x,t}$. We recall that $u_1 = u \circ \rho$, with

$$
\rho(x,t) := (x, t + \mathcal{P}^*_{\eta t} \varphi(x) - \varphi(x)) =: (x, \tau(x,t)).
$$

Thus,

$$\partial_t u_1(x,t) = J(x,t)(\partial_\tau u)(x,\tau(x,t)) \tag{3.59}$$

$$\nabla_x u_1(x,t) = \big(\nabla_x u\big)(x,\tau(x,t)) + \big(\partial_\tau u\big)(x,\tau(x,t)\big(\nabla_x \tau(x,t)\big), \tag{3.60}$$

where $J(x,t) := \partial_t \tau(x,t) = 1 + \partial_t \mathcal{P}^*_{\eta t}\varphi(x)$. Consequently,

$$(\nabla u)\circ\rho = \left(\nabla_x u_1 - \frac{\partial_t u_1}{J}\big(\nabla_x \tau\big), \frac{\partial_t u_1}{J}\right)$$

Since $J \approx 1$ in Ω_0, we have that

$$\begin{aligned}|\nabla u_1| &\lesssim \left|\left(\nabla_x u_1, \frac{\partial_t u_1}{J}\right)\right| \\ &\lesssim \left|\left(\nabla_x u_1 - \frac{\partial_t u_1}{J}\Big(\nabla_x \tau\Big), \frac{\partial_t u_1}{J}\right)\right| + |\nabla_x \tau|\,|\partial_t u_1| = |(\nabla u)\circ\rho| + |\nabla_x\tau|\,|\partial_t u_1|.\end{aligned}$$

By (2.10), the ellipticity of A, and the fact that $J \approx 1$, we have that

$$|(\nabla u)\circ\rho|^2 \lesssim A_1 \nabla u_1 \cdot \nabla u_1. \tag{3.61}$$

The latter term gives a contribution to (3.58) that may be hidden on the LHS of (3.38), if σ is chosen small enough. It remains to treat $|\nabla_x\tau|\,|\partial_t u_1|$. To this end, we make the same dyadic decomposition as in (3.41)–(3.42) to write

(3.62)

$$\iint_{\mathbb{R}^{n+1}_+} |\nabla_x\tau|^2\,|\partial_t u_1|^2\,\Psi^2\,t\,dxdt = \sum_k \sum_{Q'\in\mathbb{D}^\eta_k} \int_{2^{-k}}^{2^{-k+1}} \int_{Q'} |\nabla_x\tau|^2\,|\partial_t u_1|^2\,\Psi^2\,t\,dxdt.$$

Consider now some $t_1 \in [2^{-k}, 2^{-k+1}]$ and a cube $Q' \in \mathbb{D}^\eta_k$ for which $Q'\times\{t_1\}$ meets $\mathrm{supp}(\Psi)$, say at the point (x_1,t_1). Then $\delta(x_1) < \eta t_1/8$, by the construction of Ψ, whence by (3.42), we have $\delta(x) < \eta t_1/4$, for every $x \in Q'$. Thus, for each Q' and t_1 as above, there is a point $x_0 \in F$ and an n-disk Δ' such that (3.45), and thus also (3.46) and (3.49), hold. In particular,

$$\frac{7}{8}t < \tau(x,t) < \frac{9}{8}t, \qquad \forall (x,t) \in I(Q') := Q'\times[2^{-k},2^{-k+1}],$$

by (3.29) and the definition of $\tau(x,t)$. It then follows that for $t \in [2^{-k},2^{-k+1}]$,

$$\begin{aligned}\sup_{x\in Q'} |\partial_t u_1(x,t)| &\approx \sup_{x\in Q'} |(\partial_\tau u)(x,\tau(x,t))| \\ &\lesssim \left((\eta t)^{-n-1}\int_{2Q'}\int_{t/2}^{2t} |\partial_s u(y,s)|^2 dsdy\right)^{1/2},\end{aligned}$$

by (3.32), (3.59), Moser's interior estimates, and the t-independence of A.

We let $\mathbb{D}^\eta_k(\Psi)$ denote those $Q' \in \mathbb{D}^\eta_k$ for which $I(Q') := Q'\times[2^{-k},2^{-k+1}]$ meets $\mathrm{supp}(\Psi)$; thus, for which there is a point $(x,t) \in I(Q')$ such that $\delta(x) < \eta t/8$, by construction of Ψ. Consequently, for any such Q', by (3.42) we have that

$$\delta(y) < \mathrm{diam}(2Q') + \frac{1}{8}\eta t \le \frac{3}{16}\eta t \le \frac{3}{8}\eta\, s, \qquad \forall y \in 2Q',\ s > t/2.$$

Moreover, we have $t < 4\ell(Q)$ in $\mathrm{supp}(\Psi)$, so that $s \le 2t$ implies $s < 8\,\ell(Q)$. Set $\Omega^* := \{(y,s)\in\mathbb{R}^{n+1}_+ : \delta(y) < 3\eta s/8, 0 < s < 8\,\ell(Q)\}$. As noted above, (3.49) holds

in the present context, so that (3.62) is bounded by a constant times

$$\begin{aligned}(3.63)\qquad &\frac{1}{\eta}\sum_k\sum_{Q'\in\mathbb{D}_k^\eta}\int_{2^{-k}}^{2^{-k+1}} \fint_{Q'} |\nabla_x\tau|^2\,dx\int_{t/2}^{2t}\int_{2Q'}|\partial_s u(y,s)|^2 1_{\Omega^*}(y,s)\,dyds\,dt\\ &\leq C_{\eta,\kappa_0}\sum_k\sum_{Q'\in\mathbb{D}_k^\eta}\int_{2^{-k-1}}^{2^{-k+2}}\int_{2Q'}|\partial_s u(y,s)|^2 1_{\Omega^*}(y,s)\,s\,dyds\\ &= C_{\eta,\kappa_0}\left(\iint_{\Omega^{**}}|\partial_s u|^2\,s\,dyds+\iint_{\Omega^*\setminus\Omega^{**}}|\partial_s u|^2\,s\,dyds\right)=:\mathcal{M}+\mathcal{E},\end{aligned}$$

where

$$\Omega^{**}:=\{(y,s)\in\mathbb{R}^{n+1}_+ : \delta(y)<\eta s/18, 0<s<\ell(Q)\}.$$

We observe that by (3.34)–(3.35), we have

$$\rho^{-1}(\Omega^{**})\subset\Omega_{\eta/16}\cap(2Q\times(0,2\ell(Q)),$$

and we note that $\Psi\equiv 1$ on the latter set. Therefore, making the change of variable $s=\tau(y,t)$, we find that

$$\mathcal{M}\leq C_{\eta,\kappa_0}\iint_{\mathbb{R}^{n+1}_+}|(\partial_\tau u)\circ\rho|^2\,\Psi^2\,t\,dtdy,$$

since, as above, $J(y,t)\approx 1$. By (3.61), the latter term gives a contribution to (3.58) that may be hidden on the LHS of (3.38) if σ is chosen small enough.

To handle the error term $\mathcal{E}$, we first note that by Moser's interior estimates and the t-independence of A, we have

$$|\partial_s u(y,s)|\lesssim\frac{1}{s}\,N_*^\alpha(u)(y).$$

Thus, by definition of $\Omega^*\setminus\Omega^{**}$, we have

$$\mathcal{E}\leq C_{\eta,\kappa_0}\int_{2Q}\Big(N_*^\alpha(u)(y)\Big)^2\left(\int_{8\delta(y)/(3\eta)}^{18\delta(y)/\eta}\frac{ds}{s}+\int_{\ell(Q)}^{8\ell(Q)}\frac{ds}{s}\right)dy\leq C_{\eta,\kappa_0}(\gamma\lambda)^2|Q|,$$

where in the last step we have used (3.18) and Hölder's inequality. This concludes our treatment of the term $\mathcal{S}_1$ in (3.52). It remains only to treat the term $\mathcal{S}_2$.

To this end, we write

$$\begin{aligned}(3.64)\qquad 2\,\mathcal{S}_2&=\iint_{\mathbb{R}^{n+1}_+}u_1^2\,\partial_t\left(\frac{1}{J}\langle A\,\mathbf{p},\mathbf{p}\rangle\right)\Psi^2 dtdx\\ &=\iint_{\mathbb{R}^{n+1}_+}u_1^2\,\partial_t\left(\frac{1}{J}\right)\langle A\,\mathbf{p},\mathbf{p}\rangle\,\Psi^2 dtdx\\ &\quad+\iint_{\mathbb{R}^{n+1}_+}u_1^2\,\frac{1}{J}\langle\,\partial_t\mathbf{p},A^*\mathbf{p}\rangle\,\Psi^2 dtdx+\iint_{\mathbb{R}^{n+1}_+}u_1^2\,\frac{1}{J}\langle A\,\mathbf{p},\partial_t\mathbf{p}\rangle\,\Psi^2 dtdx\\ &=:I+II+III,\end{aligned}$$

where we have used that A is t-independent.

We treat these terms in order. We recall that $J(x,t) = 1 + \partial_t \mathcal{P}^*_{\eta t}\varphi(x)$. Then

$$\begin{aligned} I &= -\iint_{\mathbb{R}^{n+1}_+} u_1^2\, \frac{\partial_t^2 \mathcal{P}^*_{\eta t}\varphi}{J^2}\, \langle A\,\mathbf{p},\mathbf{p}\rangle\, \Psi^2\, dtdx \\ &= \iint_{\mathbb{R}^{n+1}_+} \partial_t\left(u_1^2\right)\, \frac{\partial_t \mathcal{P}^*_{\eta t}\varphi}{J^2}\, \langle A\,\mathbf{p},\mathbf{p}\rangle\, \Psi^2\, dtdx \\ &\quad + \iint_{\mathbb{R}^{n+1}_+} u_1^2\, \frac{\partial_t \mathcal{P}^*_{\eta t}\varphi}{J^2}\, \partial_t\langle A\,\mathbf{p},\mathbf{p}\rangle\, \Psi^2\, dtdx \\ &\quad + \iint_{\mathbb{R}^{n+1}_+} u_1^2\, \partial_t \mathcal{P}^*_{\eta t}\varphi\, \partial_t\left(\frac{1}{J^2}\right)\, \langle A\,\mathbf{p},\mathbf{p}\rangle\, \Psi^2\, dtdx \\ &\quad + \iint_{\mathbb{R}^{n+1}_+} u_1^2\, \frac{\partial_t \mathcal{P}^*_{\eta t}\varphi}{J^2}\, \langle A\,\mathbf{p},\mathbf{p}\rangle\, \partial_t\left(\Psi^2\right)\, dtdx =: I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where we have used that the boundary terms vanish, since $\partial_t \mathcal{P}^*_{\eta t}\varphi\big|_{t=0} = 0$ (as may be seen by first considering φ in the domain of $L^*_\parallel := -\operatorname{div} A^*_\parallel \nabla$, and then using a density argument).

We recall that $\mathbf{p} := (\nabla_x(\mathcal{P}^*_{\eta t} - I)\varphi(x), -1) = (\nabla_x \tau(x,t), -1)$. Since $\partial_t \mathcal{P}^*_{\eta t}\varphi$ is bounded, and $J \approx 1$, in Ω_0, the term I_4 may then be handled exactly like the terms $\mathcal{E}''_{11}$ and $\mathcal{E}''_{12}$.

The other terms will require some further work. To begin,

$$|I_1| \leq \sigma \iint_{\mathbb{R}^{n+1}_+} |\partial_t u_1|^2\, |\mathbf{p}|^2\, \Psi^2\, t\, dtdx + \frac{C}{\sigma}\iint_{\mathbb{R}^{n+1}_+} u_1^2\, |\partial_t \mathcal{P}^*_{\eta t}\varphi|^2\, |\mathbf{p}|^2\, \Psi^2\, \frac{dtdx}{t}. \tag{3.65}$$

By definition of $\mathbf{p}$, the first of these terms may be handled exactly like (3.62), and hidden on the LHS of (3.38), if σ is chosen small enough. The second term is treated via the same dyadic decomposition as above:

$$\iint_{\mathbb{R}^{n+1}_+} u_1^2\, |\partial_t \mathcal{P}^*_{\eta t}\varphi|^2\, |\mathbf{p}|^2\, \Psi^2\, \frac{dtdx}{t} = \sum_k \sum_{Q'\in\mathbb{D}^\eta_k} \int_{Q'}\int_{2^{-k}}^{2^{-k+1}} u_1^2\, |\partial_t \mathcal{P}^*_{\eta t}\varphi|^2\, |\mathbf{p}|^2\, \Psi^2\, \frac{dtdx}{t},$$

and in turn we note that

$$\begin{aligned} &\int_{Q'}\int_{2^{-k}}^{2^{-k+1}} u_1^2\, |\partial_t \mathcal{P}^*_{\eta t}\varphi|^2\, |\mathbf{p}|^2\, \Psi^2\, \frac{dtdx}{t} \\ &\lesssim \int_{2^{-k}}^{2^{-k+1}} \left(\operatorname{essinf}_{Q'}(N^\alpha_*(u))\right)^2 \\ &\qquad \times \left(\int_{2Q'}\int_{2^{-k-1}}^{2^{-k+2}} |\partial_s \mathcal{P}^*_{\eta s}\varphi(y)|^2\, \frac{dyds}{s}\right) \fint_{Q'} |\mathbf{p}|^2\, \Psi^2\, \frac{dxdt}{t} \\ &\lesssim C_{\kappa_0}\left(\int_{2Q'} (N^\alpha_*(u))^2 \int_{2^{-k-1}}^{2^{-k+2}} |\partial_s \mathcal{P}^*_{\eta s}\varphi(y)|^2\, 1_{\Omega_0}\, \frac{dyds}{s}\right), \end{aligned}$$

where we have used (3.50), and Moser's parabolic local interior estimates (of course, accounting for the rescaling $t \to t^2$) in the first inequality, and (3.49) (which holds in the present situation), along with the definitions of Ψ and Ω_0 in the second. At this point, we may sum in Q' and in k, and then argue as in our treatment of $\mathcal{S}'_{12}$

above (cf. (3.57)), using (2.18) (or rather its analogue for $\mathcal{P}_t^*$), to obtain a bound on the order of $C_{\sigma,\kappa_0}(\gamma\lambda)^2|Q|$, as desired.

Next, we consider the term I_2, which by definition of $\mathbf{p}$ satisfies the bound

$$|I_2| \lesssim \iint_{\mathbb{R}^{n+1}_+} u_1^2\, |\nabla_x \partial_t \mathcal{P}^*_{\eta t}\varphi|^2\, \Psi^2\, t\, dtdx + \iint_{\mathbb{R}^{n+1}_+} u_1^2\, |\partial_t \mathcal{P}^*_{\eta t}\varphi|^2\, |\mathbf{p}|^2\, \Psi^2\, \frac{dtdx}{t}. \tag{3.66}$$

But, the terms above are both fine, since the first is the same as $\mathcal{S}'_{12}$ in (3.55), and the second is the same as the second term on the RHS of (3.65). We therefore obtain the bound $|I_2| \lesssim (\gamma\lambda)^2|Q|$.

To conclude our treatment of term I, we observe that by definition of J, we have

$$|I_3| \lesssim \iint_{\mathbb{R}^{n+1}_+} u_1^2\, |\partial_t^2\, \mathcal{P}^*_{\eta t}\varphi|^2\, \Psi^2\, t\, dtdx + \iint_{\mathbb{R}^{n+1}_+} u_1^2\, |\partial_t \mathcal{P}^*_{\eta t}\varphi|^2\, |\mathbf{p}|^2\, \Psi^2\, \frac{dtdx}{t}.$$

Except for the t-derivative in place of ∇_x in the first term, this is exactly the same bound as we had for I_2, and these terms may therefore be handled in exactly the same way.

Next we treat term II. By definition of $\mathbf{p}$, we have $\partial_t \mathbf{p} = (\nabla_x \partial_t \mathcal{P}^*_{\eta t}\varphi, 0)$, whence it follows from (2.3) that for $x \in 5Q$,

(3.67)

$$\begin{aligned}\langle \partial_t \mathbf{p}, A^*\mathbf{p}\rangle &= \langle \nabla_x \partial_t \mathcal{P}^*_{\eta t}\varphi, A^*_{\|}\nabla_x \mathcal{P}^*_{\eta t}\varphi\rangle - \langle \nabla_x \partial_t \mathcal{P}^*_{\eta t}\varphi, A^*_{\|}\nabla_x \varphi\rangle - \langle \nabla_x \partial_t \mathcal{P}^*_{\eta t}\varphi, \mathbf{c}\rangle \\ &= \langle \nabla_x \partial_t \mathcal{P}^*_{\eta t}\varphi, A^*_{\|}\nabla_x \mathcal{P}^*_{\eta t}\varphi\rangle - \langle \nabla_x \partial_t \mathcal{P}^*_{\eta t}\varphi, \mathbf{h}\rangle\end{aligned}$$

Thus,

$$\begin{aligned} II &= \iint_{\mathbb{R}^{n+1}_+} u_1^2\, \frac{1}{J}\, \langle\, \nabla_x \partial_t \mathcal{P}^*_{\eta t}\varphi, A^*_{\|}\nabla_x \mathcal{P}^*_{\eta t}\varphi\rangle\, \Psi^2 dtdx \\ &\quad - \iint_{\mathbb{R}^{n+1}_+} u_1^2\, \frac{1}{J}\, \langle\, \nabla_x \partial_t \mathcal{P}^*_{\eta t}\varphi, \mathbf{h}\rangle\, \Psi^2 dtdx =: II_1 + II_2.\end{aligned}$$

In turn,

$$\begin{aligned} II_1 &= \iint_{\mathbb{R}^{n+1}_+} u_1^2\, \frac{1}{J}\, \left(\partial_t \mathcal{P}^*_{\eta t}\varphi\right)\, \left(L^*_{\|}\mathcal{P}^*_{\eta t}\varphi\right)\, \Psi^2 dtdx \\ &\quad - \iint_{\mathbb{R}^{n+1}_+} \partial_t \mathcal{P}^*_{\eta t}\varphi \left\langle \nabla_x \left(u_1^2\, \frac{1}{J}\right),\, A^*_{\|}\nabla_x \mathcal{P}^*_{\eta t}\varphi \right\rangle \Psi^2 dtdx \\ &\quad - \iint_{\mathbb{R}^{n+1}_+} u_1^2\, \frac{1}{J}\, \partial_t \mathcal{P}^*_{\eta t}\varphi\, \langle\, \nabla_x \left(\Psi^2\right),\, A^*_{\|}\nabla_x \mathcal{P}^*_{\eta t}\varphi\rangle\, dtdx \\ &=: II_1' + II_1'' + II_1'''.\end{aligned}$$

Since $L^*_{\|}\mathcal{P}^*_{\eta t} = -(2\eta^2 t)^{-1}\partial_t \mathcal{P}^*_{\eta t}$, the term II_1' is like the second term on the RHS of (3.65), only a bit simpler, as we just have 1 in place of $\mathbf{p}$.

Distributing ∇_x, and using that $J \approx 1$, and that $\nabla_x J = \nabla_x \partial_t \mathcal{P}^*_{\eta t}\varphi$, we have that

$$\begin{aligned} |II_1''| &\leq \sigma \iint_{\mathbb{R}^{n+1}_+} |\nabla_x u_1|^2\, \Psi^2\, t\, dtdx + C \iint_{\mathbb{R}^{n+1}_+} u_1^2\, |\nabla_x \partial_t \mathcal{P}^*_{\eta t}\varphi|^2\, \Psi^2\, t\, dtdx \\ &\quad + C\left(\sigma^{-1}+1\right) \iint_{\mathbb{R}^{n+1}_+} u_1^2\, |\partial_t \mathcal{P}^*_{\eta t}\varphi|^2\, |\nabla_x \mathcal{P}^*_{\eta t}\varphi|^2\, \Psi^2 \frac{dtdx}{t}.\end{aligned} \tag{3.68}$$

The first of these terms is bounded by (3.58), and may therefore be treated in exactly the same way. The second and third terms are essentially like the two terms bounding I_2 in (3.66), since in the last term we may handle the factor $|\nabla_x \mathcal{P}^*_{\eta t}\varphi|^2$ just like $|\mathbf{p}|^2$, using (3.49).

To complete our treatment of II_1, we observe that

$$|II_1'''| \lesssim \iint_{\mathbb{R}^{n+1}_+} u_1^2\, |\nabla_x \Psi|^2\, t\, dtdx \,+\, \iint_{\mathbb{R}^{n+1}_+} u_1^2\, |\partial_t \mathcal{P}^*_{\eta t}\varphi|^2\; |\nabla_x \mathcal{P}^*_{\eta t}\varphi|^2\, \Psi^2 \frac{dtdx}{t}.$$

The first of these is the same as $\mathcal{S}'_{13}$ in (3.55), and the second is the same as the last term in (3.68).

Next, we consider II_2. Since $\mathbf{h}$ is divergence free,

$$II_2 = \iint_{\mathbb{R}^{n+1}_+} \partial_t \mathcal{P}^*_{\eta t}\varphi\, \langle\, \nabla_x \left(\frac{u_1^2}{J}\right), \mathbf{h}\rangle\, \Psi^2 dtdx + \iint_{\mathbb{R}^{n+1}_+} \frac{u_1^2}{J}\, \partial_t \mathcal{P}^*_{\eta t}\varphi\, \langle\, \nabla_x \left(\Psi^2\right), \mathbf{h}\rangle\, dtdx.$$

The first of these terms may be treated exactly like II_1'' above, and the second exactly like II_1''', since $\mathbf{h} = \mathbf{c}1_{5Q} + A^*_{\|}\nabla_x\varphi$, and therefore may be handled via (3.49), just like the factor $\nabla_x \mathcal{P}^*_{\eta t}\varphi$.

Last, we consider term III. By an identity analogous to (3.67), we have

$$\begin{aligned} III &= \iint_{\mathbb{R}^{n+1}_+} u_1^2\, \frac{1}{J}\, \langle A_{\|}\nabla_x \mathcal{P}^*_{\eta t}\varphi, \nabla_x \partial_t \mathcal{P}^*_{\eta t}\varphi\rangle\, \Psi^2 dtdx \\ &\quad - \iint_{\mathbb{R}^{n+1}_+} u_1^2\, \frac{1}{J}\, \langle\, \mathbf{b} + A_{\|}\nabla_x\varphi, \nabla_x \partial_t \mathcal{P}^*_{\eta t}\varphi\rangle\, \Psi^2 dtdx \\ &= \iint_{\mathbb{R}^{n+1}_+} u_1^2\, \frac{1}{J}\, \langle\, \nabla_x \left(\mathcal{P}^*_{\eta t}\varphi - \varphi\right), A^*_{\|}\nabla_x \partial_t \mathcal{P}^*_{\eta t}\varphi\rangle\, \Psi^2 dtdx \\ &\quad - \iint_{\mathbb{R}^{n+1}_+} u_1^2\, \frac{1}{J}\, \langle\, \mathbf{b},\, \nabla_x \partial_t \mathcal{P}^*_{\eta t}\varphi\rangle\, \Psi^2 dtdx =: III_1 + III_2. \end{aligned}$$

In turn,

$$\begin{aligned} III_1 &= -\iint_{\mathbb{R}^{n+1}_+} \left(\mathcal{P}^*_{\eta t}\varphi - \varphi\right)\, \langle\, \nabla_x \left(u_1^2\, \frac{1}{J}\, \Psi^2\right), A^*_{\|}\nabla_x \partial_t \mathcal{P}^*_{\eta t}\varphi\rangle\, dtdx \\ &\quad - \iint_{\mathbb{R}^{n+1}_+} u_1^2\, \frac{1}{J}\, \left(\mathcal{P}^*_{\eta t}\varphi - \varphi\right)\, \left(L^*_{\|}\partial_t \mathcal{P}^*_{\eta t}\varphi\right)\, \Psi^2\, dtdx =: III_1' + III_1''. \end{aligned}$$

By (3.29), we have that $|\mathcal{P}^*_{\eta t}\varphi - \varphi| \ll t$ in the support of Ψ. Thus, III_1', upon distributing ∇_x over $u_1^2, 1/J$, and Ψ^2, yields integrals that may be handled just like the terms J, S'_{12} and K, respectively, in (3.55). To handle III_1'', we first note that

$$\int_{\mathbb{R}^n} \left(\int_0^\infty |(\mathcal{P}^*_{\eta t} - I)F|^2 \frac{dt}{t^3}\right)^{p/2} dx \lesssim \|\nabla F\|^p_{L^p(\mathbb{R}^n)}, \tag{3.69}$$

as may be seen by the use of the elementary identity $\mathcal{P}^*_{\eta t} - I = \int_0^{\eta t} \partial_s P^*_s ds$, along with Hardy's inequality in t, to reduce matters to (2.18). We further note that by (3.29) and the definition of u_1,

$$\sup_{t>\delta(x)/\eta} |u_1(x,t)| \leq \sup_{t>0} |u(x,t)| \leq N^\alpha_*(u)(x).$$

Consequently,

$$III_1'' \lesssim \int_{2Q} \Big(N_*^\alpha(u)(x)\Big)^2 \left(\int_0^\infty |(\mathcal{P}_{\eta t}^* - I)\varphi|^2 \frac{dt}{t^3}\right)^{1/2} \left(\int_0^\infty |t^2 \partial_t \, \mathcal{P}_{\eta t}^* L_\parallel^* \varphi|^2 \, \frac{dt}{t}\right)^{1/2} dx$$
$$\lesssim \left(\int_{2Q} \Big(N_*^\alpha(u)(x)\Big)^{2(2+\varepsilon)/\varepsilon}\right)^{\varepsilon/(2+\varepsilon)} \|\nabla\varphi\|_{2+\varepsilon}^2 \lesssim (\gamma\lambda)^2 |Q|,$$

where we have used (2.18), (3.69), (2.4), and (3.18) (with $p_0 := 2(2+\varepsilon)/\varepsilon$.)

It remains now only to treat term III_2. To this end, we used the Hodge decomposition (2.3) to write

$$\mathbf{b} 1_{5Q} = A_\parallel \nabla\tilde{\varphi} + \widetilde{\mathbf{h}} = \big(A_\parallel \nabla_x \tilde{\varphi} - A_\parallel \nabla_x \mathcal{P}_{\eta t}\tilde{\varphi}\big) + A_\parallel \nabla_x \mathcal{P}_{\eta t}\tilde{\varphi} + \widetilde{\mathbf{h}},$$

where $\mathcal{P}_{\eta t} := e^{-(\eta t)^2 L_\parallel}$, and where $\widetilde{\mathbf{h}}$ is divergence free. We recall that by construction, the various estimates that we have used for φ and $\mathcal{P}_{\eta t}^*\varphi$, hold also for $\tilde{\varphi}$ and $\mathcal{P}_{\eta t}\tilde{\varphi}$. The contribution of $\widetilde{\mathbf{h}}$ may then be handled exactly like II_2 above, while the contribution of $A_\parallel \nabla_x \mathcal{P}_{\eta t}\tilde{\varphi}$ may be handled like II_1 above i.e., by integrating by parts in x to move ∇_x away from $\partial_t \mathcal{P}_{\eta t}^*\varphi$. Finally, the contribution of $\big(A_\parallel \nabla_x \tilde{\varphi} - A_\parallel \nabla_x \mathcal{P}_{\eta t}\tilde{\varphi}\big)$ in term III_2 equals

$$\iint_{\mathbb{R}^{n+1}_+} u_1^2 \, \frac{1}{J} \, \langle \, (\nabla_x \tilde{\varphi} - \nabla_x \mathcal{P}_{\eta t}\tilde{\varphi}) \, , \, A_\parallel^* \nabla_x \partial_t \mathcal{P}_{\eta t}^* \varphi \rangle \, \Psi dt dx,$$

which can then be handled like III_1.

3.3. **Step 3: from large p to arbitrary p.** At this point, we observe that our work in the previous two subsections yields the $S < N$ bound (1.8) for all finite $p > p_0$, where as above $p_0 = 2(2+\varepsilon)/\varepsilon$, and $2+\varepsilon$ is the exponent in the Hodge decomposition (2.3)–(2.5) (cf. (3.16)). We now proceed to remove the restriction on p, following [FS]. Let us observe that the standard pullback mechanism, as used in the proof of Corollary 1.17, implies that on any Lipschitz graph domain Ω_ψ as in (1.2), we obtain from (3.16) the bound

$$\|S_\psi(u)\|_{L^p(\partial\Omega_\psi)} \leq C_p \|N_{*,\psi}(u)\|_{L^p(\partial\Omega_\psi)}, \qquad p\,(\|\nabla\psi\|_\infty) < p < \infty, \tag{3.70}$$

for $Lu = 0$ in Ω_ψ, where $S_\psi(u)$, $N_{*,\psi}(u)$ are the square function and non-tangential maximal function relative to Ω_ψ (cf. (1.21)–(1.22)). For the moment, the range of p depends upon the Lipschitz constant of ψ, because the ellipticity of the pullback matrix depends upon this Lipschitz constant, and in turn, the parameter ε that appears in the Hodge decomposition and in the definition of p_0, depends upon ellipticity. The conclusion of Theorem 1.7 then follows immediately from (3.70) and the following:

Lemma 3.71. *Suppose that for every Lipschitz graph domain Ω_ψ, and every elliptic t-independent matrix A with real bounded measurable coefficients, there exists constants C and $q \in (0,\infty)$, depending on dimension, ellipticity, and $\|\nabla\psi\|_\infty$, such that any solution u to the equation $-\mathrm{div}_{x,t} A\nabla_{x,t} u = 0$ in Ω_ψ satisfies*

$$\|S_\psi u\|_{L^q(\partial\Omega_\psi)} \leq C \|N_{*,\psi} u\|_{L^q(\partial\Omega_\psi)}. \tag{3.72}$$

Then, the $S < N$ estimate (1.8) *is valid for all $p \in (0,q)$.*

Proof. We follow the argument of [FS]. Set the aperture of the cone defining $N_*(u)$ to be 2. Fix any $\lambda > 0$ and let

$$F_\lambda := \{x \in \mathbb{R}^n : N_* u(x) \leq \lambda\}.$$

Then, the distribution function $\tau_{N_* u}(\lambda) := |F_\lambda^c|$. Denote by $\mathcal{R}$ an (infinite) saw-tooth region above F_λ, i.e., $\mathcal{R} = \mathcal{R}(F_\lambda) := \cup_{x \in F_\lambda} \Gamma(x)$, where the vertical cones $\Gamma(x)$ have aperture 1 and vertex at $x \in \mathbb{R}^n$. Clearly, $\mathcal{R}$ is a Lipschitz graph domain (with boundary given by the graph of $\psi(x) := \operatorname{dist}(x, F_\lambda)$), with Lipschitz constant 1, so, in particular (3.72) holds in $\mathcal{R}(F_\lambda)$ for some $q < \infty$. Furthermore, we may take the cones defining S_ψ and $N_{*,\psi}$ to have aperture $1/2$.

Let $\tau_{S(u)} := \{x \in \mathbb{R}^n : S(u) > \lambda\}$, where we have fixed the aperture of the cone defining $S(u)$ to be $1/2$. Then

$$\begin{aligned}\tau_{Su}(\lambda) &= |\{x \in F_\lambda : Su(x) > \lambda\}| + |\{x \in F_\lambda^c : Su(x) > \lambda\}| \\ &\leq \frac{C}{\lambda^q} \int_{F_\lambda} (Su(x))^q \, dx + \tau_{N_* u}(\lambda).\end{aligned}$$

However, due to (3.72) on $\mathcal{R}(F_\lambda)$,

$$\begin{aligned}\int_{F_\lambda} (Su(x))^q \, dx &\leq \int_{\partial\mathcal{R}(F_\lambda)} (S_\psi u(x))^q \, d\sigma(x) \leq \int_{\partial\mathcal{R}(F_\lambda)} (N_{*,\psi} u(x))^q \, d\sigma(x) \\ &\lesssim \int_{F_\lambda} (N_* u(x))^q \, dx + \int_{\partial\mathcal{R}(F_\lambda)\setminus F_\lambda} (N_{*,\psi} u(x))^q \, d\sigma(x),\end{aligned}$$

where $d\sigma$ is surface measure on the Lipschitz graph $t = \psi(x)$. However,

$$\int_{F_\lambda} (N_* u(x))^q \, dx \leq C \int_0^\lambda t^{q-1} \tau_{N_* u}(t) \, dt.$$

Furthermore, any point $x \in \mathcal{R}(F_\lambda)$ belongs to some cone with a vertex in F_λ. Since $N_* u \leq \lambda$ on F_λ, it follows that $|u(x)| \leq \lambda$ for any $x \in \mathcal{R}(F_\lambda)$, and therefore, $N_{*,\psi} u(x) \leq \lambda$ for any $x \in \partial\mathcal{R}(F_\lambda)$. Hence,

$$\int_{\partial\mathcal{R}(F_\lambda)\setminus F_\lambda} (N_{*,\psi} u(x))^q \, d\sigma(x) \leq C\lambda^q |\partial\mathcal{R}(F_\lambda)\setminus F_\lambda| \leq C\lambda^q |F_\lambda^c| = C\lambda^q \tau_{N_* u}(\lambda).$$

All in all, we have

$$\tau_{Su}(\lambda) \leq C\tau_{N_* u}(\lambda) + C\lambda^{-q} \int_0^\lambda t^{q-1} \tau_{N_* u}(t) \, dt.$$

Consequently,

$$\begin{aligned}\|Su\|_{L^p(\mathbb{R}^n)}^p &= C \int_0^\infty \lambda^{p-1} \tau_{Su}(\lambda) \, d\lambda \\ &\leq C \int_0^\infty \lambda^{p-1} \tau_{N_* u}(\lambda) \, d\lambda + C \int_0^\infty \lambda^{p-q-1} \int_0^\lambda t^{q-1} \tau_{N_* u}(t) \, dt d\lambda \\ &\leq C \|N_* u\|_{L^p(\mathbb{R}^n)}^p,\end{aligned}$$

provided that $p < q$. □

4. Proof of Theorem 1.14: local "$N < S$" bounds

In this section, taking the global S/N bounds, as expressed in (1.18) and (1.19), as our starting point, we shall establish the local $N < S$ estimate as stated in Theorem 1.14 following the proof of [KKPT, Theorem 3.18] very closely. We shall prove Theorem 1.14 in the special case that the bounded solution u is continuous on the closure of $\mathbb{R}^{n+1}_+$. Of course, we shall obtain the desired estimate (1.15) with bounds depending only on dimension and ellipticity. Eventually, in Section 5, we shall see that, in order to prove Theorem 1.23, it is enough to verify (1.15) in the sense of an *a priori* bound, for solutions that are continuous up to the boundary. On the other hand, *a posteriori*, with Theorem 1.23 in hand, the interested reader could revisit the arguments of the present section, which continue to work with continuity at the boundary replaced by non-tangential convergence a.e. (dx), to obtain Theorem 1.14 in the general case. We omitted the details, but noted that by the Fatou theorem of [CFMS] (whose proof carries over, *mutatis mutandi*, to the case of non-symmetric coefficients), a bounded solution has a non-tangential trace a.e. $(d\omega)$, and thus, in the presence of Theorem 1.23, also a.e. (dx).

Consider now a solution u of the equation $Lu = 0$ in $\mathbb{R}^{n+1}_+$, which is bounded and continuous on $\overline{\mathbb{R}^{n+1}_+}$. We fix a cube $Q \subset \mathbb{R}^n$, a constant $\theta \in (0,1)$, and recall that θQ is the cube concentric with Q, of side length $\theta\,\ell(Q)$. We further fix constants $\theta_0, \theta_1 \ldots, \theta_6$ satisfying $0 < \theta < \theta_0 < \theta_1 < \cdots < \theta_6 < 1$. Define a Lipschitz function $\psi : \mathbb{R}^n \to [0,\infty)$ such that $\|\nabla\psi\|_\infty \leq \varepsilon_0$, where ε_0 is a small positive number to be chosen, $\psi \equiv 0$ on $\theta_0 Q$ and on $\mathbb{R}^n \backslash Q$, and $\psi > 0$ on $\theta_6 Q \backslash \overline{\theta_0 Q}$. In addition, we may suppose that $\psi(x) \approx \ell(Q)$ on $\theta_5 Q \backslash \theta_1 Q$ (with the implicit constants depending on ε_0). In this section, we shall find it convenient to work with the following variant of the non-tangential maximal function:

$$\widetilde{N}_*(w)(x) = \widetilde{N}^\gamma_*(w)(x) := \sup_{t>0} \frac{1}{|B_\gamma(x,t)|} \iint_{B_\gamma(x,t)} |w(y,s)|\, dyds, \tag{4.1}$$

where $B_\gamma(x,t)$ is the ball with center (x,t) and radius γt, with $0 < \gamma < 1$. We note that by Moser's interior estimates, if $Lu = 0$ in $\mathbb{R}^{n+1}_+$, then $N_*(u) \lesssim \widetilde{N}_*(u)$ pointwise, provided that the aperture of the cone defining $N_*(u)$ is sufficiently small, depending on γ.

We recall that R_Q is the "short Carleson box" above Q (cf. (1.13)), and we consider the domain $\Omega \subset \Omega_\psi$ (where Ω_ψ is the usual graph domain as in (1.2)), given by

$$\Omega := \{(x,t) : x \in \theta_5 Q,\, \psi(x) < t < \psi(x) + \theta_5 \ell(Q)/2\}\,.$$

We observe that $\Omega \subset R_Q$, provided that ε_0 is chosen sufficiently small, depending upon dimension and θ_5. Let $K := \partial\Omega \backslash \{(x,\psi(x)) : x \in \theta_1 Q\}$. We note that $K \subset\subset R_Q$, with $\operatorname{dist}(K, \partial R_Q) \approx \ell(Q)$ (again provided that ε_0 is small enough.) Let $\Phi_1 \in C^\infty_0(\theta_2 Q)$, with $0 \leq \Phi_1 \leq 1$, and $\Phi_1 \equiv 1$ on $\theta_1 Q$. We split $u = u_1 + u_2$ in Ω, where $Lu_i = 0$ in Ω and where u_1, u_2 are continuous and bounded in $\overline{\Omega}$, with

$$u_2\big|_{\partial\Omega} = u(x,\psi(x))\,\Phi_1(x),$$

on $\{(x,\psi(x))\} \cap \partial\Omega$, and zero otherwise on $\partial\Omega$. Note that

$$\sup_\Omega |u_1| \leq \sup_{\partial\Omega} |u_1| \leq \sup_K |u|. \tag{4.2}$$

Consequently,

$$\fint_{\theta Q} \widetilde{N}_{*,Q}(u_1)^2\,dx \leq \Big(\sup_K |u|\Big)^2, \tag{4.3}$$

where the "truncated" maximal function $\widetilde{N}_{*,Q}(u)$ is defined as in (4.1), except that we now consider a restricted supremum over $0 < t \lesssim \ell(Q)$. Moreover, by Fubini's theorem and Caccioppoli's inequality at the boundary,

$$\fint_{\theta Q} S_Q(u_1)^2(x)\,dx \lesssim \sup_{0<t<c\ell(Q)} \fint_{\theta_0 Q} |u_1(y,t)|^2 dy \lesssim \Big(\sup_K |u|\Big)^2, \tag{4.4}$$

where S_Q is defined with respect to cones $\Gamma_Q(x)$, which have been truncated at height $\approx \ell(Q)$, so that $\Gamma_Q(x) \subset \Omega$, for $x \in \theta Q$, and where the implicit constants depend upon θ and θ_0.

We now consider u_2. Let $\Phi \in C_0^\infty(\theta_4 Q)$, with $0 \leq \Phi \leq 1$, and $\Phi \equiv 1$ on $\theta_3 Q$. Let $\mu \in C_0^\infty(\mathbb{R})$, with μ supported in $|t| < \theta_4\ell(Q)/4$, $\mu(t) \equiv 1$ for $|t| < \theta_4\ell(Q)/8$, and set

$$v(x,t) := \Phi(x)\,\mu(t-\psi(x))\,u_2(x,t).$$

As above, let $\Omega_\psi = \{t > \psi(x)\}$, and decompose $v = v_1 + v_2$ in Ω_ψ, where v_1 is bounded and continuous in $\overline{\Omega_\psi}$, and solves

$$\begin{cases} Lv_1 = 0 \quad \text{in } \Omega_\psi \\ v_1\big|_{\partial\Omega_\psi} = v\big|_{\partial\Omega_\psi}, \end{cases} \tag{4.5}$$

while $Lv_2 = Lv$ in Ω_ψ, with $v_2|_{\partial\Omega_\psi} = 0$. We note that the solution v_1 may be constructed so that $v_1 \to 0$ at infinity, since its boundary data has compact support. We now claim that there is a set $F \subset \Omega$, with $\operatorname{dist}(F, \partial R_Q) \approx \ell(Q)$, such that

$$\int_{\partial\Omega_\psi} \Big(\widetilde{N}_{*,\psi}(v_2)^2 + S_\psi(v_2)^2\Big)\,dx \lesssim |Q|\,\Big(\sup_F |u_2|\Big)^2, \tag{4.6}$$

where $\widetilde{N}_{*,\psi}$, S_ψ are defined relative to Ω_ψ (cf. (4.1) and (1.21); in the case of $\widetilde{N}_{*,\psi}$, the ball $B_\gamma(x,t)$ now has radius equal to $\gamma(t-\psi(x))$, with γ sufficiently small depending on $\|\nabla\psi\|_\infty$.) Let us momentarily take this claim for granted. By (1.19), we have

$$\int_{\partial\Omega_\psi} \widetilde{N}_{*,\psi}(v_1)^2 \lesssim \int_{\partial\Omega_\psi} S_\psi(v_1)^2 \lesssim \int_{\partial\Omega_\psi} S_\psi(v)^2 + \int_{\partial\Omega_\psi} S_\psi(v_2)^2,$$

where we have used the pointwise bound $\widetilde{N}_{*,\psi}(w) \leq N_{*,\psi}(w)$. We observe that

$$\nabla v = \Phi(x)\,\mu(t-\psi(x))\,\nabla u_2(x,t) + \nabla\big(\Phi(x)\,\mu(t-\psi(x))\big)\,u_2(x,t) =: \mathbf{V}_1 + \mathbf{V}_2,$$

and in turn,

$$\begin{aligned} &\nabla\big(\Phi(x)\,\mu(t-\psi(x))\big) \\ &\quad = \big(\nabla_x\Phi(x)\,\mu(t-\psi(x)) - \Phi(x)\,\mu'(t-\psi(x))\,\nabla_x\psi(x), \Phi(x)\,\mu'(t-\psi(x))\big). \end{aligned}$$

Thus, $\nabla(\Phi(x)\,\mu(t-\psi(x)))$ (restricted to Ω_ψ), and hence also $\mathbf{V}_2$, are supported in

$$\begin{aligned} &\big\{(x,t) : x \in \theta_4 Q \backslash \theta_3 Q,\, 0 < t-\psi(x) < \theta_4\ell(Q)/4\big\} \\ &\qquad \cup \big\{(x,t) : x \in \theta_4 Q,\, \theta_4\ell(Q)/8 < t-\psi(x) < \theta_4\ell(Q)/4\big\} =: E_1 \cup E_2. \end{aligned} \tag{4.7}$$

Consequently, there is a set $F \subset \Omega$, with $\operatorname{dist}(F, \partial R_Q) \approx \ell(Q)$, such that $|\mathbf{V}_2| \lesssim \ell(Q)^{-1} \sup_F |u_2|$, whence it follows that

$$\int_{\partial\Omega_\psi} S_\psi(v)^2 \, d\sigma \lesssim \int_{\theta_5 Q} S_Q(u_2)^2(x)\, dx + |Q| \Big(\sup_F |u_2| \Big)^2, \tag{4.8}$$

provided that the constant ε_0 (which controls $\|\nabla\psi\|_\infty$) is sufficiently small. Moreover

$$\fint_{\theta Q} \widetilde{N}^\gamma_{*,Q}(u_2)^2(x)\, dx \lesssim \frac{1}{|Q|} \int_{\partial\Omega_\psi} \widetilde{N}^\gamma_\psi(v)^2 d\sigma + \Big(\sup_F |u_2| \Big)^2,$$

if γ is sufficiently small. Indeed, in that case, for $x \in \theta Q$, and $0 < t \lesssim \ell(Q)$, we have that $(u_2 - v))1_{B_\gamma(x,t)}$ is supported in a region of Whitney type i.e., so that $t \approx \ell(Q)$ inside Ω. Gathering these estimates, we obtain

$$\begin{aligned}
\fint_{\theta Q} \widetilde{N}_{*,Q}(u_2)^2(x)\, dx &\lesssim \Big(\sup_F |u_2| \Big)^2 + \frac{1}{|Q|} \int_{\partial\Omega_\psi} \widetilde{N}_{*,\psi}(v_1)^2 d\sigma + \frac{1}{|Q|} \int_{\partial\Omega_\psi} \widetilde{N}_{*,\psi}(v_2)^2 d\sigma \\
&\lesssim \Big(\sup_F |u_2| \Big)^2 + \fint_{\theta_5 Q} S_Q(u_2)^2(x)\, dx \\
&\quad + \frac{1}{|Q|} \int_{\partial\Omega_\psi} S_\psi(v_2)^2 + \frac{1}{|Q|} \int_{\partial\Omega_\psi} \widetilde{N}_{*,\psi}(v_2)^2 \\
&\lesssim \Big(\sup_F |u_2| \Big)^2 + \fint_{\theta_5 Q} S_Q(u_2)^2(x)\, dx,
\end{aligned}$$

where in the last step we have used the claim (4.6) (and where we have also used that the set F may be taken to be the same in (4.6) and (4.8): just take the union of the two, or, see the proof of (4.6) below.) Combining the latter estimate with (4.2)–(4.4), and setting $K_Q := F \cup K$, we have

$$\begin{aligned}
\fint_{\theta Q} \widetilde{N}_{*,Q}(u)^2(x)\, dx &\lesssim \fint_{\theta Q} \widetilde{N}_{*,Q}(u_1)^2(x)\, dx + \fint_{\theta Q} \widetilde{N}_{*,Q}(u_2)^2(x)\, dx \\
&\lesssim \Big(\sup_K |u| \Big)^2 + \Big(\sup_F |u_2| \Big)^2 + \fint_{\theta_5 Q} S_Q(u_2)^2(x)\, dx \\
&\lesssim \Big(\sup_{K_Q} |u| \Big)^2 + \Big(\sup_F |u_1| \Big)^2 + \fint_{\theta_5 Q} S_Q(u)^2(x)\, dx \\
&\quad + \fint_{\theta_5 Q} S_Q(u_1)^2(x)\, dx \\
&\lesssim \Big(\sup_{K_Q} |u| \Big)^2 + \fint_{\theta_5 Q} S_Q(u)^2(x)\, dx,
\end{aligned}$$

whence (1.15), the conclusion of Theorem 1.14, follows directly.

It remains to prove the claim (4.6). To this end, we shall require the following lemma. For notational convenience, we write $X = (x, t)$ to denote points in $\mathbb{R}^{n+1}$.

Lemma 4.9. *Let $x_0 \in \mathbb{R}^n$, $r > 0$, and set $X_0 := (x_0, 0)$ and $B := B(X_0, r)$. Let κB denote the concentric dilate of B by a factor of κ. Suppose that w is bounded and continuous on $\overline{\mathbb{R}^{n+1}_+}$, with $w \to 0$ at infinity, that $Lw = 0$ in $\mathbb{R}^{n+1}_+ \backslash B$, and that $w|_{\mathbb{R}^n \backslash B} \equiv 0$. Set*

$$\mathcal{M} := \|w\|_{L^\infty\left(\mathbb{R}^{n+1}_+ \cap (3B \backslash 2B)\right)}.$$

Then there exist constants C and $\nu > 0$, depending only upon dimension and ellipticity, such that

$$|w(X)| \leq C\mathcal{M}\left(\frac{r}{|X-X_0|}\right)^{n-1+\nu}, \qquad |X-X_0| \geq 3r.$$

Remark 4.10. We note that in the case that L is symmetric, Lemma 4.9 is a well-known classical result of Serrin and Weinberger [SW]. However, their proof does not carry over to the non-symmetric case, therefore we shall supply a proof below.

We defer for the moment the proof of the lemma.

Recall that $Lv_2 = Lv$ in Ω_ψ, that $Lu_2 = 0$ in Ω, and that $\Phi(x)\mu(t-\psi(x))1_{\Omega_\psi}(x,t)$ is supported in Ω. Therefore,

$$Lv_2 = \operatorname{div}\Big(A\nabla\big(\Phi\,\mu\big)\,u_2\Big) + \nabla(\Phi\mu)\cdot A\nabla u_2 =: \operatorname{div}\mathbf{f} + g. \tag{4.11}$$

Recall also that $\nabla(\Phi\mu)$ (restricted to Ω_ψ) is supported in the union $E_1 \cup E_2$ of the sets defined in (4.7). We observe that, by construction, $u_2|_{\partial\Omega}$ is supported in $\partial\Omega \cap \partial\Omega_\psi$, and $\operatorname{supp} u_2(x,\psi(x)) \subset \theta_2 Q$, while $E_1 \cup E_2 \subset \Omega$, with $\operatorname{dist}(E_2, \partial\Omega \cap \partial\Omega_\psi) \approx \ell(Q)$, and $1_{E_1}(x,\psi(x) \subset \theta_4 Q \backslash \theta_3 Q$. Therefore, by Caccioppoli's inequality at the boundary, we have that

$$\begin{aligned}\iint |g|^2 dxdt &\lesssim \ell(Q)^{-2}\iint_{E_1\cup E_2}|\nabla u_2|^2 dxdt \lesssim \ell(Q)^{-4}\iint_{E_1^*\cup E_2^*}|u_2|^2 dxdt \\ &\lesssim \ell(Q)^{n-3}\Big(\sup_F |u_2|\Big)^2,\end{aligned} \tag{4.12}$$

where $E_i^* \subset \Omega$ is a slightly fattened version of E_i, with $E_1^* \cup E_2^* \subseteq F \subset \Omega$, and $F \subset\subset R_Q$, with $\operatorname{dist}(F, \partial R_Q) \approx \ell(Q)$. Moreover, we have that

$$\|\mathbf{f}\|_\infty \lesssim \ell(Q)^{-1}\sup_F |u_2|. \tag{4.13}$$

Since v_2 vanishes on $\partial\Omega_\psi$, it follows from (4.11) that

$$v_2 = L_D^{-1}(\operatorname{div}\mathbf{f} + g),$$

where L_D is the operator L with Dirichlet boundary condition in Ω_ψ. Now, $\nabla L_D^{-1}\operatorname{div}$ is bounded on $L^2(\Omega_\psi)$, and $\nabla L_D^{-1} : L^{2_*}(\Omega_\psi) \to L^2(\Omega_\psi)$, where $2_* := (2n+2)/(n+3)$ is the $(n+1)$-dimensional Sobolev exponent. Therefore, since $\mathbf{f}$ and g are supported in $\Omega \subset R_Q$, we have

$$\begin{aligned}\iint_{\Omega_\psi}|\nabla v_2|^2 &\lesssim \iint_\Omega |\mathbf{f}|^2 + \left(\iint_\Omega |g|^{2_*}\right)^{2/2_*} \\ &\leq |R_Q|\,\|f\|_\infty^2 + |R_Q|^{-1+2/2_*}\iint|g|^2 \lesssim \ell(Q)^{n-1}\Big(\sup_F|u_2|\Big)^2,\end{aligned} \tag{4.14}$$

where in the last step we have used (4.12)–(4.13). Consequently,

$$\begin{aligned}\int_{\partial\Omega_\psi}\left(S_\psi\left(v_2\,1_{t\lesssim\ell(Q)}\right)\right)^2 d\sigma &\approx \int_{\mathbb{R}^n}\int_{\psi(x)}^{C\ell(Q)}|\nabla v_2(x,t)|^2\big(t-\psi(x)\big)\,dtdx \\ &\lesssim \ell(Q)\iint_{\Omega_\psi}|\nabla v_2|^2 \lesssim |Q|\Big(\sup_F|u_2|\Big)^2.\end{aligned} \tag{4.15}$$

Moreover, since v_2 vanishes on $\partial\Omega_\psi$, we have that

$$\int_{\mathbb{R}^n}\int_{\psi(x)}^{C\ell(Q)} |v_2(x,t)|^2\,dtdx = \int_{\mathbb{R}^n}\int_{\psi(x)}^{C\ell(Q)} \left|\int_{\psi(x)}^{t} \partial_s v_2(x,s)\,ds\right|^2 dtdx \tag{4.16}$$
$$\lesssim \ell(Q)^2 \int_{\mathbb{R}^n}\int_{\psi(x)}^{C\ell(Q)} |\nabla v_2(x,s)|^2\,dsdx$$
$$\lesssim \ell(Q)^{n+1}\left(\sup_F |u_2|\right)^2,$$

by (4.14). We let x_Q denote the center of Q, and set $r_Q = C_1\ell(Q)$, with C_1 chosen large enough that $T_Q \subset B_Q := B(x_Q, r_Q)$. Since $Lv_2 = 0$ in $\mathbb{R}^{n+1}_+\backslash B_Q$, and $v_2 = 0$ in $(\mathbb{R}^n\times\{0\})\backslash B_Q$, by Moser's estimates we have that

$$\mathcal{M}_Q := \|v_2\|_{L^\infty(\Omega_\psi\cap(3B_Q\setminus 2B_Q))} \lesssim |B_Q|^{-1/2}\,\|v_2\|_{L^2(\Omega_\psi\cap 4B_Q)} \lesssim \sup_F |u_2|, \tag{4.17}$$

where in the last step we have used (4.16). We observe that $-v_2 = v_1$ in $\Omega_\psi\backslash B_Q$. We may therefore apply Lemma 4.9 to v_2, with $r = r_Q$, $x_0 = x_Q$, to obtain

(4.18)
$$\int_{\partial\Omega_\psi}\left(S_\psi\left(v_2\,1_{t\gtrsim\ell(Q)}\right)\right)^2 d\sigma \approx \int_{\mathbb{R}^n}\int_{C\ell(Q)}^{\infty} |\nabla v_2(x,t)|^2\,t\,dtdx$$
$$\approx \sum_{k=k_0}^{\infty} 2^k\ell(Q)\int_{2^k\ell(Q)}^{2^{k+1}\ell(Q)}\int_{\mathbb{R}^n} |\nabla v_2(x,t)|^2\,dtdx$$
$$\lesssim \sum_{k=k_0-1}^{\infty}\frac{1}{2^k\ell(Q)}\int_{2^k\ell(Q)}^{2^{k+1}\ell(Q)}\int_{\mathbb{R}^n} |v_2(x,t)|^2\,dxdt$$
$$\lesssim \mathcal{M}_Q^2\sum_{k=k_0-1}^{\infty}\frac{1}{2^k\ell(Q)}\,(\ell(Q))^{2n-2+2\nu}$$
$$\times\int_{2^k\ell(Q)}^{2^{k+1}\ell(Q)}\int_{\mathbb{R}^n} |X-x_Q|^{-2(n-1+\nu)}\,dX$$
$$\lesssim |Q|\left(\sup_F |u_2|\right)^2\sum_k 2^{-k(n-2+2\nu)} \lesssim |Q|\left(\sup_F |u_2|\right)^2,$$

since $n\geq 2$ and $\nu > 0$, where in the third, fourth and fifth lines, respectively, we have used Caccioppoli's inequality, Lemma 4.9, and (4.17). Combining (4.15) and (4.18), we produce the desired bound for $S_\psi(v_2)$.

We now turn to $\widetilde{N}_{*,\psi}(v_2)$. By Lemma 4.9, it is enough to establish (4.6) for $\widetilde{N}_{*,\psi,Q}(v_2)$, where the latter is defined by restricting the supremum to values of $t \leq 3C_1\ell(Q)$. To this end, we fix $(x,t)\in\Omega_\psi$, with $t\lesssim\ell(Q)$, and a ball $B_\gamma(x,t)$, centered at (x,t), of radius $\gamma(t-\psi(x)$. Our goal is to show that

$$\frac{1}{B_\gamma(x,t)}\iint_{B_\gamma(x,t)} |v_2(y,\tau)|dyd\tau \lesssim M\left(\int_{\psi(\cdot)}^{C\ell(Q)} |\nabla v(\cdot,s)|\,ds\right)(x), \tag{4.19}$$

where M denotes the Hardy-Littlewood operator acting in the "horizontal" (i.e., x) variable. Momentarily taking (4.19) for granted, we find that

$$\begin{aligned}\int_{\partial\Omega_\psi}\left(\widetilde{N}_{*,\psi,Q}(v_2)\right)^2 d\sigma &\lesssim \int_{\mathbb{R}^n}\left(M\left(\int_{\psi(\cdot)}^{C\ell(Q)}|\nabla v(\cdot,s)|\,ds\right)(x)\right)^2 dx\\ &\lesssim \ell(Q)\int_{\mathbb{R}^n}\int_{\psi(x)}^{C\ell(Q)}|\nabla v(\cdot,s)|^2\,dsdx\\ &\lesssim \ell(Q)\iint_{\partial\Omega_\psi}|\nabla v_2|^2 \lesssim |Q|\left(\sup_F|u_2|\right)^2,\end{aligned}$$

as desired, by (4.14). Turning to the proof of (4.19), we observe that since v_2 vanishes on $\partial\Omega_\psi$, the left hand side of (4.19) equals

$$\begin{aligned}&\frac{1}{B_\gamma(x,t)}\iint_{B_\gamma(x,t)}\left|\int_{\psi(y)}^{\tau}\partial_s v_2(y,s)\,ds\right|dyd\tau\\ &\lesssim \fint_{|x-y|<C(t-\psi(x))}\int_{\psi(y)}^{C\ell(Q)}|\nabla v_2(y,s)|\,dsdy,\end{aligned}$$

whence (4.19) follows immediately. This concludes the proof of Theorem 1.14 (for solutions that are continuous up to the boundary of $\mathbb{R}^{n+1}_+$), modulo the proof of Lemma 4.9.

Proof of Lemma 4.9. Let us make several elementary reductions, as follows: By dilation and translation invariance of the class of operators under consideration, we may suppose that B is the unit ball centered at 0, i.e., that $x_0 = 0$ and that $r = 1$. Furthermore, by renormalizing, we may suppose that $\mathcal{M} = 1$, i.e., that $|w| \le 1$ on $\mathbb{R}^{n+1}_+\cap(3B\backslash 2B)$. Finally, we claim that without loss of generality, we may suppose that $w \ge 0$. Indeed, let $\Omega' := \mathbb{R}^{n+1}_+\backslash 2B$ and set $f := w|_{\partial\Omega'}$. Let $f = f^+ - f^-$ be the splitting of f into its positive and negative parts, and observe that

$$\max(f^+, f^-) = |f| \le \|w\|_{L^\infty\left(\mathbb{R}^{n+1}_+\cap(3B\backslash 2B)\right)} = \mathcal{M} = 1, \tag{4.20}$$

by our renormalization, since f vanishes on $\partial\Omega'\cap(\mathbb{R}^n\times\{0\})$. We then may construct solutions w_+, w_- in Ω', continuous up to the boundary of Ω', with compactly supported data f^+, f^-, respectively, which decay to 0 at infinity. By the maximum principle, $w = w_+ - w_-$ in Ω', and furthermore by (4.20), we have that

$$\max(\|w_+\|_{L^\infty(\Omega')}, \|w_-\|_{L^\infty(\Omega')}) \le \mathcal{M} = 1.$$

Therefore, by treating separately w_+, w_-, we may suppose that w is a non-negative solution in Ω', with $\|w\|_{L^\infty(\Omega')} \le 1$.

Let $\Gamma(X,0)$ be the fundamental solution for L with pole at the origin, so that $\Gamma(X,0) \approx |X|^{1-n}$ in $\mathbb{R}^{n+1}\backslash\{0\}$. Set $w_0(X) := C_0\,\Gamma(X,0)$, where we choose the constant C_0, depending only upon dimension and ellipticity, so that $w(X) \le w_0(X)$ for $X \in \mathbb{R}^{n+1}_+\cap(3B\backslash 2B)$. By the decay of w at infinity, it follows by the maximum principle that $w(X) \le w_0(X)$ for $X \in \mathbb{R}^{n+1}_+\backslash 2B$. We now make the following claim.

Claim 4.21. Suppose that w_1 is continuous and bounded in $\overline{\mathbb{R}^{n+1}_+\backslash 2B}$, with $w_1 \ge 0$, $Lw_1 = 0$ in $\mathbb{R}^{n+1}_+\backslash 2B$, $w_1 \to 0$ at infinity, and $w_1|_{\mathbb{R}^n\backslash B} = 0$. Suppose further that $w_1(X) \le w_0(X)$ for $X \in \mathbb{R}^{n+1}_+\backslash 2^jB$, for some integer $j \ge 1$. Then $w_1(X) \le$

$(1-\delta)\, w_0(X)$, in $\mathbb{R}^{n+1}_+ \backslash 2^{j+1}B$, for some $\delta > 0$ depending only upon dimension and ellipticity.

Since $w_0(X) \approx |X|^{1-n}$, the conclusion of Lemma 4.9 follows from the claim by a straightforward iteration argument, whose details we omit. Therefore, it remains only to establish the claim. To this end, we fix j such that $w_1(X) \leq w_0(X)$ for $X \in \mathbb{R}^{n+1}_+ \backslash 2^j B$. We note that by Hölder continuity at the boundary, and the fact that $w_0(Y) \approx 2^{j(1-n)}$ in $2^{j+2}B \backslash 2^j B$, there is a constant η_0 depending only on ellipticity and dimension, such that for $X = (x, 0)$, with $|X| = 2^{j+1}$, we have

$$w_1(Y) \leq \frac{1}{2} w_0(Y), \qquad \forall Y \in B(X, \eta_0 2^j) \cap \mathbb{R}^{n+1}_+.$$

Set $h := w_0 - w_1$. Then $h \geq 0$, $Lh = 0$ in $\mathbb{R}^{n+1}_+ \backslash B$, and

$$h(Y) \geq \frac{1}{2} w_0(Y), \qquad \forall Y \in B(X, \eta_0 2^j) \cap \mathbb{R}^{n+1}_+,$$

and for all $X = (x, 0)$ with $|X| = 2^{j+1}$. Therefore, by Harnack's inequality, there is some constant $\delta > 0$ depending only upon ellipticity and dimension such that

$$h(Y) \geq \delta\, w_0(Y) \qquad \forall Y \in \mathbb{R}^{n+1}_+ \text{ with } |Y| = 2^{j+1},$$

i.e., $w_1(Y) \leq (1-\delta) w_0(Y)$ for all $Y \in \mathbb{R}^{n+1}_+$ with $|Y| = 2^{j+1}$. The claim now follows by the maximum principle. □

5. ϵ-approximability and the proof of Theorem 1.23

In order to prove Theorem 1.23, it is enough, by [KKPT, Theorem 2.3], to show that if u is bounded in $\mathbb{R}^{n+1}_+$, with $\|u\|_\infty \leq 1$, and $Lu = 0$ in $\mathbb{R}^{n+1}_+$, then u enjoys the following "ϵ-approximability" property, for every $\epsilon > 0$:

Definition 5.1. Let $u \in L^\infty(\mathbb{R}^{n+1}_+)$, with $\|u\|_\infty \leq 1$. Given $\epsilon > 0$, we say that u is **ϵ-approximable** if for every cube $Q_0 \subset \mathbb{R}^n$, there is a $\varphi = \varphi_{Q_0} \in W^{1,1}(T_{Q_0})$ such that

$$\|u - \varphi\|_{L^\infty(T_{Q_0})} < \epsilon, \tag{5.2}$$

and

$$\sup_{Q \subset Q_0} \frac{1}{|Q|} \iint_{T_Q} |\nabla \varphi(x,t)|\, dxdt \leq C_\epsilon, \tag{5.3}$$

where C_ϵ depends also upon dimension and ellipticity, but not on Q_0.

Actually, the definition of ϵ-approximability given in [KKPT], is stated in terms of the existence of a smooth, globally defined φ, but the version above is in fact all that is needed in the proof of Theorem 2.3 of that paper. Moreover, the arguments of [KKPT] do not require ϵ-approximability for all bounded solutions, but only for solutions whose boundary data is the characteristic function of a bounded Borel set. We shall return to this point below.

In this section, we shall assume that u satisfies the following pair of estimates. Given a cube $Q \subset \mathbb{R}^n$, with center x_Q, we let $P_Q := (x_Q, (1-\eta)\ell(Q))$ denote the "Corkscrew point" relative to Q, where $\eta > 0$ is a small number to be chosen. Note that, if $\psi : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, $\|\nabla \psi\|_\infty \leq M$, and $0 \leq \psi \leq \frac{1}{8}\ell(Q)$ in Q, then $\operatorname{dist}(P_Q, \partial\Omega_\psi \cap T_Q) > \frac{\eta}{2}\, \ell(Q)$, provided that η is sufficiently small. Here, as usual, $\Omega_\psi := \{(x,t) : t > \psi(x)\}$.

Estimate 1. *Let $Lu = 0$ in $\mathbb{R}^{n+1}_+$, $\|u\|_\infty < \infty$. We say that Estimate 1 holds if for every cube $Q \subset \mathbb{R}^n$, and every ψ as above, we have:*

$$\int_{(1-s_n)Q} |u\big(x,\psi(x)\big) - u(P_Q)|^2 dx \leq C_{M,\eta} \iint_{\Omega_\psi \cap T_Q} |\nabla u|^2 \, t \, dtdx, \tag{5.4}$$

for some $s_n < 1$ sufficiently small, where $C_{M,\eta}$ depends also on dimension and ellipticity.

Estimate 2. *Let L, u be as in Estimate 1, $\|u\|_\infty \leq 1$. We say that Estimate 2 holds if*

$$\sup_Q \frac{1}{|Q|} \iint_{T_Q} |\nabla u(x,t)|^2 \, t \, dtdx \leq C. \tag{5.5}$$

Remark 5.6. For bounded null solutions of t-independent operators, Estimate 2 has already been proved in general: indeed, it is simply a re-statement of Corollary 1.10. Moreover, at this point, we have verified Estimate 1 for solutions u that are continuous up to the boundary. Indeed, Estimate 1 follows easily from (1.20), for every $s_n \in (0,1)$, by interior estimates for solutions, since $\psi \geq 0$ and thus $t - \psi(x) \leq t$. In turn, by the pull-back mechanism described in the proof of Corollary 1.17, (1.20) for continuous u follows directly from (1.15) for continuous u, and we have established the latter in Section 4. As discussed at the beginning of Section 4, this will be enough to establish Theorem 1.23 as we shall see momentarily.

The main result in this section is:

Theorem 5.7. *Assume that $Lu = 0$ in $\mathbb{R}^{n+1}_+$, $\|u\|_\infty \leq 1$, and that Estimate 1 and Estimate 2 hold for u. Then, for each $\epsilon > 0$, u is ϵ-approximable. The constant C_ϵ in the Carleson measure condition* (5.3) *depends also on dimension, ellipticity and the constants in Estimate 1 and Estimate 2, but not on Q_0.*

Before proving the theorem, let us use it to complete the proof of Theorem 1.23.

Proof of Theorem 1.23. As noted above, in order to obtain the conclusion of Theorem 1.23 via the program of [KKPT], it is enough to establish ϵ-approximability for solutions with boundary data of the form $u(x,0) = 1_{\mathcal{B}}$, where $\mathcal{B}$ is a bounded Borel set. Thus, given Theorem 5.7, it is enough to establish Estimate 1 for such solutions (since we already know that Estimate 2 holds for bounded solutions in general). Moreover, it is enough to do this for a t-independent operator L with smooth coefficients, as long as the bound in (5.4) depends only upon the stated parameters. Indeed, to prove Theorem 1.23, we may then proceed initially under the qualitative assumption that the coefficients are smooth, to obtain the A_∞ property of L-harmonic measure, but with A_∞ constants depending only on dimension and ellipticity. We may then deduce the A_∞ conclusion in the general case (i.e., without *a priori* smoothness of the coefficients), by an approximation argument as in [KKPT, pp. 256–257].

Therefore, we suppose that the coefficients of L are smooth, and we fix a bounded Borel set $\mathcal{B}$. For $Y = (y,s) \in \mathbb{R}^{n+1}_+$, set $u(Y) := \omega^Y(\mathcal{B})$, the solution of the Dirichlet problem with data $1_{\mathcal{B}}$. Let us first suppose that $\mathcal{B}$ is open. Let $X = (x,t)$ be a fixed point in $\mathbb{R}^{n+1}_+$. By the inner regularity of L-harmonic measure, and Urysohn's lemma, we may find a sequence $\{f_k\}$ of continuous functions, and closed sets $F_1 \subset F_2 \subset \cdots \subset F_k \subset \cdots \subset \mathcal{B}$, such that $f_k \equiv 1$ on F_k, $f_k \equiv 0$ on $\mathcal{B}^c$, and

such that $u_k(Y) \leq u(Y)$ for all $Y \in \mathbb{R}^{n+1}_+$, with $u_k(X) \to u(X)$ as $k \to \infty$, where u_k denotes the solution with data f_k. Thus, by Harnack's inequality,

$$u_k \to u, \quad \text{uniformly on compacta in } \mathbb{R}^{n+1}_+. \tag{5.8}$$

Our goal at the moment is to show that (5.4) holds for u. To this end, fix a small number $\delta > 0$, and given a Lipschitz function ψ, we set $\psi_\delta(x) := \max(\psi(x), \delta)$. We note that $\|\nabla\psi_\delta\|_\infty \leq \|\nabla\psi\|_\infty = M$ uniformly in δ. Since (5.4) holds for solutions that are continuous up to the boundary, we have for each $\delta > 0$, and for every cube Q, that

$$\begin{aligned}
\int_{(1-s_n)Q} |u\big(x,\psi_\delta(x)\big) - u(P_Q)|^2 dx &= \lim_{k\to\infty} \int_{(1-s_n)Q} |u_k\big(x,\psi_\delta(x)\big) - u_k(P_Q)|^2 dx \\
&\lesssim \limsup_{k\to\infty} \iint_{\Omega_{\psi_\delta}\cap T_Q} |\nabla u_k|^2\, t\, dtdx \\
&\lesssim \iint_{\Omega_{\psi_\delta}\cap T_Q} |\nabla u|^2\, t\, dtdx \\
&\quad + \limsup_{k\to\infty} \iint_{\Omega_{\psi_\delta}\cap T_Q} |\nabla(u_k - u)|^2\, t\, dtdx \\
&\lesssim \iint_{\Omega_{\psi}\cap T_Q} |\nabla u|^2,
\end{aligned}$$

since $\Omega_{\psi_\delta} \subset \Omega_\psi$, where the implicit constants depend only upon the stated parameters, and where the first limit holds by (5.8), and the second by Cacciopoli's inequality and (5.8). Recall that at this point we have assumed qualitatively that our coefficients are smooth, so that L-harmonic measure and Lebesgue measure dx on the boundary are mutually absolutely continuous. Thus, by the results of [CFMS] (which, as we have observed above, remain valid in the setting of non-symmetric coefficients), the bounded solution u converges non-tangentially to its boundary data a.e. (dx). We may therefore take a limit as $\delta \to 0$ to obtain (5.4) for solutions with boundary data given by the characteristic function of a bounded open set. To establish (5.4) when $u(x,0) = 1_{\mathcal{B}}$ is the characteristic function of a general bounded Borel set $\mathcal{B}$, we simply use outer regularity of harmonic measure, and repeat the previous argument, but now with $u_k(Y) := \omega^Y(\mathcal{O}_k)$, where $\{\mathcal{O}_k\}$ is a nested sequence of open sets containing $\mathcal{B}$. We omitted the details. □

We have now reduced matters to proving Theorem 5.7.

Proof of Theorem 5.7. We fix a cube $Q_0 \subset \mathbb{R}^n$, and by dilation and translation invariance, we may suppose that $Q_0 = \{0 \leq x_j \leq 1\}$ is the unit cube in $\mathbb{R}^n$. Then $T(Q_0) := T_{Q_0} = \{0 \leq x_j \leq 1, 0 \leq t \leq 1\}$ is the associated Carleson box. For large N (to be chosen to depend only on n) we let $S(Q_0) := \big\{0 \leq x_j \leq 1, 2^{-N} \leq t \leq 1\big\}$ be a "rectangle". As above, we let $P_{Q_0} = \big(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, 1-\eta\big)$ be the "Corkscrew point" relative to Q_0, where $0 < \eta < 1/4$ is to be chosen later, and set $\widetilde{P}_{Q_0} = \big(\frac{1}{2}, \ldots, \frac{1}{2}, 1\big)$. Thus, $|P_{Q_0} - \widetilde{P}_{Q_0}| = \eta l(Q_0) = \eta$.

For each $k = 1, 2, \ldots$ we partition Q_0 into 2^{knn} dyadic sub-cubes Q^k_j, with $l(Q^k_j) = 2^{-kN} l\,(Q_0) = 2^{-kN}$. By abuse of language, we call the collection $\big\{Q^k_j\big\}_{j,k}$ "the dyadic" sub-cubes of Q_0 (of course, they are dyadic, but they are not all of the dyadics). For Q a "dyadic" sub-cube of Q_0, we define $T(Q), S(Q), P_Q, \widetilde{P}_Q$

analogously. Note that for each "dyadic" Q, the "rectangles" $S(Q')$ such that $Q' \subset Q$ and Q' is "dyadic", form a "Whitney" tiling of $T(Q)$. For $\kappa > 1$ near 1 (depending on N, n) we let $\widetilde{S}(Q)$ be the rectangle obtained by expanding $S(Q)$ around its center by a factor of κ. If κ is close enough to 1 (depending on N, n), $Q = Q_j^k$, we still have $\mathrm{dist}(\widetilde{S}(Q), \mathbb{R}^n \times \{0\}) \approx 2^{-Nk}$. Moreover, we have

1) $\{\widetilde{S}(Q)\}$ have bounded overlaps.
2) If we fix Q_1, a "dyadic" cube, and consider $\{S(Q)\}$, $Q \subset Q_1$, Q "dyadic", then this is a "Whitney" tiling of $T(Q_1)$; moreover, $\{\widetilde{S}(Q)\}$ are all contained in $T(\widetilde{Q_1})$ where $\widetilde{Q_1}$ is the κ expansion of Q_1. We fix such a κ from now on.

We now fix an operator of the form $L = -\mathrm{div} A(x)\nabla$, where $(x,t) \in \mathbb{R}^{n+1}_+, x \in \mathbb{R}^n$, and $A(x)$ is an $(n+1) \times (n+1)$ real, elliptic, t-independent matrix, not necessarily symmetric, with ellipticity constant $\lambda > 0$. For solutions of $Lu = 0$ in $\mathbb{R}^{n+1}_+$, we have the following classical estimates:

5.1. **Preliminary estimates.** For the reader's convenience, we state here some classical estimates that we shall use repeatedly, in the form that we shall use them, i.e., stated for "dyadic" Q.
(Cacciopoli:)

$$\iint_{S(Q)} |\nabla u|^2 \leq \frac{C_{\lambda,n,N,\kappa}}{l(Q)^2} \iint_{\widetilde{S}(Q)} |u|^2 \tag{5.9}$$

(Regularity:)

$$\text{For } x, y \in S(Q),\ |u(x) - u(y)| \leq C_{\lambda,n,N,\kappa} \left(\frac{|x-y|}{l(Q)}\right)^{\alpha} \cdot l(Q) \left(\left|\widetilde{S}(Q)\right|^{-1} \iint_{\widetilde{S}(Q)} |\nabla u|^2\right)^{\frac{1}{2}}, \tag{5.10}$$

$\alpha = \alpha(\lambda, n) > 0$.
(Regularity bis:)

$$\text{For } x, y \in S(Q),\ |u(x) - u(y)| \leq C_{\lambda,n,N,\kappa} \left(\frac{|x-y|}{l(Q)}\right)^{\alpha} \left(\left|\widetilde{S}(Q)\right|^{-1} \iint_{\widetilde{S}(Q)} |u|^2\right)^{\frac{1}{2}}, \tag{5.11}$$

$\alpha = \alpha(\lambda, n) > 0$.

We now return to the proof of Theorem 5.7. Fix Q "dyadic", $Q \subset Q_0$, $\widetilde{P}_Q$ as before. Let $\widetilde{Q}$ be the cube (in $\mathbb{R}^n \times \{l(Q)\}$) centered at $\widetilde{P}_Q$, with $\mathrm{diam}(\widetilde{Q}) = 2\eta l(Q)$, so that $\widetilde{Q} \subset \mathrm{top}\, S(Q)$. Note that $H^n(\widetilde{Q}) = c_n \eta^{(n)} |Q|$.

Claim 5.12. Assume $\epsilon > 0$ is given, Assume that for some constant A, $|A| \leq 1$, we have $|u(P_Q) - A| \geq \frac{\epsilon}{10}$. Then $\forall X \in \widetilde{Q}$, we have $|u(X) - A| \geq \frac{\epsilon}{20}$, provided $\eta = \eta(\epsilon, \lambda, n)$ is small enough.

Indeed, by (5.11),

$$|u(X) - u(P_Q)| \leq C_{\lambda,n,N,\kappa} \left(\frac{|X - P_Q|}{l(Q)}\right)^{\alpha} \cdot \left(⨍\!\!⨍_{\widetilde{S}(Q)} u^2\right)^{\frac{1}{2}} \leq C_{\lambda,n,N,\kappa} \cdot \eta^{\alpha} \leq \frac{\epsilon}{20}$$

if $X \in \widetilde{Q}$, and if η is small.

Now, given $\epsilon > 0$ as in Thm 5.7, we choose and fix η as in Claim 5.12.

5.2. **Stopping time construction, part I.** We will now define "generation" cubes. We set $G_0 = \{Q_0\}$. Fix $\epsilon > 0$, and define the first generation, $G_1 = G_1(Q_0)$ to be the maximal "dyadic" $Q \subset Q_0$, for which $|u(P_Q) - u(P_{Q_0})| \geq \frac{\epsilon}{10}$. The "dyadic" cubes in $G_1(Q_0)$ have pairwise disjoint interiors. For $Q \in G_1(Q_0)$ we define $G_1(Q)$ in the same way. We set $G_2 = G_2(Q_0) = \cup\{G_1(Q) : Q \in G_1\}$. Later generations, $G_3, G_4, \ldots$ are defined inductively. Note that each $Q \in G_{p+1}$ is contained in a unique $Q' \in G_p$ and $|u(P_Q) - u(P_{Q'})| \geq \frac{\epsilon}{10}$.

Lemma 5.13. *There exist $0 < \mu < 1$, and $N = N(\lambda, n, \mu)$ such that*

$$\sum_{Q_j \in G_1} |Q_j| \leq C_{\epsilon,\lambda,n,(5.4),\mu} \iint_{T(Q_0)\setminus \cup_{Q_j \in G_1} T(Q_j)} t|\nabla u|^2 dxdt + (1-\mu)|Q_0|,$$

and more generally, if $Q' \in G_p$, we have

$$\sum_{Q_j \in G_1(Q')} |Q_j| \leq C_{\epsilon,\lambda,\mu,(5.4),\mu} \iint_{T(Q')\setminus \cup_{Q_j \in G_1(Q')} T(Q_j)} t|\nabla u|^2 dxdt + (1-\mu)|Q'|.$$

Proof. We prove the first estimate, the proof of the second one being the same. Consider the infinite downward cone, $\Gamma_\delta := \{(x,t) : |x| < -\delta t, t < 0\}$, where $\delta > 0$ is small. Let $U_1 = \cup T(Q_j)$, $Q_j \in G_1$. Consider $\Omega_- = (\cup_{P \in U_1}(P + \Gamma_\delta)) \cap T(Q_0)$ and also $\Omega_+ = (T(Q_0)\setminus\Omega_-)^\circ$.

We begin with several observations. If $Q \in G_1$, then $l(Q) \leq 2^{-N}$, by the definition of "dyadic" and the fact that $Q_0 \notin G_1$. Also, $\Omega = \cup_{P \in U_1}(\Gamma_\delta + P)$ is a domain given as the domain below the graph of a Lipschitz function Ψ_1, whose Lipschitz constant is less than $\frac{1}{\delta}$. (One way to see this is that Ω verifies the uniform infinite exterior and interior cone conditions with respect to uniform vertical cones, since U_1 is given by a graph). The next observation is that, for $N > 2, 0 \leq \Psi_1 \leq \frac{1}{4}$ on Q_0. Another observation is that $\Omega_+ \cap U_1 = \emptyset$. Let $Q_{i_0}, Q_{i_1} \in G_1$ be given. We say that "Q_{i_0} partially covers" Q_{i_1} if $Q_{i_0} \neq Q_{i_1}$, and

$$\big[\big[\cup_{P \in T(Q_{i_0})}(\Gamma_\delta + P)\big] \cap T(Q_{i_0}))\big] \cap \operatorname{top} T(Q_{i_1}) \neq \emptyset,$$

where we note that $\operatorname{top} T(Q_{i_1}) = \operatorname{top} S(Q_{i_1})$, and $\cup_{P \in T(Q_{i_0})}(\Gamma_\delta + P) \cap T(Q_0) = \cup_{P \in \operatorname{top} T(Q_{i_0})}(\Gamma_\delta + P) \cap T(Q_0)$.

Note that if Q_{i_0} partially covers Q_{i_1}, we must have $l(Q_{i_1}) < l(Q_{i_0})$.

We say that $Q_{i_0}, Q_{i_1}, \ldots, Q_{i_k} \in G_1$ are such that $(Q_{i_0}, Q_{i_1}, \ldots, Q_{i_k})$ forms a chain starting at Q_{i_0} and ending at Q_{i_k}, if for each $0 \leq j \leq k-1$, Q_{i_j} partially covers $Q_{i_{j+1}}$. Fix $Q_{i_0} \in G_1$. We define $T_r(Q_{i_0})$, the tree with top Q_{i_0}, by

$$\begin{aligned} &T_r(Q_{i_0}) \\ &\quad := \{\text{all cubes } Q_j \in G_1 : \text{there exists a chain starting at } Q_{i_0}, \text{ending at } Q_j\} \\ &\quad \cup Q_{i_0}. \end{aligned}$$

Finally, we say that $Q_{j_0} \in G_1$ is "uncovered" if there exists no $Q \in G_1$ with Q partially covering Q_{j_0}.

Fact 1. *For small δ, $T_r(Q_{j_0}) \subset 8Q_{j_0}$, for any $Q_{j_0} \in G_1$, where $8Q_{j_0}$ is the cube with length $8l(Q_{j_0})$ and same center as Q_{j_0}.*

Proof. Let $Q_j \neq Q_{j_0} \in T_r(Q_{j_0})$. Then there exists a chain $(Q_{j_0}, Q_{j_1}, \ldots, Q_{j_k})$ with $Q_{j_k} = Q_j$. Note that since Q_{j_s} partially covers $Q_{j_{s+1}}, s \geq 0, l(Q_{j_{s+1}}) \leq 2^{-N} l(Q_{j_s})$. Also note that if Q_{i_0} partially covers $Q_{i_1}, \exists y_1 \in Q_{i_1}$ with

$$(y_1, l(Q_{i_1})) \in \big[\cup_{P \in T(Q_{i_0})}(\Gamma_\delta + P) \cap T(Q_0)\big] = \big[\cup_{P \in \operatorname{top} T(Q_{i_0})}(\Gamma_\delta + P) \cap T(Q_0)\big].$$

Since $(y_1, l(Q_{i_1})) \in T(Q_0)$, (because $Q_{i_1} \subset Q_0$), there exists $P = (x, l(Q_{i_0}))\,, x \in Q_{i_0}$ such that $(y_1, l(Q_{i_1})) \in \Gamma_\delta + P$, i.e., $|y_1 - x| < -\delta\,[l(Q_{i_1}) - l(Q_{i_0})] = \delta[l(Q_{i_0}) - l(Q_{i_1})]$.

Let $x_{i_0} =$ center of Q_{i_0}, $r_{i_0} = \max_{x\in Q_{i_0}} |x - x_{i_0}|$, so that $r_{i_0} = c_n l(Q_{i_0})$. For any $\tilde{y} \in Q_{i_1}$,

$$\begin{aligned} |x_{i_0} - \tilde{y}| &\le |x - x_{i_0}| + |x - y_1| + |y_1 - \tilde{y}| \\ &\le c_n l(Q_{i_0}) + \delta\,[l(Q_{i_0}) - l(Q_{i_1})] + 2c_n l(Q_{i_1}) \le 4c_n l(Q_{i_0}), \end{aligned}$$

if δ is small.

Next note that if $(Q_{j_0} \ldots, Q_{j_k})$ is a chain, then $l(Q_{j_k}) \le 2^{-kN} l(Q_{j_0})$. Suppose now that $\tilde{y} \in Q_{j_k} = Q_j$. Then, $|\tilde{y} - x_{j_{k-1}}| \le 4c_n l(Q_{j_{k-1}})$ by the previous estimate and

$$|x_{j_s} - x_{j_{s-1}}| \le 4c_n l(Q_{j_{s-1}}), s = 1, \ldots, k$$

Hence,

$$\begin{aligned} |\tilde{y} - x_{j_0}| &\le |\tilde{y} - x_{j_{k-1}}| + |x_{j_{k-1}} - x_{j_{k-2}}| + \cdots + |x_{j_1} - x_{j_0}| \\ &\le 4c_n l(Q_{j_{k-1}}) + 4c_n l(Q_{j_{k-2}}) + \cdots + 4c_n l(Q_{j_0}) \\ &\le 4c_n l(Q_{j_0}) \left[1 + \frac{1}{2^N} + \frac{1}{2^{2N}} + \cdots + \frac{1}{2^{(k-1)N}}\right] \\ &\le 8c_n l(Q_{j_0}), \end{aligned}$$

where we used $l(Q_{j_s}) \le 2^{-Ns} l(Q_{j_0})$, which follows because $(Q_{j_0}, \ldots, Q_{j_s})$ is a chain. Fact 1 follows. □

Fact 2.

$$|\cup_{Q\in T_r(Q_{j_0})} Q| \le c_n |Q_{j_0}|.$$

Follows from Fact 1 and the disjointness of the intervals in G_1.

Fact 3. *Assume that $Q_{j_0} \in G_1$ is "uncovered". Then, $(x, l(Q_{j_0})), x \in Q_{j_0}$, belongs to the graph of Ψ_1, i.e. $\Psi_1(x) = l(Q_{j_0})$, $x \in Q_{j_0}$, and hence to the boundary of $\Omega_+ \cap T(Q_0)$.*

This is immediate from the definition of Ω_-, Ψ_1 and the definition of "partially covers" and "uncovered".

Let now $\tilde{G}_1 = \{Q \in G_1 : Q$ is "uncovered"$\}$.

Fact 4.

$$G_1 = \cup_{Q\in\tilde{G}_1} T_r(Q).$$

It suffices to show $G_1 \subset \cup_{Q\in\tilde{G}_1} T_r(Q)$. Define $i_1 = \min\left\{i : l(Q) = 2^{-iN}, Q \in G_1\right\}$. Let $G_{1,1} = G_1, \tilde{G}_{1,1} = \left\{Q \in G_1 : l(Q) = 2^{-i_1 N}\right\}$. Note that if $Q \in \tilde{G}_{1,1}$, then Q is "uncovered", because if Q' partially covers Q, $l(Q) \le 2^{-N} l(Q')$ which is impossible for $Q \in \tilde{G}_{1,1}$ since $l(Q)$ is maximal among lengths in G_1. Note also that $i_1 \ge 1$. Next, let $G_{1,2} = G_1 \setminus \cup_{Q\in\tilde{G}_{1,1}} T_r(Q)$. Let $i_2 = \min\left\{i : l(Q) = 2^{-iN}, Q \in G_{1,2}\right\}$, unless $G_{1,2} = \emptyset$, in which case the process stops. Note that unless the process stops, $i_2 > i_1$. Let now $\tilde{G}_{1,2} = \left\{Q \in G_{1,2} : l(Q) = 2^{-i_2 N}\right\}$. We claim that if $Q_1 \in \tilde{G}_{1,2}$, then Q_1 is "uncovered". Suppose not, let $Q' \in G_1$ partially cover Q_1. Then, $l(Q_1) < l(Q')$, so Q' cannot belong to $G_{1,2}$. Hence, $Q' \in \cup_{Q\in\tilde{G}_{1,1}} T_r(Q)$. Thus, there exists $Q \in \tilde{G}_{1,1}$ such that $Q' \in T_r(Q)$, i.e., $\exists\,(Q_{i_0}, \ldots, Q_{i_k})$ a chain,

with $Q = Q_i \in \tilde{G}_{1,1}, Q_{i_k} = Q'$. But then, since $(Q_{i_0}, \ldots, Q_{i_k}, Q_1)$ is a chain, $Q_1 \in T_r(Q_{i_0}), Q_0 \in \tilde{G}_{1,1}$, which contradicts the fact that $Q_1 \in G_{1,2}$. Thus, Q_1 is "uncovered". Next, we define

$$G_{1,3} = G_{1,2} \Big\backslash \bigcup_{Q \in \tilde{G}_{1,2}} T_r(Q) = G_1 \Big\backslash \left[\cup_{Q \in \tilde{G}_{1,1}} T_r(Q) \bigcup \cup_{Q \in \tilde{G}_{1,2}} T_r(Q) \right].$$

Let $i_3 = \min\{i : l(Q) = 2^{-iN}, Q \in G_{1,3}\}$ (unless $G_{1,3} = \emptyset$ in which case the process stops). If the process does not stop, we let $\tilde{G}_{1,3} = \{Q \in G_{1,3} : l(Q) = 2^{-i_3 N}\}$. We claim that if $Q_1 \in \tilde{G}_{1,3}$ then Q_1 is "uncovered". If not, $\exists Q' \in G_1$, with Q' partially covering Q_1, so that $l(Q_1) < l(Q')$. Hence, Q' cannot belong to $G_{1,3}$. If $Q' \in \cup_{Q \in \tilde{G}_{1,2}} T_r(Q)$, we reach a contradiction as before. Hence, Q' cannot belong to $G_{1,2}$, since $G_{1,2} = G_{1,3} \cup [\cup_{Q \in \tilde{G}_{1,2}} T_r(Q)]$. Since $G_1 = G_{1,2} \bigcup \cup_{Q \in \tilde{G}_{1,1}} T_r(Q), Q' \in \cup_{Q \in \tilde{G}_{1,1}} T_r(Q)$. But then $Q_1 \in \cup_{Q \in \tilde{G}_{1,1}} T_r(Q)$, a contradiction. We continue inductively in this manner. If the process stops at stage k, we have

$$G_1 \subset \bigcup_{Q \in \tilde{G}_{1,k-1}} T_r(Q) \cup \bigcup_{Q \in \tilde{G}_{1,k-2}} T_r(Q) \cup \cdots \cup \bigcup_{Q \in \tilde{G}_{1,1}} T_r(Q),$$

and Fact 4 follows. If the process never stops, $i_k \uparrow \infty$ and it is also easy to verify Fact 4.

Fact 5.

$$\Sigma_{Q_j \in G_1} |Q_j| \leq c_n \sum_{Q_j \in \tilde{G}_1} |Q_j|, \qquad (c_n > 1).$$

Let $O_1 = \cup_{Q_j \in G_1} Q_j$, $|O_1| = \sum_{Q_j \in G_1} |Q_j|$. Now use Fact 4 and Fact 2.

End of the proof of Lemma 5.13: For μ to be chosen, N to be chosen, consider now:

Case 5.14.

$$\sum_{Q_j \in G_1} |Q_j| \leq (1 - \mu)|Q_0|.$$

In this case Lemma 5.13 clearly holds.

Case 5.15.

$$\sum_{Q_j \in G_1} |Q_j| > (1 - \mu)|Q_0|.$$

Consider now $\tilde{G}_1' = \{Q_j \in \tilde{G}_1 : Q_j \subset (1 - s_n)Q_0\}$. Let

$$\tilde{G}_1'' := \{Q_j \in \tilde{G}_1 : Q_j \cap (Q_0 \backslash (1 - s_n)Q_0) \neq \emptyset\}.$$

We claim that if $Q_j \in \tilde{G}_1''$, then, if N is large enough, $Q_j \cap (1 - 2s_n)Q_0 = \emptyset$. Let $x_0 =$ center of $Q_0, x_1 \in Q_j \cap (Q_0 \backslash (1 - s_n)Q_0), x \in Q_j$. Then,

$$|x - x_0| \geq |x_1 - x_0| - |x - x_1| \geq d_n(1 - s_n)l(Q_0) - 2d_n 2^{-N} l(Q_0) \geq d_n(1 - 2s_n)l(Q_0),$$

for N large, where d_n is chosen so that if Q is a cube with center x_Q and length $l(Q)$, then, for $x \in Q$, $|x - x_Q| \leq d_n l(Q)$.

Because of the claim, $\sum_{Q_j\in\tilde{G}_1''}|Q_j| \leq [1-(1-2s_n)^n]\,|Q_0|$. But then, if we choose s_n so small that, with c_n as in Fact 5, we have $c_n\,[1-(1-2s_n)^n] \leq \delta_n$, where $2\delta_n < 1$ and $\mu = \delta_n$, then

$$\begin{aligned}(1-\mu)|Q_0| &\leq \sum_{Q_j\in G_1}|Q_j| \leq c_n \sum_{Q_j\in\tilde{G}_1}|Q_j| \\ &\leq c_n \sum_{Q_j\in\tilde{G}_1'}|Q_j| + c_n \sum_{Q_j\in\tilde{G}_1''}|Q_j| \\ &\leq c_n \sum_{Q_j\in\tilde{G}_1'}|Q_j| + c_n\,[1-(1-2s_n)^n]\,|Q_0| \\ &\leq c_n \sum_{Q_j\in\tilde{G}_1'}|Q_j| + \delta_n|Q_0|.\end{aligned}$$

Then $(1-2\delta_n)|Q_0| \leq c_n \sum_{Q_j\in\tilde{G}_1'}|Q_j|$, and so

$$\sum_{Q_j\in G_1}|Q_j| \leq \frac{c_n}{(1-2\delta_n)}\sum_{Q_j\in\tilde{G}_1'}|Q_j|.$$

Hence, using estimate (5.4), the construction of generation cubes, the claim at the start of the proof of 5.7 and Fact 3, we get

$$|u(P_{Q_0}) - u(X)| \geq \frac{\epsilon}{100}, \quad X\in\widetilde{Q}_j.$$

$$\begin{aligned}\int_{\{(x,\Psi_1(x)):x\in(1-s_n)Q_0\}}|u-u(P_{Q_0})|^2 &\leq C_{\delta,\eta,\lambda,n,(5.4)}\iint_{\Omega_+\cap T(Q_0)} t|\nabla u|^2dxdt \\ &\leq C_{\delta,\eta,\lambda,n,(5.4)}\iint_{{}^cU_1\cap T(Q_0)} t|\nabla u|^2dxdt,\end{aligned}$$

since $\Omega_+\cap T(Q_0)\subset {}^cU_1\cap T(Q_0)$. Thus,

$$\frac{\epsilon^2}{100^2}\sum_{Q_j\in\tilde{G}_1'}|Q_j| \leq C_{\delta,\eta,\lambda,n,(5.4)}\iint_{T(Q_0)\setminus\cup_{Q\in G_1}T(Q)} t|\nabla u|^2,$$

which shows that in case 5.15, $\frac{\epsilon^2}{100^2}\sum_{Q_j\in G_1}|Q_j| \leq C_{\delta,n,\lambda,\mu,(5.4)}\iint_{T(Q_0)\setminus\cup_{Q\in G_1}T(Q)} t|\nabla u|^2$, finishing the proof of Lemma 5.13. □

Recall that Q is a "generation cube" if $Q\in G_p$ for some $p\geq 1$. We define $G_0 = \{Q_0\}$.

Lemma 5.16. *"Packing property" Let Q be a "dyadic" cube $\subset Q_0$. Then*

$$\sum_{Q_j\subset Q, Q_j \text{ a generation cube}}|Q_j| \leq C_{\lambda,n,\epsilon,\eta,N,\mu,(5.4),(5.5)}|Q|.$$

Proof. Let $M(Q) = \{\text{maximal generation cubes contained in } Q\}$, i.e., $Q_1\in M(Q)$ if Q_1 is a generation cube and $\nexists Q'$, a generation cube, $Q'\subset Q$ with $Q_1\subsetneq Q'$. Note that the cubes in $M(Q)$ are pairwise disjoint, and any generation cube Q_j contained in Q is contained in a unique maximal $Q_1\in M(Q)$. By disjointness, $\sum_{Q_1\in M(Q)}|Q_1| \leq |Q|$.

By the construction, we must have

$$\{Q_j : Q_j\subset Q, Q_j \text{ is a generation cube}\} = \cup_{Q_1\in M(Q)}\cup_{p\geq 0} G_p(Q_1).$$

Fix Q and fix a maximal generation cube contained in Q, Q_1. We define $G_0 := G_0(Q_1) = \{Q_1\}$, and $G_1 := G_1(Q_1), G_2 := G_2(Q_1), \ldots$, etc., analogously to $G_p(Q_0)$ above. We define $U_0 = Q_1, U_1 = \cup_{Q' \in G_1(Q_1)} Q', U_2 = \cup_{Q' \in G_2(Q_1)} Q'$, etc., and note that

$$U_{p+1} = \cup_{Q' \in G_p(Q_1)} U_1(Q').$$

Thus, $|U_{p+1}| = \sum_{Q' \in G_p} |U_1(Q')|$. By Lemma 5.13, for $p = 0, 1, \ldots$, we have

$$\begin{aligned} |U_{p+1}| &\leq C \sum_{Q' \in G_p} \iint_{T(Q') \setminus \cup_{Q'' \in G_1(Q')} T(Q'')} t|\nabla u|^2 + (1-\mu) \sum_{Q' \in G_p} |Q'| \\ &= C \sum_{Q' \in G_p} \iint_{T(Q') \setminus \cup_{Q'' \in G_1(Q')} T(Q'')} t|\nabla u|^2 + (1-\mu)|U_p|. \end{aligned}$$

Thus, using the disjointness of the regions $T(Q') \setminus \cup_{Q'' \in G_1(Q')} T(Q'')$ for each fixed p, in Q' and for consecutive p's, and summing in p, we obtain:

$$\sum_{p=0}^{\infty} |U_{p+1}| \leq C \iint_{T(Q_1)} t|\nabla u|^2 + (1-\mu) \sum_{p=0}^{\infty} |U_p|$$

Thus, $\mu \sum_{p=1}^{\infty} |U_p| \leq C \iint_{T(Q_1)} t|\nabla u|^2 + (1-\mu)|Q_1|$ and using (5.5), we obtain $\sum_{p=1}^{\infty} |U_p| \leq C|Q_1|$ or, for each $Q_1 \in M(Q)$,

$$\sum_{p \geq 0} \sum_{Q_j \in G_p(Q_1)} |Q_j| \leq C_{\lambda, n, \epsilon, \eta, N, \mu, (5.4), (5.5)} |Q_1|.$$

If we now sum over $Q_1 \in M(Q)$, Lemma 5.16 follows. □

5.3. **The stopping time construction, part II.** For each generation cube Q, we define the corresponding Carleson box $T(Q)$ and the "rectangle" $S = S(Q)$. We call the resulting $T(Q)$'s "generation boxes". For each generation box $T(Q)$, we define the "dyadic sawtooth region" $\Omega(Q) = T(Q) \setminus \cup_{Q_i \in G_1(Q)} T(Q_i)$.

Note that if $Q' \subset Q_0$ is a "dyadic" sub-cube, then $S = S(Q')$ is contained in a unique $\Omega(Q)$. The uniqueness comes from the fact that if two generation intervals Q_j, Q_i are distinct, their associated regions $\Omega(Q_j)$, $\Omega(Q_i)$ have disjoint interiors. The fact that S is contained in some $\Omega(Q)$ is due to the fact that if $l_p = \max\{l(Q) : Q \in G_p\}$, then $l_p \to 0$.

Next, relative to $\mathbb{R}^{n+1}_+$, for each generation cube Q, $\partial\Omega(Q)$ consists of horizontal and vertical "segments". The intersection of these "segments" with any box $T(Q')$ have H^n measure adding up to at most $c_n|Q'|$, since $H^n(\partial T(Q)) = c_n|Q|$. Also, the $\{Q_j\}$ in $G_1(Q)$, Q a generation cube, are non-overlapping, by maximality. For each generation cube Q_j, including the unit cube Q_0, we define $\varphi_1(z) = u(P_{Q_j})$ on the interior of $\Omega(Q_j)$. Thus,

$$\varphi_1(z) = \sum_{p=0}^{\infty} \sum_{Q_j \in G_p} u(P_{Q_j}) \chi_{\mathring{\Omega}(Q_j)}.$$

We consider now $|\nabla \varphi_1(z)|$. As a distribution on $\mathbb{R}^{n+1}_+$,

$$\nabla \varphi_1 = \sum_{p=0}^{\infty} \sum_{Q_j \in G_p} u(P_{Q_j}) \nabla \chi_{\mathring{\Omega}(Q_j)}.$$

It is easy to see that $|\nabla \chi_{\Omega(Q_j)}^{\circ}| = dH^n \lfloor_{\Sigma_j}$, where $\Sigma_j = \{t > 0\} \cap \partial\Omega(Q_j)$. Since $|u(P_{Q_j})| \leq 1, |\nabla\varphi_1| \leq \sum_{p=0}^{\infty} \sum_{Q_j \in G_p} |\nabla \chi_{\Omega(Q_j)}^{\circ}|$. Thus, for fixed Q, we have

$$\iint_{T(Q)} |\nabla\varphi_1| \leq \sum_{p,j} H^n(T(Q) \cap \Sigma_j).$$

Claim 5.17.

$$\sum_{p,j} H^n(T(Q) \cap \Sigma_j) \leq C_{\epsilon,\lambda,\mu,(5.4),(5.5)} |Q|.$$

To see this, first consider those Q_j such that

$$T(Q) \cap \Sigma_j = T(Q) \cap \partial\Omega(Q_j) \cap \{t > 0\} \neq \emptyset,$$

but such that $Q_j \not\subseteq Q$. In this case, assume first that $l(Q_j) \leq l(Q)$. Then, $T(Q) \cap \Sigma_j$ is a union of "intervals" along a "vertical" side of $T(Q)$. These "intervals" are pairwise disjoint, so they contribute at most $c_n H^n(\partial T(Q))$. If $l(Q_j) > l(Q)$, there are at most c_n such cubes, each contributes at most $c_n H^n(\partial T(Q))$. Next we consider generation cubes such that $Q_j \subset Q$. Then,

$$\sum_{Q_j \subset Q} H^n(T(Q) \cap \Sigma_j) \leq c_n \sum_{Q_j \subset Q} |Q_j| \leq C_{\epsilon,\lambda,n,(5.4),(5.5)} |Q|,$$

by Lemma 5.16. Thus, $|\nabla\varphi_1|$ is a Carleson measure.

We now say that $S = S(Q)$ is a blue "rectangle" if

$$\sup_{X,Y \in S} |u(X) - u(Y)| \leq \frac{\epsilon}{10}.$$

Otherwise, we say that S is a red "rectangle". Assume that $S = S(Q)$ is a blue "rectangle". Let Q_j be the unique generation cube so that $S(Q) \subset \Omega(Q_j)$. Because $S(Q) \subset \Omega(Q_j)$, $|u(P_Q) - u(P_{Q_j})| < \frac{\epsilon}{10}$. Since $P_Q \in S(Q)$, if $X \in S(Q)$, then $|u(P_Q) - u(X)| < \frac{\epsilon}{10}$. Hence, $|u(X) - u(P_{Q_j})| \leq \frac{\epsilon}{5}$ for $X \in S(Q)$. But, $\varphi_1(X) = u(P_{Q_j})$ on $\Omega(Q_j)$, so that $|\varphi_1(X) - u(X)| \leq \frac{\epsilon}{5}$ on every blue S.

The final step is to correct φ_1 in the red rectangles. $S = S(Q)$ is red if there exists $X_0, Y_0 \in S$ such that

$$|u(X_0) - u(Y_0)| > \frac{\epsilon}{10}.$$

Let $\widetilde{S}$ be the slightly larger version of S, as at the start of this section. By (5.10),

$$\begin{aligned} \frac{\epsilon^2}{100} &\leq C_{\lambda,n}^2 \left(\frac{|X_0 - Y_0|}{l(Q)} \right)^{2\alpha} l(Q)^2 \fint\!\!\!\fint_{\widetilde{S}} |\nabla u|^2 \\ &\leq C_{\lambda,n}^2 \frac{1}{l(Q)^n} \iint_{\widetilde{S}} t\, |\nabla u|^2 \end{aligned}$$

or

$$|Q| \leq \frac{C_{\lambda,n}^2}{\epsilon^2} \iint_{\widetilde{S}} t\, |\nabla u|^2.$$

By the bounded overlap of $\{\widetilde{S}\}$, we have:

$$\sum_{Q_k \subset Q : S(Q_k) \text{ red}} |Q_k| \leq \frac{C_{\lambda,n,(5.5)}^2}{\epsilon^2} |Q|,$$

in view of estimate (5.5). Also, if S is red, then by (5.9) (with $S = S(Q)$),

$$\begin{aligned}\iint_S |\nabla u| &\le \left(\iint_S |\nabla u|^2\right)^{\frac{1}{2}} l(Q)^{\frac{n+1}{2}} \\ &\le \frac{C_{\lambda,n}}{l(Q)} \left(\iint_{\widetilde{S}} |u|^2\right)^{\frac{1}{2}} l(Q)^{\frac{n+1}{2}} \\ (\text{since } \|u\|_\infty \le 1) &\le \frac{C_{\lambda,n}}{l(Q)} \cdot l(Q)^{n+1} \le \frac{C_{\lambda,n}}{\epsilon^2} \iint_{\widetilde{S}} t|\nabla u|^2,\end{aligned}$$

by the previous estimate.

Then, if $\mathcal{R} = \cup_{S=S(Q'),\, S \text{ red}} S$, and we consider $|\nabla u|\, \chi_{\mathcal{R}}$, also note that $T(Q) \cap \mathcal{R} = \cup_{Q': S(Q') \subset T(Q),\, S(Q') \text{ red}} S(Q')$. Then,

$$\begin{aligned}\iint_{T(Q)} |\nabla u|\, \chi_{\mathcal{R}} &= \sum_{S=S(Q')\subset T(Q),\, S \text{ red}} \iint_{S(Q')} |\nabla u| \\ &\le \sum \frac{C_{\lambda,n}}{\epsilon^2} \iint_{\widetilde{S}(Q')} t|\nabla u|^2 \le \frac{C_{\lambda,n}}{\epsilon^2} \iint_{T(\widetilde{Q})} t|\nabla u|^2 \\ &\le C_{\lambda,n,(5.5)}\, |Q|\end{aligned}$$

by (5.5), so that $|\nabla u|\, \chi_{\mathcal{R}}$ is a Carleson measure.

Define now

$$\varphi_2(z) = \begin{cases} \varphi_1(z), & z \notin \mathcal{R} \\ u(z), & z \in \mathcal{R} \end{cases}$$

We clearly have $|u(z) - \varphi_2(z)| \le \epsilon$. Also, $\nabla\varphi_2(z) = \chi_{\mathcal{R}} \nabla u + \chi_{(T(Q_0)\setminus\mathcal{R})} \nabla\varphi_1 + J$, where J accounts for the jumps of φ_2 as z crosses $\partial\mathcal{R} \cap \mathbb{R}^{n+1}_+$. Since $|\varphi_2| \le 1 + \epsilon$, J is a measure dominated by $(1+\epsilon)\, dH^n \lfloor_{\partial\mathcal{R}}$. This last measure is Carleson by a previous estimate. This proves Theorem 5.7. □

References

[A] Pascal Auscher, *On necessary and sufficient conditions for L^p-estimates of Riesz transforms associated to elliptic operators on $\mathbb{R}^n$ and related estimates*, Mem. Amer. Math. Soc. **186** (2007), no. 871, xviii+75, DOI 10.1090/memo/0871. MR2292385 (2007k:42025)

[AA] Pascal Auscher and Andreas Axelsson, *Weighted maximal regularity estimates and solvability of non-smooth elliptic systems I*, Invent. Math. **184** (2011), no. 1, 47–115, DOI 10.1007/s00222-010-0285-4. MR2782252 (2012c:35111)

[AAAHK] M. Angeles Alfonseca, Pascal Auscher, Andreas Axelsson, Steve Hofmann, and Seick Kim, *Analyticity of layer potentials and L^2 solvability of boundary value problems for divergence form elliptic equations with complex L^∞ coefficients*, Adv. Math. **226** (2011), no. 5, 4533–4606, DOI 10.1016/j.aim.2010.12.014. MR2770458 (2012g:35055)

[AHLMcT] Pascal Auscher, Steve Hofmann, Michael Lacey, Alan McIntosh, and Ph. Tchamitchian, *The solution of the Kato square root problem for second order elliptic operators on $\mathbb{R}^n$*, Ann. of Math. (2) **156** (2002), no. 2, 633–654, DOI 10.2307/3597201. MR1933726 (2004c:47096c)

[AHLT] Pascal Auscher, Steve Hofmann, John L. Lewis, and Philippe Tchamitchian, *Extrapolation of Carleson measures and the analyticity of Kato's square-root operators*, Acta Math. **187** (2001), no. 2, 161–190, DOI 10.1007/BF02392615. MR1879847 (2004c:47096a)

[AT] Pascal Auscher and Philippe Tchamitchian, *Square root problem for divergence operators and related topics* (English, with English and French summaries), Astérisque **249** (1998), viii+172. MR1651262 (2000c:47092)

[BLP] Alain Bensoussan, Jacques-Louis Lions, and George Papanicolaou, *Asymptotic analysis for periodic structures*, Studies in Mathematics and its Applications, vol. 5, North-Holland Publishing Co., Amsterdam, 1978. MR503330 (82h:35001)

[CFMS] L. Caffarelli, E. Fabes, S. Mortola, and S. Salsa, *Boundary behavior of nonnegative solutions of elliptic operators in divergence form*, Indiana Univ. Math. J. **30** (1981), no. 4, 621–640, DOI 10.1512/iumj.1981.30.30049. MR620271 (83c:35040)

[CF] R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250. MR0358205 (50 #10670)

[CMS] R. R. Coifman, Y. Meyer, and E. M. Stein, *Some new function spaces and their applications to harmonic analysis*, J. Funct. Anal. **62** (1985), no. 2, 304–335, DOI 10.1016/0022-1236(85)90007-2. MR791851 (86i:46029)

[D] Björn E. J. Dahlberg, *Approximation of harmonic functions* (English, with French summary), Ann. Inst. Fourier (Grenoble) **30** (1980), no. 2, vi, 97–107. MR584274 (82i:31010)

[DJK] Björn E. J. Dahlberg, David S. Jerison, and Carlos E. Kenig, *Area integral estimates for elliptic differential operators with nonsmooth coefficients*, Ark. Mat. **22** (1984), no. 1, 97–108, DOI 10.1007/BF02384374. MR735881 (85h:35021)

[FS] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), no. 3-4, 137–193. MR0447953 (56 #6263)

[G] John B. Garnett, *Bounded analytic functions*, Pure and Applied Mathematics, vol. 96, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1981. MR628971 (83g:30037)

[GT] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 224, Springer-Verlag, Berlin, 1983. MR737190 (86c:35035)

[H] Steve Hofmann, *A local Tb theorem for square functions*, Perspectives in partial differential equations, harmonic analysis and applications, Proc. Sympos. Pure Math., vol. 79, Amer. Math. Soc., Providence, RI, 2008, pp. 175–185. MR2500492 (2010b:42025)

[HKMP2] Hofmann S., Kenig C., Mayboroda S., and Pipher J., *The Regularity problem for second order elliptic operators with complex-valued bounded measurable coefficients.* preprint.

[HLMc] Steve Hofmann, Michael Lacey, and Alan McIntosh, *The solution of the Kato problem for divergence form elliptic operators with Gaussian heat kernel bounds*, Ann. of Math. (2) **156** (2002), no. 2, 623–631, DOI 10.2307/3597200. MR1933725 (2004c:47096b)

[HMaMo] Hofmann S., Mayboroda S., and Mourgoglou M., *L^p and endpoint solvability results for divergence form elliptic equations with complex L^∞ coefficients.* preprint.

[HMiMo] Hofmann S., Mitrea M., and Morris A., *The method of layer potentials in L^p and endpoint spaces for elliptic operators with L^∞ Coefficients.* preprint.

[JK] David S. Jerison and Carlos E. Kenig, *The Dirichlet problem in nonsmooth domains*, Ann. of Math. (2) **113** (1981), no. 2, 367–382, DOI 10.2307/2006988. MR607897 (84j:35076)

[Ka] Tosio Kato, *Perturbation theory for linear operators*, Die Grundlehren der mathematischen Wissenschaften, Band 132, Springer-Verlag New York, Inc., New York, 1966. MR0203473 (34 #3324)

[Ke] Carlos E. Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems*, CBMS Regional Conference Series in Mathematics, vol. 83, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1994. MR1282720 (96a:35040)

[KKPT] C. Kenig, H. Koch, J. Pipher, and T. Toro, *A new approach to absolute continuity of elliptic measure, with applications to non-symmetric equations*, Adv. Math. **153** (2000), no. 2, 231–298, DOI 10.1006/aima.1999.1899. MR1770930 (2002f:35071)

[KLS] Carlos E. Kenig, Fanghua Lin, and Zhongwei Shen, *Homogenization of elliptic systems with Neumann boundary conditions*, J. Amer. Math. Soc. **26** (2013), no. 4, 901–937, DOI 10.1090/S0894-0347-2013-00769-9. MR3073881

[KP] Carlos E. Kenig and Jill Pipher, *The Neumann problem for elliptic equations with nonsmooth coefficients*, Invent. Math. **113** (1993), no. 3, 447–509, DOI 10.1007/BF01244315. MR1231834 (95b:35046)

[KS1] Carlos E. Kenig and Zhongwei Shen, *Homogenization of elliptic boundary value problems in Lipschitz domains*, Math. Ann. **350** (2011), no. 4, 867–917, DOI 10.1007/s00208-010-0586-3. MR2818717 (2012m:35039)

[KS2] Carlos E. Kenig and Zhongwei Shen, *Layer potential methods for elliptic homogenization problems*, Comm. Pure Appl. Math. **64** (2011), no. 1, 1–44, DOI 10.1002/cpa.20343. MR2743875 (2011i:35009)

[R] Andreas Rosén, *Layer potentials beyond singular integral operators*, Publ. Mat. **57** (2013), no. 2, 429–454, DOI 10.5565/PUBLMAT_57213_08. MR3114777

[SW] James Serrin and H. F. Weinberger, *Isolated singularities of solutions of linear elliptic equations*, Amer. J. Math. **88** (1966), 258–272. MR0201815 (34 #1697)

[V] Varopoulos N., *A remark on BMO and bounded harmonic functions* (1970).

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