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HEIGHT ESTIMATE FOR SPECIAL WEINGARTEN SURFACES OF ELLIPTIC TYPE IN $\mathbb{M}^2(c) \times \mathbb{R}$

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ABSTRACT. In this article we provide a vertical height estimate for compact special Weingarten surfaces of elliptic type in $\mathbb{M}^2(c) \times \mathbb{R}$, i.e. surfaces whose mean curvature H and extrinsic Gauss curvature K_e satisfy $H = f(H^2 - K_e)$ with $4x(f'(x))^2 < 1$, for all $x \in [0, +\infty)$. The vertical height estimate generalizes a result by Rosenberg and Sa Earp and applies only to surfaces verifying a height estimate condition. When $c < 0$, using also a horizontal height estimate, we show a non-existence result for properly embedded Weingarten surfaces of elliptic type in $\mathbb{H}^2(c) \times \mathbb{R}$ with finite topology and one end.

1. INTRODUCTION

In this work we will consider special Weingarten surfaces of elliptic type in $\mathbb{M}^2(c) \times \mathbb{R}$. Here $\mathbb{M}^2(c) = \mathbb{S}^2(c), \mathbb{R}^2$ or $\mathbb{H}^2(c)$ depending on the sign of the sectional curvature c. If H and K_e denote the mean curvature and the extrinsic Gauss curvature of a surface Σ respectively, then Σ is called a special Weingarten surface if the following identity holds:

$$H = f(H^2 - K_e), \tag{1}$$

with $f \in \mathcal{C}^0([0, +\infty))$. Furthermore if $f \in \mathcal{C}^1([0, +\infty))$ and $4x(f'(x))^2 < 1 \, \forall x \in [0, +\infty)$, then f is said to be elliptic and Σ is said to be a special Weingarten surface of elliptic type, henceforth called a SWET surface.

The study of Weingarten surfaces started with H. Hopf [7], P. Hartman and W. Wintner [8] and S. S. Chern [3], who considered compact Weingarten surfaces in $\mathbb{R}^3$. H. Rosenberg and R. Sa Earp [14] showed that compact special Weingarten surfaces in $\mathbb{R}^3$ and $\mathbb{H}^3$ satisfy an a priori height estimate, assuming also that f satisfies a height estimate condition, and used this fact to prove that the annular ends of a properly embedded special Weingarten surface M are cylindrically bounded. Moreover, if such an M is non-compact and has finite topological type, then M must have more than one end; if M has two ends, then it must be a rotational surface; and if M has three ends, it is contained in a slab. They followed the ideas of Meeks [11] and Korevaar-Kusner-Solomon [10] for non-zero constant mean curvature surfaces in $\mathbb{R}^3$. Recently, Aledo-Espinar-Gálvez in [2] obtained a geometric height estimate for SWET surfaces with $f(0) \neq 0$ in $\mathbb{M}^3(c)$, $c \leq 0$, with no other hypothesis on f.

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R. Sa Earp and E. Toubiana in [16–18] studied rotational special Weingarten surfaces in $\mathbb{R}^3$ and $\mathbb{H}^3$. In the case $f(0) \neq 0$ (constant mean curvature type), they determined necessary and sufficient conditions for existence and uniqueness of examples whose geometric behaviour is the same as the one of Delaunay surfaces in $\mathbb{R}^3$, i.e. unduloids (embedded) and nodoids (non-embedded), which have non-zero constant mean curvature. In the case $f(0) = 0$ (minimal type), they established the existence of examples whose geometric behaviour is the same as those of the catenoid of $\mathbb{R}^3$, which is the only rotational minimal surface in $\mathbb{R}^3$.

By arguments similar to those used by Sa Earp and Toubiana, the author and M. Rodriguez in [12] determined necessary and sufficient conditions for existence and uniqueness of rotational SWET surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ of minimal type ($f(0) = 0$).

The reason we focus on SWET surfaces is that the ellipticity of f ensures that the operator obtained by linearization of (1) is elliptic in the sense of Hopf [7] and solutions to (1) satisfy an interior and a boundary maximum principle.

Furthermore we show that an estimate for the height (defined below), similar to one given in [14], holds for SWET surfaces in product manifolds of dimension three under additional assumptions of f.

Let Σ be a connected orientable hypersurface immersed in $\mathbb{M}^2(c) \times \mathbb{R}$. The height function, denoted by h, of Σ is defined as the restriction to Σ of the projection $t : \mathbb{M}^2(c) \times \mathbb{R} \to \mathbb{R}$.

Theorem 1.1 (Height estimate)**.** *Let Σ be a compact SWET surface embedded in $\mathbb{M}^2(c) \times \mathbb{R}$ which is a graph over $\mathbb{M}^2(c) \times \{0\}$ with $\partial\Sigma \subset \mathbb{M}^2(c) \times \{0\}$. Let $x = H^2 - K_e$. If $f > 0$, $f - 2xf' > 0$ and $f^2 + c + x(1 - 4ff') > 0$, then*

$$|h| \leqslant 1/l$$

where

$$l = \min_{\Sigma} \left(2f + \frac{c - K_e}{f(1 - 2ff') + 2K_e f'} \right). \tag{2}$$

Corollary 1.2 (Height estimate for cmc surfaces)**.** *Let Σ be a compact surface having constant mean curvature which is embedded in $\mathbb{M}^2(c) \times \mathbb{R}$ and a graph over $\mathbb{M}^2(c) \times \{0\}$ such that $\partial\Sigma \subset \mathbb{M}^2(c) \times \{0\}$. Suppose that*

(1) $c \leq 0$, $H > \sqrt{\frac{\max_\Sigma K_e - c}{2}}$, *or*

(2) $c > 0$, $\max_\Sigma K_e - c > 0$, $H > \sqrt{\frac{\max_\Sigma K_e - c}{2}}$, *or*

(3) $c > 0$, $\max_\Sigma K_e - c \leq 0$, $H > 0$;

then

$$|h| \leqslant \frac{H}{2H^2 + c - \max_\Sigma K_e}.$$

The technique used in [2] to prove a geometric height estimate for SWET surfaces in $\mathbb{M}^3(c)$, $c \leq 0$, does not apply to our setting.

Theorem 6.2 in [6] provides a horizontal height estimate for compact surfaces Σ with constant curvature or constant mean curvature in $\mathbb{H}^2(c) \times \mathbb{R}$ and boundary contained in a vertical plane. The case of surfaces in $\mathbb{S}^2(c) \times \mathbb{R}$ is not considered to be $\mathbb{S}^2(c)$ compact.

It is possible to prove a similar estimate for SWET surfaces.

Theorem 1.3 (Horizontal height estimate)**.** *Let P denote a vertical plane in $\mathbb{H}^2(c) \times \mathbb{R}$. Let Σ be a compact SWET surface in $\mathbb{H}^2(c) \times \mathbb{R}$, with $\partial\Sigma \subset P$. Assume that the elliptic function $f \geq H_0 > \frac{\sqrt{-c}}{2}$. Then for every $p \in \Sigma$, the horizontal distance in $\mathbb{H}^2(c) \times \mathbb{R}$ of p to P is bounded by a constant C which does not depend on Σ.*

The proof of Theorem 6.2 in [6] applies verbatim to our setting, with a unique exception: the proof uses the maximum principle to compare Σ to a surface Σ_0 that in our case has to be the sphere of constant mean curvature equal to H_0.

Combining Theorems 1.1 and 1.3 we are in order to prove the following non-existence result.

Theorem 1.4. *There are no properly embedded SWET surfaces in $\mathbb{H}^2(c) \times \mathbb{R}$ with finite topology, one end and whose elliptic function f satisfies the hypotheses of Theorems* 1.1*,* 1.3*.*

Such a theorem generalizes Theorem 7.2 of [6], which applies to surfaces with constant mean curvature $H > 1/2$ or constant curvature $K > 0$ which are properly embedded in $\mathbb{H}^2 \times \mathbb{R}$.

2. Proof of Theorem 1.1

2.1. **Preliminaries.** Let Σ be a oriented connected Riemannian m-manifold and let $F : \Sigma \to \mathbb{M}^{m+1}$ be an isometric immersion of Σ into an orientable Riemannian $(m+1)$-manifold $\mathbb{M}^{m+1}$. We choose a normal unit vector field N along Σ and define the shape operator A associated with the second fundamental form of Σ; that is, for any $p \in \Sigma$,

$$\langle A(X), Y\rangle = -\langle \overline{\nabla}_X N, Y\rangle,\ X, Y \in T_p\Sigma,$$

where $\overline{\nabla}$ is the Riemannian connection of $\mathbb{M}^{m+1}$.

Let $k_1, \ldots, k_m$ denote the eigenvalues of A. For $1 \leqslant r \leqslant m$, let S_r denote the r-th symmetric function of $k_1, \ldots, k_m$ and T_r be the r-th Newton transformation: $T_0 = I$, $T_r = S_r I - AT_{r-1}$.

If H_r denotes the r-th mean curvature of Σ, then $H_r = S_r/C_m^r$, where $C_m^r = \frac{m!}{r!(m-r)!}$.

Let us consider a domain $D \subset \Sigma$ such that its closure $\overline{D}$ is compact with smooth boundary.

Definition 2.1. A variation of D is a differentiable map $\phi : (-\varepsilon, \varepsilon) \times \Sigma \to \mathbb{M}^{m+1}$, where $\varepsilon > 0$, such that for each $s \in (-\varepsilon, \varepsilon)$ the map $\phi_s : \Sigma \to \mathbb{M}^{m+1}$ defined by $\phi_s(p) = \phi(s, p)$ is an immersion and $\phi_0(p) = F(p)$ for every $p \in \Sigma$ (we recall that F denotes the immersion of Σ in $\mathbb{M}^{m+1}$) and $\phi_s(p) = F(p)$ for $p \in \Sigma \setminus \overline{D}$ and $s \in (-\varepsilon, \varepsilon)$.

We set

$$E_s(p) = \frac{\partial \phi}{\partial s}(s, p) \quad \text{and } f_s = \langle E_s, N_s\rangle,$$

where N_s is the unit normal vector field along $\phi_s(\Sigma)$. E is called the variational vector field of ϕ. Let $A_s(p)$ be the shape operator of $\phi_s(\Sigma)$ at the point p and $S_r(s, p)$ the r-th symmetric function of the eigenvalues of $A_s(p)$.

Definition 2.2. Let $g \in \mathcal{C}^2(\Sigma)$. We define $L_r(g) = div(T_r \nabla g)$, $0 \leqslant r \leqslant m$.

In [5] M.F. Elbert proved that, for $1 \leqslant r \leqslant m$,

$$\frac{\partial S_r}{\partial s} = L_{r-1}(f_s) + f_s(S_1 S_r - (r+1)S_{r+1}) + f_s tr(T_{r-1}\overline{R}_N) + E_s^T(S_r), \tag{3}$$

where $\overline{R}_N$ is defined as $\overline{R}_N(X) = \overline{R}(N,X)N$, $\overline{R}$ is the curvature tensor of $\mathbb{M}^{m+1}$ and E_s^T denotes the tangent part of E_s.

In the sequel we will consider the case where $\mathbb{M}^{m+1}$ has a special structure: $\mathbb{M}^{m+1} = \mathbb{M}^m \times \mathbb{R}$, where $\mathbb{M}^m$ is an m-dimensional Riemannian manifold.

Definition 2.3. Let Σ be a connected orientable hypersurface immersed in $\mathbb{M}^m \times \mathbb{R}$. The height function, denoted by h, of Σ in $\mathbb{M}^m \times \mathbb{R}$ is defined as the restriction to Σ of the projection $t : \mathbb{M}^m \times \mathbb{R} \to \mathbb{R}$.

The following result has been proved in [4] by X. Cheng and H. Rosenberg.

Lemma 2.4. *Let Σ be an immersed orientable hypersurface in $\mathbb{M}^m \times \mathbb{R}$ (with or without boundary) and N be its normal unit vector field. Then*

$$L_r(h) = (r+1)S_{r+1}n,$$

for $0 \leqslant r \leqslant m$, where h denotes the height function of Σ, and $n = \langle \frac{\partial}{\partial t}, N\rangle$.

Lemma 2.5. *Let Σ be an immersed hypesurface in $\mathbb{M}^m \times \mathbb{R}$. Then we have*

$$L_r(n) = -n(S_1 S_{r+1} - (r+2)S_{r+2} + tr(T_r\overline{R}_N)) - E_0^T(S_{r+1}).$$

Proof. The proof uses the same argument as the proof of Lemma 4.2 in [4], with the only difference being that in our case S_r is not assumed to be constant on Σ. □

Remark 2.6. If either $r = 0$ or $r = 1$, we get respectively for $m = 2$ the following formulae:

$$L_0(n) = \Delta n = -n(S_1^2 - 2S_2 + tr(T_0\overline{R}_N)) - E_0^T(S_1),$$
$$L_1(n) = -n(S_1 S_2 + tr(T_1\overline{R}_N)) - E_0^T(S_2).$$

If the manifold $\mathbb{M}^m$ has constant sectional curvature, then we are able to express all terms of $L_{r-1}(n)$ given by Lemma 2.5 in terms of the curvatures S_r.

We denote by X^h the horizontal component of $X \in T_p(\mathbb{M}^m(c) \times \mathbb{R})$, by e_i the principal directions of A and by A_i the restriction of A to the $(m-1)$-dimensional space normal to e_i.

Lemma 2.7. *Let Σ denote a hypersurface immersed in $\mathbb{M}^m(c) \times \mathbb{R}$. For $0 \leqslant r \leqslant m$, the following holds:*

$$tr(T_r\overline{R}_N) = c(m-r)S_r.$$

Proof.

$$tr(T_r\overline{R}_N)(p) = \sum_i \langle e_i, T_{r+1}\overline{R}_N(e_i)\rangle(p) = \sum_i S_{r+1}(A_i)\overline{R}(N, e_i, N, e_i)(p)$$
$$= \sum_i S_{r+1}(A_i)K(e_i^h, N^h)|e_i^h \wedge N^h|^2(p) = c\sum_i S_{r+1}(A_i) = c(m-r)S_r.$$

□

Remark 2.8. If either $r = 0$ or $r = 1$, we get respectively for $m = 2$ the following formulae:

$$tr(T_0\overline{R}_N) = 2cS_0 = 2c,$$
$$tr(T_1\overline{R}_N) = cS_1 = 2cH_1.$$

In the next section we will use the results presented here to show that the special Weingarten surfaces of elliptic type satisfy an interior and a boundary maximum principle and a height estimate under additional conditions.

2.2. **Maximum principle for special Weingarten surfaces.** Let Σ be an oriented connected hypersurface immersed in $\mathbb{M}^m(c) \times \mathbb{R}$ and $f \in \mathcal{C}^1([0,\infty))$. Let us suppose that the first and second mean curvatures $H_1(s), H_2(s)$ of $\phi_s(\Sigma)$ (see Definition 2.1) satisfy

$$H_1 - f(H_1^2 - H_2) = 0. \tag{4}$$

The first variation of the left member of this identity at $s = 0$ gives us

$$\left((1 - 2H_1 f'(H_1^2 - H_2))\frac{\partial H_1}{\partial s} + f'(H_1^2 - H_2)\frac{\partial H_2}{\partial s}\right)_{|\{s=0\}} = 0. \tag{5}$$

From (3), the principal parts of $\partial_s H_1(0) = \frac{1}{m}\partial_s S_1(0)$ and $\partial_s H_2(0) = \frac{2}{m(m-1)}\partial_s S_2(0)$ are respectively L_0/m and $\frac{2}{m(m-1)}L_1$.

When $m = 2$ the linearized operator of (4) reduces to

$$L_f = \left(\frac{1 - 2ff'}{2}\right)\Delta + f' L_1.$$

As in [14, page 294], we can prove the following lemma.

Lemma 2.9. *If the function f is elliptic, that is, $4x(f'(x))^2 < 1$ for all $x \geq 0$, then the eigenvalues of the operator L_f are positive. In other terms L_f is elliptic.*

Remark 2.10. Lemma 2.9 says that $H_1 = f(H_1^2 - H_2)$ is elliptic in the sense of Hopf and the solutions of this equation satisfy an interior and a boundary maximum principle (see [7, pages 156-158]).

Let Σ_1, Σ_2 be two oriented special Weingarten surfaces in $\mathbb{M}^2(c) \times \mathbb{R}$ satisfying (1) for the same function f, whose unit normal vectors coincide at a common point p. For $i = 1, 2$, we can write Σ_i locally around p as a graph of a function u_i over a domain in $T_p\Sigma_1 = T_p\Sigma_2$ (in exponential coordinates). We will say that Σ_1 is above Σ_2 in a neighbourhood of p, and we will write $\Sigma_1 \geq \Sigma_2$ if $u_1 \geq u_2$.

Proposition 2.11 (Maximum Principle [7]). *Let Σ_1, Σ_2 be two special Weingarten surfaces in $\mathbb{M}^2(c) \times \mathbb{R}$ with respect to the same elliptic function f. Let us suppose that*

- *Σ_1 and Σ_2 are tangent at an interior point $p \in \Sigma_1 \cap \Sigma_2$ or*
- *there exists $p \in \partial\Sigma_1 \cap \partial\Sigma_2$ such that both $T_p\Sigma_1 = T_p\Sigma_2$ and $T_p\partial\Sigma_1 = T_p\partial\Sigma_2$.*

Also suppose that the unit normal vectors of Σ_1, Σ_2 coincide at p. If $\Sigma_1 \geq \Sigma_2$ in a neighbourhood U of p, then $\Sigma_1 = \Sigma_2$ in U. In the case Σ_1, Σ_2 have no boundary, $\Sigma_1 = \Sigma_2$.

To show the main theorem we need the following result.

Lemma 2.12. *If $H_1(s)$ and $H_2(s)$ verify $H_1 - f(H_1^2 - H_2) = 0$ for each s, then*

$$(1 - 2ff')E_s^T(H_1) + f'E_s^T(H_2) = 0.$$

Proof. It is sufficient to find the explicit expression of $E_s^T(H_1 - f(H_1^2 - H_2))$ and to use the fact that it vanishes. It holds that

$$E_s^T(H_1) - E_s^T f(H_1^2 - H_2) = E_s^T(H_1) - 2ff' E_s^T(H_1) + f' E_s^T(H_2) = 0.$$

That is, $(1 - 2ff')E_s^T(H_1) + f' E_s^T(H_2) = 0$. □

Now we give the proof of Theorem 1.1.

Proof. Let m denote the maximum of h on Σ. It is sufficient to give the proof assuming that Σ is a graph on the slice $\{t = 0\}$, where t denotes the coordinate on $\mathbb{R}$. Indeed by coming from infinity with horizontal slices and applying the Alexandrov reflection to Σ, we see that the part of Σ above the plane $t = m/2$ is a graph over a domain in this plane. The estimate we are going to prove is $|h| \leqslant 1/l$ when Σ is a graph. We can assume $h \geqslant 0$ on Σ; otherwise we apply the following argument to the part of Σ above $\{t = 0\}$ and to the part of Σ below $\{t = 0\}$. We orient the unit normal vector to Σ so that $n \leqslant 0$. We set $\varphi = lh + n$. On $\partial\Sigma$ we have $\varphi = n \leqslant 0$. If we show that $L_f\varphi \geqslant 0$ on Σ, from ellipticity of L_f and the maximum principle we get that $\varphi \leqslant 0$ on Σ, that is, $h \leqslant -n/l \leqslant 1/l$. By Lemma 2.4, Remarks 2.6 and 2.8, and Lemma 2.12 we get

$$\begin{aligned}
L_f(\varphi) &= L_f(lh + n) = lL_f(h) + L_f(n) \\
&= l\left(\left(\frac{1-2ff'}{2}\right)\Delta h + f' L_1 h\right) + \left(\frac{1-2ff'}{2}\right)\Delta n + f' L_1 n \\
&= l\left(\left(\frac{1-2ff'}{2}\right) 2H_1 n + 2f' H_2 n\right) \\
&\quad - \left(\frac{1-2ff'}{2}\right)((4H_1^2 - 2H_2 + 2c)n + 2E_0^T(H_1)) - f'((2H_1H_2 + 2cH_1)n + E_0^T(H_2)) \\
&= -n\left((1-2ff')\left(2H_1^2 - H_2 + c\right) + f'\left(2H_1H_2 + 2cH_1\right)\right) \\
&\quad + l\left((1-2ff')H_1 n + 2f' H_2 n\right) - (1-2ff')E_0^T(H_1) + f' E_0^T(H_2) \\
&= -n\left((1-2ff')\left(2H_1^2 - H_2 + c - lH_1\right) + f'\left(2H_1H_2 + 2cH_1 - 2lH_2\right)\right).
\end{aligned}$$

We replace H_1 by f and $H_1^2 - H_2$ by x. We get

$$\begin{aligned}
&-n\left((1-2ff')\left(f^2 + x + c - lf\right) + f'\left(2f(f^2 - x) + 2cf - 2l(f^2 - x)\right)\right) \\
&= -n\left(f^2 - lf + x + 2xlf' - 4xff' + c\right).
\end{aligned}$$

Such a quantity can be written as

$$-n\left(f(f-l)(1 - 4xf'^2) + c + x(1-2ff')^2 + 2xlf'(1-2ff')\right),$$

which is non-negative if

$$\begin{aligned}
&f(f-l)(1-4xf'^2) + c + x(1-2ff')^2 + 2xlf'(1-2ff') \\
&= f^2 + c + x(1-4ff') - l[f(1-4xf'^2) - 2xf'(1-2ff')] \\
&= f^2 + c + x(1-4ff') - l[f - 2xf'] \geqslant 0.
\end{aligned}$$

Suppose that

$$f - 2xf' > 0.$$

Then

$$l \leqslant \frac{f^2 + c + x(1-4ff')}{f - 2xf'} = f + \frac{c + x(1-2ff')}{f - 2xf'}.$$

As $x = f^2 - K_e$ we have
$$l \leqslant f + \frac{c - K_e + f^2 - 2(f^2 - K_e)ff'}{f - 2(f^2 - K_e)f'} = 2f + \frac{c - K_e}{f - 2(f^2 - K_e)f'} =: L.$$
If we assume $f^2 + c + x(1 - 4ff') > 0$, then $L > 0$. We conclude that the best constant l is
$$l = \min_{\Sigma} \left(2f + \frac{c - K_e}{f - 2(f^2 - K_e)f'}\right) = \min_{\Sigma} \left(2f + \frac{c - K_e}{f(1 - 2ff') + 2K_e f'}\right).$$
□

Now we give the proof of Corollary 1.2.

Proof. If $f' = 0$, that is, Σ has constant mean curvature, we get the estimate
$$h \leqslant \frac{1}{l} = \max_{\Sigma} \frac{H}{2H^2 + c - K_e} = \frac{H}{2H^2 + c - \max_{\Sigma} K_e}$$
under the assumption $2H^2 + c - \max_{\Sigma} K_e > 0$. □

Remark 2.13. If $c < 0$, then the estimate of Corollary 1.2 holds if $H > \sqrt{\frac{\max_{\Sigma} K_e - c}{2}}$. We observe that $\sqrt{\frac{\max_{\Sigma} K_e - c}{2}} \geqslant \sqrt{\frac{-c}{2}}$. As a consequence such an estimate is not sharp because the smallest value of the mean curvature of a compact surface in $\mathbb{H}^2(c) \times \mathbb{R}$ equals $\sqrt{-c}/2$. Optimal estimates have been derived in [1]. If $c = 0$ and Σ is a semisphere (in particular $H^2 = K_e$ holds), we get the well known estimate $|h| \leqslant 1/H$.

2.3. **Proof of Theorem 1.4.**

Proof. Let us suppose by absurdity that S is a properly embedded SWET surface with finite topology and one end with respect to a function f which also satisfies the hypotheses of Theorems 1.1, 1.3.

Let us denote by D a constant bigger than $\max\{C, d\}$, where C is the bound for the horizontal height given by Theorem 1.3 and d is the horizontal diameter of the sphere of constant mean curvature equal to H_0.

Let p denote a point in S and γ a horizontal geodesic containing p. Let p_1, p_2 be two points in γ such that $dist_{\mathbb{H}^2(c)}(p, p_1) = dist_{\mathbb{H}^2(c)}(p_1, p_2) = D$, $dist_{\mathbb{H}^2(c)}(p, p_2) = 2D$. Let P_1, P_2 be two vertical totally geodesic planes intersecting orthogonally with γ at p_1, p_2 respectively.

We now use the following variant of the Plane Separation Lemma proved in [13]. Its proof is exactly the same.

Lemma 2.14. *Let S be a properly embedded SWET annulus in $\mathbb{H}^2(c) \times \mathbb{R}$. Suppose that $f \geq H_0 > \frac{\sqrt{-c}}{2}$. Let P_1 and P_2 be two vertical totally geodesic planes. Assume that the distance between P_1 and P_2 is bigger than the horizontal diameter of the sphere of constant mean curvature H_0. Denote by P_1^+ and P_2^+ the components of $\mathbb{H}^2(c) \times \mathbb{R} \setminus P_j$ such that $P_1^+ \cap P_2^+ = \emptyset$. Then all the connected components of $S \cap P_1^+$ or $S \cap P_2^+$ are compact.*

The distance between P_1 and P_2 equals $D > d$; then the previous lemma applies. Suppose that all of the connected components of $S \cap P_1^+$ are compact. By construction the plane P_1 is at distance D to the point $p \in S \cap P_1^+$. As $D > C$, C being the bound for the horizontal distance, this would contradict Theorem 1.3.

Then all the connected components of $S \cap P_2^+$ are compact. By Theorem 1.3 the points of $S \cap P_2^+$ are at a distance to P_2 which has to be smaller than C.

If we use the same argument after replacing γ by every other horizontal geodesic line passing by p, we can prove that S is located at finite distance to $\{p\} \times \mathbb{R}$. In other words, S is contained in a vertical cylinder.

As by hypothesis S has exactly one end, we can assume that S is contained in the halfspace $\{t \leq 0\}$ and is tangent to $\{t = 0\}$.

For $z < 0$ we consider the reflection in $\{t = z\}$ of the compact piece S_z of S contained in $\{z \leq t \leq 0\}$. We will show that S_z is a vertical graph on $\{t = z\}$ for any $z < 0$. We denote by S_z^R the reflection of S_z in $\{t = z\}$. We observe that S_z^R does not have common tangent points with S. Otherwise by the Maximum Principle, Proposition 2.11, we could conclude that S is compact. For the same reason S_z^R and S_z are not orthogonal to $\{t = z\}$ for any $z < 0$. This proves that S_z is a graph.

Now we can choose $z_0 < 0$ with $|z_0|$ big enough so that the height of the compact graph S_{z_0} (having boundary on the plane $\{t = z_0\}$) is arbitrarily big. This contradicts Theorem 1.1. □

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