QUASI-MORPHISMS ON THE GROUP OF AREA-PRESERVING
DIFFEOMORPHISMS OF THE 2-DISK VIA BRAID GROUPS

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Abstract. Recently Gambaudo and Ghys proved that there exist infinitely
many quasi-morphisms on the group \( \text{Diff}_{\infty}^\omega(D^2, \partial D^2) \) of area-preserving diffeo-
morphisms of the 2-disk \( D^2 \). For the proof, they constructed a homomorphism
from the space of quasi-morphisms on the braid group to the space of quasi-
morphisms on \( \text{Diff}_{\infty}^\omega(D^2, \partial D^2) \). In this paper, we study this homomorphism
and prove its injectivity.

1. Introduction

For a group \( G \), a function \( \phi: G \to \mathbb{R} \) is called a quasi-morphism if the real valued
function on \( G \times G \) defined by
\[
(g, h) \mapsto \phi(gh) - \phi(g) - \phi(h)
\]
is bounded. The real number
\[
D(\phi) = \sup_{g, h \in G} |\phi(gh) - \phi(g) - \phi(h)|
\]
is called the defect of \( \phi \). We denote the \( \mathbb{R} \)-vector space of quasi-morphisms on the
group \( G \) by \( \hat{Q}(G) \). By definition, bounded functions on groups are quasi-morphisms.
Hence we denote the set of bounded functions on the group \( G \) by \( C^{1}_b(G; \mathbb{R}) \)
and consider the quotient space \( Q(G) = \hat{Q}(G) / C^{1}_b(G; \mathbb{R}) \). A quasi-morphism \( \phi: G \to \mathbb{R} \)
is said to be homogeneous if the equation
\[
\phi(g^p) = p \phi(g)
\]
holds for any \( g \in G \) and \( p \in \mathbb{Z} \). For any quasi-morphism \( \phi \), a homogeneous quasi-
morphism \( \tilde{\phi} \) is defined by setting
\[
\tilde{\phi}(g) = \lim_{p \to \infty} \frac{1}{p} \phi(g^p).
\]
The limit always exists for each element \( g \) of \( G \). The new function \( \tilde{\phi} \) is in fact a quasi-
morphism equal to the original quasi-morphism \( \phi \) as an element of \( Q(G) \). Thus we
can identify the vector space of homogeneous quasi-morphisms on the group \( G \) with
\( Q(G) \). Homogeneous quasi-morphisms are invariant under conjugation. Therefore
we are interested in \( Q(G) \) rather than \( \hat{Q}(G) \).

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Let $\text{Diff}^\infty_\Omega(D^2, \partial D^2)$ be the group of area-preserving $C^\infty$-diffeomorphisms of the 2-disk $D^2$, which are the identity on a neighborhood of the boundary. On the vector space $Q(\text{Diff}^\infty_\Omega(D^2, \partial D^2))$, the following theorem is known.

**Theorem 1.1** (Entov-Polterovich [3], Gambaudo-Ghys [5]). The vector space $Q(\text{Diff}^\infty_\Omega(D^2, \partial D^2))$ is infinite dimensional.

To prove Theorem 1.1 Entov and Polterovich explicitly constructed countably many quasi-morphisms on $\text{Diff}^\infty_\Omega(D^2, \partial D^2)$, which are linearly independent. After that Gambaudo and Ghys constructed countably many quasi-morphisms on $\text{Diff}^\infty_\Omega(D^2, \partial D^2)$ using a different idea, which is to consider the suspension of area-preserving diffeomorphisms of the disk and average the value of the signature of the braids appearing in the suspension. By generalizing their strategy Brandenbursky [4] defined the homomorphism

$$\Gamma_n : Q(P_n(D^2)) \to Q(\text{Diff}^\infty_\Omega(D^2, \partial D^2)),$$

which we review in Section 2. Here, $P_n(D^2)$ denotes the pure braid group on $n$-strands.

Let $B_n(D^2)$ be the braid group on $n$-strands. The natural inclusion $i : P_n(D^2) \to B_n(D^2)$ induces the homomorphism $Q(i) : Q(B_n(D^2)) \to Q(P_n(D^2))$. In this paper, we study the homomorphism $\Gamma_n$ and prove the following theorem.

**Theorem 1.2.** The composition

$$\Gamma_n \circ Q(i) : Q(B_n(D^2)) \to Q(\text{Diff}^\infty_\Omega(D^2, \partial D^2))$$

is injective.

2. Gambaudo and Ghys’ construction and proof of the main theorem

In this section, we review Gambaudo and Ghys’ construction [5] of quasi-morphisms on the group $\text{Diff}^\infty_\Omega(D^2, \partial D^2)$ in a generalized form and prove Theorem 1.2.

Let $X_n(D^2)$ be the configuration space of ordered $n$-tuples in the 2-disk $D^2$ and $x^0 = (x_1^0, \ldots, x_n^0)$ its base point. For any $g \in \text{Diff}^\infty_\Omega(D^2, \partial D^2)$ and for almost all $x = (x_1, \ldots, x_n) \in X_n(D^2)$, we define the pure braid $\gamma(g; x)$ as the following. First we set the loop $l(g; x) : [0, 1] \to X_n(D^2)$ by

$$l(g; x)(t) = \begin{cases} 
\{(1 - 3t)x_1^0 + 3tx_1 \} & (0 \leq t \leq \frac{1}{3}), \\
\{gt_{3-1}(x_i) \} & (\frac{1}{3} \leq t \leq \frac{2}{3}), \\
\{(3 - 3t)g(x_i) + (3t - 2)x_i^0 \} & (\frac{2}{3} \leq t \leq 1), 
\end{cases}$$

where $\{g_t\}_{t \in [0, 1]}$ is a Hamiltonian isotopy such that $g_0$ is the identity and $g_1 = g$. We define the pure braid $\gamma(g; x)$ to be the braid represented by the loop $l(g; x)$. For almost every $x$, the braid $\gamma(g; x)$ is well-defined. Furthermore, the braid $\gamma(g; x)$ is independent of the choice of the flow $\{g_t\}$. This is because of the fact that the group $\text{Diff}^\infty_\Omega(D^2, \partial D^2)$ is contractible, which is easily proved from the contractibility of the diffeomorphism group $\text{Diff}^\infty(D^2, \partial D^2)$ of $D^2$ [8] and the homotopy equivalence between $\text{Diff}^\infty(D^2, \partial D^2)$ and $\text{Diff}^\infty_\Omega(D^2, \partial D^2)$ [7]. For a
quasi-morphism $\phi$ on the pure braid group $P_n(D^2)$ on $n$-strands, we define the function $\hat{\Gamma}_n(\phi): \text{Diff}_{\Omega}^\infty(D^2, \partial D^2) \rightarrow \mathbb{R}$ by

$$\hat{\Gamma}_n(\phi)(g) = \int_{x \in X_n(D^2)} \phi(\gamma(g; x))dx.$$ 

For any $\phi \in Q(P_n(D^2))$ and $g \in \text{Diff}_{\Omega}^\infty(D^2, \partial D^2)$ the function $\phi(\gamma(g; \cdot))$ is integrable and thus the map $\hat{\Gamma}_n: \hat{Q}(P_n(D^2)) \rightarrow \hat{Q}(\text{Diff}_{\Omega}^\infty(D^2, \partial D^2))$ is well-defined \[2\]. The obtained function $\hat{\Gamma}_n(\phi): \text{Diff}_{\Omega}^\infty(D^2, \partial D^2) \rightarrow \mathbb{R}$ is also a quasi-morphism, and the map $\hat{\Gamma}_n: \hat{Q}(P_n(D^2)) \rightarrow \hat{Q}(\text{Diff}_{\Omega}^\infty(D^2, \partial D^2))$ is clearly $\mathbb{R}$-linear. Moreover, it is easily checked that any bounded function on $P_n(D^2)$ is mapped to a bounded function on $\text{Diff}_{\Omega}^\infty(D^2, \partial D^2)$, and thus the homomorphism

$$\hat{\Gamma}_n: \hat{Q}(P_n(D^2)) \rightarrow \hat{Q}(\text{Diff}_{\Omega}^\infty(D^2, \partial D^2))$$

induces the homomorphism $\Gamma_n: Q(P_n(D^2)) \rightarrow Q(\text{Diff}_{\Omega}^\infty(D^2, \partial D^2))$.

**Remark 2.1.** We see that the homomorphism $\Gamma_n: Q(P_n(D^2)) \rightarrow Q(\text{Diff}_{\Omega}^\infty(D^2, \partial D^2))$ maps the classical linking number homomorphism $\text{lk}_n: B_n(D^2) \rightarrow \mathbb{R}$ on the braid group to a homomorphism on $\text{Diff}_{\Omega}^\infty(D^2, \partial D^2)$. In fact, the image of $\text{lk}: B_n(D^2) \rightarrow \mathbb{R}$ by the homomorphism $\Gamma_n(\text{lk}_n)$ coincides with a constant multiple of the classical Calabi homomorphism on $\text{Diff}_{\Omega}^\infty(D^2, \partial D^2)$ \[4\], and in this sense quasi-morphisms obtained in this way can be considered as generalizations of the Calabi homomorphism. By an argument of Brandenburcy which verifies that the homomorphism $\Gamma: \hat{Q}(P_n(D^2)) \rightarrow \hat{Q}(\text{Diff}_{\Omega}^\infty(D^2, \partial D^2))$ is well-defined, it is observed that quasi-morphisms obtained by the homomorphism $\hat{\Gamma}_n: \hat{Q}(P_n(D^2)) \rightarrow \hat{Q}(\text{Diff}_{\Omega}^\infty(D^2, \partial D^2))$ can be defined on the group of area-preserving $C^1$-diffeomorphisms of $D^2$, as well as the Calabi homomorphism.

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let us suppose that a homogeneous quasi-morphism $\phi \in \hat{Q}(B_n(D^2))$ is non-trivial. Then there exists a braid $\beta \in B_n(D^2)$ such that $\phi(\beta) \neq 0$. We may assume that $\beta$ is pure. It is sufficient to prove that the homogeneous quasi-morphism $\hat{\Gamma}_n(\phi) \in \hat{Q}(\text{Diff}_{\Omega}^\infty(D^2, \partial D^2))$ is also non-trivial. That is, there exists an area-preserving diffeomorphism $g \in \text{Diff}_{\Omega}^\infty(D^2, \partial D^2)$ such that

$$\lim_{p \rightarrow \infty} \frac{1}{p} \Gamma_n(\phi)(g^p) \neq 0.$$ 

Let $A_{i,j}$ be the pure braid which twists only the $i$-th and the $j$-th strands for $1 \leq i < j \leq n$ (see Figure 1). Since the braid $\beta$ is pure, it can be written as a composition of $A_{i,j}$’s and their inverses. We take $n$ disjoint subsets $U_i$ of $D^2$. Furthermore, for a pair of $(i, j)$, we take subsets $V_{i,j}$ and $W_{i,j}$ of $D^2$ such that $U_i \cup U_j \subset W_{i,j} \subset V_{i,j}$, $U_k \cap V_{i,j} = \emptyset$ if $k \neq i, j$ and $V_{i,j}, W_{i,j}$ are diffeomorphic to $D^2$. Let $\{h_t\}_{t \in [0,1]}$ be a path in $\text{Diff}_{\Omega}^\infty(D^2, \partial D^2)$ such that the support of $h_t$ is contained in the interior of $V_{i,j}$ and rotates $W_{i,j}$ once. Taking paths $\{h_t\}$ constructed above for all the $A_{i,j}$’s which present $\beta$ and composing them, we have a path $\{g_t\}_{t \in [0,1]}$ in $\text{Diff}_{\Omega}^\infty(D^2, \partial D^2)$ with $g_0 = \text{id}$ which twists the $U_i$’s in the form of the pure
braid $\beta$. If we set $g = g_1$, then $g$ is the identity on the $U_i$'s and $\gamma(g; (x_1, \ldots, x_n)) = \beta$ for $x_i \in U_i$. Then by setting $U = U_1 \cup \cdots \cup U_n$, we have

$$
\lim_{p \to \infty} \frac{1}{p} \hat{\Gamma}_n(\phi)(g^p) = \lim_{p \to \infty} \frac{1}{p} \left( \int_{x \in X_n(U)} \phi(\gamma(g^p; x))dx + \int_{x \in X_n(D^2) \setminus X_n(U)} \phi(\gamma(g^p; x))dx \right)
$$

$$
= \int_{x \in X_n(U)} \phi(\gamma(g; x))dx + \lim_{p \to \infty} \frac{1}{p} \int_{x \in X_n(D^2) \setminus X_n(U)} \phi(\gamma(g^p; x))dx.
$$

If we denote the first term of the equation by $Y$ and set $a_i = \text{area}(U_i)$ and $[n] = \{1, \ldots, n\}$, then $Y$ is written as

$$
\int_{x \in X_n(U)} \phi(\gamma(g; x))dx = \sum_{F: [n] \to [n]} \left( \prod_{i=1}^n a_{F(i)} \right) x_F,
$$

where $x_F = \phi(\gamma(g; x))$ for $x$ in the case when each $x_i$ is in $U_{F(i)}$. The real numbers $x_F$ have the following properties:

(i) For two maps $F$ and $G: [n] \to [n]$, if $\#F^{-1}(i) = \#G^{-1}(i)$ for each $1 \leq i \leq n$, then $x_F = x_G$.

(ii) If a map $F: [n] \to [n]$ is bijective, then $x_F$ is non-zero.

Property (i) follows from the invariance of $\phi$ under conjugation, and property (ii) follows because $\phi(\beta)$ is non-zero. Therefore, the coefficient of $a_1 \ldots a_n$ in $Y$ is non-zero. Since the polynomial $Y$ is not identically 0, we can choose the $a_i$’s such that $Y$ is non-zero.

Note that if we replace the $a_i$’s by bigger ones, by fixing the ratio of any two of them the term $Y$ stays non-zero. On the other hand, the value $\phi(\gamma(g; x))$ is bounded because of the construction of $g$, and we thus have

$$
\lim_{p \to \infty} \frac{1}{p} \int_{x \notin X_n(U)} \phi(\gamma(g^p; x))dx \to 0 \quad (\text{as } a_1 + \cdots + a_n \to \text{area}(D^2)).
$$

This completes the proof. \qed
As we noted in Remark 2.1, the homomorphism $\hat{\Gamma}_n$ maps any homomorphism on $P_n(D^2)$ to a homomorphism on $\text{Diff}_G^\infty(D^2, \partial D^2)$. Hence the homomorphism

$$Q(P_n(D^2))/H^1(P_n(D^2); \mathbb{R}) \to Q(\text{Diff}_G^\infty(D^2, \partial D^2))/H^1(\text{Diff}_G^\infty(D^2, \partial D^2); \mathbb{R})$$

is also induced. By an argument similar to the proof of Theorem 1.2, the following proposition holds.

**Proposition 2.2.** The map

$$Q(B_n(D^2))/H^1(B_n(D^2); \mathbb{R}) \to Q(\text{Diff}_G^\infty(D^2, \partial D^2))/H^1(\text{Diff}_G^\infty(D^2, \partial D^2); \mathbb{R})$$

induced by the composition $\Gamma_n \circ Q(i): Q(B_n(D^2)) \to Q(\text{Diff}_G^\infty(D^2, \partial D^2))$ is injective.

The homomorphism $\Gamma_n: Q(P_n(D^2)) \to Q(\text{Diff}_G^\infty(D^2, \partial D^2))$ can also be defined for the 2-sphere $S^2$ instead of $D^2$ as Gambaudo and Ghys mentioned in their paper. Let $\text{Diff}_G^\infty(S^2)_0$ be the identity component of the group of area-preserving diffeomorphisms of $S^2$. Then we can choose a pure braid $\gamma(g; x) \in P_n(S^2)$ for any $g \in \text{Diff}_G^\infty(S^2)_0$ and for almost every $x \in X_n(S^2)$ as in the case of the 2-disk. Since the group $\text{Diff}_G^\infty(S^2)_0$ is homotopy equivalent to $SO(3)$ [7, 8] and its fundamental group has order 2, for any element $g$ of $\text{Diff}_G^\infty(S^2)_0$ there exist two homotopy classes of paths connecting the identity and $g$ in $\text{Diff}_G^\infty(S^2)_0$. However, for any homogeneous quasi-morphism $\phi$ on $P_n(S^2)$, the value $\phi(\gamma(g; x))$ is independent of the choice of the path. In fact, the braid obtained from a path which represents the generator of $\pi_1(\text{Diff}_G^\infty(S^2)_0)$ has order 2 and is in the center of $P_n(S^2)$. Hence the homomorphism $\Gamma_n: Q(P_n(S^2)) \to Q(\text{Diff}_G^\infty(S^2)_0)$ is defined. Since the braid group $B_n(S^2)$ of the 2-sphere on $n$-strands can be considered as a quotient group of the braid group $B_n(D^2)$, by an argument similar to the proof of Theorem 1.2 we obtain the following theorem.

**Theorem 2.3.** The composition

$$\Gamma_n \circ Q(i): Q(B_n(S^2)) \to Q(\text{Diff}_G^\infty(S^2)_0)$$

is injective.

The homomorphism $Q(i)$ in the statement of Theorem 2.3 is the one induced from the inclusion $i: P_n(S^2) \to B_n(S^2)$.

3. Kernel of the homomorphism $\Gamma_n$

The homomorphism $\Gamma_n: Q(P_n(D^2)) \to Q(\text{Diff}_G^\infty(D^2, \partial D^2))$ itself is not injective although Theorem 1.2 holds. In this section we study the kernel of the homomorphism $\Gamma_n$.

Let $G$ be a group and $H$ its finite index subgroup. We denote by $\overline{\beta}$ the image of an element $\beta \in G$ by the natural projection $G \to G/H$. For each left coset $\sigma \in G/H$ of $G$ modulo $H$, we fix an element $\gamma_{\sigma} \in G$ such that $\overline{\gamma_{\sigma}} = \sigma$ and for any $\phi \in \hat{Q}(H)$ define the function $\hat{T}(\phi): G \to \mathbb{R}$ by

$$\hat{T}(\phi)(\beta) = \frac{1}{[G:H]} \sum_{\sigma \in G/H} \phi(\gamma_{\overline{\beta\gamma_{\sigma}}}^{-1}\beta_{\gamma_{\sigma}}).$$

Since $\gamma_{\overline{\beta\gamma_{\sigma}}}^{-1}\beta_{\gamma_{\sigma}}$ is in $H$, the function $\hat{T}(\phi)$ is well-defined on $G$. 
Lemma 3.1. For any quasi-morphism $\phi$ on $H$, the function $\hat{T}(\phi) : G \to \mathbb{R}$ is also a quasi-morphism.

Proof. Since the equality
\[
\gamma_1^{-1} \beta_1 \gamma_2^{-1} \beta_2 \gamma_1 = (\gamma_1^{-1} \gamma_2^{-1} \beta_1 \gamma_2^{-1} \beta_2 \gamma_1) (\gamma_2^{-1} \beta_2 \gamma_1)
\]
holds, we have the inequality
\[
|\hat{T}(\phi)(\beta_1 \beta_2) - \hat{T}(\phi)(\beta_1) - \hat{T}(\phi)(\beta_2)|
\]
\[
= \frac{1}{(G : H)} \sum_{\sigma \in G/H} \left\{ \phi((\gamma_1^{-1} \beta_1 \gamma_2^{-1} \beta_2 \gamma_1) (\gamma_2^{-1} \beta_2 \gamma_1)) \right. \\
- \phi(\gamma_1^{-1} \beta_1 \gamma_2) - \phi(\gamma_2^{-1} \beta_2 \gamma_1) \left. \right\}
\]
\[
= \frac{1}{(G : H)} \sum_{\sigma \in G/H} \left\{ \phi((\gamma_1^{-1} \beta_1 \gamma_2^{-1} \beta_2 \gamma_1) (\gamma_2^{-1} \beta_2 \gamma_1)) \right. \\
- \phi(\gamma_1^{-1} \beta_1 \gamma_2) - \phi(\gamma_2^{-1} \beta_2 \gamma_1) \left. \right\}
\]
\[
\leq D(\phi).
\]
Hence the function $\hat{T}(\phi) : G \to \mathbb{R}$ is also a quasi-morphism. \hfill \square

The map $\hat{T} : \hat{Q}(H) \to \hat{Q}(G)$ is clearly $\mathbb{R}$-linear and induces a homomorphism $\mathcal{T} : \hat{Q}(P_n(D^2)) \to \hat{Q}(B_n(D^2))$. Furthermore, the following proposition holds.

Proposition 3.2. The homomorphism $\mathcal{T} : \hat{Q}(H) \to \hat{Q}(G)$ is independent of the choice of $\gamma_\sigma$'s.

Proof. Suppose that $\phi$ is a homogeneous quasi-morphism on $H$. If an element $\beta$ is in $H$, then $\gamma_\sigma \beta = \sigma$ for each $\sigma \in G/H$. For any $\beta \in G$ there exists an integer $k$ such that $\beta^k$ is in $H$ and we have
\[
\lim_{p \to \infty} \frac{1}{p} \hat{T}(\phi)(\beta^p) = \lim_{p' \to \infty} \frac{1}{kp'} \hat{T}(\phi)(\beta^{kp'})
\]
\[
= \lim_{p' \to \infty} \frac{1}{(G : H)kp'} \sum_{\sigma \in G/H} \phi(\gamma_\sigma^{-1} \beta^k \gamma_\sigma) p'
\]
\[
= \frac{1}{(G : H)k} \sum_{\sigma \in G/H} \phi(\gamma_\sigma^{-1} \beta^k \gamma_\sigma).
\]
(3.1)
Since $\phi$ is invariant under conjugations in $H$, the value $\phi(\gamma_\sigma^{-1} \beta^k \gamma_\sigma)$ depends only on $\sigma$. \hfill \square

Let $Q(i) : Q(G) \to Q(H)$ be the homomorphism induced by the inclusion $i : H \to G$. As a corollary to equality (3.1), we have the following.

Corollary 3.3. The composition $\mathcal{T} \circ Q(i) : Q(G) \to Q(G)$ is the identity on $Q(G)$. Furthermore, we have the decomposition
\[
Q(H) = \text{Ker}(\mathcal{T}) \oplus \text{Im}(Q(i))
\]
as vector spaces.
Remark 3.4. Of course, the homomorphism \( \hat{T}(\phi): G \to \mathbb{R} \) can be defined using the right coset \( H\backslash G \) instead of \( G/H \) by
\[
\hat{T}(\phi)(\beta) = \frac{1}{(G : H)} \sum_{\sigma \in G/H} \phi(\gamma_\sigma \beta \gamma_{\sigma^{-1}}^{-1}).
\]

By an argument similar to the proof of Lemma 3.1 and Proposition 3.2, it is verified that this alternative definition is also well-defined and induces the same homomorphism \( T: Q(H) \to Q(G) \).

Remark 3.5. The homomorphism \( T: Q(H) \to Q(G) \) is just a straightforward generalization of the transfer map, and it is also introduced in \([8, 9]\).

Since the pure braid groups \( P_n(D^2) \) and \( P_n(S^2) \) are finite index subgroups of the braid groups \( B_n(D^2) \) and \( B_n(S^2) \), respectively, the homomorphisms
\[
T: Q(P_n(D^2)) \to Q(B_n(D^2)) \text{ and } T: Q(P_n(S^2)) \to Q(B_n(S^2))
\]
can be defined and Corollary 3.3 is true for \( G = B_n(D^2), H = P_n(D^2) \) and \( G = B_n(S^2), H = P_n(S^2) \), respectively.

The following proposition is the main result of this section.

**Proposition 3.6.** The composition
\[
\Gamma_n \circ Q(i) \circ T: Q(P_n(D^2)) \to Q(Diff^\infty(D^2, \partial D^2))
\]
coincides with \( \Gamma_n \). In particular, \( \text{Ker}(\Gamma_n) = \text{Ker}(T) \) and \( \text{Im}(\Gamma_n) = \text{Im}(\Gamma_n \circ Q(i)) \).

**Proof.** Let \( \mathfrak{S}_n \) be the symmetric group of \( n \) symbols. By equality (3.1), for any homogeneous quasi-morphism \( \phi \in Q(P_n(D^2)) \) and any area-preserving diffeomorphism \( g \in Diff^\infty(D^2, \partial D^2) \),
\[
(3.2) \lim_{p \to \infty} \frac{1}{p} \Gamma_n \circ Q(i) \circ \hat{T}(\phi)(g^p) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \lim_{p \to \infty} \frac{1}{p} \int_{x \in X_n(D^2)} \phi(\gamma_\sigma \gamma(g^p; x)\gamma_{\sigma^{-1}}^{-1}) dx.
\]

For any \( \sigma \in \mathfrak{S}_n \) and almost all \( x \in D^2 \), we set the path \( l: [0, 1] \to X_n(D^2) \) by
\[
l(t) = \begin{cases} 
(1 - 2t)x_i^0 + 2tx_i & (0 \leq t \leq \frac{1}{2}), \\
(2 - 2t)x_i + (2t - 1)x_{\sigma(i)}^0 & (\frac{1}{2} \leq t \leq 1).
\end{cases}
\]

Considering the path \( l \) as a loop in the quotient space \( X_n(D^2)/\mathfrak{S}_n \), we define the braid \( \beta(\sigma; x) \) to be the braid represented by the loop \( l \). Then by definition,
\[
\beta(\sigma; x)\gamma(g; \sigma^{-1}(x))\beta(\sigma; g_\ast x)^{-1} = \gamma(g; x),
\]
where the symmetric group \( \mathfrak{S}_n \) acts on \( X_n(D^2) \) by the permutation
\[
\sigma(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).
\]

Since the homomorphism \( T: Q(P_n(D^2)) \to Q(B_n(D^2)) \) is defined independently of the choice of braids \( \gamma_\sigma \)'s, we may choose \( \gamma_\sigma \) to be \( \beta(\sigma; x) \). Hence we have
\[
\gamma_\sigma \gamma(g; \sigma^{-1}(x))\gamma_{\sigma^{-1}}^{-1} = \beta(\sigma; x)\gamma(g; \sigma^{-1}(x))\beta(\sigma; x)^{-1} = \gamma(g; x)\beta(\sigma; g_\ast(x))\beta(\sigma; x)^{-1}.
\]
Since the function $\phi(\beta(\sigma;\cdot)) : D^2 \to \mathbb{R}$ is bounded on $D^2$, we have

$$\lim_{p \to \infty} \frac{1}{p} \int_{x \in X_n(D^2)} \phi(\gamma \sigma \gamma(g^p; x) \gamma^{-1}_\sigma) dx$$

$$= \lim_{p \to \infty} \frac{1}{p} \int_{x \in X_n(D^2)} \phi(\gamma \sigma \gamma(g^p; x) \gamma^{-1}_\sigma) dx$$

$$= \lim_{p \to \infty} \frac{1}{p} \int_{x \in X_n(D^2)} \phi(g^p; x) dx.$$ 

Therefore, by equality (3.2),

$$\lim_{p \to \infty} \frac{1}{p} \hat{\Gamma}_n \circ Q(i) \circ T(\phi)(g^p) = \frac{1}{n!} \sum_{\sigma \in S_n} \lim_{p \to \infty} \frac{1}{p} \int_{x \in X_n(D^2)} \phi(g^p; x) dx$$

$$= \lim_{p \to \infty} \frac{1}{p} \hat{\Gamma}_n(\phi)(g^p),$$

and thus we have $\Gamma_n \circ Q(i) \circ T = \Gamma_n$.

Then obviously $\text{Ker}(T) \subseteq \text{Ker}(\Gamma_n)$ and $\text{Im}(\Gamma_n) = \text{Im}(\Gamma_n \circ Q(i))$ hold. If $\phi \in \text{Ker}(\Gamma_n)$, then

$$\Gamma_n \circ Q(i) \circ T(\phi) = \Gamma_n(\phi) = 0,$$

and hence $T(\phi) = 0$ by Theorem 1.2. Thus we have $\text{Ker}(\Gamma_n) \subseteq \text{Ker}(T)$. \hfill $\square$

Remark 3.7. Proposition 3.6 also holds for $P_n(S^2)$ and $\text{Diff}^\infty_\Omega(S^2)_0$ instead of $P_n(D^2)$ and $\text{Diff}^\infty_\Omega(D^2, \partial D^2)$, respectively.

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