

CONTINUOUS LEAFWISE HARMONIC FUNCTIONS ON CODIMENSION ONE TRANSVERSELY ISOMETRIC FOLIATIONS

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ABSTRACT. Let \mathcal{F} be a codimension one foliation on a closed manifold M which admits a transverse dimension one Riemannian foliation. Then any continuous leafwise harmonic functions are shown to be constant on leaves.

1. INTRODUCTION

Let M be a closed C^2 manifold, and let \mathcal{F} be a continuous leafwise C^2 foliation on M . This means that M is covered by a finite union of continuous foliation charts and the transition functions are continuous, together with their leafwise partial derivatives up to order 2. Let g be a continuous leafwise¹ C^2 leafwise Riemannian metric. In this paper such a triplet (M, \mathcal{F}, g) is simply referred to as a *leafwise C^2 foliation*. For simplicity, we assume throughout the paper that the manifold M and the foliation \mathcal{F} are oriented. For a continuous leafwise² C^2 real valued function h on M , the leafwise Laplacian Δh is defined using the leafwise metric g . It is a continuous function of M .

Definition 1.1. A continuous leafwise C^2 function h is called *leafwise harmonic* if $\Delta h = 0$.

Definition 1.2. A leafwise C^2 foliation (M, \mathcal{F}, g) is called *Liouville* if any continuous leafwise harmonic function is leafwise constant.

As an example, if \mathcal{F} is a foliation by compact leaves, then (M, \mathcal{F}, g) is Liouville. Moreover, there is an easy observation.

Proposition 1.3. *If \mathcal{F} admits a unique minimal set, then (M, \mathcal{F}, g) is Liouville.*

This can be seen as follows. Let m_1 (resp. m_2) be the maximum (resp. minimum) value of the continuous leafwise harmonic function h on M . Assume h takes the maximum value m_1 at $x \in M$. Then by the maximum principle, $h = m_1$ on the

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¹This means that the leafwise partial derivatives up to order 2 of the components of g in each foliation chart are continuous in the chart.

²The leafwise partial derivatives of h up to order 2 in each foliation chart are continuous in the chart.

leaf F_x which passes through x . Now the closure of F_x contains the unique minimal set X . Therefore $h = m_1$ on X . The same argument shows that $h = m_2$ on X , finishing the proof that h is constant on M .

A first example of non-Liouville foliations is obtained by R. Feres and A. Zeghib in a beautiful and simple construction [FZ]. It is a foliated S^2 -bundle over a hyperbolic surface, with two compact leaves. There are also examples in codimension one. B. Deroin and V. Kleptsyn [DK] have shown that a codimension one foliation \mathcal{F} is non-Liouville if \mathcal{F} is transversely C^1 , admits no transverse invariant measure, and possesses more than one minimal set, and they have constructed such a foliation.

A codimension one foliation \mathcal{F} is called \mathbb{R} -covered if the leaf space of its lift to the universal covering space is homeomorphic to \mathbb{R} . See [F] or [FFP]. It is shown in [F] and [DKNP] that an \mathbb{R} -covered foliation without compact leaves admits a unique minimal set. Therefore the above example of a codimension one non-Liouville foliation is not \mathbb{R} -covered. This led the authors of [FFP] to the study of the Liouville property for \mathbb{R} -covered foliations. The purpose of this paper is to generalize a result of [FFP].

Definition 1.4. A codimension one leafwise C^2 foliation (M, \mathcal{F}, g) is called *transversely isometric* if there is a continuous dimension one foliation ϕ transverse to \mathcal{F} such that the holonomy map of ϕ sending a (part of a) leaf of \mathcal{F} to another leaf is C^2 and preserves the leafwise metric g .

Notice that a transversely isometric foliation is \mathbb{R} -covered. Our main result is the following.

Theorem 1. *A leafwise C^2 transversely isometric codimension one foliation is Liouville.*

In [FFP], the above theorem is proved in the case where the leafwise Riemannian metric is negatively curved. Undoubtedly this is the most important case. But the general case may equally be of interest.

If a transversely isometric foliation \mathcal{F} does not admit a compact leaf, then, being \mathbb{R} -covered, it admits a unique minimal set, and Theorem 1 holds true by Proposition 1.3. Therefore we consider only the case where \mathcal{F} admits a compact leaf. In this case the union X of compact leaves is closed. Let U be a connected component of $M \setminus X$, and let N be the metric completion of U . Then N is a foliated interval bundle, since the one dimensional transverse foliation ϕ is Riemannian.

Therefore we are led to consider the following situation. Let K be a closed C^2 manifold of dimension ≥ 2 , equipped with a C^2 Riemannian metric g_K . Let $N = K \times I$, where I is the interval $[0, 1]$. Denote by $\pi : N \rightarrow K$ the canonical projection. Consider a continuous foliation \mathcal{L} which is transverse to the fibers $\pi^{-1}(y)$, $\forall y \in K$. Although \mathcal{L} is only continuous, its leaf has a C^2 differentiable structure as a covering space of K by the restriction of π . Also, \mathcal{L} admits a leafwise Riemannian metric g obtained as the lift of g_K to each leaf by π . Such a triplet (N, \mathcal{L}, g) is called a *leafwise C^2 foliated I -bundle* in this paper. Now Theorem 1 reduces to the following theorem.

Theorem 2. *Assume a leafwise C^2 foliated I -bundle (N, \mathcal{L}, g) does not admit a compact leaf in the interior $\text{Int}(N)$. Then any continuous leafwise harmonic function is constant on N .*

Sections 2–4 are devoted to the proof of Theorem 2. The proof is by contradiction. Throughout these sections, (N, \mathcal{L}, g) denotes a leafwise C^2 foliated I -bundle without interior compact leaves, and we assume that there is a continuous leafwise harmonic function f such that $f(K \times \{i\}) = i$, $i = 0, 1$. As remarked in [FFP], this is not a loss of generality. Also notice that for any point $x \in \text{Int}(N)$, we have $0 < f(x) < 1$.

The same phenomenon for discrete group actions is discussed in [FR]. We give an alternative proof of it in Section 5.

2. PRELIMINARIES

In this section, we recall fundamental facts about Brownian motions, needed in the next section.

Let us denote by Ω the space of continuous leafwise paths $\omega : [0, \infty) \rightarrow N$. For any $t \geq 0$, a random variable $X_t : \Omega \rightarrow N$ is defined by $X_t(\omega) = \omega(t)$. Let \mathcal{B} be the σ -algebra of Ω generated by X_t ($0 \leq t < \infty$). As is well known and easy to show, \mathcal{B} coincides with the σ -algebra generated by the compact open topology on Ω . A bounded function $\phi : \Omega \rightarrow \mathbb{R}$ is called a *Borel function* if ϕ is \mathcal{B} -measurable, i.e. if for any Borel subset $B \subset \mathbb{R}$, the inverse image $\phi^{-1}(B)$ belongs to \mathcal{B} .

For any point $x \in N$, the Wiener probability measure P^x is defined using the leafwise Riemannian metric g . Notice that $P^x\{X_0 = x\} = 1$. For any bounded Borel function $\phi : \Omega \rightarrow \mathbb{R}$, the expectation of ϕ w.r.t. P^x is denoted by $E^x[\phi]$. The following proposition is well known.

Proposition 2.1. *Let f be a bounded Borel function defined on a leaf L of \mathcal{L} . Then f is harmonic on L if and only if*

$$(2.1) \quad E^x[g(X^t)] = g(x), \quad \forall t \geq 0, \quad \forall x \in L.$$

Let \mathcal{F} be the completion of \mathcal{B} by the measure P^x . For any $t \geq 0$, let \mathcal{B}_t be the σ -algebra generated by X_s ($0 \leq s \leq t$). Its completion is denoted by \mathcal{F}_t . Notice that unlike \mathcal{B} and \mathcal{B}_t , \mathcal{F} and \mathcal{F}_t depend strongly on $x \in M$, but we suppress the dependence in the notation.

The following fact is well known and can be shown easily using the Radon-Nikodým theorem.

Proposition 2.2. *Given any bounded \mathcal{F} -measurable function $\phi : \Omega \rightarrow \mathbb{R}$ and $t > 0$, there is a unique bounded \mathcal{F}_t -measurable function ϕ_t such that for any bounded \mathcal{F}_t -measurable function ψ , we have*

$$E^x[\psi\phi_t] = E^x[\psi\phi].$$

The \mathcal{F}_t -measurable function ϕ_t is called the *conditional expectation of ϕ with respect to \mathcal{F}_t* and is denoted by $E^x[\phi | \mathcal{F}_t]$.

For any $t > 0$, let $\theta_t : \Omega \rightarrow \Omega$ be the shift map by t , defined by

$$\theta_t(\omega)(s) = \omega(t + s), \quad \forall s \in [0, \infty).$$

The following proposition is known as the Markov property. See for example [O].

Proposition 2.3. *Let $\phi : \Omega \rightarrow \mathbb{R}$ be a bounded \mathcal{F} -measurable function. Then we have*

$$(2.2) \quad E^x[\phi \circ \theta_t | \mathcal{F}_t] = E^{X_t}[\phi].$$

A family ϕ_t of uniformly bounded \mathcal{F}_t -measurable functions, $t \geq 0$, is called a *bounded P^x -martingale* if

$$(2.3) \quad E^x[\phi_{t+h} \mid \mathcal{F}_t] = \phi_t, \quad \forall t \geq 0, \forall h > 0.$$

We have the following martingale convergence theorem. See [O], Appendix C.

Theorem 2.4. *Let $\{\phi_t\}$ be a bounded P^x -martingale. Then there is a bounded \mathcal{F} -measurable function ϕ such that $\phi_t \rightarrow \phi$ as $t \rightarrow \infty$, P^x -almost surely.*

We shall raise two applications of the above facts, which will be useful in the next section. Let $f : N \rightarrow [0, 1]$ be the continuous leafwise harmonic function defined at the end of Section 1. Then $f(X_t)$ is a Borel function defined on Ω .

Lemma 2.5. *For any $x \in N$, there is an \mathcal{F} -measurable function $\phi : \Omega \rightarrow [0, 1]$ such that $f(X_t) \rightarrow \phi$ as $t \rightarrow \infty$, P^x -almost surely.*

Proof. We only need to show that $f(X_t)$ is a bounded P^x -martingale. For this, we have

$$E^x[f(X_{t+h}) \mid \mathcal{F}_t] = E^x[f(X_h) \circ \theta_t \mid \mathcal{F}_t] = E^{X_t}[f(X_h)] = f(X_t),$$

where the second equality is by the Markov property, and the last by the leafwise harmonicity of f . \square

Lemma 2.6. *Let $\phi : \Omega \rightarrow [0, 1]$ be a Borel function such that $\phi \circ \theta_t = \phi$ for any $t > 0$. Then $x \mapsto E^x[\phi]$ is a harmonic function on each leaf L of \mathcal{L} .*

Proof. By Proposition 2.1, we only need to show that

$$E^x[E^{X_t}[\phi]] = E^x[\phi], \quad \forall x \in L, \forall t > 0,$$

but we have

$$E^x[E^{X_t}[\phi]] = E^x[E^x[\phi \circ \theta_t \mid \mathcal{F}_t]] = E^x[E^x[\phi \mid \mathcal{F}_t]] = E^x[\phi]. \quad \square$$

3. PROOF OF THEOREM 2

Again let f be a continuous leafwise harmonic function defined at the end of Section 1. A probability measure μ on N is called *stationary* if $\langle \mu, \Delta h \rangle = 0$ for any continuous leafwise C^2 function h .

Proposition 3.1. *There does not exist a stationary measure μ such that*

$$\mu(\text{Int}(N)) > 0.$$

Proof. Denote by X the union of leaves on which f is constant. The subset X is closed in N . L. Garnett [G] has shown that $\mu(X) = 1$ for any stationary measure μ . Therefore if $\mu(\text{Int}(N)) > 0$, there is a leaf L in $\text{Int}(N)$ on which f is constant. But since we are assuming that there are no interior compact leaves, the closure of L must contain both boundary components of N . This is a contradiction to the continuity of f . \square

Given $0 < \alpha < 1$, let $V = K \times (\alpha, 1]$, a neighbourhood of the upper boundary component $K \times \{1\}$. Let Ω_V be the subset of Ω defined by

$$\Omega_V = \{X_{t_i} \in V, \exists t_i \rightarrow \infty\}.$$

Thus Ω_V consists of those paths which visit V at arbitrarily large time. Let ϕ be the characteristic function of Ω_V . Clearly ϕ is a Borel function on Ω and satisfies $\phi \circ \theta_t = \phi$ for any $t > 0$. Thus by Lemma 2.6, the function $p : M \rightarrow [0, 1]$ defined by $p(x) = E^x[\phi]$ is leafwise harmonic.

Another important feature of the function p is that p is nondecreasing along the fiber $\pi^{-1}(y)$, $\forall y \in K$, since our leafwise Brownian motion is synchronized; i.e. it is the lift of the Brownian motion on K . Notice that $p = i$ on $K \times \{i\}$, $i = 0, 1$.

Lemma 3.2. *The function p is constant on $\text{Int}(N)$.*

The proof is the same as the proof of Proposition 9.1 of [FFP]. In short, if we assume the contrary, we can construct a stationary measure μ such that $\mu(\text{Int}(N)) > 0$, which is contrary to Proposition 3.1. The proof is included in Section 4 for completeness.

Lemma 3.3. *The function p is 1 on $\text{Int}(N)$.*

Proof. Assume $p < 1$ on $\text{Int}(N)$. For any $x \in N$, P^x -almost surely the limit $\phi = \lim_{t \rightarrow \infty} f(X_t)$ exists by Lemma 2.5. Choose a constant $0 < a < 1$ so that $f^{-1}[a, 1]$ is contained in V . Then we have for any $x \in \text{Int}(N)$,

$$q_x = P^x\{\phi \geq a\} \leq p.$$

Therefore

$$\begin{aligned} E^x[\phi] &\leq 1 \cdot P^x\{\phi \geq a\} + a \cdot P^x\{\phi < a\} \\ &= q_x + a(1 - q_x) = a + (1 - a)q_x \leq a + (1 - a)p < 1. \end{aligned}$$

Now by the dominated convergence theorem we have

$$E^x[\phi] = E^x[\lim_{t \rightarrow \infty} f(X_t)] = \lim_{t \rightarrow \infty} E^x[f(X_t)] = \lim_{t \rightarrow \infty} f(x) = f(x).$$

Since x is an arbitrary point in $\text{Int}(N)$, this shows that f cannot take a value greater than $a + (1 - a)p$ in $\text{Int}(N)$, contradicting the continuity of f . \square

Proof of Theorem 2. Fix $x \in \text{Int}(N)$. Now for any small neighbourhood V of $K \times \{1\}$, we have $P^x(\Omega_V) = 1$. This shows that $\limsup_{t \rightarrow \infty} f(X_t) = 1$, P^x -almost surely.

Likewise, considering small neighbourhoods of $K \times \{0\}$, we have $\liminf_{t \rightarrow \infty} f(X_t) = 0$, P^x -almost surely. But this contradicts Lemma 2.5. We are done with the proof of Theorem 2.

4. PROOF OF LEMMA 3.2

The projection $\pi : N \rightarrow K$ has a distinguished role since the leafwise metric g is the lift of g_K by π . Let B be an open ball in the base manifold K , and consider an inclusion $\iota : B \times I \subset M$ such that $\iota(\{y\} \times I) = \pi^{-1}(y)$, $\forall y \in B$ and $\iota(B \times \{t\})$ is contained in a leaf of \mathcal{L} , $\forall t \in I$. Such an inclusion ι , or its domain $B \times I$, is called a *distinguished chart*.

Using a distinguished chart, let us define $\hat{p} : N \rightarrow [0, 1]$ by

$$\begin{aligned} \hat{p}(y, t) &= \lim_{h \downarrow 0} p(y, t + h) \quad \text{if } t < 1, \\ \hat{p}(y, 1) &= p(y, 1). \end{aligned}$$

Of course \hat{p} does not depend on the choice of the distinguished charts and is defined on the whole N .

The function \hat{p} is right semicontinuous on each fiber $\pi^{-1}(y)$, $\forall y \in K$, and is leafwise harmonic just as p . Since \hat{p} is nondecreasing on each fiber $\pi^{-1}(y)$, $y \in K$, we get a probability measure ν_y on $\pi^{-1}(y)$ by

$$\begin{aligned}\nu_y(\{y\} \times (s, t]) &= \hat{p}(y, t) - \hat{p}(y, s), \\ \nu_y(\{y\} \times \{0\}) &= \hat{p}(y, 0) - p(y, 0)\end{aligned}$$

for any $0 \leq s < t \leq 1$. Again ν_y does not depend on the choice of the distinguished charts.

Our goal is to show that $\nu_y(\text{Int}(\pi^{-1}(y))) = 0$ for any $y \in K$. Clearly this shows that \hat{p} , and hence p , is constant on $\text{Int}(N)$. Since we do not use the definition of p in what follows, we write p for \hat{p} for simplicity.

We first show that ν_y does not have an atom in $\text{Int}(\pi^{-1}(y))$, $\forall y \in K$. Assume on the contrary that there is $y_0 \in K$ such that ν_{y_0} has an atom in $\text{Int}(\pi^{-1}(y_0))$. Then in a distinguished chart, there is $t_0 \in (0, 1)$ such that

$$q(y_0, t_0) = p(y_0, t_0) - \lim_{t \uparrow t_0} p(y_0, t) > 0.$$

Define a positive function q on the plaque $B \times \{t_0\}$ by

$$q(y, t_0) = p(y, t_0) - \lim_{t \uparrow t_0} p(y, t).$$

Using other distinguished charts, we can define q on the whole leaf L which passes through (y_0, t_0) . The function q is positive harmonic on L . Clearly we have

$$\sum_{x \in L \cap \pi^{-1}(y)} q(x) \leq 1,$$

for any $y \in K$.

Define a measure μ on N by

$$(4.1) \quad \mu = \int_K \left(\sum_{x \in L \cap \pi^{-1}(y)} q(x) \delta_x \right) dy,$$

where δ_x is the Dirac mass at x and dy denotes the volume form on K . Precisely (4.1) means that for any continuous function f on M , we have

$$\int_M f(z) d\mu(z) = \int_K \left(\sum_{x \in L \cap \pi^{-1}(y)} q(x) f(x) \right) dy.$$

Then according to a criterion in [G], μ is a stationary measure, contradicting Proposition 3.1.

Now p is continuous on $\text{Int}(N)$, since there is no atom of ν_y . Next assume that there is $y_0 \in K$ and an interval J in $\pi^{-1}(y_0)$ such that $\nu_{y_0}(J) = 0$. Let J be a maximal such interval and write it as $J = \{y_0\} \times [a, b]$ in a distinguished chart $B \times I$ such that $y_0 \in B$. Then in that chart the function $y \mapsto p(y, b) - p(y, a)$ is nonnegative harmonic and takes value 0 at y_0 . Therefore we have

$$p(y, b) = p(y, a), \quad \forall y \in B.$$

This equality holds also in neighbouring distinguished charts and therefore over all N . Thus if we delete the saturation of $\text{Int}(J)$ from N , the function p is still well defined and continuous. Deleting all such open saturated sets, we get a new manifold, still denoted by N , and a new foliation, still denoted by \mathcal{L} .

For the new foliation \mathcal{L} , the function p is continuous and strictly increasing on the interior of each fiber $\pi^{-1}(y)$. It may not be continuous on the boundary.

However it is possible to extend the function $p|_{\text{Int}(N)}$ to the boundary by continuity, thanks to the leafwise harmonicity of p . The new function p is still constant on each boundary component of N . After normalization, one may assume that $p = i$ on $K \times \{i\}$, $i = 0, 1$. Let B be an open ball in K centered at y_0 . Define a special kind of distinguished chart $B \times I$ by using the value p along the fiber $\pi^{-1}(y_0)$. That is, $p(y_0, t) = t$, $\forall t \in I$. Notice that ν_{y_0} is the Lebesgue measure dt in this chart.

Given two values $0 \leq t' < t'' \leq 1$, the function $q(y) = p(y, t'') - p(y, t')$ is a positive harmonic function on B . By the Harnack inequality, there is $C_1 > 0$ independent of t' and t'' such that

$$|\log q(y) - \log q(y_0)| < C_1 d(y, y_0) < C_2.$$

Since $q(y_0) = t'' - t'$, this shows that there is $C > 1$ such that

$$(4.2) \quad C^{-1}|t'' - t'| \leq |p(y, t'') - p(y, t')| \leq C|t'' - t'|,$$

for any $y \in B$ and $0 \leq t' < t'' \leq 1$. Then $t \mapsto p(y, t)$ is Lipschitz and thus differentiable dt -almost everywhere. Precisely we have

$$(4.3) \quad \text{for any } y \in B, \text{ there is a Lebesgue full measure set } I_y \text{ of } I \text{ such that} \\ \text{the partial derivative } p_t(y, t) \text{ exists for any } t \in I_y.$$

On the other hand, since the space of harmonic functions taking values in $[C^{-1}, C]$ is compact, (4.2) shows that the upper partial derivative $p_t^+(y, t)$ defined by limsup and the lower partial derivative $p_t^-(y, t)$ exists for any $(y, t) \in B \times I$ and is a harmonic function of y . Notice also that

$$C^{-1} \leq p_t^\pm(y, t) \leq C.$$

By (4.3), we have $p_t^+(y, t) = p_t^-(y, t)$ for any $t \in I_y$. Then by Fubini, there is a Lebesgue full measure set I_* of I such that for any $t \in I_*$,

$$(4.4) \quad p_t^+(y, t) = p_t^-(y, t)$$

holds for dy -almost all $y \in B$. But then (4.4) holds for any $y \in B$, since $p_t^\pm(y, t)$ is harmonic in y .

Writing the common value by $p_t(y, t)$ as usual, $\forall t \in I_*$, we can define a measure μ on $B \times I$ by

$$\mu = \int_{I_*} (p_t(y, t) dy) dt.$$

This measure does not depend on the choice of the center y_0 of B . Thus μ can be defined on the whole N . Again by a criterion in [G], μ is a stationary measure. This contradicts Proposition 3.1, finishing the proof of Lemma 3.2.

5. DISCRETE GROUP ACTIONS

Let Γ be a group with a finite symmetric generating set S . We put a probability measure μ on S such that for any $g \in S$,

$$(1) \mu(g) > 0 \quad \text{and} \quad (2) \mu(g^{-1}) = \mu(g).$$

For completeness we write $\mu(g) = 0$ if $g \notin S$.

Assume Γ acts continuously on I , preserving the orientation. A continuous function $f : I \rightarrow \mathbb{R}$ is called μ -harmonic if for any $x \in I$ we have

$$f(x) = \sum_{g \in \Gamma} \mu(g) f(gx).$$

The purpose of this section is to give an alternative proof to the following theorem due to R. Feres and E. Ronshausen. Our proof is parallel to a probabilistic proof of the nonexistence of bounded harmonic functions on the real line.

Theorem 5.1. *Assume there is no global fixed point in $\text{Int}(I)$. Then any continuous μ -harmonic function on I is constant.*

Let us define

$$\Omega = S^{\mathbb{N}} = \{\omega = (\omega_1, \omega_2, \omega_3, \dots) \mid \omega_i \in S\}.$$

Given $n \geq 1$ and $g_1, \dots, g_n \in S$, the n -th cylinder set $[g_1, \dots, g_n] \subset \Omega$ is defined by

$$[g_1, \dots, g_n] = \{\omega \in \Omega \mid \omega_1 = g_1, \dots, \omega_n = g_n\}.$$

Denote by \mathcal{B}_n the (finite) σ -algebra generated by all the n -th cylinder sets. Let \mathcal{B} be the σ -algebra generated by $\cup_n \mathcal{B}_n$. Define a probability measure \mathbb{P} on (Ω, \mathcal{B}) by $\mathbb{P} = \mu^{\otimes \mathbb{N}}$. That is,

$$\mathbb{P}([g_1, \dots, g_n]) = \mu(g_1) \cdots \mu(g_n).$$

Let \mathcal{F} be the completion of \mathcal{B} by the measure \mathbb{P} . For a bounded \mathcal{F} -measurable function ϕ , the conditional expectation by \mathbb{P} is denoted by $\mathbb{E}[\phi \mid \mathcal{B}_n]$. It is a \mathcal{B}_n -measurable function, that is, simply a function constant on each n -th cylinder set. Thus it can be thought of as a function of the n -ple product S^n of S . Now any $\omega \in [g_1, \dots, g_n]$ has the form

$$\omega = (g_1, \dots, g_n, \omega') \text{ for some } \omega' \in \Omega,$$

and the probability conditioned on $[g_1, \dots, g_n]$ is $\mathbb{P}(d\omega')$. Therefore we have

$$\mathbb{E}[\phi \mid \mathcal{B}_n](g_1, \dots, g_n) = \int_{\Omega} \phi(g_1, \dots, g_n, \omega') \mathbb{P}(d\omega').$$

If furthermore the function ϕ is \mathcal{B}_{n+1} -measurable, i.e. a function on S^{n+1} , then we have

$$\mathbb{E}[\phi \mid \mathcal{B}_n](g_1, \dots, g_n) = \sum_{h \in \Gamma} \mu(h) \phi(g_1, \dots, g_n, h).$$

For $n \geq 1$, define a random variable $X_n : \Omega \rightarrow \Gamma$ by

$$X_n(\omega) = \omega_n \cdots \omega_1.$$

Now let us embark upon the proof of Theorem 5.1. Assume for contradiction that there is a continuous μ -harmonic function $f : I \rightarrow [0, 1]$ such that $f(i) = i$, $i = 0, 1$.

Lemma 5.2. *For any $x \in I$, the family $\{f(X_n \cdot x)\}$ is a martingale; i.e. we have for any $n \in \mathbb{N}$*

$$\mathbb{E}[f(X_{n+1} \cdot x) \mid \mathcal{B}_n] = f(X_n \cdot x).$$

Proof. We have

$$\begin{aligned} \mathbb{E}[f(X_{n+1} \cdot x) \mid \mathcal{B}_n](g_1, \dots, g_n) &= \sum_{h \in S} \mu(h) f(h \cdot g_n \cdots g_1 \cdot x) \\ &= f(g_n \cdots g_1 \cdot x) = f(X_n \cdot x)(g_1, \dots, g_n), \end{aligned}$$

where the second equality follows from the μ -harmonicity of f . \square

Therefore by the discrete martingale convergence theorem ([O], Appendix C) we have the following.

Lemma 5.3. *For any $x \in I$, the random variable $f(X_n \cdot x)$ converges as $n \rightarrow \infty$, \mathbb{P} -almost surely.*

But it is shown in [DKNP] that $\liminf_{n \rightarrow \infty} X_n \cdot x = 0$ and $\limsup_{n \rightarrow \infty} X_n \cdot x = 1$, \mathbb{P} -almost surely, provided the probability measure μ satisfies (1) and (2). The contradiction shows Theorem 5.1.

Finally we have the following consequence of Theorem 5.1.

Corollary 5.4. *Assume that the probability μ satisfies (1) and (2), and the group Γ acts on S^1 continuously, preserving the orientation. Then any μ -harmonic function on S^1 is constant on any orbit.*

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REFERENCES

- [DK] Bertrand Deroin and Victor Kleptsyn, *Random conformal dynamical systems*, Geom. Funct. Anal. **17** (2007), no. 4, 1043–1105, DOI 10.1007/s00039-007-0606-y. MR2373011 (2010j:37012)
- [DKNP] B. Deroin, V. Kleptsyn, A. Navas, and K. Parwani, *Symmetric random walks on $\text{Homeo}^+(\mathbf{R})$* , Ann. Probab. **41** (2013), no. 3B, 2066–2089, DOI 10.1214/12-AOP784. MR3098067
- [F] Sérgio R. Fenley, *Foliations, topology and geometry of 3-manifolds: \mathbf{R} -covered foliations and transverse pseudo-Anosov flows*, Comment. Math. Helv. **77** (2002), no. 3, 415–490, DOI 10.1007/s00014-002-8348-9. MR1933786 (2003g:57044)
- [FFP] S. Fenley, R. Feres, and K. Parwani, *Harmonic functions on \mathbb{R} -covered foliations*, Ergodic Theory Dynam. Systems **29** (2009), no. 4, 1141–1161, DOI 10.1017/S0143385708000643. MR2529643 (2010e:53051)
- [FR] Renato Feres and Emily Ronshausen, *Harmonic functions over group actions*, Geometry, rigidity, and group actions, Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, 2011, pp. 59–71. MR2807829 (2012e:60011)
- [FZ] R. Feres and A. Zeghib, *Dynamics on the space of harmonic functions and the foliated Liouville problem*, Ergodic Theory Dynam. Systems **25** (2005), no. 2, 503–516, DOI 10.1017/S0143385704000793. MR2129108 (2005m:30047)
- [G] Lucy Garnett, *Foliations, the ergodic theorem and Brownian motion*, J. Funct. Anal. **51** (1983), no. 3, 285–311, DOI 10.1016/0022-1236(83)90015-0. MR703080 (84j:58099)
- [O] B. Oksendal, *Stochastic differential equations*, sixth edition, Universitext, Springer-Verlag, Berlin, 2003. MR2001996 (2004e:60102)

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