ARC INDEX OF PRETZEL KNOTS OF TYPE $(-p, q, r)$

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Abstract. We computed the arc index for some of the pretzel knots $K = P(-p, q, r)$ with $p, q, r \geq 2$, $r \geq q$ and at most one of $p, q, r$ is even. If $q = 2$, then the arc index $\alpha(K)$ equals the minimal crossing number $c(K)$. If $p \geq 3$ and $q = 3$, then $\alpha(K) = c(K) - 1$. If $p \geq 5$ and $q = 4$, then $\alpha(K) = c(K) - 2$.

1. Arc presentation

Let $D$ be a diagram of a knot or a link $L$. Suppose that there is a simple closed curve $C$ meeting $D$ in $k$ distinct points which divide $D$ into $k$ arcs $\alpha_1, \alpha_2, \ldots, \alpha_k$ with the following properties:

1. Each $\alpha_i$ has no self-crossing.
2. If $\alpha_i$ crosses over $\alpha_j$ at a crossing, then $i > j$ and it crosses over $\alpha_j$ at any other crossings with $\alpha_j$.
3. For each $i$, there exists an embedded disk $d_i$ such that $\partial d_i = C$ and $\alpha_i \subset d_i$.
4. $d_i \cap d_j = C$, for distinct $i$ and $j$.

Then the pair $(D, C)$ is called an arc presentation of $L$ with $k$ arcs, and $C$ is called the axis of the arc presentation. Figure 1 shows an arc presentation of the trefoil knot. The thick round curve is the axis. It is known that every knot or link has an arc presentation [3,4]. For a given knot or link $L$, the minimal number of arcs in all arc presentations of $L$ is called the arc index of $L$, denoted by $\alpha(L)$.

![Figure 1. An arc presentation of the right-handed trefoil knot](image)

By removing a point $P$ from $C$ away from $L$, we may identify $C \setminus P$ with the $z$-axis and each $d_i \setminus P$ with a vertical half plane along the $z$-axis. This shows that an arc presentation is equivalent to an open-book presentation.

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Given a link \( L \), let \( c(L) \) denote the minimal crossing number of \( L \).

**Theorem 1.1 (Jin-Park).** A prime link \( L \) is nonalternating if and only if \( \alpha(L) \leq c(L) \).

## 2. Kauffman Polynomial

The Kauffman polynomial \( F_L(a, z) \) of an oriented knot or link \( L \) is defined by

\[
F_L(a, z) = a^{-w(D)} \Lambda_D(a, z)
\]

where \( D \) is a diagram of \( L \), \( w(D) \) the writhe of \( D \) and \( \Lambda_D(a, z) \) the polynomial determined by the rules K1, K2 and K3.

1. (K1) \( \Lambda_O(a, z) = 1 \) where \( O \) is the trivial knot diagram.
2. (K2) For any four diagrams \( D_+, D_-, D_0 \) and \( D_\infty \) which are identical outside a small disk in which they differ as shown below,

\[
\begin{align*}
D_+ & \quad D_- & \quad D_0 & \quad D_\infty
\end{align*}
\]

we have the relation

\[
\Lambda_{D_+}(a, z) + \Lambda_{D_-}(a, z) = z(\Lambda_{D_0}(a, z) + \Lambda_{D_\infty}(a, z)).
\]

3. (K3) For any three diagrams \( D_+, D \) and \( D_- \) which are identical outside a small disk in which they differ as shown below,

\[
\begin{align*}
D_+ & \quad D & \quad D_-
\end{align*}
\]

we have the relation

\[
a \Lambda_{D_+}(a, z) = \Lambda_D(a, z) = a^{-1} \Lambda_{D_-}(a, z).
\]

For a connected sum and a split union of two diagrams, \( \Lambda \) satisfies the following properties:

1. (K4) If \( D \) is a connected sum of \( D_1 \) and \( D_2 \), then

\[
\Lambda_D(a, z) = \Lambda_{D_1}(a, z) \Lambda_{D_2}(a, z).
\]

2. (K5) If \( D \) is the split union of \( D_1 \) and \( D_2 \), then

\[
\Lambda_D(a, z) = (z^{-1} a - 1 + z^{-1} a^{-1}) \Lambda_{D_1}(a, z) \Lambda_{D_2}(a, z).
\]
The Laurent degree in the variable $a$ of the Kauffman polynomial $F_L(a,z)$ is denoted by $\text{spread}_a(F_L)$ and defined by the formula

$$\text{spread}_a(F_L) = \max - \deg_a(F_L) - \min - \deg_a(F_L).$$

Notice that $\text{spread}_a(F_L) = \text{spread}_a(\Lambda_D)$ for any diagram $D$ of $L$. The following theorem gives an important lower bound for the arc index.

**Theorem 2.1** (Morton-Beltrami). Let $L$ be a link. Then

$$\alpha(L) \geq \text{spread}_a(F_L) + 2.$$

If $L$ is nonsplit and alternating, then the equality holds so that $\alpha(L) = c(L) + 2$. This is shown by Bae and Park using arc presentations in the form of wheel diagrams.

### 3. Pretzel knots

Given a sequence of integers $p_1, p_2, \ldots, p_n$, we connect two disjoint disks by $n$ bands with $p_i$ half twists, $i = 1, 2, \ldots, n$, so that the boundary of the resulting surface is a link as shown in Figure 3. This link is called the *pretzel link* of type $(p_1, p_2, \ldots, p_n)$ and denoted by $P(p_1, p_2, \ldots, p_n)$.

![Figure 3. Pretzel links $P(p_1, p_2, \ldots, p_n)$ and $P(-p, q, r)$](image)

In the case $n = 3$, the pretzel links satisfy the following properties:

**Proposition 3.1.** Let $p$, $q$, and $r$ be nonzero integers.

1. The link type of $P(p, q, r)$ is independent of the order of $p, q, r$.
2. $P(p, q, r)$ is a knot if and only if at most one of $p, q, r$ is an even number.

In this work, we compute the arc index for the pretzel knots $K = P(-p, q, r)$ with $p, q, r \geq 2$. By Proposition 3.1(1), we may assume that $r \geq q$. By Theorem 3.2, we know that $P(-p, q, r)$ is a minimal crossing diagram of $K$, i.e., $c(K) = p + q + r$.

**Theorem 3.2** (Lickorish-Thistlethwaite). If a link $L$ admits a reduced Montesinos diagram having $n$ crossings, then $L$ cannot be projected with fewer than $n$ crossings.

This work was motivated by Theorem 3.3 which is a special case of Theorem 1.1.

**Theorem 3.3** (Beltrami-Cromwell). If $K = P(-p, q, r)$ is a knot with $p, q, r \geq 2$, then

$$\alpha(K) \leq c(K) = p + q + r.$$

By computing $\text{spread}_a(F_K)$ and finding arc presentations of $K = P(-p, q, r)$ with the minimum number of arcs for various values of $p$, $q$ and $r$, we obtained sharper results.
4. Main results

Theorem 4.1. If $K = P(-2, q, r)$ is a knot with $3 \leq q \leq r$, then

$$\alpha(K) \leq c(K) - 1.$$ 

Theorem 4.2. If $K = P(-p, 2, r)$ is a knot with $p \geq 3$, $r \geq 3$, then

$$\alpha(K) = c(K).$$

Theorem 4.3. If $K = P(-p, 3, r)$ is a knot with $p \geq 3$, $r \geq 3$, then

$$\alpha(K) = c(K) - 1.$$ 

Theorem 4.4. If $K = P(-p, 4, r)$ is a knot with $p \geq 5$, $r \geq 5$, then

$$\alpha(K) = c(K) - 2.$$ 

Theorem 4.5. If $K = P(-3, 4, r)$ is a knot with $r \geq 7$, then

$$c(K) - 4 \leq \alpha(K) \leq c(K) - 2.$$ 

5. Arc presentations of $P(-p, q, r)$

Proposition 5.1. If $K = P(-2, q, r)$ is a knot with $3 \leq q \leq r$, then $K$ has an arc presentation with $q + r + 1$ arcs.

Proof. Figure 4 shows a pretzel diagram of $P(-2, q, r)$ and its arc presentation with $q + r + 1$ arcs. The thick curve is the axis of the arc presentation which cuts the knot at 1 place in the leftmost box, $q - 1$ places in the second, 2 places in the third, and $r - 1$ places in the fourth. The $q + r + 1$ arcs of the knot satisfy the four properties of an arc presentation. \(\Box\)

Proposition 5.2. If $K = P(-p, q, r)$ is a knot with $p \geq 3$ and $2 \leq q \leq 3 \leq r$, then $K$ has an arc presentation with $p + r + 2$ arcs.

Proof. For each of $q = 2, 3$, Figure 5 shows a pretzel diagram of $P(-p, q, r)$ and its arc presentation with $p + r + 2$ arcs. The thick curve is the axis of the arc presentation which cuts the knot at $p - 1$ places in the leftmost box, 4 places in the second, and $r - 1$ places in the third. The $p + r + 2$ arcs of the knot satisfy the four properties of an arc presentation. \(\Box\)
Proposition 5.3. If $K = P(-p, q, r)$ is a knot with $p \geq 3$ and $4 \leq q \leq r$, then $K$ has an arc presentation with $p + q + r - 2$ arcs.

Proof. In Figure 6, the diagram (a) shows a pretzel diagram of $P(-p, q, r)$ with $p \geq 3$ and $4 \leq q \leq r$. The diagram (b) is obtained from (a) by two applications of the Reidemeister move of type 3. The diagram (c) shows an arc presentation with $p + q + r - 1$ arcs. The diagram (d) is obtained from (c) by isotoping the arc labeled $x$ over the axis so that there are only $p + q + r - 2$ arcs. Each of the seven boxes, from left to right, contains 1, $p - 3$, 2, 3, $q - 3$, 1, and $r - 3$ arcs, respectively. \(\square\)
6. The Kauffman Polynomial of the Pretzel Knots $P(-p, q, r)$

For any link diagram $D$, the polynomial $\Lambda_D$ is of the form

$$\Lambda_D(a, z) = \sum_{i=m}^{n} f_i(z)a^i$$

where $m, n$ are integers with $m \leq n$, and $f_i(z)$'s are polynomials in $z$ with integer coefficients such that $f_m(z) \neq 0$ and $f_n(z) \neq 0$. To simplify our computation of spread $a(\Lambda_D)$ we use the notation

$$f_m(z) = \langle k_m z^{h_m} \rangle, \quad f_n(z) = \langle k_n z^{h_n} \rangle,$$

where $k_m z^{h_m}$ and $k_n z^{h_n}$ are the highest degree terms in $f_m(z)$ and $f_n(z)$, respectively. For example, we write

$$z(z^2 - 1)a^{-1} + z^2a^{-2} - 2za^{-3} = \left[ (z^3)a^{-1}, \langle -2z \rangle a^{-3} \right].$$

We also use the notation $\Lambda_{(p_1, p_2, \ldots, p_n)}$ for $\Lambda_D$ when $D = P(p_1, p_2, \ldots, p_n)$.

![Figure 7. Links $P(-m, 0)$, $P(0, n)$, and $P(-m, n)$](image)

Let the polynomial $\Phi_i(z)$ be defined by $\Phi_0(z) = 1, \Phi_1(z) = z$ and

$$\Phi_{i+1}(z) = z\Phi_i(z) - \Phi_{i-1}(z)^1$$

Lemma 6.1. Let $m, n$ be nonnegative integers. Then

$$\Lambda_{(-m, n)} = \begin{cases} 
\left[ (z)^{a_k^{-1}}, (z^{k-1})a^{-1} \right] & \text{if } k = m - n > 1, \\
\langle z \rangle^{a^{-1}} & \text{if } k = m - n = 1, \\
\langle z \rangle^{a - 1} & \text{if } k = n - m = 0, \\
\langle z \rangle^{a} & \text{if } k = n - m = 1, \\
\left[ (z^{k-1})a, \langle z \rangle a^{-(k-1)} \right] & \text{if } k = n - m > 1.
\end{cases}$$

More precisely,

$$\Lambda_{(-m, n)} = \begin{cases} 
za^{m-1}, \langle z \rangle^{m-1}a^{-1} & \text{if } m \geq 3, \ n = 0, \\
\langle z \rangle^{n} & \text{if } m = 0, \ n \geq 3.
\end{cases}$$

$^1\Phi_i(2z)$'s are the Chebyshev's polynomials of the second type.
Proof. Using the skein relations K1, K2, and K3, we have the following

\[ \Lambda_{(1,0)} = a^{-1}, \quad \Lambda_{(0,1)} = a \]

\[ \Lambda_{(0,0)} = z^{-1}a - 1 + z^{-1}a^{-1}. \]

Let \( D_v \) be a vertical integer tangle which has \(|v| \) times half-twists in the positive or negative direction according to the sign of \( v \). \( D_{+1}, D_{-1}, D_0, \) and \( D_{\infty} \) are exemplified in (K2). In the sublemma below, we use the following simplified notation:

\[ \Lambda_v = \Lambda_{D_v}, \quad \Lambda_\infty = \Lambda_{D_\infty}. \]

Sublemma. For \( m \geq 2 \) and \( n \geq 2 \), we have

\[
\begin{align*}
(6.1) \quad \Lambda_{-m} &= -\Phi_{m-2}(z) \Lambda_0 + \Phi_{m-1}(z) \Lambda_{-1} + \left( \sum_{i=0}^{m-2} z\Phi_i(z)a^{m-1-i} \right) \Lambda_\infty, \\
(6.2) \quad \Lambda_n &= -\Phi_{n-2}(z) \Lambda_0 + \Phi_{n-1}(z) \Lambda_{+1} + \left( \sum_{j=0}^{n-2} z\Phi_j(z)a^{-(n-1-j)} \right) \Lambda_\infty.
\end{align*}
\]

Proof of Sublemma. As shown below, equation (6.1) holds when \( m = 2, 3 \).

\[
\begin{align*}
\Lambda_{-2} &= -\Lambda_0 + z\Lambda_{-1} + za\Lambda_\infty \\
&= -\Phi_0(z) \Lambda_0 + \Phi_1(z) \Lambda_{-1} + z\Phi_0(z)a \Lambda_\infty \\
\Lambda_{-3} &= -\Lambda_{-1} + z\Lambda_{-2} + za^2\Lambda_\infty \\
&= -\Phi_0(z) \Lambda_{-1} + z(-\Phi_0(z) \Lambda_0 + \Phi_1(z) \Lambda_{-1} + z\Phi_0(z)a \Lambda_\infty) + za^2\Lambda_\infty \\
&= -\Phi_1(z) \Lambda_0 + \Phi_2(z) \Lambda_{-1} + \left( \sum_{i=0}^{1} z\Phi_i(z)a^{2-i} \right) \Lambda_\infty.
\end{align*}
\]

Suppose that equation (6.1) holds for \( 2 \leq m \leq k - 1 \) for some \( k \geq 4 \). Then

\[
\begin{align*}
\Lambda_{-k} &= -\Lambda_{-(k-2)} + z\Lambda_{-(k-1)} + za^{k-1}\Lambda_\infty \\
&= -\left( -\Phi_{k-4}(z) \Lambda_0 + \Phi_{k-3}(z) \Lambda_{-1} + \left( \sum_{i=0}^{k-4} z\Phi_i(z)a^{k-3-i} \right) \Lambda_\infty \right) \\
&\quad + z \left( -\Phi_{k-3}(z) \Lambda_0 + \Phi_{k-2}(z) \Lambda_{-1} + \left( \sum_{i=0}^{k-3} z\Phi_i(z)a^{k-2-i} \right) \Lambda_\infty \right) + za^{k-1}\Lambda_\infty \\
&= -\Phi_{k-2}(z) \Lambda_0 + \Phi_{k-1}(z) \Lambda_{-1} + \left( \sum_{i=0}^{k-2} z\Phi_i(z)a^{k-1-i} \right) \Lambda_\infty.
\end{align*}
\]

This proves that equation (6.1) holds for \( m \geq 2 \). In a similar manner, we can prove that equation (6.2) holds for \( n \geq 2 \). \( \square \)
For \( m \geq 2 \) and \( n \geq 2 \), using equations (6.1) and (6.2) on \( m \) and \( n \) respectively, we obtain

\[
\Lambda_{(-m,0)} = -\Phi_{m-2}(z)\Lambda_{(0,0)} + \Phi_{m-1}(z)\Lambda_{(-1,0)} + \sum_{i=0}^{m-2} z\Phi_i(z)a^{m-1-i}
\]

\[
= \begin{cases} 
[(z - z^{-1})a, (z - z^{-1})a^{-1}] & \text{if } m = 2, \\
za^{m-1}, \{\Phi_{m-1}(z) - z^{-1}\Phi_{m-2}(z)a^{-1}\} & \text{if } m > 2
\end{cases}
\]

\[
= \begin{cases} 
[z^{-1}\Phi_2(z)a, z^{-1}\Phi_2(z)a^{-1}] & \text{if } m = 2, \\
za^{m-1}, z^{-1}\Phi_m(z)a^{-1} & \text{if } m > 2.
\end{cases}
\]

\[
\Lambda_{(0,n)} = -\Phi_{n-2}(z)\Lambda_{(0,0)} + \Phi_{n-1}(z)\Lambda_{(0,1)} + \sum_{j=0}^{n-2} z\Phi_j(z)a^{-(n-1-j)}
\]

\[
= \begin{cases} 
[(z - z^{-1})a, (z - z^{-1})a^{-1}] & \text{if } n = 2, \\
\{\Phi_{n-1}(z) - z^{-1}\Phi_{n-2}(z)a^{-1}\}a, za^{-(n-1)} & \text{if } n > 2
\end{cases}
\]

\[
= \begin{cases} 
[z^{-1}\Phi_2(z)a, z^{-1}\Phi_2(z)a^{-1}] & \text{if } n = 2, \\
[z^{-1}\Phi_n(z)a, za^{-(n-1)}] & \text{if } n > 2.
\end{cases}
\]

Since \( \Lambda_D(a,z) \) is an invariant under regular isotopy of diagrams, we have

\[
\Lambda_{(-m,n)} = \begin{cases} 
\Lambda_{(-k,0)} & \text{if } k = m - n \geq 1, \\
\Lambda_{(0,0)} & \text{if } k = m - n = 0, \\
\Lambda_{(0,k)} & \text{if } k = n - m \geq 1.
\end{cases}
\]

This completes the proof.

Lemma 6.2.

\[
\Lambda_{(-p,0,r)} = \begin{cases} 
1 & \text{if } p = r = 1, \\
[(z)a^p, (z^{p-1})a^0] & \text{if } p > 1, r = 1, \\
[(z^r)a^p, (z^p)a^{-r}] & \text{if } p > 1, r > 1.
\end{cases}
\]

More precisely, for \( p \geq 3 \) and \( r \geq 3 \)

\[
\Lambda_{(-p,0,r)} = [\Phi_r(z)a^p, \Phi_p(z)a^{-r}]
\]

and for \( r \geq 2 \)

\[
\Lambda_{(-p,1,r)} = \begin{cases} 
[z^{-1}\Phi_{r+2}(z)a^2, \Phi_1(z)a^{-r}] & \text{if } p = 2, \\
\Phi_{r+1}(z)a^p, \Phi_{p-1}(z)a^{-r} & \text{if } p > 2.
\end{cases}
\]

Proof. Since \( P(-p,0,r) = P(-p,0)\overline{z}P(0,r) \), we have \( \Lambda_{(-p,0,r)} = \Lambda_{(-p,0)}\Lambda_{(0,r)} \).

Therefore the formula about \( \Lambda_{(-p,0,r)} \) follows from Lemma 6.1.
Now we consider the formula about $\Lambda_{(-p,1,r)}$. Three cases with $p = 0$ or $p = 1$ follow from K1, K2, K3, and Lemma 6.1. The other case is derived by equation (6.1). For $p \geq 2$, we have

$$
\Lambda_{(-p,1,r)} = -\Phi_{p-2}(z)\Lambda_{(0,1,r)} + e^{p-1}(z)\Lambda_{(-1,1,r)} + \sum_{i=0}^{p-2} z\Phi_i(z)a^{p-1-i}\Lambda_{(1,r)}
$$

$$
= -\Phi_{p-2}(z)a\Lambda_{(0,r)} + e^{p-1}(z)a^{-r} + \sum_{i=0}^{p-2} z\Phi_i(z)a^{p-1-i}\Lambda_{(0,r+1)}
$$

$$
= \begin{cases} 
\left\{ (z^{-1})^2 \Phi_r(z) + \Phi_{r+1}(z) \right\} a^2, & \text{if } p = 2, \\
\left\{ \Phi_{r+1}(z)a^p, \Phi_{p-1}(z)a^{-r} \right\} & \text{if } p > 2
\end{cases}
$$

This completes the proof.

**Proposition 6.3.** $\text{spread}_a(\Lambda_{(-p,2,r)}(a,z)) = p + r$ for $p \geq 3$, and $r \geq 3$.

**Proof.** For $p \geq 3$, we show that

$$
\Lambda_{(-p,2,r)} = \begin{cases} 
\left\{ (z^3)^2 a^p, (z^{p-1})a^{-r} \right\} & \text{if } r = 1, \\
\left\{ (z^4)^2 a^p, (3z^{p-2})a^{-r} \right\} & \text{if } r = 2, \\
\left\{ (z^{r+2})a^p, (z^{p-2})a^{-r} \right\} & \text{if } r \geq 3.
\end{cases}
$$

Using K1, K2, K3 and Lemmas 6.1 and 6.2, we obtain

$$
\Lambda_{(-p,2,1)} = -\Lambda_{(-p,0,1)} + z\Lambda_{(-p,1,1)} + za^{-1}\Lambda_{(-p,1)}
$$

$$
= - \left\{ (z)a^p, (z^{p-1})a^0 \right\} + z \left\{ (z^2)a^p, (z^{p-1})a^{-1} \right\} + za^{-1} \left\{ (z)a^{p-2}, (z^{p-2})a^{-1} \right\}
$$

$$
= \left\{ (z^3)^2 a^p, (z^{p-1})a^{-2} \right\},
$$

$$
\Lambda_{(-p,2,2)} = -\Lambda_{(-p,0,2)} + z\Lambda_{(-p,1,2)} + za^{-1}\Lambda_{(-p,2)}
$$

$$
= - \left\{ za^{p-1}, z^{-1}\Phi_p(z)a^{-1} \right\} \left\{ z^{-1}\Phi_2(z)a, z^{-1}\Phi_2(z)a^{-1} \right\}
$$

$$
+ z \left\{ \Phi_3(z)a^p, \Phi_{p-1}(z)a^{-2} \right\}
$$

$$
+ za^{-1} \begin{cases} 
\left\{ (z^{-1})^2 \Phi_2(z)a, z^{-1}\Phi_2(z)a^{-1} \right\} & \text{if } p = 3, \\
\left\{ za^{-3}, z^{-1}\Phi_{p-2}(z)a^{-1} \right\} & \text{if } p \geq 5
\end{cases}
$$

$$
= - \left\{ \Phi_2(z)a^p, z^{-2}\Phi_2(z)\Phi_p(z)a^{-2} \right\} + z \left\{ \Phi_3(z)a^p, \Phi_{p-1}(z)a^{-2} \right\}
$$

$$
+ za^{-2} \begin{cases} 
za^{-2} & \text{if } p = 3, \\
za^{-1} \left\{ (z^{-1})^2 \Phi_2(z)a, z^{-1}\Phi_2(z)a^{-1} \right\} & \text{if } p = 4, \\
za^{-1} \left\{ za^{-3}, z^{-1}\Phi_{p-2}(z)a^{-1} \right\} & \text{if } p \geq 5
\end{cases}
$$

$$
= \left\{ \Phi_1(z)a^p, (3z^{p-2})a^{-2} \right\},
$$
which prove the first two cases of (6.3). Now we show the third case of (6.3) by an induction on \( r \). For \( r = 3 \), we have

\[
\Lambda_{(-p,2,3)} = -\Lambda_{(-p,2,1)} + z\Lambda_{(-p,2,2)} + za^{-2}\Lambda_{(-p,2)}
\]

\[
= -[\langle z^3 \rangle a^p, \langle z^{p-1} \rangle a^{-2}] + z [\langle z^4 \rangle a^p, \langle 3z^{p-2} \rangle a^{-2}]
\]

\[
+ \begin{cases} za^{-3} & \text{if } p = 3, \\ [\langle z^2 \rangle a^{p-5}, \langle z^{p-2} \rangle a^{-3}] & \text{if } p \geq 4 \end{cases}
\]

\[
= [\langle z^5 \rangle a^p, \langle z^{p-2} \rangle a^{-3}]
\]

and for \( r \geq 4 \), inductively, we have

\[
\Lambda_{(-p,2,r)} = -\Lambda_{(-p,2,r-2)} + z\Lambda_{(-p,2,r-1)} + za^{-(r-1)}\Lambda_{(-p,2)}
\]

\[
= -[\langle z^r \rangle a^p, (\ast) a^{-(r-2)}] + z [\langle z^{r+1} \rangle a^p, \langle z^{p-2} \rangle a^{-(r-1)}]
\]

\[
+ \begin{cases} za^{-r} & \text{if } p = 3, \\ [\langle z^2 \rangle a^{p-r-2}, \langle z^{p-2} \rangle a^{-r}] & \text{if } p \geq 4 \end{cases}
\]

\[
= [\langle z^{r+2} \rangle a^p, \langle z^{p-2} \rangle a^{-r}]
\]

where \((\ast) a^{-(r-2)}\) indicates that the lowest \( a \)-degree of \( \Lambda_{(-p,2,r-2)} \) is not smaller than \((r-2)\).

This completes the proof. \( \square \)

**Proposition 6.4.** \( \text{spread}_a(\Lambda_{(-p,3,r)}(a, z)) = p + r \) for \( p \geq 3 \) and \( r \geq 3 \).

**Proof.** We show that

\[
\Lambda_{(-p,3,r)} = \begin{cases}
[\langle z^6 \rangle a^p, \langle 2z^{p-3} \rangle a^{-3}] & \text{if } r = 3, \\
[\langle z^{r+3} \rangle a^p, \langle z^{p-3} \rangle a^{-r}] & \text{if } r \geq 4.
\end{cases}
\]

Using equation (6.2) and Lemmas 6.1 and 6.2 we obtain

\[
\Lambda_{(-p,3,3)} = -\Phi_1(z)\Lambda_{(-p,0,3)} + \Phi_2(z)\Lambda_{(-p,1,3)} + (za^{-2} + z^2a^{-1})\Lambda_{(-p,3)}
\]

\[
= -\Phi_1(z) [za^{p-1}, z^{-1}\Phi_p(z)a^{-1}] [z^{-1}\Phi_3(z)a, za^{-2}]
\]

\[
+ \Phi_2(z) [\Phi_4 a^p, \Phi_{p-1}(z)a^{-3}]
\]

\[
+ (za^{-2} + z^2a^{-1}) \begin{cases} (z^{-1}a - 1 + z^{-1}a^{-1}) & \text{if } p = 3, \\
\langle z \rangle a^{p-4}, \langle z^{p-4} \rangle a^{-1} & \text{if } p = 4, \\
\langle z \rangle a^{p-4}, \langle z^{p-4} \rangle a^{-1} & \text{if } p \geq 5 \end{cases}
\]

\[
= [\langle z^6 \rangle a^p, \Phi_{p-3}(z)a^{-3}] + \begin{cases} \langle z, a^{-3} \rangle & \text{if } p = 3, \\
\langle z^2a^{-2}, za^{-3} \rangle & \text{if } p = 4, \\
\langle z^3a^{p-5}, z^{p-3}a^{-3} \rangle & \text{if } p \geq 5 \end{cases}
\]

\[
= [\langle z^6 \rangle a^p, \langle 2z^{p-3} \rangle a^{-3}].
\]
Using K1, K2, K3 and the results above, we obtain
\[
\Lambda(-p,3,4) = -\Lambda(-p,2,3) + z\Lambda(-p,3,3) + za^{-3}\Lambda(-p,3) \quad (\because \Lambda(-p,3,2) = \Lambda(-p,2,3))
\]
\[
= -\left(\langle z^5 \rangle a^p, \langle z^{p-2} \rangle a^{-2} \right) + z \left(\langle z^6 \rangle a^p, \langle 2z^{p-3} \rangle a^{-3} \right)
\]
\[
+ za^{-3} \left\{ \begin{array}{ll}
(z^{-1}a - 1 + z^{-1}a^{-1}) & \text{if } p = 3, \\
a^{-1} & \text{if } p = 4, \\
\left[\langle z \rangle a^{p-4}, \langle z^{p-4} \rangle a^{-1} \right] & \text{if } p \geq 5 \end{array} \right.
\]
\[
= \left[\langle z^7 \rangle a^p, \langle z^{p-3} \rangle a^{-4} \right],
\]
and, for \( r \geq 5 \), inductively, we have
\[
\Lambda(-p,3,r) = -\Lambda(-p,3,r-2) + z\Lambda(-p,3,r-1) + za^{-(r-1)}\Lambda(-p,3)
\]
\[
= -\left(\langle z^{r+1} \rangle a^p, \langle * \rangle a^{-(r-2)} \right) + z \left(\langle z^{r+2} \rangle a^p, \langle z^{p-3} \rangle a^{-(r-1)} \right)
\]
\[
+ za^{-(r-1)} \left\{ \begin{array}{ll}
(z^{-1}a - 1 + z^{-1}a^{-1}) & \text{if } p = 3, \\
a^{-1} & \text{if } p = 4, \\
\left[\langle z \rangle a^{p-4}, \langle z^{p-4} \rangle a^{-1} \right] & \text{if } p \geq 5 \end{array} \right.
\]
\[
= \left[\langle z^{r+3} \rangle a^p, \langle z^{p-3} \rangle a^{-4} \right].
\]
This completes the proof. \( \square \)

**Proposition 6.5.** \( \text{spread}_a(\Lambda(-3,4,r)(a, z)) = r + 1 \) for \( r \geq 7 \).

**Proof.**
\[
\Lambda(-3,4,0) = \Lambda(-3,0)\Lambda(0,4) = \left[\Phi_4(z)a^3, \Phi_3(z)a^{-4} \right].
\]
Using equation (6.1) and K1, K3 and Lemma 6.1 we obtain
\[
\Lambda(-3,4,1) = -\Phi_1(z)\Lambda(0,4,1) + \Phi_2(z)\Lambda(-1,4,1) + \left(\sum_{i=0}^{1} z\Phi_i(z)a^{2-i} \right) \Lambda(4,1)
\]
\[
= -za\Lambda(0,4) + \Phi_2(z)a^{-4} + (za^2 + z^2a)\Lambda(0,5)
\]
\[
= -za \left[ z^{-1}\Phi_4(z)a, za^{-3} \right] + \Phi_2(z)a^{-4} + (za^2 + z^2a) \left[ z^{-1}\Phi_5(z)a, za^{-4} \right]
\]
\[
= \left[\Phi_5(z)a^3, \Phi_2(z)a^{-4} \right].
\]
Using equation (6.2) and the above two formulas, for \( r \geq 7 \) we obtain
\[
\Lambda(-3,4,r) = -\Phi_{r-2}(z)\Lambda(-3,4,0) + \Phi_{r-1}(z)\Lambda(-3,4,1) + \sum_{i=0}^{r-2} z\Phi_i(z)a^{-(r-1-i)}\Lambda(-3,4)
\]
\[
= -\Phi_{r-2}(z) \left[\Phi_4(z)a^3, \Phi_3(z)a^{-4} \right]
\]
\[
+ \Phi_{r-1}(z) \left[\Phi_5(z)a^3, \Phi_2(z)a^{-4} \right] + \sum_{i=0}^{r-2} z\Phi_i(z)a^{-(r-2-i)}
\]
\[
= \left[\langle z^{r+4} \rangle a^3, za^{-(r-2)} \right].
\]
This completes the proof. \( \square \)

**Proposition 6.6.** \( \text{spread}_a(\Lambda(-p,4,r)(a, z)) = p + r \) for \( p \geq 5 \) and \( r \geq 5 \).
Proof. Using equation (6.2) and Lemmas 6.1 and 6.2, we obtain

\[
\Lambda_{(-p,4,r)} = -\Phi_2(z)\Lambda_{(-p,0,r)} + \Phi_3(z)\Lambda_{(-p,1,r)} + \sum_{i=0}^{2} z\Phi_i(z)a^{-(3-i)}\Lambda_{(-p,r)}
\]

\[
= -(z^2 - 1) [\Phi_r(z)a^p, \Phi_p(z)a^{-r}] + (z^3 - 2z) [\Phi_{r+1}(z)a^p, \Phi_{p+1}(z)a^{-r}] + \sum_{i=0}^{2} z\Phi_i(z)a^{-i} \Lambda_{(-p,r)}
\]

\[
= \left\{ \begin{array}{ll}
\langle z \rangle a^{k-1}, \langle z \rangle a^{k-1} & \text{if } k = p - r > 1,
\langle z \rangle a^{-1} & \text{if } k = p - r = 1,
\langle z \rangle a^{-1} & \text{if } k = p - r = 0,
\langle z \rangle a & \text{if } k = r - p = 1,
\langle z \rangle a^{-1} & \text{if } k = r - p > 1
\end{array} \right.
\]

\[
= \langle z^{r+4} \rangle a^p, \{-(z^2 - 1)\Phi_p(z) + (z^3 - 2z)\Phi_{p-1}(z)\}a^{-r}
\]

\[
= \langle z^{r+4} \rangle a^p, \Phi_{p-4}(z)a^{-r} \}
\]

This completes the proof. □

7. PROOFS OF MAIN RESULTS AND COMMENTS

Theorem 4.1 is proved by Proposition 5.1. Table 1 shows that the upper bound ‘c(K) − 1’ for the arc index in Theorem 4.1 is best possible. It also shows that the lower bound ‘spread\textsubscript{a}(F\textsubscript{K}) + 2’ in Theorem 2.1 is best possible.

<table>
<thead>
<tr>
<th>Pretzel knot K</th>
<th>DT Name</th>
<th>spread\textsubscript{a}(F\textsubscript{K}) + 2</th>
<th>arc index</th>
<th>c(K) − 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(−2, 3, 3)</td>
<td>8n3</td>
<td>6</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>P(−2, 3, 5)</td>
<td>10n21</td>
<td>6</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>P(−2, 3, 7)</td>
<td>12n242</td>
<td>9</td>
<td>9</td>
<td>11</td>
</tr>
<tr>
<td>P(−2, 5, 5)</td>
<td>12n725</td>
<td>10</td>
<td>10</td>
<td>11</td>
</tr>
</tbody>
</table>

The proof of Theorem 4.2 is a combination of Propositions 5.2 and 6.3. The proof of Theorem 4.3 is a combination of Propositions 5.2 and 6.4. The proof of Theorem 4.4 is a combination of Propositions 5.3 and 6.6. The proof of Theorem 4.5 is a combination of Propositions 5.3 and 6.5. Table 2 shows that the upper bound ‘c(K) − 2’ is best possible but the lower bound ‘c(K) − 4’ may not be best possible.

<table>
<thead>
<tr>
<th>Pretzel knot K</th>
<th>DT Name</th>
<th>c(K) − 4</th>
<th>arc index</th>
<th>c(K) − 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(−3, 4, 5)</td>
<td>12n475</td>
<td>8</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>P(−3, 4, 7)</td>
<td>14n12205</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
</tbody>
</table>

\textsuperscript{2}The Dowker-Thistlethwaite name. See [10].
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