EXTENSION PROBLEM OF SUBSET-CONTROLLED QUASIMORPHISMS

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Abstract. Let \((G, H)\) be \((\text{Ham}(\mathbb{R}^{2n}), \text{Ham}(\mathbb{R}^{2n}))\) or \((B_{\infty}, B_n)\). We conjecture that any semi-homogeneous subset-controlled quasimorphism on \([G, G]\) can be extended to a homogeneous subset-controlled quasimorphism on \(G\) and also give a theorem supporting this conjecture by using a Bavard-type duality theorem on conjugation invariant norms.

1. Problems and results

To state our conjecture, we introduce the notion of subset-controlled quasimorphism (partial quasimorphism, pre-quasimorphism) which is a generalization of quasimorphism.

Definition 1.1. Let \(G\) be a group and let \(H\) be a subset of \(G\). We define the fragmentation norm \(q_H\) with respect to \(H\) for an element \(f\) of \(G\),
\[
q_H(f) = \min\{k; \exists g_1, \ldots, g_k \in G, \exists h_1, \ldots, h_k \in H \text{ such that } f = g_1^{-1}h_1g_1 \cdots g_k^{-1}h_kg_k\}.
\]
If there is no such decomposition of \(f\) as above, we put \(q_H(f) = \infty\).

\(H\) \(c\)-generates \(G\) if such decomposition as above exists for any \(f \in G\).

Definition 1.2. Let \(H, G'\) be subgroups of a group \(G\). A function \(\mu : G' \to \mathbb{R}\) is called an \(H\)-quasimorphism on \(G'\) if there exists a positive number \(C\) such that for any elements \(f, g\) of \(G'\),
\[
|\mu(fg) - \mu(f) - \mu(g)| < C \cdot \min\{q_H(f), q_H(g)\}.
\]

\(\mu\) is called homogeneous if \(\mu(f^n) = n\mu(f)\) holds for any element \(f\) of \(G'\) and any \(n \in \mathbb{Z}\). \(\mu\) is called semi-homogeneous if \(\mu(f^n) = n\mu(f)\) holds for any element \(f\) of \(G'\) and any \(n \in \mathbb{Z}_{\geq 0}\).

Such generalization as above of quasimorphism appeared first in [EP06]. For a symplectic manifold \((M, \omega)\), let \(\text{Ham}(M)\) denote the group of Hamiltonian diffeomorphisms with compact support and \((\mathbb{R}^{2n}, \omega_0)\), \((\mathbb{B}^{2n}, \omega_0)\) denote the 2\(n\)-dimensional Euclidean space, ball with the standard symplectic form, respectively. Let \(B_n\) denote the \(n\)-braid group and \(B_{\infty}\) denote the infinite braid group \(\bigcup_n B_n\).

We pose the following conjecture. For a group \(G\), let \([G, G]\) denote the commutator subgroup (the subgroup generated by the set \(\{[a, b] = aba^{-1}b^{-1}a, b \in G\}\) of commutators).

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Conjecture 1.3. Let \((G,H)\) be \((\text{Ham}(\mathbb{R}^{2n}),\text{Ham}(\mathbb{B}^{2n}))\) or \((B_{\infty},B_{n})\). For a semi-homogeneous \(H\)-quasimorphism \(\mu\) on \([G,G]\), there exists a homogeneous \(H\)-quasimorphism \(\hat{\mu}\) on \(G\) such that \(\hat{\mu}|_{[G,G]} = \mu\). In particular, any semi-homogeneous \(H\)-quasimorphism on \([G,G]\) is a homogeneous \(H\)-quasimorphism.

The author ([Ka1]) and Kimura ([Ki1]) constructed a non-trivial \(H\)-quasimorphism on \(G\) when \((G,H) = (\text{Ham}(\mathbb{R}^{2n}),\text{Ham}(\mathbb{B}^{2n}))\), \((G,H) = (B_{\infty},B_{n})\), respectively. Kimura also proved that the dimension of the linear space of \(H\)-quasimorphisms on \(G\) is infinite when \((G,H) = (B_{\infty},B_{n})\) ([Ki2]).

We give examples of semi-homogeneous subset-controlled quasimorphisms which are not homogeneous. Let \(\mathbb{T}^2\) be a 2-torus. By Proposition 3.1 of [Ka3], we see that the asymptotic Oh-Schwarz invariant \(\mu: \text{Ham}(\mathbb{T}^2) \to \mathbb{R}\) with respect to the fundamental class \([\mathbb{T}^2]\) is a semi-homogeneous \(\text{Ham}(U)\)-quasimorphism for any open subset \(U\) of \(\mathbb{T}^2\) whose closure \(\bar{U}\) is contractible. Since a meridian curve \(M\) in the 2-torus \(\mathbb{T}^2\) is heavy but not superheavy in the sense of Entov and Polterovich ([EP09]), we see that \(\mu\) is not homogeneous. Let \(\Lambda\) be an annulus embedded to \(\mathbb{T}^2\) such that \(M,U \subset \Lambda\). By restricting \(\mu\) to \(\text{Ham}(\Lambda)\), we can construct a semi-homogeneous subset-controlled quasimorphism which is not homogeneous on \(\text{Ham}(\mathbb{B}^{2n})\).

However, the author does not know whether there is a semi-homogeneous subset-controlled quasimorphism which is not homogeneous on \(\text{Ham}(\mathbb{B}^{2n})\).

Our main theorem is the following one which supports the above conjecture.

**Theorem 1.4.** Let \((G,H)\) be \((\text{Ham}(\mathbb{R}^{2n}),\text{Ham}(\mathbb{B}^{2n}))\) or \((B_{\infty},B_{n})\). For a semi-homogeneous \(H\)-quasimorphism \(\mu\) on \([G,G]\) and an element \(g\) of \([G,G]\) such that \(\mu(g) \neq 0\), there exists a homogeneous \(H\)-quasimorphism \(\hat{\mu}_g\) on \(G\) such that \(\hat{\mu}_g(g) \neq 0\).

In Section 2, we prepare some notions and statements. We prove Theorem 1.4 when \((G,H) = (B_{\infty},B_{n}), (\text{Ham}(\mathbb{R}^{2n}),\text{Ham}(\mathbb{B}^{2n}))\) in Sections 3 and 4 respectively.

## 2. Preliminaries

Let \(G'\) be a \(G\)-invariant subgroup of a group \(G\) i.e. \(g^{-1}g'g \in G'\) holds for any \(g' \in G'\) and any \(g \in G\). A function \(\nu: G' \to \mathbb{R}_{\geq 0}\) is called a \(G\)-invariant norm on \(G'\) if \(\nu\) is a conjugation-invariant norm on \(G'\) (see [BIP]) and \(\nu(g^{-1}g') = \nu(g')\) holds for any \(g' \in G'\) and any \(g \in G\). A \(G\)-invariant norm \(\nu_0\) on \(G'\) is called \(G\)-extremal if for any \(G\)-invariant norm \(\nu\) on \(G'\), there exist \(a,b \in \mathbb{R}_{\geq 0}\) such that \(\nu(g') - b < \nu_0(g')\) holds for any \(g' \in G'\).

Let \(G\) be a group and \(H\) a subgroup of \(G\) and \(p,q \in \mathbb{Z}_{>0} \cup \{\infty\}\). We define the \((H,p,q)\)-commutator subgroup \([G,G]^{H}_{p,q}\) of \(G\) with a subgroup \(H\) to be the subgroup generated by commutators \([f,g]\) such that \(q_H(f) \leq p, q_H(g) \leq q\). We also define the \((H,p,q)\)-commutator length \(c^H_{p,q}: [G,G]^{H}_{p,q} \to \mathbb{R}\) by

\[
c^H_{p,q}(h) = \min\{k \mid \exists f_1,\ldots,f_k,g_1,\ldots,g_k; q_H(f_i) \leq p, q_H(g_j) \leq q (i,j=1,\ldots,k); h = [f_1,g_1]\cdots[f_k,g_k]\}.
\]

We can easily prove that \([G,G]^{H}_{p,q}\) is a \(G\)-invariant subgroup and \(c^H_{p,q}\) is a \(G\)-invariant norm on \([G,G]^{H}_{p,q}\). To prove Theorem 1.4, we use the following propositions.

**Proposition 2.1 ([Ka1], [Ki1]).** Let \((G,H)\) be \((\text{Ham}(\mathbb{R}^{2n}),\text{Ham}(\mathbb{B}^{2n}))\) or \((B_{\infty},B_{n})\). Then \([G,G]^{H}_{p,q} = [G,G]\) holds for any \(p,q \in \mathbb{Z}_{>0} \cup \{\infty\}\).
For a conjugation-invariant norm \( \nu \) on a group \( G \), let \( s\nu \) denote the stabilization of \( \nu \) i.e. \( s\nu(g) = \lim_{n \to \infty} \frac{\nu(g^n)}{n} \) (this limit exists by Fekete’s Lemma).

**Proposition 2.2 ([Ka2]).** If there exists a semi-homogeneous \( H \)-quasimorphism \( \mu \) on \([G, G]^H_{p, q}\) with \( \mu(g) \neq 0 \) for some \( g \in [G, G]^H_{p, q} \) then \( scl^H_{p, q}(g) > 0 \) holds for any \( p, q \in \mathbb{Z}_{>0} \cup \{\infty\} \).

Bavard ([Bav]) gave some duality theorem between stable commutator length and quasimorphisms which generalizes Matsumoto and Morita’s famous work ([MM]). We use the following Bavard-type duality theorem.

**Theorem 2.3 ([Ka2]).** Let \( (G, H) \) be \((\text{Ham}(\mathbb{R}^{2n}), \text{Ham}(\mathbb{B}^{2n}))\) or \( (B_\infty, B_n) \) and let \( \nu \) be a conjugation-invariant norm on \( G \). Then, for any element \( g \) of \( G \) such that \( s\nu(g) > 0 \), there exists a homogeneous \( H \)-quasimorphism \( \mu: G \to \mathbb{R} \) such that \( \mu(g) > 0 \).

3. Proof on braid group

In the present section, let \( G, H \) denote \( B_\infty, B_n \), respectively.

Theorem 1.4 when \((G, H) = (B_\infty, B_n)\) immediately follows from Proposition 2.2. Theorem 2.3 and the following proposition. Let \( \sigma_1 \) denote the first standard Artin generator of \( B_\infty \). It is known that \( \{\sigma_1^{\pm 1}\} \) c-generates \( G \).

**Proposition 3.1 ([Ki1]).** The restriction of \( q_{\{\sigma_1^{\pm 1}\}} \) to \([G, G]\) is \( G \)-extremal.

**Proof of Theorem 1.4 when \((G, H) = (B_\infty, B_n)\).** Let \( g \) be an element of \([G, G]\) and let \( \mu \) be a semi-homogeneous \( H \)-quasimorphism on \([G, G]\) with \( \mu(g) \neq 0 \). Since \( \mu(g) \neq 0 \), Proposition 2.2 implies \( scl^H_{p, q}(g) > 0 \). Thus, by Proposition 2.1 and Proposition 3.1, \( sq_{\{\sigma_1^{\pm 1}\}}(g) > 0 \). Then Theorem 2.3 implies that there exists a homogeneous \( H \)-quasimorphism \( \hat{\mu}_g \) on \( G \) such that \( \hat{\mu}_g(g) \neq 0 \).

4. Proof on Hamiltonian diffeomorphism group

In the present section, let \( G, H \) denote \( \text{Ham}(\mathbb{R}^{2n}), \text{Ham}(\mathbb{B}^{2n}) \), respectively. We follow the notion of [E] and thus let \( \phi_F^t \) denote the time-\( t \) map of the Hamiltonian flow generated by \( F \) for a (time-dependent) Hamiltonian function \( F: \mathbb{R}^{2n} \times [0, 1] \to \mathbb{R} \).

**Definition 4.1 ([C, Ban]).** The Calabi homomorphism \( \text{Cal}: \text{Ham}(\mathbb{R}^{2n}) \to \mathbb{R} \) is defined by

\[
\text{Cal}(h) = \int_0^1 \int_M H\omega_0^n dt \text{ for a Hamiltonian diffeomorphism } h,
\]

where \( H: \mathbb{R}^{2n} \times [0, 1] \to \mathbb{R} \) is a Hamiltonian function which generates \( h \). \( \text{Cal}(h) \) does not depend on the choice of generating Hamiltonian function \( H \) and thus the functional \( \text{Cal} \) is a well-defined homomorphism.

For proving Theorem 1.4 when \((G, H) = (\text{Ham}(\mathbb{R}^{2n}), \text{Ham}(\mathbb{B}^{2n}))\), it is important to construct a Hamiltonian analogue of \( q_{\{\sigma_1^{\pm 1}\}} \). Let \( F: \mathbb{R}^{2n} \to \mathbb{R} \) be a (time-independent) Hamiltonian function such that \( \phi_F^1 \notin \text{Ker}(\text{Cal}) \) and let \( h \) be an element of \( \text{Ker}(\text{Cal}) \). Note that \( \text{Cal}(\phi_F^t) = t\text{Cal}(\phi_F^1) \). We define the conjugation-invariant norm \( \nu_{F, h} \) by \( \nu_{F, h} = q_{\{\phi_F^t\}_{t \in \mathbb{R}}}^{h \pm 1} \). Since \([G, G]\) is a simple group and...
\[ [G, G] = \ker(\text{Cal}) \] (Ban), the subset \( \{\phi^*_F\}_{t \in \mathbb{R}} \cup \{h^{\pm 1}\} \) c-generates \( G \). Thus \( \nu_{F,h} \) is a conjugation-invariant norm on \( G \).

We use the following proposition which is a Hamiltonian analogue of Proposition 3.1.

**Proposition 4.3.** The restriction of \( \nu_{F,h} \) to \([G, G]\) is \( G \)-extremal.

To prove Proposition 4.2 we use the following lemma.

**Lemma 4.3.** Let \( \nu \) be a \( G \)-invariant norm on \([G, G]\). There exists a positive constant \( C_{F,\nu} \) which depends only on \( F \) and \( \nu \) such that \( \nu([g, \phi^*_F]) < C_{F,\nu} \) holds for any element \( g \) of \( G \).

**Proof.** Let \( R \) be a sufficient large number such that \( \text{Supp}(F) \subset Q_R \) where \( Q_R = [-R, R]^{2n} \subset \mathbb{R}^{2n} \). Let \( h_0 \) be an element of \([G, G]\) such that \( Q_R \cap h_0(Q_R) = \emptyset \). Note that \( \nu(h_0) \) depends only on \( F \) and \( \nu \). Fix an element \( g \) of \( G \) and take an element \( h_g \) of \( G \) such that \( h_g(Q_R) = Q_R \) and \( h_g h_0(Q_R) \cap (Q_R \cup \text{Supp}(g)) = \emptyset \). Then \((h_g h_0 h_g^{-1})(\phi^*_F)^{-1} (h_g h_0 h_g^{-1})^{-1}\) commutes with \( \phi^*_F \) and \( g \) and thus \([g, \phi^*_F] = [g, \phi^*_F, h_g h_0 h_g^{-1}]\). Since \( \nu \) is a \( G \)-invariant norm on \([G, G]\),

\[
\nu([g, \phi^*_F]) \leq \nu(g(\phi^*_F, h_g h_0 h_g^{-1})g^{-1}) + \nu((\phi^*_F, h_g h_0 h_g^{-1})^{-1})
\]

\[
= 2\nu((\phi^*_F, h_g h_0 h_g^{-1})^{-1}) \leq 2(\nu(\phi^*_F (h_g h_0 h_g^{-1}))(\phi^*_F)^{-1} + \nu((h_g h_0 h_g^{-1})^{-1}))
\]

\[
= 4\nu(h_g h_0 h_g^{-1}) = 4\nu(h_0).
\]

\[ \square \]

**Proof of Proposition 4.2.** Let \( \phi \) be an element of \([G, G]\) and \( m \) a natural number such that \( \nu_{F,h}(\phi) \leq m \). Then, by the definition of \( \nu_{F,h} \), there exist \( f_1, \ldots, f_m \in \{\phi^*_F\}_{t \in \mathbb{R}} \cup \{h^{\pm 1}\} \) and \( g_1, \ldots, g_m \in G \) such that \( \phi = g_1^{-1} f_1 g_1 \cdots g_m^{-1} f_m g_m \). We define a function \( \tau : \{\phi^*_F\}_{t \in \mathbb{R}} \cup \{h^{\pm 1}\} \rightarrow \mathbb{R} \) by

\[
\tau(f) = \begin{cases} t & \text{(if } f = \phi^*_F) , \\ 0 & \text{(if } f \in \{h^{\pm 1}\}) \end{cases}
\]

We define real numbers \( T_k \) \((k = 1, \ldots, m+1)\) by \( T_k = \sum_{i=1}^{k-1} \tau(f_i) \) and set \( T_1 = 0 \). Then we define elements \( \alpha_k \) \((k = 1, \ldots, m)\) of \( \ker(\text{Cal}) = [G, G] \) by

\[
\alpha_k = \begin{cases} [\phi^*_F g_k^{-1}, \phi^*_F] & \text{(if } f_k = \phi^*_F) , \\ [\phi^*_F g_k^{-1}, \phi^*_F(\phi^*_F g_k^{-1})^{-1}] & \text{(if } f_k \in \{h^{\pm 1}\}) \end{cases}
\]

Fix a \( G \)-invariant norm \( \nu \) on \([G, G]\). Note that Lemma 4.3 implies \( \nu(\alpha_k) \leq \max\{C_{F,\nu}, \nu(h)\} \) holds for any \( k \). Since \( \phi^*_F g_k^{-1} f_k g_k = \alpha_k \phi^*_F \) holds for any \( k \),

\[
\phi = \phi^*_F g_1^{-1} f_1 g_1 \cdots g_m^{-1} f_m g_m = \alpha_1 \phi^*_F g_2^{-1} f_2 g_2 \cdots g_m^{-1} f_m g_m = \alpha_1 \alpha_2 \phi^*_F g_3^{-1} f_3 g_3 \cdots g_m^{-1} f_m g_m = \cdots = \alpha_1 \cdots \alpha_m \phi^*_F g_{m+1}^{-1} f_{m+1} g_{m+1} \nu(\phi) \leq \max\{C_{F,\nu}, \nu(h)\} \cdot m \) holds. Hence \( \nu(\phi) \leq \max\{C_{F,\nu}, \nu(h)\} \cdot \nu_{F,h}(\phi) \) holds for any element \( \phi \) of \([G, G]\). \[ \square \]
The proof of Theorem 1.4 when $(G,H) = (\text{Ham}(\mathbb{R}^{2n}), \text{Ham}(\mathbb{B}^{2n}))$ is completely similar to the one when $(G,H) = (B_{\infty}, B_n)$ if we replace Proposition 3.1 by Proposition 4.2.

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