PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 1, Pages 135–147 (December 5, 2014)
S 2330-1511(2014)00015-8

# ARC INDEX OF PRETZEL KNOTS OF TYPE $(-p, q, r)$

HWA JEONG LEE AND GYO TAEK JIN

(Communicated by Daniel Ruberman)

Abstract. We computed the arc index for some of the pretzel knots $K = P(-p, q, r)$ with $p, q, r \geq 2$, $r \geq q$ and at most one of $p, q, r$ is even. If $q = 2$, then the arc index $\alpha(K)$ equals the minimal crossing number $c(K)$. If $p \geq 3$ and $q = 3$, then $\alpha(K) = c(K) - 1$. If $p \geq 5$ and $q = 4$, then $\alpha(K) = c(K) - 2$.

## 1. Arc presentation

Let $D$ be a diagram of a knot or a link $L$. Suppose that there is a simple closed curve $C$ meeting $D$ in $k$ distinct points which divide $D$ into $k$ arcs $\alpha_1, \alpha_2, \ldots, \alpha_k$ with the following properties:

(1) Each $\alpha_i$ has no self-crossing.
(2) If $\alpha_i$ crosses over $\alpha_j$ at a crossing, then $i > j$ and it crosses over $\alpha_j$ at any other crossings with $\alpha_j$.
(3) For each $i$, there exists an embedded disk $d_i$ such that $\partial d_i = C$ and $\alpha_i \subset d_i$.
(4) $d_i \cap d_j = C$, for distinct $i$ and $j$.

Then the pair $(D, C)$ is called an *arc presentation* of $L$ with $k$ arcs, and $C$ is called the *axis* of the arc presentation. Figure 1 shows an arc presentation of the trefoil knot. The thick round curve is the axis. It is known that every knot or link has an arc presentation [3, 4]. For a given knot or link $L$, the minimal number of arcs in all arc presentations of $L$ is called the *arc index* of $L$, denoted by $\alpha(L)$.

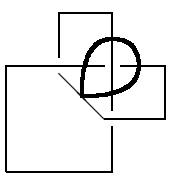


Figure 1. An arc presentation of the right-handed trefoil knot

By removing a point $P$ from $C$ away from $L$, we may identify $C \setminus P$ with the $z$-axis and each $d_i \setminus P$ with a vertical half plane along the $z$-axis. This shows that an arc presentation is equivalent to an open-book presentation.

Received by the editors April 3, 2012 and, in revised form, September 12, 2012, August 24, 2013 and December 20, 2013.

2010 *Mathematics Subject Classification.* Primary 57M27; Secondary 57M25.

The first author was supported in part by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2010-0024630).

The second author was supported in part by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2011-0027989).

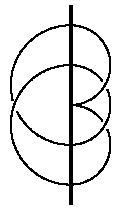


FIGURE 2. An open-book presentation of the right-handed trefoil knot

Given a link $L$, let $c(L)$ denote the minimal crossing number of $L$.

**Theorem 1.1** (Jin-Park). *A prime link $L$ is nonalternating if and only if*

$$\alpha(L) \leq c(L).$$

## 2. Kauffman polynomial

The *Kauffman polynomial* $F_L(a,z)$ of an oriented knot or link $L$ is defined by

$$F_L(a,z) = a^{-w(D)}\Lambda_D(a,z)$$

where $D$ is a diagram of $L$, $w(D)$ the writhe of $D$ and $\Lambda_D(a,z)$ the polynomial determined by the rules K1, K2 and K3.

(K1) $\Lambda_O(a,z) = 1$ where $O$ is the trivial knot diagram.

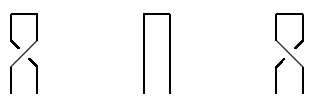
(K2) For any four diagrams $D_+$, $D_-$, $D_0$ and $D_\infty$ which are identical outside a small disk in which they differ as shown below,

$D_+$ $D_-$ $D_0$ $D_\infty$

we have the relation

$$\Lambda_{D_+}(a,z) + \Lambda_{D_-}(a,z) = z(\Lambda_{D_0}(a,z) + \Lambda_{D_\infty}(a,z)).$$

(K3) For any three diagrams $D_+$, $D$ and $D_-$ which are identical outside a small disk in which they differ as shown below,



$D_+$ $D$ $D_-$

we have the relation

$$a\,\Lambda_{D_+}(a,z) = \Lambda_D(a,z) = a^{-1}\Lambda_{D_-}(a,z).$$

For a connected sum and a split union of two diagrams, $\Lambda$ satisfies the following properties:

(K4) If $D$ is a connected sum of $D_1$ and $D_2$, then

$$\Lambda_D(a,z) = \Lambda_{D_1}(a,z)\,\Lambda_{D_2}(a,z).$$

(K5) If $D$ is the split union of $D_1$ and $D_2$, then

$$\Lambda_D(a,z) = (z^{-1}a - 1 + z^{-1}a^{-1})\,\Lambda_{D_1}(a,z)\,\Lambda_{D_2}(a,z).$$

The Laurent degree in the variable $a$ of the Kauffman polynomial $F_L(a, z)$ is denoted by $\text{spread}_a(F_L)$ and defined by the formula

$$\text{spread}_a(F_L) = \text{max-deg}_a(F_L) - \text{min-deg}_a(F_L).$$

Notice that $\text{spread}_a(F_L) = \text{spread}_a(\Lambda_D)$ for any diagram $D$ of $L$. The following theorem gives an important lower bound for the arc index.

**Theorem 2.1** (Morton-Beltrami)**.** *Let $L$ be a link. Then*

$$\alpha(L) \geq \text{spread}_a(F_L) + 2.$$

If $L$ is nonsplit and alternating, then the equality holds so that $\alpha(L) = c(L) + 2$. This is shown by Bae and Park [1] using arc presentations in the form of wheel diagrams.

## 3. Pretzel knots

Given a sequence of integers $p_1, p_2, \ldots, p_n$, we connect two disjoint disks by $n$ bands with $p_i$ half twists, $i = 1, 2, \ldots, n$, so that the boundary of the resulting surface is a link as shown in Figure 3. This link is called the *pretzel link* of type $(p_1, p_2, \ldots, p_n)$ and denoted by $P(p_1, p_2, \ldots, p_n)$.

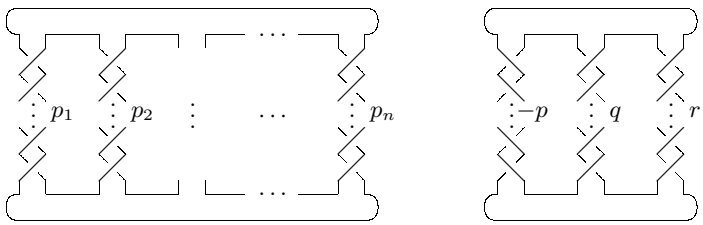



Figure 3. Pretzel links $P(p_1, p_2, \ldots, p_n)$ and $P(-p, q, r)$

In the case $n = 3$, the pretzel links satisfy the following properties:

**Proposition 3.1.** *Let $p$, $q$, and $r$ be nonzero integers.*

(1) *The link type of $P(p, q, r)$ is independent of the order of $p, q, r$.*
(2) *$P(p, q, r)$ is a knot if and only if at most one of $p, q, r$ is an even number.*

In this work, we compute the arc index for the pretzel knots $K = P(-p, q, r)$ with $p, q, r \geq 2$. By Proposition 3.1(1), we may assume that $r \geq q$. By Theorem 3.2, we know that $P(-p, q, r)$ is a minimal crossing diagram of $K$, i.e., $c(K) = p + q + r$.

**Theorem 3.2** (Lickorish-Thistlethwaite)**.** *If a link $L$ admits a reduced Montesinos diagram having $n$ crossings, then $L$ cannot be projected with fewer than $n$ crossings.*

This work was motivated by Theorem 3.3 which is a special case of Theorem 1.1.

**Theorem 3.3** (Beltrami-Cromwell)**.** *If $K = P(-p, q, r)$ is a knot with $p, q, r \geq 2$, then*

$$\alpha(K) \leq c(K) = p + q + r.$$

By computing $\text{spread}_a(F_K)$ and finding arc presentations of $K = P(-p, q, r)$ with the minimum number of arcs for various values of $p$, $q$ and $r$, we obtained sharper results.

## 4. Main results

**Theorem 4.1.** *If $K = P(-2, q, r)$ is a knot with $3 \leq q \leq r$, then*

$$\alpha(K) \leq c(K) - 1.$$

**Theorem 4.2.** *If $K = P(-p, 2, r)$ is a knot with $p \geq 3$, $r \geq 3$, then*

$$\alpha(K) = c(K).$$

**Theorem 4.3.** *If $K = P(-p, 3, r)$ is a knot with $p \geq 3$, $r \geq 3$, then*

$$\alpha(K) = c(K) - 1.$$

**Theorem 4.4.** *If $K = P(-p, 4, r)$ is a knot with $p \geq 5$, $r \geq 5$, then*

$$\alpha(K) = c(K) - 2.$$

**Theorem 4.5.** *If $K = P(-3, 4, r)$ is a knot with $r \geq 7$, then*

$$c(K) - 4 \leq \alpha(K) \leq c(K) - 2.$$

## 5. Arc presentations of $P(-p, q, r)$

**Proposition 5.1.** *If $K = P(-2, q, r)$ is a knot with $3 \leq q \leq r$, then $K$ has an arc presentation with $q + r + 1$ arcs.*

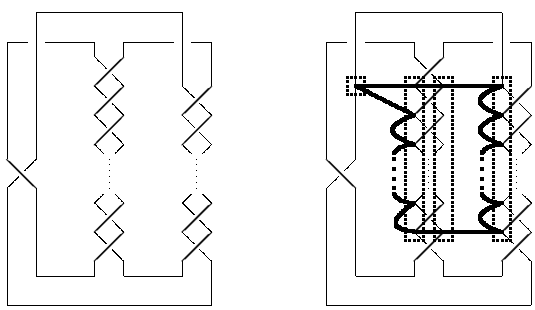
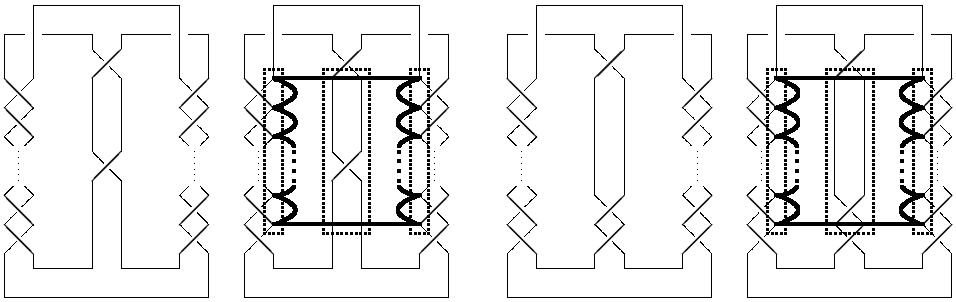


Figure 4. An arc presentation of $P(-2, q, r)$

*Proof.* Figure 4 shows a pretzel diagram of $P(-2, q, r)$ and its arc presentation with $q+r+1$ arcs. The thick curve is the axis of the arc presentation which cuts the knot at 1 place in the leftmost box, $q - 1$ places in the second, 2 places in the third, and $r - 1$ places in the fourth. The $q + r + 1$ arcs of the knot satisfy the four properties of an arc presentation. □

**Proposition 5.2.** *If $K = P(-p, q, r)$ is a knot with $p \geq 3$ and $2 \leq q \leq 3 \leq r$, then $K$ has an arc presentation with $p + r + 2$ arcs.*

*Proof.* For each of $q = 2, 3$, Figure 5 shows a pretzel diagram of $P(-p, q, r)$ and its arc presentation with $p + r + 2$ arcs. The thick curve is the axis of the arc presentation which cuts the knot at $p - 1$ places in the leftmost box, 4 places in the second, and $r - 1$ places in the third. The $p + r + 2$ arcs of the knot satisfy the four properties of an arc presentation. □



$q = 2$ $q = 3$

FIGURE 5. Arc presentations of $P(-p, q, r)$ for $q = 2, 3$

**Proposition 5.3.** *If $K = P(-p, q, r)$ is a knot with $p \geq 3$ and $4 \leq q \leq r$, then $K$ has an arc presentation with $p + q + r - 2$ arcs.*

*Proof.* In Figure 6, the diagram (a) shows a pretzel diagram of $P(-p, q, r)$ with $p \geq 3$ and $4 \leq q \leq r$. The diagram (b) is obtained from (a) by two applications of the Reidemeister move of type 3. The diagram (c) shows an arc presentation with $p+q+r-1$ arcs. The diagram (d) is obtained from (c) by isotoping the arc labeled $x$ over the axis so that there are only $p + q + r - 2$ arcs. Each of the seven boxes, from left to right, contains 1, $p - 3$, 2, 3, $q - 3$, 1, and $r - 3$ arcs, respectively. □

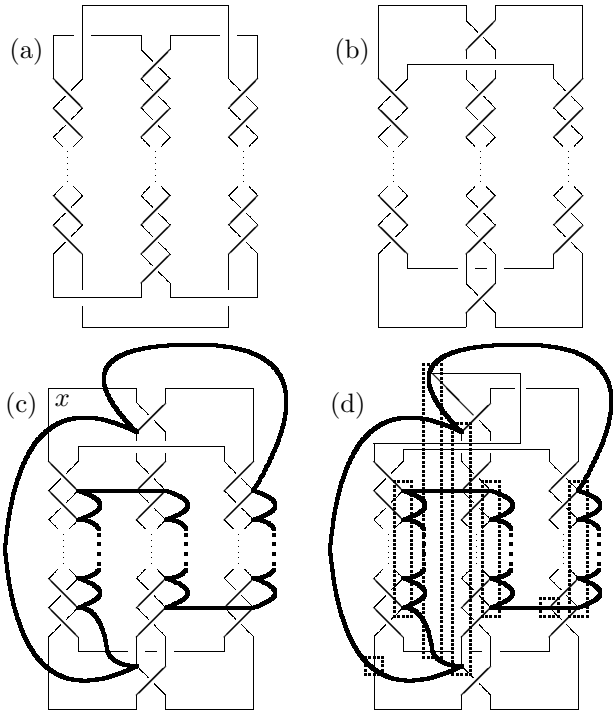



FIGURE 6. Arc presentations of $P(-p, q, r)$ with $p \geq 3$ and $4 \leq q \leq r$

## 6. The Kauffman polynomial of the pretzel knots $P(-p,q,r)$

For any link diagram $D$, the polynomial $\Lambda_D$ is of the form

$$\Lambda_D(a,z) = \sum_{i=m}^{n} f_i(z)a^i$$

where $m,n$ are integers with $m \le n$, and $f_i(z)$'s are polynomials in $z$ with integer coefficients such that $f_m(z) \neq 0$ and $f_n(z) \neq 0$. To simplify our computation of $\mathrm{spread}_a(\Lambda_D)$ we use the notation

$$f_m(z) = \langle k_m z^{h_m} \rangle,$$

$$f_n(z) = \langle k_n z^{h_n} \rangle,$$

$$\sum_{i=m}^{n} f_i(z)a^i = \left[\langle k_n z^{h_n}\rangle a^n, \langle k_m z^{h_m}\rangle a^m\right], \quad (m<n)$$

where $k_m z^{h_m}$ and $k_n z^{h_n}$ are the highest degree terms in $f_m(z)$ and $f_n(z)$, respectively. For example, we write

$$z(z^2-1)a^{-1} + z^2a^{-2} - 2za^{-3} = \left[\langle z^3\rangle a^{-1}, \langle -2z\rangle a^{-3}\right].$$

We also use the notation $\Lambda_{(p_1,p_2,\ldots,p_n)}$ for $\Lambda_D$ when $D = P(p_1,p_2,\ldots,p_n)$.

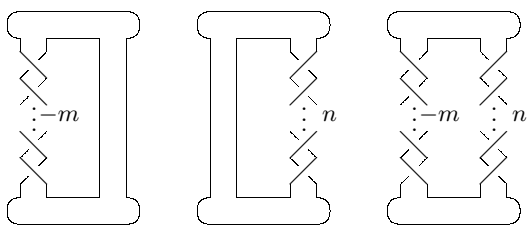



Figure 7. Links $P(-m,0)$, $P(0,n)$, and $P(-m,n)$

Let the polynomial $\Phi_i(z)$ be defined by $\Phi_0(z) = 1, \Phi_1(z) = z$ and

$$\Phi_{i+1}(z) = z\Phi_i(z) - \Phi_{i-1}(z).^1$$

**Lemma 6.1.** *Let $m,n$ be nonnegative integers. Then*

$$\Lambda_{(-m,n)} = \begin{cases} \left[\langle z\rangle a^{k-1}, \langle z^{k-1}\rangle a^{-1}\right] & \text{if } k = m-n > 1, \\ a^{-1} & \text{if } k = m-n = 1, \\ z^{-1}a - 1 + z^{-1}a^{-1} & \text{if } k = n-m = 0, \\ a & \text{if } k = n-m = 1, \\ \left[\langle z^{k-1}\rangle a, \langle z\rangle a^{-(k-1)}\right] & \text{if } k = n-m > 1. \end{cases}$$

*More precisely,*

$$\Lambda_{(-m,n)} = \begin{cases} \left[za^{m-1}, z^{-1}\Phi_m(z)a^{-1}\right] & \text{if } m \ge 3, \quad n = 0, \\ \left[z^{-1}\Phi_n(z)a, za^{-(n-1)}\right] & \text{if } m = 0, \quad n \ge 3. \end{cases}$$

[1] $\Phi_i(2z)$'s are the Chebyshev's polynomials of the second type.

*Proof.* Using the skein relations K1, K2, and K3, we have the following

$$\Lambda_{(-1,0)} = a^{-1}, \quad \Lambda_{(0,1)} = a$$

$$\Lambda_{(0,0)} = z^{-1}a - 1 + z^{-1}a^{-1}.$$

Let $D_v$ be a vertical integer tangle which has $|v|$ times half-twists in the positive or negative direction according to the sign of $v$ for an integer $v$. $D_{+1}$, $D_{-1}$, $D_0$, and $D_\infty$ are exemplified in (K2). In the sublemma below, we use the following simplified notation:

$$\Lambda_v = \Lambda_{D_v}, \quad \Lambda_\infty = \Lambda_{D_\infty}.$$

**Sublemma.** *For $m \geq 2$ and $n \geq 2$, we have*

$$\Lambda_{-m} = -\Phi_{m-2}(z)\,\Lambda_0 + \Phi_{m-1}(z)\,\Lambda_{-1} + \left(\sum_{i=0}^{m-2} z\Phi_i(z)a^{m-1-i}\right)\Lambda_\infty, \tag{6.1}$$

$$\Lambda_n = -\Phi_{n-2}(z)\,\Lambda_0 + \Phi_{n-1}(z)\,\Lambda_{+1} + \left(\sum_{j=0}^{n-2} z\Phi_j(z)a^{-(n-1-j)}\right)\Lambda_\infty. \tag{6.2}$$

*Proof of Sublemma.* As shown below, equation (6.1) holds when $m = 2, 3$.

$$\begin{aligned}
\Lambda_{-2} &= -\Lambda_0 + z\Lambda_{-1} + za\Lambda_\infty \\
&= -\Phi_0(z)\,\Lambda_0 + \Phi_1(z)\,\Lambda_{-1} + z\Phi_0(z)a\,\Lambda_\infty \\
\Lambda_{-3} &= -\Lambda_{-1} + z\Lambda_{-2} + za^2\Lambda_\infty \\
&= -\Phi_0(z)\,\Lambda_{-1} + z\left(-\Phi_0(z)\,\Lambda_0 + \Phi_1(z)\,\Lambda_{-1} + z\Phi_0(z)a\,\Lambda_\infty\right) + za^2\Lambda_\infty \\
&= -\Phi_1(z)\,\Lambda_0 + \Phi_2(z)\,\Lambda_{-1} + \left(\sum_{i=0}^{1} z\Phi_i(z)a^{2-i}\right)\Lambda_\infty.
\end{aligned}$$

Suppose that equation (6.1) holds for $2 \leq m \leq k-1$ for some $k \geq 4$. Then

$$\begin{aligned}
\Lambda_{-k} &= -\Lambda_{-(k-2)} + z\Lambda_{-(k-1)} + za^{k-1}\Lambda_\infty \\
&= -\left(-\Phi_{k-4}(z)\,\Lambda_0 + \Phi_{k-3}(z)\,\Lambda_{-1} + \left(\sum_{i=0}^{k-4} z\Phi_i(z)a^{k-3-i}\right)\Lambda_\infty\right) \\
&\quad + z\left(-\Phi_{k-3}(z)\,\Lambda_0 + \Phi_{k-2}(z)\,\Lambda_{-1} + \left(\sum_{i=0}^{k-3} z\Phi_i(z)a^{k-2-i}\right)\Lambda_\infty\right) + za^{k-1}\Lambda_\infty \\
&= -\Phi_{k-2}(z)\,\Lambda_0 + \Phi_{k-1}(z)\,\Lambda_{-1} + \left(\sum_{i=0}^{k-2} z\Phi_i(z)a^{k-1-i}\right)\Lambda_\infty.
\end{aligned}$$

This proves that equation (6.1) holds for $m \geq 2$. In a similar manner, we can prove that equation (6.2) holds for $n \geq 2$. □

□

For $m \geq 2$ and $n \geq 2$, using equations (6.1) and (6.2) on $m$ and $n$ respectively, we obtain

$$\begin{aligned}\Lambda_{(-m,0)} &= -\Phi_{m-2}(z)\Lambda_{(0,0)} + \Phi_{m-1}(z)\Lambda_{(-1,0)} + \sum_{i=0}^{m-2} z\Phi_i(z)a^{m-1-i} \\ &= \begin{cases} \left[(z-z^{-1})a, (z-z^{-1})a^{-1}\right] & \text{if } m=2, \\ \left[za^{m-1}, \{\Phi_{m-1}(z) - z^{-1}\Phi_{m-2}(z)\}a^{-1}\right] & \text{if } m>2 \end{cases} \\ &= \begin{cases} \left[z^{-1}\Phi_2(z)a, z^{-1}\Phi_2(z)a^{-1}\right] & \text{if } m=2, \\ \left[za^{m-1}, z^{-1}\Phi_m(z)a^{-1}\right] & \text{if } m>2. \end{cases}\end{aligned}$$

$$\begin{aligned}\Lambda_{(0,n)} &= -\Phi_{n-2}(z)\Lambda_{(0,0)} + \Phi_{n-1}(z)\Lambda_{(0,1)} + \sum_{j=0}^{n-2} z\Phi_j(z)a^{-(n-1-j)} \\ &= \begin{cases} \left[(z-z^{-1})a, (z-z^{-1})a^{-1}\right] & \text{if } n=2, \\ \left[\{\Phi_{n-1}(z) - z^{-1}\Phi_{n-2}(z)\}a, za^{-(n-1)}\right] & \text{if } n>2 \end{cases} \\ &= \begin{cases} \left[z^{-1}\Phi_2(z)a, z^{-1}\Phi_2(z)a^{-1}\right] & \text{if } n=2, \\ \left[z^{-1}\Phi_n(z)a, za^{-(n-1)}\right] & \text{if } n>2. \end{cases}\end{aligned}$$

Since $\Lambda_D(a,z)$ is an invariant under regular isotopy of diagrams, we have

$$\Lambda_{(-m,n)} = \begin{cases} \Lambda_{(-k,0)} & \text{if } k = m-n \geq 1, \\ \Lambda_{(0,0)} & \text{if } k = m-n = 0, \\ \Lambda_{(0,k)} & \text{if } k = n-m \geq 1. \end{cases}$$

This completes the proof. □

**Lemma 6.2.**

$$\Lambda_{(-p,0,r)} = \begin{cases} 1 & \textit{if } p=r=1, \\ \left[\langle z\rangle a^p, \langle z^{p-1}\rangle a^0\right] & \textit{if } p>1,\ r=1, \\ \left[\langle z^r\rangle a^p, \langle z^p\rangle a^{-r}\right] & \textit{if } p>1,\ r>1. \end{cases}$$

$$\Lambda_{(-p,1,r)} = \begin{cases} a^2 & \textit{if } p=0,\ r=1, \\ \left[\langle z^{r-1}\rangle a^2, \langle z\rangle a^{-(r-2)}\right] & \textit{if } p=0,\ r>1, \\ a^{-r} & \textit{if } p=1,\ r\geq 1, \\ \left[\langle z^{r+1}\rangle a^p, \langle z^{p-1}\rangle a^{-r}\right] & \textit{if } p>1,\ r\geq 1. \end{cases}$$

*More precisely, for $p \geq 3$ and $r \geq 3$*

$$\Lambda_{(-p,0,r)} = \left[\Phi_r(z)a^p,\ \Phi_p(z)a^{-r}\right]$$

*and for $r \geq 2$*

$$\Lambda_{(-p,1,r)} = \begin{cases} \left[z^{-1}\Phi_{r+2}(z)a^2,\ \Phi_1(z)a^{-r}\right] & \textit{if } p=2, \\ \left[\Phi_{r+1}(z)a^p,\ \Phi_{p-1}(z)a^{-r}\right] & \textit{if } p>2. \end{cases}$$

*Proof.* Since $P(-p,0,r) = P(-p,0)\sharp P(0,r)$, we have $\Lambda_{(-p,0,r)} = \Lambda_{(-p,0)}\,\Lambda_{(0,r)}$. Therefore the formula about $\Lambda_{(-p,0,r)}$ follows from Lemma 6.1.

Now we consider the formula about $\Lambda_{(-p,1,r)}$. Three cases with $p=0$ or $p=1$ follow from K1, K2, K3, and Lemma 6.1. The other case is derived by equation (6.1). For $p \geq 2$, we have

$$\begin{aligned}
\Lambda_{(-p,1,r)} &= -\Phi_{p-2}(z)\Lambda_{(0,1,r)} + \Phi_{p-1}(z)\Lambda_{(-1,1,r)} + \sum_{i=0}^{p-2} z\Phi_i(z)a^{p-1-i}\Lambda_{(1,r)} \\
&= -\Phi_{p-2}(z)a\Lambda_{(0,r)} + \Phi_{p-1}(z)a^{-r} + \sum_{i=0}^{p-2} z\Phi_i(z)a^{p-1-i}\Lambda_{(0,r+1)} \\
&= \begin{cases} \left[\{-z^{-1}\Phi_r(z)+\Phi_{r+1}(z)\}a^2,\ \Phi_1(z)a^{-r}\right] & \text{if } p=2, \\ \left[\Phi_{r+1}(z)a^p,\ \Phi_{p-1}(z)a^{-r}\right] & \text{if } p>2 \end{cases} \\
&= \begin{cases} \left[z^{-1}\Phi_{r+2}(z)a^2,\ \Phi_{p-1}(z)a^{-r}\right] & \text{if } p=2, \\ \left[\Phi_{r+1}(z)a^p,\ \Phi_{p-1}(z)a^{-r}\right] & \text{if } p>2. \end{cases}
\end{aligned}$$

This completes the proof. □

**Proposition 6.3.** $\mathrm{spread}_a(\Lambda_{(-p,2,r)}(a,z)) = p+r$ *for* $p \geq 3$, *and* $r \geq 3$.

*Proof.* For $p \geq 3$, we show that

$$\Lambda_{(-p,2,r)} = \begin{cases} \left[\langle z^3\rangle a^p, \langle z^{p-1}\rangle a^{-2}\right] & (\text{if } r=1), \\ \left[\langle z^4\rangle a^p, \langle 3z^{p-2}\rangle a^{-2}\right] & (\text{if } r=2), \\ \left[\langle z^{r+2}\rangle a^p, \langle z^{p-2}\rangle a^{-r}\right] & (\text{if } r\geq 3). \end{cases} \tag{6.3}$$

Using K1, K2, K3 and Lemmas 6.1 and 6.2, we obtain

$$\begin{aligned}
\Lambda_{(-p,2,1)} &= -\Lambda_{(-p,0,1)} + z\Lambda_{(-p,1,1)} + za^{-1}\Lambda_{(-p,1)} \\
&= -\left[\langle z\rangle a^p, \langle z^{p-1}\rangle a^0\right] + z\left[\langle z^2\rangle a^p, \langle z^{p-1}\rangle a^{-1}\right] + za^{-1}\left[\langle z\rangle a^{p-2}, \langle z^{p-2}\rangle a^{-1}\right] \\
&= \left[\langle z^3\rangle a^p, \langle z^{p-1}\rangle a^{-2}\right], \\
\Lambda_{(-p,2,2)} &= -\Lambda_{(-p,0,2)} + z\Lambda_{(-p,1,2)} + za^{-1}\Lambda_{(-p,2)} \\
&= -\left[za^{p-1}, z^{-1}\Phi_p(z)a^{-1}\right]\left[z^{-1}\Phi_2(z)a, z^{-1}\Phi_2(z)a^{-1}\right] \\
&\qquad + z\left[\Phi_3(z)a^p, \Phi_{p-1}(z)a^{-2}\right] \\
&\qquad + za^{-1}\begin{cases} a^{-1} & (\text{if } p=3), \\ \left[z^{-1}\Phi_2(z)a, z^{-1}\Phi_2(z)a^{-1}\right] & (\text{if } p=4), \\ \left[za^{p-3}, z^{-1}\Phi_{p-2}(z)a^{-1}\right] & (\text{if } p\geq 5) \end{cases} \\
&= -\left[\Phi_2(z)a^p, z^{-2}\Phi_2(z)\Phi_p(z)a^{-2}\right] + z\left[\Phi_3(z)a^p, \Phi_{p-1}(z)a^{-2}\right] \\
&\qquad + \begin{cases} za^{-2} & (\text{if } p=3), \\ za^{-1}\left[z^{-1}\Phi_2(z)a, z^{-1}\Phi_2(z)a^{-1}\right] & (\text{if } p=4), \\ za^{-1}\left[za^{p-3}, z^{-1}\Phi_{p-2}(z)a^{-1}\right] & (\text{if } p\geq 5) \end{cases} \\
&= \left[\Phi_4(z)a^p, \langle 3z^{p-2}\rangle a^{-2}\right],
\end{aligned}$$

which prove the first two cases of (6.3). Now we show the third case of (6.3) by an induction on $r$. For $r = 3$, we have

$$\begin{aligned}
\Lambda_{(-p,2,3)} &= -\Lambda_{(-p,2,1)} + z\Lambda_{(-p,2,2)} + za^{-2}\Lambda_{(-p,2)} \\
&= -\left[\langle z^3\rangle a^p, \langle z^{p-1}\rangle a^{-2}\right] + z\left[\langle z^4\rangle a^p, \langle 3z^{p-2}\rangle a^{-2}\right] \\
&\qquad + \begin{cases} za^{-3} & (\text{if } p = 3), \\ \left[\langle z^2\rangle a^{p-5}, \langle z^{p-2}\rangle a^{-3}\right] & (\text{if } p \geq 4) \end{cases} \\
&= \left[\langle z^5\rangle a^p, \langle z^{p-2}\rangle a^{-3}\right],
\end{aligned}$$

and for $r \geq 4$, inductively, we have

$$\begin{aligned}
\Lambda_{(-p,2,r)} &= -\Lambda_{(-p,2,r-2)} + z\Lambda_{(-p,2,r-1)} + za^{-(r-1)}\Lambda_{(-p,2)} \\
&= -\left[\langle z^r\rangle a^p, \langle *\rangle a^{-(r-2)}\right] + z\left[\langle z^{r+1}\rangle a^p, \langle z^{p-2}\rangle a^{-(r-1)}\right] \\
&\qquad + \begin{cases} za^{-r} & (\text{if } p = 3), \\ \left[\langle z^2\rangle a^{p-r-2}, \langle z^{p-2}\rangle a^{-r}\right] & (\text{if } p \geq 4) \end{cases} \\
&= \left[\langle z^{r+2}\rangle a^p, \langle z^{p-2}\rangle a^{-r}\right]
\end{aligned}$$

where $\langle *\rangle a^{-(r-2)}$ indicates that the lowest $a$-degree of $\Lambda_{(-p,2,r-2)}$ is not smaller than $-(r-2)$.

This completes the proof. □

**Proposition 6.4.** $\operatorname{spread}_a(\Lambda_{(-p,3,r)}(a,z)) = p + r$ *for* $p \geq 3$ *and* $r \geq 3$.

*Proof.* We show that

$$\Lambda_{(-p,3,r)} = \begin{cases} \left[\langle z^6\rangle a^p, \langle 2z^{p-3}\rangle a^{-3}\right] & (\text{if } r = 3), \\ \left[\langle z^{r+3}\rangle a^p, \langle z^{p-3}\rangle a^{-r}\right] & (\text{if } r \geq 4). \end{cases}$$

Using equation (6.2) and Lemmas 6.1 and 6.2, we obtain

$$\begin{aligned}
\Lambda_{(-p,3,3)} &= -\Phi_1(z)\Lambda_{(-p,0,3)} + \Phi_2(z)\Lambda_{(-p,1,3)} + (za^{-2} + z^2a^{-1})\Lambda_{(-p,3)} \\
&= -\Phi_1(z)\left[za^{p-1}, z^{-1}\Phi_p(z)a^{-1}\right]\left[z^{-1}\Phi_3(z)a, za^{-2}\right] \\
&\qquad + \Phi_2(z)\left[\Phi_4 a^p, \Phi_{p-1}(z)a^{-3}\right] \\
&\qquad + (za^{-2} + z^2a^{-1}) \begin{cases} (z^{-1}a - 1 + z^{-1}a^{-1}) & (\text{if } p = 3), \\ a^{-1} & (\text{if } p = 4), \\ \left[\langle z\rangle a^{p-4}, \langle z^{p-4}\rangle a^{-1}\right] & (\text{if } p \geq 5) \end{cases} \\
&= \left[\langle z^6\rangle a^p, \Phi_{p-3}(z)a^{-3}\right] + \begin{cases} \left[z, a^{-3}\right] & (\text{if } p = 3), \\ \left[z^2a^{-2}, za^{-3}\right] & (\text{if } p = 4), \\ \left[\langle z^3\rangle a^{p-5}, \langle z^{p-3}\rangle a^{-3}\right] & (\text{if } p \geq 5) \end{cases} \\
&= \left[\langle z^6\rangle a^p, \langle 2z^{p-3}\rangle a^{-3}\right].
\end{aligned}$$

Using K1, K2, K3 and the results above, we obtain

$$\begin{aligned}
\Lambda_{(-p,3,4)} &= -\Lambda_{(-p,2,3)} + z\Lambda_{(-p,3,3)} + za^{-3}\Lambda_{(-p,3)} \qquad (\because \Lambda_{(-p,3,2)} = \Lambda_{(-p,2,3)}) \\
&= -\left[\langle z^5\rangle a^p, \langle z^{p-2}\rangle a^{-2}\right] + z\left[\langle z^6\rangle a^p, \langle 2z^{p-3}\rangle a^{-3}\right] \\
&\qquad + za^{-3}\begin{cases} (z^{-1}a - 1 + z^{-1}a^{-1}) & (\text{if } p=3), \\ a^{-1} & (\text{if } p=4), \\ \left[\langle z\rangle a^{p-4}, \langle z^{p-4}\rangle a^{-1}\right] & (\text{if } p\geq 5) \end{cases} \\
&= \left[\langle z^7\rangle a^p, \langle z^{p-3}\rangle a^{-4}\right],
\end{aligned}$$

and, for $r \geq 5$, inductively, we have

$$\begin{aligned}
\Lambda_{(-p,3,r)} &= -\Lambda_{(-p,3,r-2)} + z\Lambda_{(-p,3,r-1)} + za^{-(r-1)}\Lambda_{(-p,3)} \\
&= -\left[\langle z^{r+1}\rangle a^p, \langle *\rangle a^{-(r-2)}\right] + z\left[\langle z^{r+2}\rangle a^p, \langle z^{p-3}\rangle a^{-(r-1)}\right] \\
&\qquad + za^{-(r-1)}\begin{cases} (z^{-1}a - 1 + z^{-1}a^{-1}) & (\text{if } p=3), \\ a^{-1} & (\text{if } p=4), \\ \left[\langle z\rangle a^{p-4}, \langle z^{p-4}\rangle a^{-1}\right] & (\text{if } p\geq 5) \end{cases} \\
&= \left[\langle z^{r+3}\rangle a^p, \langle z^{p-3}\rangle a^{-r}\right].
\end{aligned}$$

This completes the proof. □

**Proposition 6.5.** $\mathrm{spread}_a(\Lambda_{(-3,4,r)}(a,z)) = r+1$ *for* $r \geq 7$.

*Proof.*

$$\Lambda_{(-3,4,0)} = \Lambda_{(-3,0)}\Lambda_{(0,4)} = \left[\Phi_4(z)a^3, \Phi_3(z)a^{-4}\right].$$

Using equation (6.1) and K1, K3 and Lemma 6.1, we obtain

$$\begin{aligned}
\Lambda_{(-3,4,1)} &= -\Phi_1(z)\Lambda_{(0,4,1)} + \Phi_2(z)\Lambda_{(-1,4,1)} + \left(\sum_{i=0}^{1} z\Phi_i(z)a^{2-i}\right)\Lambda_{(4,1)} \\
&= -za\Lambda_{(0,4)} + \Phi_2(z)a^{-4} + (za^2 + z^2a)\Lambda_{(0,5)} \\
&= -za\left[z^{-1}\Phi_4(z)a, za^{-3}\right] + \Phi_2(z)a^{-4} + (za^2 + z^2a)\left[z^{-1}\Phi_5(z)a, za^{-4}\right] \\
&= \left[\Phi_5(z)a^3, \Phi_2(z)a^{-4}\right].
\end{aligned}$$

Using equation (6.2) and the above two formulas, for $r \geq 7$ we obtain

$$\begin{aligned}
\Lambda_{(-3,4,r)} &= -\Phi_{r-2}(z)\Lambda_{(-3,4,0)} + \Phi_{r-1}(z)\Lambda_{(-3,4,1)} + \sum_{i=0}^{r-2} z\Phi_i(z)a^{-(r-1-i)}\Lambda_{(-3,4)} \\
&= -\Phi_{r-2}(z)\left[\Phi_4(z)a^3, \Phi_3(z)a^{-4}\right] \\
&\quad + \Phi_{r-1}(z)\left[\Phi_5(z)a^3, \Phi_2(z)a^{-4}\right] + \sum_{i=0}^{r-2} z\Phi_i(z)a^{-(r-2-i)} \\
&= \left[\langle z^{r+4}\rangle a^3, za^{-(r-2)}\right].
\end{aligned}$$

This completes the proof. □

**Proposition 6.6.** $\mathrm{spread}_a(\Lambda_{(-p,4,r)}(a,z)) = p+r$ *for* $p \geq 5$ *and* $r \geq 5$.

*Proof.* Using equation (6.2) and Lemmas 6.1 and 6.2, we obtain

$$\begin{aligned}
\Lambda_{(-p,4,r)} &= -\Phi_2(z)\Lambda_{(-p,0,r)} + \Phi_3(z)\Lambda_{(-p,1,r)} + \sum_{i=0}^{2} z\Phi_i(z)a^{-(3-i)}\Lambda_{(-p,r)} \\
&= -(z^2-1)\left[\Phi_r(z)a^p,\ \Phi_p(z)a^{-r}\right] \\
&\quad + (z^3-2z)\left[\Phi_{r+1}(z)a^p,\ \Phi_{p-1}(z)a^{-r}\right] \\
&\quad + \{za^{-3} + z^2a^{-2} + (z^3-z)a^{-1}\} \begin{cases} \left[\langle z\rangle a^{k-1}, \langle z^{k-1}\rangle a^{-1}\right] & \text{if } k = p-r > 1, \\ a^{-1} & \text{if } k = p-r = 1, \\ z^{-1}a - 1 + z^{-1}a^{-1} & \text{if } k = p-r = 0, \\ a & \text{if } k = r-p = 1, \\ \left[\langle z^{k-1}\rangle a, \langle z\rangle a^{-(k-1)}\right] & \text{if } k = r-p > 1 \end{cases} \\
&= \left[\langle z^{r+4}\rangle a^p, \{-(z^2-1)\Phi_p(z) + (z^3-2z)\Phi_{p-1}(z)\}a^{-r}\right] \\
&= \left[\langle z^{r+4}\rangle a^p, \Phi_{p-4}(z)a^{-r}\right].
\end{aligned}$$

This completes the proof. □

## 7. Proofs of main results and comments

Theorem 4.1 is proved by Proposition 5.1. Table 1 shows that the upper bound '$c(K)-1$' for the arc index in Theorem 4.1 is best possible. It also shows that the lower bound '$\mathrm{spread}_a(F_K)+2$' in Theorem 2.1 is best possible.

Table 1. Examples of Theorem 4.1

| Pretzel knot $K$ | DT Name[2] | $\mathrm{spread}_a(F_K)+2$ | arc index | $c(K)-1$ |
|---|---|---|---|---|
| $P(-2,3,3)$ | $8n3$ | 6 | 7 | 7 |
| $P(-2,3,5)$ | $10n21$ | 6 | 8 | 9 |
| $P(-2,3,7)$ | $12n242$ | 9 | 9 | 11 |
| $P(-2,5,5)$ | $12n725$ | 10 | 10 | 11 |

The proof of Theorem 4.2 is a combination of Propositions 5.2 and 6.3. The proof of Theorem 4.3 is a combination of Propositions 5.2 and 6.4. The proof of Theorem 4.4 is a combination of Propositions 5.3 and 6.6. The proof of Theorem 4.5 is a combination of Propositions 5.3 and 6.5. Table 2 shows that the upper bound '$c(K)-2$' is best possible but the lower bound '$c(K)-4$' may not be best possible.

Table 2. Examples of Theorem 4.5

| Pretzel knot $K$ | DT Name | $c(K)-4$ | arc index | $c(K)-2$ |
|---|---|---|---|---|
| $P(-3,4,5)$ | $12n475$ | 8 | 10 | 10 |
| $P(-3,4,7)$ | $14n12205$ | 10 | 11 | 12 |

[2]The Dowker-Thistlethwaite name. See [10].

## Acknowledgment

The authors would like to thank the referee for valuable comments and suggestions.

## References

[1] Yongju Bae and Chan-Young Park, *An upper bound of arc index of links*, Math. Proc. Cambridge Philos. Soc. **129** (2000), no. 3, 491–500, DOI 10.1017/S0305004100004576. MR1780500 (2002f:57009)

[2] Elisabetta Beltrami and Peter R. Cromwell, *Minimal arc-presentations of some nonalternating knots*, Topology Appl. **81** (1997), no. 2, 137–145, DOI 10.1016/S0166-8641(97)00024-2. MR1481138 (98h:57012)

[3] H. Brunn, *Über verknotete Kurven*, Mathematiker-Kongresses Zurich 1897, Leipzig (1898) 256–259.

[4] Peter R. Cromwell, *Embedding knots and links in an open book. I. Basic properties*, Topology Appl. **64** (1995), no. 1, 37–58, DOI 10.1016/0166-8641(94)00087-J. MR1339757 (96g:57006)

[5] Peter R. Cromwell and Ian J. Nutt, *Embedding knots and links in an open book. II. Bounds on arc index*, Math. Proc. Cambridge Philos. Soc. **119** (1996), no. 2, 309–319, DOI 10.1017/S0305004100074181. MR1357047 (96m:57011)

[6] Gyo Taek Jin and Wang Keun Park, *Prime knots with arc index up to 11 and an upper bound of arc index for non-alternating knots*, J. Knot Theory Ramifications **19** (2010), no. 12, 1655–1672, DOI 10.1142/S0218216510008595. MR2755494 (2012d:57006)

[7] W. B. R. Lickorish and M. B. Thistlethwaite, *Some links with nontrivial polynomials and their crossing-numbers*, Comment. Math. Helv. **63** (1988), no. 4, 527–539, DOI 10.1007/BF02566777. MR966948 (90a:57010)

[8] Hugh R. Morton and Elisabetta Beltrami, *Arc index and the Kauffman polynomial*, Math. Proc. Cambridge Philos. Soc. **123** (1998), no. 1, 41–48, DOI 10.1017/S0305004197002090. MR1474862 (98i:57015)

[9] Knotscape, `http://www.math.utk.edu/~morwen/knotscape.html`

[10] Table of Knot Invariants, `http://www.indiana.edu/~knotinfo/`

Department of Mathematics, Chung-Ang University, 221 Heukseok-dong, Dongjak-gu, Seoul 156-756, Korea

*E-mail address*: `hjwith@cau.ac.kr`

*Current address*: Department of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon 305-701, Korea

*E-mail address*: `hjwith@kaist.ac.kr`

Department of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon 305-701, Korea

*E-mail address*: `trefoil@kaist.ac.kr`