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BOUNDS FOR THE TORNHEIM DOUBLE ZETA FUNCTION

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Abstract. In the present paper, we give bounds for the Tornheim double zeta function $T(s_1, s_2, s_3)$ when $|t_1|, |t_2|, |t_3| \ge 1$, $|t_1+t_2|, |t_2+t_3|, |t_3+t_1| \ge 1$ and $|t_1+t_2+t_3| \ge 1$ with $\sigma_1, \sigma_2, \sigma_3 > -K$ and $\sigma_1+\sigma_2, \sigma_2+\sigma_3, \sigma_3+\sigma_1 > 1-K$, where K is a positive integer, from bounds for the Hurwitz zeta function which are shown by Bourgain's bounds for exponential sums.

1. Introduction

The study of the order of the Riemann zeta function has a long history. For $\sigma, t \in \mathbb{R}$, put $s = \sigma + it$, where i is the imaginary unit. Let $\varepsilon > 0$, $0 \le D < 1/2$ and

$$g_{\varepsilon,D}(\sigma) := \begin{cases} 1/2 - \sigma & \sigma < 0, \\ 1/2 - (1-2D)\sigma + \varepsilon & 0 \le \sigma \le 1/2, \\ 2D(1-\sigma) + \varepsilon & 1/2 \le \sigma \le 1, \\ 0 & \sigma > 1. \end{cases} \tag{1.1}$$

It is well-known that the Phragmèn-Lindelöf convexity principal, the Dirichlet series expression and the functional equation of the Riemann zeta function $\zeta(s)$ imply

$$\zeta(s) \ll |t|^{g_{\varepsilon,1/4}(\sigma)}$$

(e.g. [19, Chapter 5.1]). The case $\sigma = 1/2$ which determines the value D in $g_{\varepsilon,D}$ is the most important in the theory of the Riemann zeta function. The Lindelöf hypothesis says that we can take $D = 0$. The first non-trivial result $\zeta(1/2 + it) \ll |t|^{1/6+\varepsilon}$, in other words, $\zeta(\sigma + it) \ll |t|^{g_{\varepsilon,1/6}(\sigma)}$, is proved by Hardy and Littlewood (e.g. [19, Theorem 5.5]). Huxley [8, Theorem 1] obtained the bound $\zeta(1/2 + it) \ll |t|^{32/205+\varepsilon}$. The best known result till date, which was proved by Bourgain [4, Theorem 5], for the order estimation is

$$\zeta(1/2 + it) \ll |t|^{13/84+\varepsilon}.$$

It is natural to consider order estimations for other zeta functions. Applying Huxley's bounds for exponential sums, Garunkštis [6, Theorem 3] showed a bound for the Lerch zeta function defined as $L(\lambda, s, a) := \sum_{n=0}^{\infty} e^{2\pi i n\lambda}(n+a)^{-s}$, where $0 < \lambda, a \le 1$. Note that his theorem implies

$$L(1, 1/2 + it, a) - a^{-1/2-it} \ll |t|^{32/205+\varepsilon}$$

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(see also [2, Theorem 12.23] and [13, Theorem 3.1.3]). Let r be a natural number and put

$$\zeta_r(s_1,\ldots,s_r) := \sum_{0<n_1<\cdots<n_r} \frac{1}{n_1^{s_1}\cdots n_r^{s_r}},$$

where $s_1,\ldots,s_r$ are complex variables. This infinite series is called the Euler-Zagier r-ple zeta function (see [20, Sections 8 and 9]). In [9, Theorem 1], Ishikawa and Matsumoto gave an upper bound of $|\zeta_r(s_1,\ldots,s_r)|$ for general $r \in \mathbb{N}$ by using the Mellin-Barnes integral formula. For example, they showed

$$\zeta_2(it, i\alpha t) \ll |t|^{3/2+\varepsilon}, \qquad \pm 1 \ne \alpha \in \mathbb{R}.$$

Afterwards, Kiuchi and Tanigawa [11] showed an order estimation of $|\zeta_2(s_1,s_2)|$ in the strip $0 \le \sigma_1,\sigma_2 \le 1$ by using the Euler-Maclaurin summation formula and theory of double exponential sums of van der Corput's type. For instance, when $t_1 \ll t_2 \ll t_1$ and $|t_1+t_2| \ge 1$ they proved

$$\zeta_2(\sigma_1+it_1,\sigma_2+it_2) \ll |t_1|^{1-2(\sigma_1+\sigma_2)/3}\log^2|t_1|, \qquad 0 \le \sigma_1,\sigma_2 \le 1/2$$

in [11, Theorem 1.1]. They also proved an order estimation of $|\zeta_3(s_1,s_2,s_3)|$ in the strip $0 \le \sigma_1,\sigma_2,\sigma_3 \le 1$ by using the Euler-Maclaurin summation formula and van der Corput's method of multiple exponential sums in [12, Theorem]. By using Perron's formula, Banerjee, Minamide and Tanigawa [3, Theorems 3 and 4] proved

$$\zeta_2(\sigma_1+it_1,\sigma_2+it_2) \ll |t_1|^{g_{\varepsilon,D}(\sigma_1)}|t_2|^{g_{\varepsilon,D}(\sigma_2)}$$

when $\zeta(s) \ll |t|^{g_{\varepsilon,D}(\sigma)}$ for $0<\sigma_1,\sigma_2<1$ and $|t_1|,|t_2| \ge 1$ under certain conditions. Note that we can take $D = 13/84$ in the formula above according to Bourgain [4, Theorem 5].

2. Main results

For $j=1,2,3$, put $s_j = \sigma_j + it_j$, where $\sigma_j, t_j \in \mathbb{R}$. Then we define the Tornheim double zeta function by

$$T(s_1,s_2,s_3) := \sum_{m,n=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3}} \tag{2.1}$$

in the region of absolute convergence $\sigma_1+\sigma_3>1$, $\sigma_2+\sigma_3>1$ and $\sigma_1+\sigma_2+\sigma_3>2$. In [14, Theorem 1] and [15, Theorem 6.1], it is proved that $T(s_1,s_2,s_3)$ can be continued meromorphically and its true singularities are only on the hyperplanes given by one of the equations below:

$$s_1+s_3 \in \mathbb{Z}_{\le 1}, \qquad s_2+s_3 \in \mathbb{Z}_{\le 1}, \qquad s_1+s_2+s_3 = 2. \tag{2.2}$$

Particular values of the Tornheim double zeta function $T(a,b,c)$ with $a,b,c \in \mathbb{N}$ were studied by Tornheim in 1950, later by Mordell in 1958, and many mathematicians since then (see e.g., [16, Section 1]). Note that $2^sT(s,s,s)$ coincides with the Witten zeta function for $\mathfrak{sl}(3)$ (see [20, Sections 7 and 8]).

In this paper, we give the following bound for $T(s_1,s_2,s_3)$. Let $k \in \mathbb{Z}_{\ge 0}$ and put

$$g_{\varepsilon,D}^{[k]}(\sigma) := g_{\varepsilon,D}(\sigma-k),$$

where $g_{\varepsilon,D}(\sigma)$ is already defined as (1.1). Moreover, let $\zeta(s,a)$ be the Hurwitz zeta function and for $K \in \mathbb{Z}_{\ge 1}$, to state our main result we need to define the following quantities.

$$U(s_1,s_2,s_3) := |t_1|^{g_{\varepsilon,D}(\sigma_1)}|t_2|^{g_{\varepsilon,D}(\sigma_2)}|t_3|^{g_{\varepsilon,D}(\sigma_3)}, \tag{2.3}$$

$$V_K(s_1,s_2,s_3) := V_K^\sharp(s_1,s_2,s_3) + V_K^\sharp(s_3,s_1,s_2) + V_K^\sharp(s_2,s_3,s_1), \tag{2.4}$$

$$W_K(s_1,s_2,s_3) := W_K^\sharp(s_1,s_2,s_3) + W_K^\sharp(s_3,s_1,s_2) + W_K^\sharp(s_2,s_3,s_1), \tag{2.5}$$

where $V_K^\sharp$ and $W_K^\sharp$ are defined by

$$V_K^\sharp(s_1,s_2,s_3) := |t_1|^{1/2-\sigma_1} \times$$
$$\left(\sum_{k=0}^{K-1} \frac{1}{|t_1|^{k+1}} \sum_{j=0}^{k} |t_2|^{g^{[j]}_{\varepsilon,D}(\sigma_2)} |t_3|^{g^{[k-j]}_{\varepsilon,D}(\sigma_3)} + \frac{1}{|t_1|^K} \sum_{j=0}^{K} |t_2|^{g^{[j]}_{\varepsilon,D}(\sigma_2)} |t_3|^{g^{[K-j]}_{\varepsilon,D}(\sigma_3)} \right)$$

and

$$W_K^\sharp(s_1,s_2,s_3) := |t_1|^{1/2-\sigma_1}|t_2|^{1/2-\sigma_2} \left(\sum_{k=0}^{K-1} \frac{|t_3|^{g^{[k]}_{\varepsilon,D}(\sigma_3)}}{|t_1+t_2|^{k+1}} + \frac{|t_3|^{g^{[K]}_{\varepsilon,D}(\sigma_3)}}{|t_1+t_2|^K} \right).$$

Theorem 2.1. *For $a \in [1/2, 3/2]$, let us suppose that the Hurwitz zeta function satisfies*

$$\left|\zeta(s,a)\right| \le B_\sigma |t|^{g_{\varepsilon,D}(\sigma)}, \qquad B_\sigma > 0, \quad |t| \ge 1 \tag{2.6}$$

and also assume that $|t_1|, |t_2|, |t_3| \ge 1$, $|t_1+t_2|, |t_2+t_3|, |t_3+t_1| \ge 1$ and $|t_1+t_2+t_3| \ge 1$, with $\sigma_1, \sigma_2, \sigma_3 > -K$ and $\sigma_1+\sigma_2, \sigma_2+\sigma_3, \sigma_3+\sigma_1 > 1-K$, then

$$T(s_1,s_2,s_3) \ll U(s_1,s_2,s_3) + V_K(s_1,s_2,s_3) + W_K(s_1,s_2,s_3), \tag{2.7}$$

where $U(s_1,s_2,s_3)$, $V_K(s_1,s_2,s_3)$ and $W_K(s_1,s_2,s_3)$ are already defined in (2.3), (2.4) *and* (2.5), *respectively.*

Corollary 2.2. *Let us suppose that the Hurwitz zeta function $\zeta(s,a)$ satisfies the same assumption of Theorem* 2.1. *Let $|t_1|, |t_2|, |t_3| \ge 1$, $|t_1+t_2|, |t_2+t_3|$, $|t_3+t_1| \ge 1$, $|t_1+t_2+t_3| \ge 1$ and $t_1 \ll t_2 \ll t_3 \ll t_1$. Then it holds that*

$$T(s_1,s_2,s_3) \ll U(s_1,s_2,s_3). \tag{2.8}$$

Especially, the (uniform) Lindelöf hypothesis of $\zeta(s,a)$ implies the Lindelöf hypothesis of $T(s_1,s_2,s_3)$ when $t_1 \ll t_2 \ll t_3 \ll t_1$.

In addition to the above results, we establish the bound in (2.6) for the Hurwitz zeta function with $D = 13/84$ by employing Bourgain's bound for exponential sums (see [4, Theorem 4]).

Proposition 2.3. *For $a \in [1/2, 3/2]$, we have*

$$\left|\zeta(s,a)\right| \le B_\sigma |t|^{g_{\varepsilon,13/84}(\sigma)}.$$

Recall that the theorems in Ishikawa & Matsumoto [9], Kiuchi & Tanigawa [11, 12] and Banerjee, Minamide & Tanigawa [3] are order estimations of not the Tornheim zeta double function but Euler-Zagier multiple zeta functions. Note that Kiuchi & Tanigawa [11, 12], and Banerjee, Minamide & Tanigawa [3] consider the bounds on Euler-Zagier double and triple zeta functions only in the case $\sigma_j \ge 0$. However, the cases not only $\sigma_j \ge 0$ but also $\sigma_j < 0$ are discussed in Ishikawa & Matsumoto [9] and this paper. The keys for the proofs of theorems in [9], [11, 12] and [3] are the Mellin-Barnes integral formula, the Euler-Maclaurin summation formula and Perron's formula, respectively. Note that the main ingredient of the proof of Theorem 2.1 is the integral representation

$$T(s_1,s_2,s_3) = \int_0^1 \sum_{l>0} \frac{e^{2\pi ila}}{l^{s_1}} \sum_{m>0} \frac{e^{2\pi ima}}{m^{s_2}} \sum_{n>0} \frac{e^{-2\pi ina}}{n^{s_3}} da, \qquad \sigma_1,\sigma_2,\sigma_3>1 \tag{2.9}$$

due to Zagier (see the footnote in [1, p. 62]).

The rest of this paper is organized as follows. In the next section, we review some results on the Hurwitz zeta and related functions. Section 4 is devoted to the proofs of Theorem 2.1 and Corollary 2.2. In Section 5, we prove Proposition 2.3.

3. Preliminaries

For $a>0$ and $\Re(s)>1$, the Hurwitz zeta function $\zeta(s,a)$ and the periodic zeta function $F(s,a)$ are defined by

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \qquad F(s,a) := \sum_{n=1}^{\infty} \frac{e^{2\pi ina}}{n^s},$$

respectively (e.g. [2, Chapter 12]). Next we define bilateral Hurwitz zeta function $Z(s,a)$, bilateral periodic zeta function $P(s,a)$, bilateral Hurwitz zeta star function $Y(s,a)$, bilateral periodic zeta star function $O(s,a)$ by

$$\begin{aligned} Z(s,a) &:= \zeta(s,a) + \zeta(s,1-a), \qquad & P(s,a) &:= F(s,a) + F(s,1-a), \\ Y(s,a) &:= \zeta(s,a) - \zeta(s,1-a), \qquad & O(s,a) &:= -i\bigl(F(s,a) - F(s,1-a)\bigr), \end{aligned}$$

respectively (see [17, Section 1.2]). Note that $\zeta(s,a)$ and $Z(s,a)$ can be meromorphically continued to the whole complex plane with a simple pole at $s=1$ whose residue is 1 and 2, respectively (e.g. [2, Theorem 12.4]). Moreover, the functions $F(s,a)$, $P(s,a)$, $O(s,a)$ and $Y(s,a)$ with $0<a<1$ can be analytically continued to the whole complex plan since their Dirichlet series converge uniformly in each compact subset of the half-plane $\Re(s)>0$ when $0<a<1$ (e.g. [13, p. 20]). When $\sigma>1$, we can easily see that

$$\frac{\partial}{\partial a}\zeta(s,a) = \sum_{n=0}^{\infty} \frac{\partial}{\partial a}(n+a)^{-s} = -s\sum_{n=0}^{\infty}(n+a)^{-s-1} = -s\zeta(s+1,a). \tag{3.1}$$

Thus, we obtain

$$\begin{aligned} \frac{\partial}{\partial a}Z(s,a) &= s\sum_{n=0}^{\infty}\Bigl((n+1-a)^{-s-1} - (n+a)^{-s-1}\Bigr) = -sY(s+1,a), \\ \frac{\partial}{\partial a}Y(s,a) &= -s\sum_{n=0}^{\infty}\Bigl((n+a)^{-s-1} + (n+1-a)^{-s-1}\Bigr) = -sZ(s+1,a), \end{aligned}$$

for $\sigma>1$. Hence, for all $1 \ne s \in \mathbb{C}$. we have

$$\frac{\partial}{\partial a}Z(s,a) = -sY(s+1,a), \qquad \frac{\partial}{\partial a}Y(s,a) = -sZ(s+1,a) \tag{3.2}$$

by the analytic continuation of $Z(s,a)$ and $Y(s,a)$. For simplicity, put

$$\Gamma_{\cos}(s) := \frac{2\Gamma(s)}{(2\pi)^s}\cos\Bigl(\frac{\pi s}{2}\Bigr), \qquad \Gamma_{\sin}(s) := \frac{2\Gamma(s)}{(2\pi)^s}\sin\Bigl(\frac{\pi s}{2}\Bigr).$$

Then, by [5, (2.2) and (2.3)], we have the functional equations

$$\Gamma_{\cos}(s)P(s,a) = Z(1-s,a), \qquad \Gamma_{\sin}(s)O(s,a) = Y(1-s,a). \tag{3.3}$$

We recall Stirling's formula given by

$$\bigl|\Gamma(s)\bigr| = (2\pi)^{1/2}(|t|+2)^{\sigma-1/2}e^{-\pi|t|/2}\bigl(1+O\bigl((|t|+2)^{-1}\bigr)\bigr).$$

Then one has that

$$|t|^{1/2-\sigma} \ll \frac{1}{\Gamma_{\cos}(s)}, \Gamma_{\cos}(1-s), \frac{1}{\Gamma_{\sin}(s)}, \Gamma_{\sin}(1-s) \ll |t|^{1/2-\sigma} \tag{3.4}$$

according to Stirling's formula above and Euler's formula in complex analysis.

4. Proofs of Theorem 2.1 and Corollary 2.2

4.1. **Key lemma.** To show Theorem 2.1, we put

$$G_Z(s,a) := \zeta(1-s,1+a)+\zeta(1-s,1-a), \quad G_Y(s,a) := \zeta(1-s,1+a)-\zeta(1-s,1-a).$$

From $\zeta(s,1+a) = a^{-s} + \zeta(s,a)$, we can easily see that

$$Z(1-s,a) = a^{s-1} + G_Z(s,a), \qquad Y(1-s,a) = a^{s-1} + G_Y(s,a). \tag{4.1}$$

Note that $G_Z(s,a)$ is a meromorphic function with a simple pole at $s = 0$ and $G_Y(s,a)$ is analytically continuable to the whole complex plan when $a \in [0,1/2]$ (see Section 3). In order to prove Lemma 4.1, we put

$$\nu_{\varepsilon,D}^{\{n\}}(\sigma) := n + g_{\varepsilon,D}(\sigma+n), \qquad G_X^{(n)}(s,a) := (\partial^n/\partial a^n)G_X(s,a),$$

where $n \in \mathbb{Z}_{\ge 0}$ and $X = Z$ or Y. Then we have the following.

Lemma 4.1. *For $a \in [1/2,3/2]$, assume that $\zeta(s,a)$ satisfies (2.6). Then we have*

$$\bigl|G_Z^{(n)}(s,a)\bigr| \le B_\sigma^{\{n\}}|t|^{\nu_{\varepsilon,D}^{\{n\}}(1-\sigma)}, \qquad \bigl|G_Y^{(n)}(s,a)\bigr| \le C_\sigma^{\{n\}}|t|^{\nu_{\varepsilon,D}^{\{n\}}(1-\sigma)}$$

for some positive constants $B_\sigma^{\{n\}}$ and $C_\sigma^{\{n\}}$ which depend on $\sigma \in \mathbb{R}$ and $n \in \mathbb{Z}_{\ge 0}$ but do not depend on $a \in [0,1/2]$.

Proof. By the definition of $G(s,a)$, we have

$$\bigl|G_Z(1-s,a)\bigr| \le \bigl|\zeta(s,1+a)\bigr| + \bigl|\zeta(s,1-a)\bigr|.$$

Thus, we obtain this lemma for $G_Z^{(0)}(s,a) = G_Z(s,a)$ by the bound (2.6). Similarly, we can show $|G_Y^{(0)}(s,a)| \le C_\sigma^{\{0\}}|t|^{\nu_{\varepsilon,D}^{\{0\}}(1-\sigma)}$. From (3.1), one has

$$\frac{\partial}{\partial a}G_Z(s,a) = (s-1)G_Y(s+1,a), \qquad \frac{\partial}{\partial a}G_Y(s,a) = (s-1)G_Z(s+1,a). \tag{4.2}$$

Therefore, we can show Lemma 4.1 inductively. □

In order to prove the main theorems, we put

$$S(s_1,s_2,s_3) := -T(s_1,s_2,s_3) + T(s_3,s_1,s_2) + T(s_2,s_3,s_1)$$

which has already appeared in [5, Proposition 2.1] and [18, Theorem 1.2]. Lemma 4.2 plays essential role in the present paper.

Lemma 4.2. *Let us suppose that $\zeta(s,a)$ satisfies the same assumption of Lemma 4.1. For $|t_1|,|t_2|,|t_3| \ge 1$, $|t_1+t_2|,|t_2+t_3|,|t_3+t_1| \ge 1$ and $|t_1+t_2+t_3| \ge 1$, with $\sigma_1,\sigma_2,\sigma_3 > -K$ and $\sigma_1+\sigma_2, \sigma_2+\sigma_3, \sigma_3+\sigma_1 > 1-K$, where K is a positive integer, we have*

$$S(s_1,s_2,s_3) \ll U(s_1,s_2,s_3) + V_K(s_1,s_2,s_3) + W_K(s_1,s_2,s_3).$$

Proof. It is widely-known that

$4\sin x \sin y \cos z = \cos(-x+y+z) + \cos(x-y+z) - \cos(x+y-z) - \cos(x+y+z)$

for $x, y, z \in \mathbb{C}$. From the definitions of $P(s,a)$ and $O(s,a)$, we have

$$P(s,a) = 2\sum_{n=1}^{\infty} \frac{\cos 2\pi na}{n^s}, \qquad O(s,a) = 2\sum_{n=1}^{\infty} \frac{\sin 2\pi na}{n^s}$$

when $\sigma > 1$. Hence, by (2.9), $Z(s,a) = Z(s,1-a)$, $Y(s,a) = -Y(s,1-a)$ and functional equations in (3.3), it holds that

$$\begin{aligned} S(s_1,s_2,s_3) &= \int_0^1 \sum_{l,m,n>0} \frac{4\sin 2\pi ila \sin 2\pi ima \cos 2\pi ina}{l^{s_1} m^{s_2} n^{s_3}} da \\ &= \int_0^1 \frac{Y(1-s_1,a)Y(1-s_2,a)Z(1-s_3,a)}{2\Gamma_{\sin}(s_1)\Gamma_{\sin}(s_2)\Gamma_{\cos}(s_3)} da \\ &= \int_0^{1/2} \frac{Y(1-s_1,a)Y(1-s_2,a)Z(1-s_3,a)}{\Gamma_{\sin}(s_1)\Gamma_{\sin}(s_2)\Gamma_{\cos}(s_3)} da \end{aligned}$$

when $\Re(s_1), \Re(s_2), \Re(s_3) > 1$. Now we consider the integral expressed as

$$\Gamma_{\sin}(s_1)\Gamma_{\sin}(s_2)\Gamma_{\cos}(s_3) S(s_1,s_2,s_3) =$$
$$\int_0^{1/2} \bigl(a^{s_1-1} + G_Y(s_1,a)\bigr)\bigl(a^{s_2-1} + G_Y(s_2,a)\bigr)\bigl(a^{s_3-1} + G_Z(s_3,a)\bigr) da$$

(see (4.1)). Clearly, it holds that

$$\int_0^{1/2} a^{s_1-1} a^{s_2-1} a^{s_3-1} da = \frac{2^{2-s_1-s_2-s_3}}{s_1+s_2+s_3-2}.$$

Second we consider the function $I_1(s_1,s_2,s_3)$ defined by

$$I_1(s_1,s_2,s_3) := \int_0^{1/2} a^{s_1-1} a^{s_2-1} G_Z(s_3,a) da.$$

The integral on the right-hand side converges when $\sigma_1 + \sigma_2 > 1$ and $s_3 \ne 0$ since $G_Z(s,a)$ has a pole at $s = 0$. Now we show that the function $I_1(s_1,s_2,s_3)$ can be meromorphically continued by using partial integration. One has

$$\begin{aligned} I_1(s_1,s_2,s_3) &= \int_0^{1/2} a^{s_1+s_2-2} G_Z(s_3,a) da = \int_0^{1/2} \left(\frac{a^{s_1+s_2-1}}{s_1+s_2-1}\right)' G_Z(s_3,a) da \\ &= \frac{2^{1-s_1-s_2}}{s_1+s_2-1} G_Z(s_3,1/2) - \int_0^{1/2} \frac{a^{s_1+s_2-1}}{s_1+s_2-1} G'_Z(s_3,a) da \end{aligned}$$

when $\sigma_1 + \sigma_2 > 1$ and $s_3 \ne 0$. Note that $2^{1-s_1-s_2} G_Z(s_3,1/2)$ is a meromorphic function. Furthermore, the integral $\int_0^{1/2} a^{s_1+s_2-1} G'_Z(s_3,a) da$ converges when $s_3 \ne 0$ and $\sigma_1 + \sigma_2 > 0$. Hence, the function $I_1(s_1,s_2,s_3)$ is continued meromorphically when $s_3 \ne 0$, $\sigma_1+\sigma_2 > 0$ and $s_1+s_2 \ne 1$. For the integral $\int_0^{1/2} a^{s_1+s_2-1} G'_Z(s_3,a) da$, it holds that

$$\int_0^{1/2} a^{s_1+s_2-1} G'_Z(s_3,a) da = \int_0^{1/2} \left(\frac{a^{s_1+s_2}}{s_1+s_2}\right)' G'_Z(s_3,a) da$$
$$= \frac{2^{-s_1-s_2}}{s_1+s_2} G'_Z(s_3,1/2) - \int_0^{1/2} \frac{a^{s_1+s_2}}{s_1+s_2} G''_Z(s_3,a) da.$$

The last integral converges when $s_3 \neq 0$, $\sigma_1 + \sigma_2 > -1$ and $s_1 + s_2 \neq 0$. Thus $I_1(s_1, s_2, s_3)$ is continued meromorphically when $s_3 \neq 0$, $\sigma_1 + \sigma_2 > -1$ and $s_1 + s_2 \neq 0, 1$. In addition, by Lemma 4.1, we have the order estimations

$$\frac{2^{1-s_1-s_2}}{s_1+s_2-1} G_Z(s_3, 1/2) \ll \frac{|t_3|^{\nu_{\varepsilon,D}^{\{0\}}(1-\sigma_3)}}{|s_1+s_2-1|}, \qquad \frac{2^{-s_1-s_2}}{s_1+s_2} G'_Z(s_3, 1/2) \ll \frac{|t_3|^{\nu_{\varepsilon,D}^{\{1\}}(1-\sigma_3)}}{|s_1+s_2|},$$

$$\int_0^{1/2} \frac{a^{s_1+s_2}}{s_1+s_2} G''_Z(s_3, a) da \leq \max_{a \in [0,1/2]} \left| G''_Z(s_3, a) \right| \int_0^{1/2} \left| \frac{a^{s_1+s_2}}{s_1+s_2} \right| da \ll \frac{|t_3|^{\nu_{\varepsilon,D}^{\{2\}}(1-\sigma_3)}}{|s_1+s_2|}$$

when $|t_3| \geq 1$, $\sigma_1 + \sigma_2 > -1$. Therefore, we have

$$I_1(s_1, s_2, s_3) \ll \frac{|t_3|^{\nu_{\varepsilon,D}^{\{0\}}(1-\sigma_3)}}{|s_1+s_2-1|_0} + \frac{|t_3|^{\nu_{\varepsilon,D}^{\{1\}}(1-\sigma_3)}}{|s_1+s_2-1|_1} + \frac{|t_3|^{\nu_{\varepsilon,D}^{\{2\}}(1-\sigma_3)}}{|s_1+s_2-1|_1},$$

where $|s|_k := |s| \cdots |s+k|$ and $k \in \mathbb{Z}_{\geq 0}$. Hence, applying partial integration repeatedly, we can see that $I_1(s_1, s_2, s_3)$ can be continued meromorphically to the hyper-half-plane $\sigma_1 + \sigma_2 > 1 - K$, where $K \in \mathbb{N}$. Furthermore, we have

$$I_1(s_1, s_2, s_3) \ll W_K^\flat(s_1, s_2, s_3) := \sum_{k=0}^{K-1} \frac{|t_3|^{\nu_{\varepsilon,D}^{\{k\}}(1-\sigma_3)}}{|s_1+s_2-1|_k} + \frac{|t_3|^{\nu_{\varepsilon,D}^{\{K\}}(1-\sigma_3)}}{|s_1+s_2-1|_{K-1}}$$

when $|t_3| \geq 1$, $\sigma_1 + \sigma_2 > 1 - K$ and $s_1 + s_2 \neq 1, 0, -1, \dots, 2 - K$. Note that $W_K^\flat(s_1, s_2, s_3) = W_K^\flat(s_2, s_1, s_3)$. Similarly, we consider the integrals

$$I_2(s_1, s_2, s_3) := \int_0^{1/2} a^{s_3+s_1-2} G_Y(s_2, a) da,$$
$$I_3(s_1, s_2, s_3) := \int_0^{1/2} a^{s_2+s_3-2} G_Z(s_1, a) da.$$

Then, by repeating partial integration and modifying the proof above, we can easily see that $I_2(s_1, s_2, s_3)$ and $I_3(s_1, s_2, s_3)$ are continued meromorphically to $\sigma_2 + \sigma_3 > 1 - K$ and $\sigma_3 + \sigma_1 > 1 - K$, respectively. Moreover, when $|t_1|, |t_2| \geq 1$, we have

$$I_2(s_1, s_2, s_3) \ll W_K^\flat(s_2, s_3, s_1), \qquad \sigma_2 + \sigma_3 > 1 - K, \quad s_2 + s_3 \neq 1, 0, \dots, 2 - K,$$
$$I_3(s_1, s_2, s_3) \ll W_K^\flat(s_3, s_1, s_2), \qquad \sigma_3 + \sigma_1 > 1 - K, \quad s_3 + s_1 \neq 1, 0, \dots, 2 - K.$$

Next we estimate the function $J_1(s_1, s_2, s_3)$ defined as

$$J_1(s_1, s_2, s_3) := \int_0^{1/2} a^{s_1-1} G_Y(s_2, a) G_Z(s_3, a) da.$$

By using partial integration, we have

$$J_1(s_1, s_2, s_3) = \int_0^{1/2} \left(\frac{a^{s_1}}{s_1} \right)' G_Y(s_2, a) G_Z(s_3, a) da$$
$$= \frac{2^{-s_1}}{s_1} G_Y(s_2, 1/2) G_Z(s_3, 1/2) - \int_0^{1/2} \frac{a^{s_1}}{s_1} \bigl(G_Y(s_2, a) G_Z(s_3, a) \bigr)' da$$

when $\sigma_1 > 0$ and $s_3 \ne 0$ because $G_Z(s,a)$ has a pole at $s=0$. For the last integral which converges if $s_3 \ne 0$, $\sigma_1 > -1$ and $s_1 \ne 0$, we have

$$\int_0^{1/2} a^{s_1} \bigl(G_Y(s_2,a)G_Z(s_3,a)\bigr)' da = \int_0^{1/2} \left(\frac{a^{s_1+1}}{s_1+1}\right)' \bigl(G_Y(s_2,a)G_Z(s_3,a)\bigr)' da$$
$$= \frac{2^{-s_1-1}}{s_1+1} \bigl(G_Y(s_2,1/2)G_Z(s_3,1/2)\bigr)' - \int_0^{1/2} \frac{a^{s_1+1}}{s_1+1} \bigl(G_Y(s_2,a)G_Z(s_3,a)\bigr)'' da.$$

From Lemma 4.1, we can estimate each term by

$$\bigl(G_Y(s_2,1/2)G_Z(s_3,1/2)\bigr)' \ll |t_2|^{\nu_{\varepsilon,D}^{\{1\}}(1-\sigma_2)} |t_3|^{\nu_{\varepsilon,D}^{\{0\}}(1-\sigma_3)} + |t_2|^{\nu_{\varepsilon,D}^{\{0\}}(1-\sigma_2)} |t_3|^{\nu_{\varepsilon,D}^{\{1\}}(1-\sigma_3)},$$

$$\begin{aligned} &\int_0^{1/2} \frac{a^{s_1+1}}{s_1+1} \bigl(G_Y(s_2,a)G_Z(s_3,a)\bigr)'' da \\ &\le \max_{a\in[0,1/2]} \Bigl| \bigl(G_Y(s_2,a)G_Z(s_3,a)\bigr)'' \Bigr| \int_0^{1/2} \left| \frac{a^{s_1+1}}{s_1+1} \right| da \\ &\ll \frac{1}{|s_1+1|} \sum_{j=0}^{2} \binom{2}{j} |t_2|^{\nu_{\varepsilon,D}^{\{j\}}(1-\sigma_2)} |t_3|^{\nu_{\varepsilon,D}^{\{2-j\}}(1-\sigma_3)} \int_0^{1/2} a^{\sigma_1+1} da \end{aligned}$$

when $|t_2|, |t_3| \ge 1$, $\sigma_1 > -2$ and $s_1 \ne -1$. Thus, by using partial integration repeatedly, we can see that the function $J_1(s_1,s_2,s_3)$ can be continued meromorphically to the half-plane $\sigma_1 > -K$. Moreover, from Lemma 4.1 and the Leibniz product rule, one has

$$J_1(s_1,s_2,s_3) \ll V_K^\flat(s_1,s_2,s_3), \qquad |t_2|, |t_3| \ge 1,$$

where $V_K^\flat(s_1,s_2,s_3)$ is given by the function

$$\begin{aligned} &\sum_{k=0}^{K-1} \frac{1}{|s_1|_k} \sum_{j=0}^{k} \binom{k}{j} |t_2|^{\nu_{\varepsilon,D}^{\{j\}}(1-\sigma_2)} |t_3|^{\nu_{\varepsilon,D}^{\{k-j\}}(1-\sigma_3)} \\ &+ \frac{1}{|s_1|_{K-1}} \sum_{j=0}^{K} \binom{K}{j} |t_2|^{\nu_{\varepsilon,D}^{\{j\}}(1-\sigma_2)} |t_3|^{\nu_{\varepsilon,D}^{\{K-j\}}(1-\sigma_3)} \end{aligned}$$

when $\sigma_1 > -K$ and $s_1 \ne 0, -1, \dots, 1-K$. Note that $V_K^\flat(s_1,s_2,s_3) = V_K^\flat(s_1,s_3,s_2)$. Similarly, consider the integrals

$$J_2(s_1,s_2,s_3) := \int_0^{1/2} a^{s_2-1} G_Z(s_3,a) G_Y(s_1,a) da,$$
$$J_3(s_1,s_2,s_3) := \int_0^{1/2} a^{s_3-1} G_Y(s_1,a) G_Y(s_2,a) da.$$

By repeating partial integration, we can show that $J_2(s_1,s_2,s_3)$ and $J_3(s_1,s_2,s_3)$ are continued meromorphically to $\sigma_2 > -K$ and $\sigma_3 > -K$, respectively. Moreover, we have

$$J_2(s_1,s_2,s_3) \ll V_K^\flat(s_2,s_3,s_1), \qquad |t_3|, |t_1| \ge 1, \quad \sigma_2 > -K, \quad s_2 \ne 0, \dots, 1-K,$$
$$J_3(s_1,s_2,s_3) \ll V_K^\flat(s_3,s_1,s_2), \qquad |t_1|, |t_2| \ge 1, \quad \sigma_3 > -K, \quad s_3 \ne 0, \dots, 1-K.$$

Furthermore, when $|t_1|, |t_2|, |t_3| \ge 1$, it holds that

$$\int_0^{1/2} G_Y(s_1,a) G_Y(s_2,a) G_Z(s_3,a) da \ll |t_1|^{\nu_{\varepsilon,D}^{\{0\}}(1-\sigma_1)} |t_2|^{\nu_{\varepsilon,D}^{\{0\}}(1-\sigma_2)} |t_3|^{\nu_{\varepsilon,D}^{\{0\}}(1-\sigma_3)}$$

from Lemma 4.1. Therefore, we obtain

$$\begin{aligned}\Gamma_{\sin}(s_1)\Gamma_{\sin}(s_2)\Gamma_{\cos}(s_3)S(s_1,s_2,s_3) &\ll |t_1|^{\nu_{\varepsilon,D}^{\{0\}}(1-\sigma_1)}|t_2|^{\nu_{\varepsilon,D}^{\{0\}}(1-\sigma_2)}|t_3|^{\nu_{\varepsilon,D}^{\{0\}}(1-\sigma_3)}\\ &+ V_K^\flat(s_1,s_2,s_3)+V_K^\flat(s_2,s_3,s_1)+V_K^\flat(s_3,s_1,s_2)\\ &+ W_K^\flat(s_1,s_2,s_3)+W_K^\flat(s_2,s_3,s_1)+W_K^\flat(s_3,s_1,s_2).\end{aligned}$$

In addition, one has $g_{\varepsilon,D}^{[k]}(\sigma)=\nu_{\varepsilon,D}^{\{k\}}(1-\sigma)+1/2-\sigma$ by

$$\begin{aligned}&\nu_{\varepsilon,D}^{\{k\}}(1-\sigma)+1/2-\sigma = 1/2-\sigma+k+g_{\varepsilon,D}(1-\sigma+k)\\ &=\begin{cases}1/2-\sigma+k+1/2-(1-\sigma+k) & 1-\sigma+k<0,\\ 1/2-\sigma+k+1/2-(1-2D)(1-\sigma+k)+\varepsilon & 0\le 1-\sigma+k\le 1/2,\\ 1/2-\sigma+k+2D(1-(1-\sigma+k))+\varepsilon & 1/2\le 1-\sigma+k\le 1,\\ 1/2-\sigma+k & 1-\sigma+k>1,\end{cases}\\ &=\begin{cases}0 & \sigma-k>1,\\ 2D(\sigma-k)+\varepsilon & 1/2\le\sigma-k\le 1,\\ 1/2-(1-2D)(\sigma-k)+\varepsilon & 0\le\sigma-k\le 1/2,\\ 1/2-\sigma+k & \sigma-k<0,\end{cases} = g_{\varepsilon,D}(\sigma-k)=g_{\varepsilon,D}^{[k]}(\sigma).\end{aligned}$$

Hence, we have Lemma 4.2 from (3.4) and $g_{\varepsilon,D}^{[k]}(\sigma)=\nu_{\varepsilon,D}^{\{k\}}(1-\sigma)+1/2-\sigma$. □

4.2. **Proofs of the main results.** Now we are in a position to prove Theorem 2.1 and Corollary 2.2.

Proof of Theorem 2.1. Replacing variables (s_1,s_2,s_3) by (s_3,s_1,s_2) and (s_2,s_3,s_1) in the definition of $S(s_1,s_2,s_3)$, we have

$$\begin{aligned}S(s_3,s_1,s_2)&=-T(s_3,s_1,s_2)+T(s_2,s_3,s_1)+T(s_1,s_2,s_3),\\ S(s_2,s_3,s_1)&=-T(s_2,s_3,s_1)+T(s_1,s_2,s_3)+T(s_3,s_1,s_2),\end{aligned}$$

respectively. Therefore, we obtain

$$2T(s_1,s_2,s_3)=S(s_3,s_1,s_2)+S(s_2,s_3,s_1).$$

From the definition in Section 2, we have $U(s_1,s_2,s_3)=U(s_3,s_1,s_2)=U(s_2,s_3,s_1)$. Furthermore, the same relation holds for the functions V_K and W_K. Hence, these equalities and Lemma 4.2 imply Theorem 2.1. □

Proof of Corollary 2.2. We can easily see that

$$g_{\varepsilon,D}^{[k]}(\sigma)-k\le g_{\varepsilon,D}(\sigma),\qquad 0\le k\le K.$$

Thus, for $t\ll t_1\ll t_2\ll t_3\ll t$, we have

$$\begin{aligned}|t_1+t_2|^{-k}|t_3|^{g_{\varepsilon,D}^{[k]}(\sigma_3)} &\ll |t|^{g_{\varepsilon,D}^{[k]}(\sigma_3)-k}\ll |t|^{g_{\varepsilon,D}(\sigma_3)},\\ |t_1|^{-k}|t_2|^{g_{\varepsilon,D}^{[j]}(\sigma_2)}|t_3|^{g_{\varepsilon,D}^{[k-j]}(\sigma_3)} &\ll |t|^{g_{\varepsilon,D}^{[j]}(\sigma_2)-j+g_{\varepsilon,D}^{[k-j]}(\sigma_3)+j-k}\ll |t|^{g_{\varepsilon,D}(\sigma_2)+g_{\varepsilon,D}(\sigma_3)}.\end{aligned}$$

In addition, one has $|t|^{1/2-\sigma}\le|t|^{g_{\varepsilon,D}(\sigma)}$ from $1/2-\sigma\le g_{\varepsilon,D}(\sigma)$. Therefore, we obtain

$$\begin{matrix}V_K^\sharp(s_1,s_2,s_3),V_K^\sharp(s_3,s_1,s_2),V_K^\sharp(s_2,s_3,s_1),\\ W_K^\sharp(s_1,s_2,s_3),W_K^\sharp(s_3,s_1,s_2),W_K^\sharp(s_2,s_3,s_1)\end{matrix}\ll U(s_1,s_2,s_3)$$

if $t_1\ll t_2\ll t_3\ll t_1$. Hence, we have Corollary 2.2 by Theorem 2.1. □

5. Proof of Proposition 2.3

First, we show the following convexity bound for $\zeta(s,a)$. We can prove this bound by Katsurada [10, Lemma 1] but give a new proof here.

Proposition 5.1 ([10, Lemma 1]). *Let $0<a<1$ and $|t|\ge 1$. Then we have*
$$\bigl|\zeta(s,a)-a^{-s}\bigr| \le B_\sigma |t|^{g_{\varepsilon,1/4}(\sigma)}.$$

Proof. By the series expression of $\zeta(s,a)$, we have
$$\bigl|(\zeta(s,a)-a^{-s})\pm(\zeta(s,1-a)-(1-a)^{-s})\bigr| = \bigl|\zeta(s,1+a)\pm\zeta(s,2-a)\bigr| \le 2\zeta(\sigma) \tag{5.1}$$
when $\sigma>1$. Moreover, if $\sigma>1$, we have
$$\bigl|F(s,a)\pm F(s,1-a)\bigr| \le \sum_{n=1}^\infty \frac{|e^{2\pi ina}\pm e^{-2\pi ina}|}{n^\sigma} \le 2\zeta(\sigma). \tag{5.2}$$
From (3.4), (5.2), the functional equations in (3.3), we have
$$\bigl|\zeta(s,a)\pm\zeta(s,1-a)\bigr| \le B_\sigma |t|^{1/2-\sigma}$$
when $\sigma<0$. In this case, clearly we have
$$\bigl|(\zeta(s,a)-a^{-s})\pm(\zeta(s,1-a)-(1-a)^{-s})\bigr| \le B_\sigma |t|^{1/2-\sigma} \tag{5.3}$$
by the assumption $\sigma<0$ which implies $|a^{-s}|\le 1$. Therefore, we have
$$\bigl|(\zeta(s,a)-a^{-s})\pm(\zeta(s,1-a)-(1-a)^{-s})\bigr| \le B_\sigma |t|^{g_{\varepsilon,1/4}(\sigma)} \tag{5.4}$$
by (5.1), (5.3) and the Phragmèn–Lindelöf convexity principal. Hence, we have the order estimation of Proposition 5.1 by the inequality $|2x|\le|x+y|+|x-y|$ with $x=\zeta(s,a)-a^{-s}$ and $y=\zeta(s,1-a)-(1-a)^{-s}$. □

Next, we show the following order estimate of $|\zeta(1/2+it,a)|$.

Proposition 5.2. *For $a\in[1/2,3/2]$, we have*
$$\bigl|\zeta(1/2+it,a)\bigr| \le B|t|^{13/84+\varepsilon}, \qquad B>0.$$

To show Proposition 5.2, we quote the following statements from [13, Theorem 4.1.1] and [4, Theorem 4], respectively.

Lemma 5.3 ([13, Theorem 4.1.1]). *Let $0<a\le 1$, $0<\sigma\le 1$, $t\ge t_0>1$, $y:=(t/2\pi)^{1/2}$, $q:=\lfloor y\rfloor$, $k:=\lfloor y-a\rfloor$ and $b:=q-k$. Then we have*
$$\begin{aligned}\zeta(s,a) &= \sum_{n=0}^{k}\frac{1}{(n+a)^s} + e^{it+\pi i/4+2\pi ia}\Bigl(\frac{2\pi}{t}\Bigr)^{\sigma-1/2+it}\sum_{n=1}^{q-1}\frac{e^{-2\pi ian}}{n^{1-s}}\\ &\quad + e^{\pi i f(a,t)}\Bigl(\frac{2\pi}{t}\Bigr)^{\sigma/2}\psi(2y-q-k-a) + O\bigl(t^{\sigma/2-1}\bigr),\end{aligned}$$
where $f(a,t)$ and $\psi(a)$ are given by
$$\begin{aligned} f(a,t) &:= -\frac{t}{2\pi}\log\frac{t}{2\pi e} + \frac{4a^2-7}{8} - ab + 2y(b-a) - \frac{q+k}{2},\\ \psi(a) &:= \frac{\cos(\pi(a^2/2-a-1/8))}{\cos(\pi a)}.\end{aligned}$$
Note that the approximate functional equation above holds uniformly in $0<a\le 1$.

Lemma 5.4 ([4, Theorem 4]). *Let F be a smooth function on $[1/2,1]$ satisfying for some constant $c \in (0,1]$, the condition*

$$\min\bigl\{|F''(x)|, |F'''(x)|, |F''''(x)|\bigr\} > c. \tag{5.5}$$

Given $T > 0$ sufficiently large, $M \geq 1$, we put $f(u) := TF(u/M)$ with $M/2 \leq u \leq M$ and

$$S := \sum_{m\sim M} \exp\bigl(2\pi i f(m)\bigr).$$

Then it holds that

$$\bigl|S\bigr| \ll M^{1/2}T^{13/84+\varepsilon} \quad \textit{when} \quad \frac{17}{42} \leq \theta := \frac{\log M}{\log T} \leq \frac{1}{2}. \tag{5.6}$$

Proof of Proposition 5.2. We modify the arguments in [4, Section 4] and [6, Section 3]. For the application to $|\zeta(1/2+it,a)|$, we show that (5.6) also holds for $0 \leq \theta \leq 17/42$. The case $0 \leq \theta \leq 13/42$ is trivial since we have $|S| \leq M$. Hence, all that remains to be done is establishing that (5.6) holds when $\theta \in (13/42, 17/42)$. To achieve this we can employ the bound

$$\bigl|S\bigr| \ll T^{(4+103\theta)/128+\varepsilon}, \qquad 12/31 < \theta \leq 1, \tag{5.7}$$

which is [7, Theorem 3], in combination with the exponent pair estimate

$$\bigl|S\bigr| \ll (T/M)^{1/9}M^{13/18} = M^{11/18}T^{1/9}, \qquad 0 < \theta \leq 1, \tag{5.8}$$

which corresponds to the exponent pair $(1/9, 13/18) = ABA^2B(0,1)$ in [19, Chapter 5.20]. Note that (5.7) and (5.8) need additional assumptions concerning the function F, beyond condition (5.5). However, this is not an obstacle to the application to

$$\sum_{n=0}^{k} \frac{1}{(n+a)^{1/2+it}} \qquad \text{and} \qquad \sum_{n=1}^{q-1} \frac{e^{-2\pi i a n}}{n^{1/2-it}}$$

appearing in Lemma 5.3, since that only requires consideration of cases in which $F(x) = \log(x+a)$ and $F(x) = 2\pi a x/T - \log x$ (see also the proof of [6, Theorem 3]). A calculation shows that (5.6) is implied by (5.7) for all $\theta \in (12/31, 332/819]$, and is implied by (5.8) for all $\theta \in [0, 11/28]$: noting that $(13/42, 17/42) \subset [0, 11/28] \cup (12/31, 332/819]$.

Hence, we have (5.6) whenever $0 \leq \theta \leq 1/2$, at least this is so in the cases $F(x) = \log(x+a)$ and $F(x) = 2\pi a x/T - \log x$. From the approximate functional equation in Lemma 5.3 and partial summation, we obtain Proposition 5.2. ☐

Proof of Proposition 2.3. Clearly, we have $|\zeta(s,a)| \leq B_\sigma |t|^{g_{\varepsilon,1/4}(\sigma)}$ if $a \in [1/2, 3/2]$ and $\sigma < 0$ or $\sigma > 1$ by Proposition 5.1. Therefore, we have Proposition 2.3 from Proposition 5.2 and Phragmèn-Lindelöf convexity principal. ☐

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