HEIGHT ESTIMATE FOR SPECIAL WEINGARTEN SURFACES OF ELLIPTIC TYPE IN $\mathbb{M}^2(c) \times \mathbb{R}$

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Abstract. In this article we provide a vertical height estimate for compact special Weingarten surfaces of elliptic type in $\mathbb{M}^2(c) \times \mathbb{R}$, i.e. surfaces whose mean curvature $H$ and extrinsic Gauss curvature $K_e$ satisfy $H = f(H^2 - K_e)$ with $4x(f'(x))^2 < 1$ for all $x \in [0, +\infty)$. The vertical height estimate generalizes a result by Rosenberg and Sa Earp and applies only to surfaces verifying a height estimate condition. When $c < 0$, using also a horizontal height estimate, we show a non-existence result for properly embedded Weingarten surfaces of elliptic type in $\mathbb{H}^2(c) \times \mathbb{R}$ with finite topology and one end.

1. Introduction

In this work we will consider special Weingarten surfaces of elliptic type in $\mathbb{M}^2(c) \times \mathbb{R}$. Here $\mathbb{M}^2(c) = S^2(c), \mathbb{R}^2$ or $\mathbb{H}^2(c)$ depending on the sign of the sectional curvature $c$. If $H$ and $K_e$ denote the mean curvature and the extrinsic Gauss curvature of a surface $\Sigma$ respectively, then $\Sigma$ is called a special Weingarten surface if the following identity holds:

$$H = f(H^2 - K_e),$$

with $f \in C^0([0, +\infty))$. Furthermore if $f \in C^1([0, +\infty))$ and $4x(f'(x))^2 < 1 \forall x \in [0, +\infty)$, then $f$ is said to be elliptic and $\Sigma$ is said to be a special Weingarten surface of elliptic type, henceforth called a SWET surface.

The study of Weingarten surfaces started with H. Hopf [7], P. Hartman and W. Wintner [8] and S. S. Chern [3], who considered compact Weingarten surfaces in $\mathbb{R}^3$. H. Rosenberg and R. Sa Earp [14] showed that compact special Weingarten surfaces in $\mathbb{R}^3$ and $\mathbb{H}^3$ satisfy an a priori height estimate, assuming also that $f$ satisfies a height estimate condition, and used this fact to prove that the annular ends of a properly embedded special Weingarten surface $M$ are cylindrically bounded. Moreover, if such an $M$ is non-compact and has finite topological type, then $M$ must have more than one end; if $M$ has two ends, then it must be a rotational surface; and if $M$ has three ends, it is contained in a slab. They followed the ideas of Meeks [11] and Korevaar-Kusner-Solomon [10] for non-zero constant mean curvature surfaces in $\mathbb{R}^3$. Recently, Aledo-Espinar-Gálvez in [2] obtained a geometric height estimate for SWET surfaces with $f(0) \neq 0$ in $\mathbb{M}^3(c), c \leq 0$, with no other hypothesis on $f$.  

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R. Sa Earp and E. Toubiana in \cite{16,18} studied rotational special Weingarten surfaces in $\mathbb{R}^3$ and $\mathbb{H}^3$. In the case $f(0) \neq 0$ (constant mean curvature type), they determined necessary and sufficient conditions for existence and uniqueness of examples whose geometric behaviour is the same as the one of Delaunay surfaces in $\mathbb{R}^3$, i.e. unduloids (embedded) and nodoids (non-embedded), which have non-zero constant mean curvature. In the case $f(0) = 0$ (minimal type), they established the existence of examples whose geometric behaviour is the same as those of the catenoid of $\mathbb{R}^3$, which is the only rotational minimal surface in $\mathbb{R}^3$.

By arguments similar to those used by Sa Earp and Toubiana, the author and M. Rodriguez in \cite{12} determined necessary and sufficient conditions for existence and uniqueness of rotational SWET surfaces in $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$ of minimal type $(f(0) = 0)$.

The reason we focus on SWET surfaces is that the ellipticity of $f$ ensures that the operator obtained by linearization of (1) is elliptic in the sense of Hopf \cite{7} and solutions to (1) satisfy an interior and a boundary maximum principle.

Furthermore we show that an estimate for the height (defined below), similar to one given in \cite{14}, holds for SWET surfaces in product manifolds of dimension three under additional assumptions of $f$.

Let $\Sigma$ be a connected orientable hypersurface immersed in $M^2(c) \times \mathbb{R}$. The height function, denoted by $h$, of $\Sigma$ is defined as the restriction to $\Sigma$ of the projection $t : M^2(c) \times \mathbb{R} \to \mathbb{R}$.

**Theorem 1.1 (Height estimate).** Let $\Sigma$ be a compact SWET surface embedded in $M^2(c) \times \mathbb{R}$ which is a graph over $M^2(c) \times \{0\}$ with $\partial \Sigma \subset M^2(c) \times \{0\}$. Let $x = H^2 - K_e$. If $f > 0$, $f - 2xf' > 0$ and $f^2 + c + x(1 - 4ff') > 0$, then

$$|h| \leq 1/l$$

where

$$l = \min_{\Sigma} \left( 2f + \frac{c - K_e}{f(1 - 2ff') + 2K_e f'} \right).$$

**Corollary 1.2 (Height estimate for cmc surfaces).** Let $\Sigma$ be a compact surface having constant mean curvature which is embedded in $M^2(c) \times \mathbb{R}$ and a graph over $M^2(c) \times \{0\}$ such that $\partial \Sigma \subset M^2(c) \times \{0\}$. Suppose that

- $c \leq 0$, $H > \sqrt{\frac{\max_{\Sigma} K_e - c}{2}}$, or
- $c > 0$, $\max_{\Sigma} K_e - c > 0$, $H > \sqrt{\frac{\max_{\Sigma} K_e - c}{2}}$, or
- $c > 0$, $\max_{\Sigma} K_e - c \leq 0$, $H > 0$;

then

$$|h| \leq \frac{H}{2H^2 + c - \max_{\Sigma} K_e}.$$
Theorem 1.3 (Horizontal height estimate). Let $P$ denote a vertical plane in $\mathbb{H}^2(c) \times \mathbb{R}$. Let $\Sigma$ be a compact SWET surface in $\mathbb{H}^2(c) \times \mathbb{R}$, with $\partial \Sigma \subset P$. Assume that the elliptic function $f \geq H_0 > \sqrt{\frac{-c}{2}}$. Then for every $p \in \Sigma$, the horizontal distance in $\mathbb{H}^2(c) \times \mathbb{R}$ of $p$ to $P$ is bounded by a constant $C$ which does not depend on $\Sigma$.

The proof of Theorem 6.2 in [6] applies verbatim to our setting, with a unique exception: the proof uses the maximum principle to compare $\Sigma$ to a surface $\Sigma_0$ that in our case has to be the sphere of constant mean curvature equal to $H_0$.

Combining Theorems 1.1 and 1.3 we are in order to prove the following non-existence result.

Theorem 1.4. There are no properly embedded SWET surfaces in $\mathbb{H}^2(c) \times \mathbb{R}$ with finite topology, one end and whose elliptic function $f$ satisfies the hypotheses of Theorems 1.1 and 1.3.

Such a theorem generalizes Theorem 7.2 of [6], which applies to surfaces with constant mean curvature $H > 1/2$ or constant curvature $K > 0$ which are properly embedded in $\mathbb{H}^2 \times \mathbb{R}$.

2. Proof of Theorem 1.1

2.1. Preliminaries. Let $\Sigma$ be a oriented connected Riemannian $m$-manifold and let $F : \Sigma \to \mathbb{M}^{m+1}$ be an isometric immersion of $\Sigma$ into an orientable Riemannian $(m+1)$-manifold $\mathbb{M}^{m+1}$. We choose a normal unit vector field $N$ along $\Sigma$ and define the shape operator $A$ associated with the second fundamental form of $\Sigma$; that is, for any $p \in \Sigma$,

$$\langle A(X), Y \rangle = -\langle \nabla_X N, Y \rangle, \quad X, Y \in T_p \Sigma,$$

where $\nabla$ is the Riemannian connection of $\mathbb{M}^{m+1}$.

Let $k_1, \ldots, k_m$ denote the eigenvalues of $A$. For $1 \leq r \leq m$, let $S_r$ denote the $r$-th symmetric function of $k_1, \ldots, k_m$ and $T_r$ be the $r$-th Newton transformation: $T_0 = I$, $T_r = S_r I - AT_{r-1}$.

If $H_r$ denotes the $r$-th mean curvature of $\Sigma$, then $H_r = S_r / C_m$, where $C_m = \frac{m!}{r!(m-r)!}$.

Let us consider a domain $D \subset \Sigma$ such that its closure $\overline{D}$ is compact with smooth boundary.

Definition 2.1. A variation of $D$ is a differentiable map $\phi : (-\varepsilon, \varepsilon) \times \Sigma \to \mathbb{M}^{m+1}$, where $\varepsilon > 0$, such that for each $s \in (-\varepsilon, \varepsilon)$ the map $\phi_s : \Sigma \to \mathbb{M}^{m+1}$ defined by $\phi_s(p) = \phi(s, p)$ is an immersion and $\phi_0(p) = F(p)$ for every $p \in \Sigma$ (we recall that $F$ denotes the immersion of $\Sigma$ in $\mathbb{M}^{m+1}$) and $\phi_s(p) = F(p)$ for $p \in \Sigma \setminus \overline{D}$ and $s \in (-\varepsilon, \varepsilon)$.

We set

$$E_s(p) = \frac{\partial \phi}{\partial s}(s, p) \quad \text{and} \quad f_s = \langle E_s, N_s \rangle,$$

where $N_s$ is the unit normal vector field along $\phi_s(\Sigma)$. $E$ is called the variational vector field of $\phi$. Let $A_s(p)$ be the shape operator of $\phi_s(\Sigma)$ at the point $p$ and $S_r(s, p)$ the $r$-th symmetric function of the eigenvalues of $A_s(p)$.

Definition 2.2. Let $g \in C^2(\Sigma)$. We define $L_r(g) = \text{div}(T_r \nabla g)$, $0 \leq r \leq m$. 

In [5] M.F. Elbert proved that, for $1 \leq r \leq m$,  
\begin{equation}
\frac{\partial S_r}{\partial s} = L_{r-1}(f_s) + f_s(S_1 S_r - (r+1)S_{r+1}) + f_s tr(T_{r-1}R_N) + E_s^T(S_r),
\end{equation}
where $R_N$ is defined as $R_N(X) = R(N,X)N$, $R$ is the curvature tensor of $M^{m+1}$ and $E_s^T$ denotes the tangent part of $E_s$.

In the sequel we will consider the case where $M^{m+1}$ has a special structure: $M^{m+1} = M^m \times \mathbb{R}$, where $M^m$ is an $m$-dimensional Riemannian manifold.

**Definition 2.3.** Let $\Sigma$ be a connected orientable hypersurface immersed in $M^m \times \mathbb{R}$. The height function, denoted by $h$, of $\Sigma$ in $M^m \times \mathbb{R}$ is defined as the restriction to $\Sigma$ of the projection $t : M^m \times \mathbb{R} \to \mathbb{R}$.

The following result has been proved in [4] by X. Cheng and H. Rosenberg.

**Lemma 2.4.** Let $\Sigma$ be an immersed orientable hypersurface in $M^m \times \mathbb{R}$ (with or without boundary) and $N$ be its normal unit vector field. Then  
\[ L_r(h) = (r+1)S_{r+1}, \]
for $0 \leq r \leq m$, where $h$ denotes the height function of $\Sigma$, and $n = (\frac{\partial}{\partial t}, N)$.

**Lemma 2.5.** Let $\Sigma$ be an immersed hypersurface in $M^m \times \mathbb{R}$. Then we have  
\[ L_r(n) = -n(S_1 S_{r+1} - (r+2)S_{r+2} + tr(T_r R_N)) - E_0^T(S_{r+1}). \]

**Proof.** The proof uses the same argument as the proof of Lemma 4.2 in [4], with the only difference being that in our case $S_r$ is not assumed to be constant on $\Sigma$.  

**Remark 2.6.** If either $r = 0$ or $r = 1$, we get respectively for $m = 2$ the following formulae:  
\[ L_0(n) = \Delta n = -n(S_1^2 \leq 2S_2 + tr(T_0 R_N) - E_0^T(S_1), \]
\[ L_1(n) = -n(S_1 S_2 + tr(T_1 R_N)) - E_0^T(S_2). \]

If the manifold $M^m$ has constant sectional curvature, then we are able to express all terms of $L_{r-1}(n)$ given by Lemma 2.5 in terms of the curvatures $S_r$.

We denote by $X^h$ the horizontal component of $X \in T_p(M^m(c) \times \mathbb{R})$, by $e_i$ the principal directions of $A$ and by $A_i$ the restriction of $A$ to the $(m-1)$-dimensional space normal to $e_i$.

**Lemma 2.7.** Let $\Sigma$ denote a hypersurface immersed in $M^m(c) \times \mathbb{R}$. For $0 \leq r \leq m$, the following holds:  
\[ tr(T_r R_N) = c(m-r)S_r. \]

**Proof.**  
\[ tr(T_r R_N)(p) = \sum_i \langle e_i, T_{r+1} R_N(e_i) \rangle(p) = \sum_i S_{r+1}(A_i) R(N, e_i, N, e_i)(p) \]
\[ = \sum_i S_{r+1}(A_i) K(e_i^h, N^h) |e_i^h \wedge N^h|^2(p) = c \sum_i S_{r+1}(A_i) = c(m-r)S_r. \]

**Remark 2.8.** If either $r = 0$ or $r = 1$, we get respectively for $m = 2$ the following formulae:  
\[ tr(T_0 R_N) = 2cS_0 = 2c, \]
\[ tr(T_1 R_N) = cS_1 = 2cH_1. \]
In the next section we will use the results presented here to show that the special Weingarten surfaces of elliptic type satisfy an interior and a boundary maximum principle and a height estimate under additional conditions.

2.2. Maximum principle for special Weingarten surfaces. Let $\Sigma$ be an oriented connected hypersurface immersed in $\mathbb{M}^m(c) \times \mathbb{R}$ and $f \in C^4([0, \infty))$. Let us suppose that the first and second mean curvatures $H_1(s), H_2(s)$ of $\phi_s(\Sigma)$ (see Definition 2.1) satisfy

\[ H_1 - f(H_1^2 - H_2) = 0. \]

The first variation of the left member of this identity at $s = 0$ gives us

\[ \left. \left( (1 - 2H_1f'(H_1^2 - H_2)) \frac{\partial H_1}{\partial s} + f'(H_1^2 - H_2) \frac{\partial H_2}{\partial s} \right) \right|_{(s=0)} = 0. \]

From (3), the principal parts of $\partial_s H_1(0) = \frac{1}{m} \partial_s S_1(0)$ and $\partial_s H_2(0) = \frac{2}{m(m-1)} \partial_s S_2(0)$ are respectively $L_0/m$ and $\frac{2}{m(m-1)} L_1$.

When $m = 2$ the linearized operator of (4) reduces to

\[ L_f = \left( \frac{1 - 2ff'}{2} \right) \Delta + f'L_1. \]

As in [45] page 294], we can prove the following lemma.

**Lemma 2.9.** If the function $f$ is elliptic, that is, $4x(f'(x))^2 < 1$ for all $x \geq 0$, then the eigenvalues of the operator $L_f$ are positive. In other terms $L_f$ is elliptic.

**Remark 2.10.** Lemma 2.9 says that $H_1 = f(H_1^2 - H_2)$ is elliptic in the sense of Hopf and the solutions of this equation satisfy an interior and a boundary maximum principle (see [7] pages 156-158).

Let $\Sigma_1, \Sigma_2$ be two oriented special Weingarten surfaces in $\mathbb{M}^2(c) \times \mathbb{R}$ satisfying (1) for the same function $f$, whose unit normal vectors coincide at a common point $p$. For $i = 1, 2$, we can write $\Sigma_i$ locally around $p$ as a graph of a function $u_i$ over a domain in $T_p \Sigma_1 = T_p \Sigma_2$ (in exponential coordinates). We will say that $\Sigma_1$ is above $\Sigma_2$ in a neighbourhood of $p$, and we will write $\Sigma_1 \geq \Sigma_2$ if $u_1 \geq u_2$.

**Proposition 2.11** (Maximum Principle [7]). Let $\Sigma_1, \Sigma_2$ be two special Weingarten surfaces in $\mathbb{M}^2(c) \times \mathbb{R}$ with respect to the same elliptic function $f$. Let us suppose that

- $\Sigma_1$ and $\Sigma_2$ are tangent at an interior point $p \in \Sigma_1 \cap \Sigma_2$ or
- there exists $p \in \partial \Sigma_1 \cap \partial \Sigma_2$ such that both $T_p \Sigma_1 = T_p \Sigma_2$ and $T_p \partial \Sigma_1 = T_p \partial \Sigma_2$.

Also suppose that the unit normal vectors of $\Sigma_1, \Sigma_2$ coincide at $p$. If $\Sigma_1 \geq \Sigma_2$ in a neighbourhood $U$ of $p$, then $\Sigma_1 = \Sigma_2$ in $U$. In the case $\Sigma_1, \Sigma_2$ have no boundary, $\Sigma_1 = \Sigma_2$.

To show the main theorem we need the following result.

**Lemma 2.12.** If $H_1(s)$ and $H_2(s)$ verify $H_1 - f(H_1^2 - H_2) = 0$ for each $s$, then

\[ (1 - 2ff')E_s^T(H_1) + f'E_s^T(H_2) = 0. \]
Proof. It is sufficient to find the explicit expression of $E_s^T(H_1 - f(H_1^2 - H_2))$ and to use the fact that it vanishes. It holds that

$$E_s^T(H_1) - E_s^T f(H_1^2 - H_2) = E_s^T(H_1) - 2ff'E_s^T(H_1) + f'E_s^T(H_2) = 0.$$ 

That is, $(1 - 2ff')E_s^T(H_1) + f'E_s^T(H_2) = 0$. □

Now we give the proof of Theorem 1.1

Proof. Let $m$ denote the maximum of $h$ on $\Sigma$. It is sufficient to give the proof assuming that $\Sigma$ is a graph on the slice $\{t = 0\}$, where $t$ denotes the coordinate on $\mathbb{R}$. Indeed by coming from infinity with horizontal slices and applying the Alexandrov reflection to $\Sigma$, we see that the part of $\Sigma$ above the plane $t = m/2$ is a graph over a domain in this plane. The estimate we are going to prove is $|h| \leq 1/l$ when $\Sigma$ is a graph. We can assume $h \geq 0$ on $\Sigma$; otherwise we apply the following argument to the part of $\Sigma$ above $\{t = 0\}$ and to the part of $\Sigma$ below $\{t = 0\}$. We orient the unit normal vector to $\Sigma$ so that $n \leq 0$. We set $\varphi = lh + n$. On $\partial \Sigma$ we have $\varphi = n \leq 0$. If we show that $L_f \varphi \geq 0$ on $\Sigma$, from ellipticity of $L_f$ and the maximum principle we get that $\varphi \leq 0$ on $\Sigma$, that is, $h \leq -n/l \leq 1/l$. By Lemma 2.3 Remarks 2.6 and 2.8 and Lemma 2.12 we get

$$L_f(h + n) = lL_f(h) + L_f(n)$$

$$= l\left(\frac{1 - 2ff'}{2}\right) \Delta h + f'L_1 h + \left(\frac{1 - 2ff'}{2}\right) \Delta n + f'L_1 n$$

$$= l\left(\frac{1 - 2ff'}{2}\right) (2H_1 n + 2f'H_2 n)$$

$$- \left(\frac{1 - 2ff'}{2}\right) ((4H_1^2 - 2H_2 + 2c)n + 2E_0^T(H_1)) - f'((2H_1H_2 + 2cH_1)n + E_0^T(H_2))$$

$$= -n ((1 - 2ff') (2H_1^2 - H_2 + c) + f' (2H_1H_2 + 2cH_1))$$

$$+ l ((1 - 2ff') H_1 n + 2f'H_2 n) - (1 - 2ff') E_0^T(H_1) + f'E_0^T(H_2)$$

$$= -n ((1 - 2ff') (2H_1^2 - H_2 + c - lH_1) + f' (2H_1H_2 + 2cH_1 - 2lH_2)).$$

We replace $H_1$ by $f$ and $H_1^2 - H_2$ by $x$. We get

$$-n ((1 - 2ff') (f^2 + x - c - lf) + f' (2f(f^2 - x) + 2cf - 2l(f^2 - x)))$$

$$= -n (f^2 - lf + x + 2xlf' - 4xlf' + c).$$

Such a quantity can be written as

$$-n (f(f - l)(1 - 4xf'^2) + c + x(1 - 2ff')^2 + 2xlf' (1 - 2ff')),\$$

which is non-negative if

$$f(f - l)(1 - 4xf'^2) + c + x(1 - 2ff')^2 + 2xlf' (1 - 2ff')$$

$$= f^2 + c + x(1 - 4ff') - l[f(1 - 4xf'^2) - 2xlf'(1 - 2ff')]$$

$$= f^2 + c + x(1 - 4ff') - l[f - 2xf'] \geq 0.$$

Suppose that

$$f - 2xf' > 0.$$

Then

$$l \leq \frac{f^2 + c + x(1 - 4ff')}{f - 2xf'} = f + \frac{c + x(1 - 2ff')}{f - 2xf'}.$$

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As $x = f^2 - K_e$ we have
\[
l \leq f + \frac{c - K_e + f^2 - 2f^2 - K_e f f'}{f - 2(2f^2 - K_e) f'} = 2f + \frac{c - K_e}{f - 2(2f^2 - K_e) f'} =: L.
\]
If we assume $f^2 + c + x(1 - 4ff') > 0$, then $L > 0$. We conclude that the best constant $l$ is
\[
l = \min_{\Sigma} \left(2f + \frac{c - K_e}{f - 2(2f^2 - K_e) f'}\right) = \min_{\Sigma} \left(2f + \frac{c - K_e}{f(1 - 2f f') + 2K_e f'}\right).
\]
\[\square\]

Now we give the proof of Corollary 1.2

**Proof.** If $f' = 0$, that is, $\Sigma$ has constant mean curvature, we get the estimate
\[
h \leq \frac{1}{l} = \max_{\Sigma} \frac{H}{2H^2 + c - K_e} = \frac{H}{2H^2 + c - \max_{\Sigma} K_e}
\]
under the assumption $2H^2 + c - \max_{\Sigma} K_e > 0$.
\[\square\]

**Remark 2.13.** If $c < 0$, then the estimate of Corollary 1.2 holds if $H > \sqrt{\frac{\max_{\Sigma} K_e - c}{2}}$.

We observe that $\sqrt{\frac{\max_{\Sigma} K_e - c}{2}} \geq \sqrt{\frac{-c}{2}}$. As a consequence such an estimate is not sharp because the smallest value of the mean curvature of a compact surface in $H^2(c) \times \mathbb{R}$ equals $\sqrt{-c}/2$. Optimal estimates have been derived in [1]. If $c = 0$ and $\Sigma$ is a semisphere (in particular $H^2 = K_e$ holds), we get the well known estimate $|h| \leq 1/H$.

**2.3. Proof of Theorem 1.4**

**Proof.** Let us suppose by absurdity that $S$ is a properly embedded SWET surface with finite topology and one end with respect to a function $f$ which also satisfies the hypotheses of Theorems 1.1, 1.3.

Let us denote by $D$ a constant bigger than $\max\{C, d\}$, where $C$ is the bound for the horizontal height given by Theorem 1.3 and $d$ is the horizontal diameter of the sphere of constant mean curvature equal to $H_0$.

Let $p$ denote a point in $S$ and $\gamma$ a horizontal geodesic containing $p$. Let $p_1, p_2$ be two points in $\gamma$ such that $dist_{H^2(c)}(p, p_1) = dist_{H^2(c)}(p_1, p_2) = D$, $dist_{H^2(c)}(p, p_2) = 2D$. Let $P_1, P_2$ be two vertical totally geodesic planes intersecting orthogonally with $\gamma$ at $p_1, p_2$ respectively.

We now use the following variant of the Plane Separation Lemma proved in [13]. Its proof is exactly the same.

**Lemma 2.14.** Let $S$ be a properly embedded SWET annulus in $H^2(c) \times \mathbb{R}$. Suppose that $f \geq H_0 > \frac{\sqrt{-c}}{2}$. Let $P_1$ and $P_2$ be two vertical totally geodesic planes. Assume that the distance between $P_1$ and $P_2$ is bigger than the horizontal diameter of the sphere of constant mean curvature $H_0$. Denote by $P_1^+$ and $P_2^+$ the components of $H^2(c) \times \mathbb{R} \setminus P_j$ such that $P_1^+ \cap P_2^+ = \emptyset$. Then all the connected components of $S \cap P_1^+$ or $S \cap P_2^+$ are compact.

The distance between $P_1$ and $P_2$ equals $D > d$; then the previous lemma applies. Suppose that all of the connected components of $S \cap P_1^+$ are compact. By construction the plane $P_1$ is at distance $D$ to the point $p \in S \cap P_1^+$. As $D > C$, $C$ being the bound for the horizontal distance, this would contradict Theorem 1.3.
Then all the connected components of $S \cap P_2^+$ are compact. By Theorem 1.3, the points of $S \cap P_2^+$ are at a distance to $P_2$ which has to be smaller than $C$.

If we use the same argument after replacing $\gamma$ by every other horizontal geodesic line passing by $p$, we can prove that $S$ is located at finite distance to $\{p\} \times \mathbb{R}$. In other words, $S$ is contained in a vertical cylinder.

As by hypothesis $S$ has exactly one end, we can assume that $S$ is contained in the halfspace $\{t \leq 0\}$ and is tangent to $\{t = 0\}$.

For $z < 0$ we consider the reflection in $\{t = z\}$ of the compact piece $S_z$ of $S$ contained in $\{z \leq t \leq 0\}$. We will show that $S_z$ is a vertical graph on $\{t = z\}$ for any $z < 0$. We denote by $S^R_z$ the reflection of $S_z$ in $\{t = z\}$. We observe that $S^R_z$ does not have common tangent points with $S$. Otherwise by the Maximum Principle, Proposition 2.11, we could conclude that $S$ is compact. For the same reason $S^R_z$ and $S_z$ are not orthogonal to $\{t = z\}$ for any $z < 0$. This proves that $S_z$ is a graph.

Now we can choose $z_0 < 0$ with $|z_0|$ big enough so that the height of the compact graph $S_{z_0}$ (having boundary on the plane $\{t = z_0\}$) is arbitrarily big. This contradicts Theorem 1.1. □

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