

ON EULER CHARACTERISTIC AND HITCHIN-THORPE INEQUALITY FOR FOUR-DIMENSIONAL COMPACT RICCI SOLITONS

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ABSTRACT. In this article, we investigate the geometry of 4-dimensional compact gradient Ricci solitons. We prove that, under an upper bound condition on the range of the potential function, a 4-dimensional compact gradient Ricci soliton must satisfy the classical Hitchin-Thorpe inequality. In addition, some volume estimates are also obtained.

1. INTRODUCTION

A complete Riemannian metric g on an n -dimensional smooth manifold M^n is called a *gradient Ricci soliton* if there exists a smooth function f on M^n such that the Ricci tensor Ric of the metric g satisfies the equation

$$(1.1) \quad \text{Ric} + \text{Hess } f = \lambda g$$

for some constant $\lambda \in \mathbb{R}$. Here, $\text{Hess } f$ denotes the Hessian of f . A gradient Ricci soliton (1.1) is called *shrinking*, *steady* or *expanding* if the real number λ is positive, zero or negative, respectively. Ricci solitons are self-similar solutions of the Ricci flows. Moreover, since they also arise as the singularity models of the Ricci flows (see [20], [33]), understanding is very important.

It was proved by Perelman [34] that every compact Ricci soliton is a gradient Ricci soliton (also see the proof in [15]). Moreover, it is known that on a compact manifold M^n , a gradient steady or expanding Ricci soliton is necessarily an Einstein metric (see [24]). On the other hand, for real dimension 4, the first example of a compact non-Einstein gradient shrinking Ricci soliton was constructed in the 1990s by Koiso [27] and Cao [3] on the compact complex surface $\mathbb{C}\mathbb{P}^2 \sharp (-\mathbb{C}\mathbb{P}^2)$, where $(-\mathbb{C}\mathbb{P}^2)$ denotes the complex projective space with the opposite orientation. Therefore, compact non-Einstein Ricci solitons must be shrinking. In dimension $n = 2$, Hamilton [21] showed that any 2-dimensional compact gradient shrinking Ricci soliton is isometric to a quotient of the sphere \mathbb{S}^2 . For $n = 3$, by the works of Ivey [25] and Perelman [33], it is known that any 3-dimensional compact gradient shrinking Ricci soliton is a finite quotient of the round sphere \mathbb{S}^3 . Even the non-compact gradient shrinking Ricci solitons have been classified in two and three dimensions.

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Unlike the cases of dimensions 2 and 3, the classification of higher dimension gradient shrinking Ricci soliton is still incomplete. For dimension 4, after the aforementioned works of Koiso [27] and Cao [3], Wang and Zhu [40] later proved the existence of a gradient Kähler-Ricci soliton on $\mathbb{C}\mathbb{P}^2 \sharp 2(-\mathbb{C}\mathbb{P}^2)$. It remains to be determined whether a compact non-Einstein gradient Ricci soliton is necessarily Kähler. In any dimension, it is known that a compact gradient shrinking Ricci soliton with constant scalar curvature must be Einstein; see [15, Eminenti, La Nave and Mantegazza]. Even a 4-dimensional non-compact gradient shrinking Ricci soliton with constant scalar curvature is rigid, too. Indeed, recently in [12] the first and third authors of the present paper proved that a 4-dimensional non-compact gradient shrinking Ricci soliton with constant scalar curvature $S = 2\lambda$ must be isometric to a finite quotient of $\mathbb{S}^2 \times \mathbb{R}^2$. This result, together with the previous results of Petersen and Wylie [35], and Fernández-López and García-Río [18], confirms that a 4-dimensional complete non-compact gradient shrinking Ricci soliton with constant scalar curvature is isometric to the Gaussian shrinking soliton \mathbb{R}^4 , a finite quotient of $\mathbb{S}^2 \times \mathbb{R}^2$, or a finite quotient of $\mathbb{S}^3 \times \mathbb{R}$. In recent years, there has been much progress concerning the classification problem of 4-dimensional gradient shrinking Ricci solitons; see, e.g., [3, 4, 6, 7, 9, 10, 12, 26, 30–32] and the references therein.

It is interesting to study the topological character of the compact 4-dimensional gradient shrinking Ricci solitons. It follows by the works of Li [28], Derdziński [14], and Fernández-López and García-Río [17] that a compact 4-dimensional gradient shrinking Ricci soliton M has finite fundamental group; see [15] for an alternative proof. Consequently, its first Betti number $b_1(M) = 0$ and hence its Euler characteristic $\chi(M)$ and signature $\tau(M)$ satisfy the inequality $\chi(M) > |\tau(M)|$ (see (2.16)). However, it is well known that for a compact 4-dimensional Einstein manifold M , the Hitchin-Thorpe inequality holds ([39], [23]; see also [1, Theorem 6.35]), that is,

$$(1.2) \quad \chi(M) \geq \frac{3}{2} |\tau(M)|.$$

This inequality provides a topological obstruction for the existence of an Einstein metric on a given compact 4-dimensional manifold. As gradient Ricci solitons are natural generalizations of Einstein manifolds and the non-trivial gradient shrinking Ricci solitons on $\mathbb{C}\mathbb{P}^2 \sharp (-\mathbb{C}\mathbb{P}^2)$ and $\mathbb{C}\mathbb{P}^2 \sharp 2(-\mathbb{C}\mathbb{P}^2)$ indeed satisfy the Hitchin-Thorpe inequality, the following question was raised (see [4, Problem 6] and [5]):

“Does the Hitchin-Thorpe inequality hold for compact 4-dimensional gradient shrinking Ricci solitons?”

In the last years, some partial answers were obtained. In this context, the assumed conditions under which the Hitchin-Thorpe inequality holds are the following, respectively.

[29, Ma]: the scalar curvature S and the volume of M satisfy $\int_M S^2 dV_g \leq 24\lambda^2 \text{Vol}(M)$;

[16, Fernández-López and García-Río]: some upper diameter bounds in terms of the Ricci curvature;

[38, Tadano]: a lower bound on the diameter involving the maximum and minimum values of the scalar curvature on M^4 , namely,

$$\left(2 + \frac{\sqrt{6}}{2}\pi\right) \frac{\sqrt{S_{\max} - S_{\min}}}{\lambda} \leq \text{diam}(M),$$

where S_{\max} and S_{\min} denote the maximum and minimum values of the scalar curvature S on M , respectively.

Zhang and Zhang [41] proved that if a given manifold has non-positive Yamabe invariant and admits long time solutions of the normalized Ricci flow equation with bounded scalar curvature, then it must satisfy the Hitchin-Thorpe inequality.

In the Kähler case, it is known that any compact Kähler gradient Ricci soliton of real dimension 4 with the natural orientation satisfies the inequality $2\chi(M) + 3\tau(M) > 0$ (see [29]; this result was generalized to Kähler Ricci almost solitons in [2]).

In this paper, we consider the question mentioned earlier. Without loss of generality, we assume that the gradient shrinking Ricci solitons satisfy the equation

$$(1.3) \quad \text{Ric} + \text{Hess } f = \frac{1}{2}g.$$

This normalization may be achieved by a scaling of the metric. We first establish the following result.

Theorem 1. *Let (M^4, g, f) be a 4-dimensional compact gradient shrinking Ricci soliton satisfying (1.3). Then it holds that*

$$(1.4) \quad 8\pi^2\chi(M) \geq \int_M |W|^2 dV_g + \frac{1}{24} \text{Vol}(M)(5 - e^{f_{\max} - f_{\min}}),$$

where f_{\min} and f_{\max} stand for the minimum and maximum of the potential function f on M^4 , respectively, $\text{Vol}(M)$ denotes the volume of M^4 and W is the Weyl tensor.

Moreover, equality holds if and only if g is an Einstein metric (in this case, f is constant).

As a consequence of Theorem 1 we obtain the following corollary.

Corollary 1. *Let (M^4, g, f) be a 4-dimensional compact gradient shrinking Ricci soliton satisfying (1.3). If $f_{\max} - f_{\min} \leq \log 5$, then the Hitchin-Thorpe inequality*

$$(1.5) \quad \chi(M) \geq \frac{3}{2}|\tau(M)|$$

holds on M .

Remark 1. The conclusion in Corollary 1 also holds if one replaces the assumption $f_{\max} - f_{\min} \leq \log 5$ by $S_{\max} - S_{\min} \leq \log 5$, where S_{\max} and S_{\min} denote the maximum and minimum of the scalar curvature S on M , respectively. Indeed, by a choice of f , the scalar curvature S of a normalized gradient shrinking Ricci soliton may satisfy

$$S + |\nabla f|^2 = f \quad \text{and} \quad S > 0.$$

At a point $p \in M$ where the function f takes the maximum, $(\nabla f)(p) = 0$. Hence, $S(p) = f(p) = f_{\max} \geq f = S + |\nabla f|^2 \geq S$. Consequently, $S_{\max} = S(p) = f_{\max}$. Let q be a point where the function f takes the minimum. Then $f_{\max} - f_{\min} = S_{\max} - S(q) \leq S_{\max} - S_{\min}$.

Remark 2. We point out that, under the choice of f as in Remark 1, a normalized compact 4-dimensional gradient shrinking Ricci soliton (M^4, g, f) with $f_{\max} \leq 3$ must satisfy the Hitchin-Thorpe inequality. Indeed, since $S > 0$ and $S + \Delta f = 2$, one obtains that

$$\int_M S^2 dV_g \leq S_{\max} \int_M S dV_g = 2 S_{\max} \text{Vol}(M).$$

Therefore, taking into account that $S_{\max} = f_{\max} \leq 3$, we have $\int_M S^2 dV_g \leq 6 \text{Vol}(M)$. So, it suffices to use the result obtained by Ma [29] to conclude the Hitchin-Thorpe inequality holds on M^4 .

Again, as an application of Theorem 1, we deduce the following volume upper bounds depending on the range of the potential function.

Theorem 2. *Let (M^4, g, f) be a 4-dimensional compact gradient shrinking Ricci soliton satisfying (1.3). Then the following assertions hold:*

$$(1.6) \quad \text{Vol}(M) (5 - e^{f_{\max} - f_{\min}}) \leq 384\pi^2.$$

Equality holds if and only if (M, g) is a sphere \mathbb{S}^4 with the radius $\sqrt{6}$.

$$(1.7) \quad \text{Vol}(M) (5 - e^{f_{\max} - f_{\min}}) \leq \mathcal{Y}(M, [g])^2,$$

where $\mathcal{Y}(M, [g])$ stands for the Yamabe invariant of (M^4, g) . Moreover, equality holds if and only if g is an Einstein metric.

2. NOTATIONS AND PRELIMINARIES

In this section we review some basic facts and present some lemmas that will be used for the establishment of the main results. Throughout this paper, we adopt the following convention for the curvatures:

$$\begin{aligned} \text{Rm}(X, Y) &= \nabla_{Y,X}^2 - \nabla_{X,Y}^2, \quad \text{Rm}(X, Y, Z, W) = g(\text{Rm}(X, Y)Z, W), \\ K(e_i, e_j) &= \text{Rm}(e_i, e_j, e_i, e_j), \quad \text{Ric}(X, Y) = \text{tr Rm}(X, \cdot, Y, \cdot), \\ R_{ij} &= \text{Ric}(e_i, e_j), \quad S = \text{tr Ric}. \end{aligned}$$

Besides, the Weyl tensor W is defined by the following decomposition formula:

$$(2.1) \quad \begin{aligned} R_{ijkl} &= W_{ijkl} + \frac{1}{n-2} (R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}) \\ &\quad - \frac{R}{(n-1)(n-2)} (g_{jl}g_{ik} - g_{il}g_{jk}), \end{aligned}$$

where R_{ijkl} stands for the Riemann curvature tensor of (M^n, g) .

Now, let (M^n, g, f) be an n -dimensional gradient shrinking Ricci soliton satisfying

$$(2.2) \quad \text{Ric} + \text{Hess } f = \frac{1}{2}g.$$

Tracing the soliton equation (2.2) we get

$$(2.3) \quad S + \Delta f = \frac{n}{2},$$

where S denotes the scalar curvature of M .

Moreover, it is known that $S + |\nabla f|^2 - f$ is constant (see [21]) and hence, by adding a constant to f if necessary, we have the equation

$$(2.4) \quad S + |\nabla f|^2 = f.$$

It follows from (2.3) and (2.4) that

$$(2.5) \quad \Delta_f f = \frac{n}{2} - f,$$

where $\Delta_f \cdot := \Delta \cdot - \nabla_{\nabla f} \cdot$ is the drifted Laplacian.

In the sequel we recall the useful equations for the curvatures of a gradient shrinking Ricci soliton. For their proofs, we refer the reader to [15, 36].

$$(2.6) \quad \nabla_l R_{ijkl} = R_{ijkl} f_l = \nabla_j R_{ik} - \nabla_i R_{jk},$$

$$(2.7) \quad \nabla_j R_{ij} = R_{ij} f_j = \frac{1}{2} \nabla_i S,$$

$$(2.8) \quad \Delta_f \text{Rm} = \text{Rm} + \text{Rm} * \text{Rm},$$

$$(2.9) \quad \Delta_f R_{ij} = R_{ij} - 2R_{ikjl} R_{kl},$$

$$(2.10) \quad \Delta_f S = S - 2|\text{Ric}|^2 = \langle \text{Ric}, g - 2\text{Ric} \rangle.$$

In this paper, we assume that M is compact. In [11], Chen proved that every complete gradient shrinking Ricci soliton has positive scalar curvature unless it is flat. Hence, $S > 0$ when M is compact.

In the rest of this section, we focus on dimension $n = 4$. It is known that, on a four-dimensional oriented Riemannian manifold M^4 , the bundle of 2-forms Λ^2 can be invariantly decomposed as a direct sum

$$(2.11) \quad \Lambda^2 = \Lambda^+ \oplus \Lambda^-,$$

where Λ^\pm is the (± 1) -eigenspace of the Hodge star operator $*$. This decomposition is conformally invariant. In particular, let $\{e_i\}_{i=1}^4$ be an oriented orthonormal basis of the tangent space at any fixed point $p \in M^4$. Then it gives rise to bases of Λ^\pm

$$(2.12) \quad \left\{ e^1 \wedge e^2 \pm e^3 \wedge e^4, e^1 \wedge e^3 \pm e^4 \wedge e^2, e^1 \wedge e^4 \pm e^2 \wedge e^3 \right\},$$

where each bi-vector has length $\sqrt{2}$. Moreover, the decomposition (2.11) allows us to conclude that the Weyl tensor W is an endomorphism of Λ^2 such that

$$(2.13) \quad W = W^+ \oplus W^-,$$

where $W^\pm : \Lambda^\pm M \rightarrow \Lambda^\pm M$ are called the *self-dual* part and *anti-self-dual* part of the Weyl tensor W , respectively. Thereby, we may fix a point $p \in M^4$ and diagonalize W^\pm such that w_i^\pm , $1 \leq i \leq 3$, are their respective eigenvalues. Also, these eigenvalues satisfy

$$(2.14) \quad w_1^\pm \leq w_2^\pm \leq w_3^\pm \quad \text{and} \quad w_1^\pm + w_2^\pm + w_3^\pm = 0.$$

Hence, the following inequality holds:

$$(2.15) \quad \det W^+ \leq \frac{\sqrt{6}}{18} |W^+|^3.$$

Moreover, equality holds in (2.15) if and only if $w_1^+ = w_2^+$.

By Poincaré duality, it follows that, for all compact oriented 4-dimensional manifolds, the *Euler characteristic* and *signature* of M^4 are given by

$$\chi(M) = 2 - 2b_1(M) + b_2(M) \quad \text{and} \quad \tau(M) = b_+(M) - b_-(M),$$

where $b_1(M)$ and $b_2(M) = b_+(M) + b_-(M)$ are the first and second Betti numbers of M^4 , respectively. It turns out that

$$(2.16) \quad \chi(M) \geq |\tau(M)| - 2b_1(M) + 2.$$

The curvature and topology of a compact 4-dimensional manifold are connected via the classical Gauss-Bonnet-Chern formula

$$(2.17) \quad \chi(M) = \frac{1}{8\pi^2} \int_M \left(|W^+|^2 + |W^-|^2 + \frac{S^2}{24} - \frac{1}{2} |\mathring{Ric}|^2 \right) dV_g$$

and the Hirzebruch's theorem

$$(2.18) \quad \tau(M) = \frac{1}{12\pi^2} \int_M (|W^+|^2 - |W^-|^2) dV_g,$$

where $\mathring{Ric} = Ric - \frac{S}{4}g$; for more details, see [1, Chapter 13]. It is easy to check from (2.17) and (2.18) that every compact 4-dimensional Einstein manifold must satisfy the Hitchin-Thorpe inequality.

We recall some useful expressions for the Euler characteristic $\chi(M)$ and give their proof for the sake of completeness.

Lemma 1. *Let (M^4, g, f) be a compact 4-dimensional gradient shrinking Ricci soliton satisfying (2.2). Then*

$$(2.19) \quad 8\pi^2\chi(M) = \int_M |W|^2 dV_g + \frac{1}{6} \text{Vol}(M) - \frac{1}{12} \int_M \langle \nabla S, \nabla f \rangle dV_g,$$

$$(2.20) \quad 8\pi^2\chi(M) = \int_M |W|^2 dV_g + \frac{1}{2} \text{Vol}(M) - \frac{1}{12} \int_M S^2 dV_g.$$

Proof. It follows from (2.3) that

$$(2.21) \quad \int_M S dV_g = 2 \text{Vol}(M).$$

Hence, by the Cauchy inequality, we have

$$4 \text{Vol}(M)^2 = \left(\int_M S dV_g \right)^2 \leq \text{Vol}(M) \int_M S^2 dV_g.$$

Consequently,

$$(2.22) \quad \int_M S^2 dV_g \geq 4 \text{Vol}(M).$$

Moreover, equality holds in (2.22) if and only if $S = 2$ and in this case, M^4 must be Einstein (see, e.g., [15]). At the same time, observe that

$$(2.23) \quad \begin{aligned} \int_M S^2 dV_g &= \int_M S(2 - \Delta f) dV_g \\ &= 2 \int_M S dV_g - \int_M S \Delta f dV_g \\ &= 4 \text{Vol}(M) + \int_M \langle \nabla S, \nabla f \rangle dV_g. \end{aligned}$$

Therefore, (2.22) and (2.23) give that $\int_M \langle \nabla S, \nabla f \rangle dV_g \geq 0$. On the other hand, integrating (2.10) and then using (2.21), we get

$$(2.24) \quad \begin{aligned} 2 \int_M |Ric|^2 dV_g &= \int_M (S - \Delta S + \langle \nabla S, \nabla f \rangle) dV_g \\ &= 2 \text{Vol}(M) + \int_M \langle \nabla S, \nabla f \rangle dV_g. \end{aligned}$$

Substituting (2.23) and (2.24) into the Gauss-Bonnet-Chern formula (2.17) yields

$$\begin{aligned}
 8\pi^2\chi(M) &= \int_M \left(|W|^2 + \frac{S^2}{24} - \frac{1}{2}|Ric|^2 \right) dV_g \\
 &= \int_M \left(|W|^2 + \frac{S^2}{6} - \frac{1}{2}|Ric|^2 \right) dV_g \\
 (2.25) \qquad &= \int_M |W|^2 dV_g + \frac{1}{6} \text{Vol}(M) - \frac{1}{12} \int_M \langle \nabla S, \nabla f \rangle dV_g,
 \end{aligned}$$

which is (2.19). Finally, plugging (2.23) into (2.25) gives (2.20). \square

3. PROOF OF THE MAIN RESULTS

For a gradient shrinking Ricci soliton (M^n, g, f) , as in [8] by Cao and Zhou, we consider the sub-level set of the potential function:

$$D(t) := \{x \in M; f(x) < t\}.$$

In this section, we first discuss the absolute continuity of the integral of a bound function on $D(t)$.

3.1. Absolute continuity of a integral on $D(t)$. Recently, Colding and Minicozzi [13, Lemma 1.1] proved the properties of the critical set and the level sets of a proper function f satisfying $\Delta_f f = \frac{n}{2} - f$ on the set $\{x \in M; f(x) \geq \frac{n}{2}\}$, where M is an n -dimensional Riemannian manifold. In this paper, we need a version of this lemma for the whole manifold M . More precisely, we have the following lemma.

Lemma 2. *Let (M^n, g) be an n -dimensional complete (not necessarily compact) Riemannian manifold. Suppose that f is a proper and non-constant function satisfying $\Delta_f f = \frac{n}{2} - f$ on M . Let \mathcal{C} be the set of critical points of f . Then the following assertions occur:*

- (a) *The critical set \mathcal{C} in M is locally contained in a smooth $(n-1)$ -dimensional manifold.*
- (b) *Each level set $\{f(x) = c\}$ has n -dimensional Hausdorff measure $\mathcal{H}^n(\{f = c\}) = 0$.*
- (c) *The regular set $\partial D(t) \setminus \mathcal{C}$ is dense in $\partial D(t)$.*

Proof. From (2.5), $\mathcal{C} \cap \{f = \frac{n}{2}\}$ is the singular set of the eigenfunction and hence has locally finite $(n-2)$ -dimensional Hausdorff measure (see Theorem 1.1 in [22]). On the other hand, Lemma 1.1 in [13] asserts that \mathcal{C} in $\{f(x) > \frac{n}{2}\}$ is locally contained in a smooth $(n-1)$ -dimensional manifold. Hence, it remains to prove the conclusion in (a) for \mathcal{C} in $\{f < \frac{n}{2}\}$. This may be done by noticing that (2.5) implies $\Delta f > 0$ on $\mathcal{C} \cap \{f < \frac{n}{2}\}$ and following the argument in [13] for $\mathcal{C} \cap \{f > \frac{n}{2}\}$ with the appropriate adaptation. So, (a) is proved.

For any value c satisfying $\{f = c\} \neq \emptyset$, the set $\{f = c\} \setminus \mathcal{C}$ is a countable union of $(n-1)$ -manifolds. This property together with (a) gives (b).

Now, we confirm (c). For $t > \frac{n}{2}$, it is the assertion (3) in Lemma 1.1 in [13]. For $t < \frac{n}{2}$, similar to the proof of (a), the assertion in this case follows by using (2.5) and making the corresponding modifications of the argument of [13]. In the case $t = \frac{n}{2}$, the assertion follows from the properties that $\mathcal{C} \cap \{f = \frac{n}{2}\}$ has locally finite $(n-2)$ -dimensional Hausdorff measure and the set $\{f = \frac{n}{2}\} \setminus \mathcal{C}$ is locally a smooth $(n-1)$ -manifold. Thus, the proof is finished. \square

We recall a result proved in [13] by Colding and Minicozzi.

Lemma 3 ([13, Lemma 1.3]). *Suppose that b is a proper C^n function and $\mathcal{H}^n(\{|\nabla b| = 0\}) = 0$ in $\{b \geq r_0\}$ for some fixed r_0 . If g is a bounded function and $Q(r) = \int_{r_0 < b < r} g$, then Q is absolutely continuous and $Q'(r) = \int_{b=r} \frac{g}{|\nabla b|}$ a.e.*

Using Lemmas 2 and 3, the following lemma may be established.

Lemma 4. *Let (M^n, g, f) be an n -dimensional complete (not necessarily compact) gradient shrinking Ricci soliton satisfying (2.2), where f is non-constant. Suppose that h is a bounded measurable function. Then we have*

- (1) *the set of the critical points of f and each level set of f satisfy $\mathcal{H}^n(\{|\nabla f| = 0\}) = 0$ and $\mathcal{H}^n(\{f = c\}) = 0$, respectively.*
- (2) *$F(t) := \int_{D(t)} h dV_g$ is absolutely continuous and $F'(t) = \int_{f=t} \frac{g}{|\nabla f|}$ a.e., where $D(t) = \{x \in M; f(x) < t\}$.*

Proof. First note that f satisfies (2.5), that is,

$$\Delta_f f = \frac{n}{2} - f.$$

Also in [8], Cao and Zhou proved that, when M is non-compact, there exists some number $r_0 > 0$ so that for all $r(x) \geq r_0$,

$$(3.1) \quad \frac{1}{4}(r(x) - c)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c)^2.$$

This implies that f is proper. So, (1) follows from (a) and (b) in Lemma 2.

Next, we deal with the second assertion. Indeed, by using that $S > 0$ and (2.4), we know $f > 0$. The properness of f together with $f > 0$ implies that f may take the positive minimum f_{\min} . This fact and (a) in Lemma 2 imply that

$$F(t) = \int_{f_{\min} \leq f < t} h dV_g = \int_{f_{\min} < f < t} h dV_g.$$

Thus, (2) follows from (1) and Lemma 3. \square

Now we will present the proof of the main results.

3.2. Proof of Theorem 1.

Proof. If f is constant, then M^4 is Einstein and (2.19) implies that the equality in (1.4) holds. Thus, we consider the case that f is non-constant.

Let a and b be the minimum and maximum of f on M^4 , respectively. Then, by using (2) in Lemma 4, we obtain that

$$(3.2) \quad \begin{aligned} \int_{D(s)} \langle \nabla S, \nabla f \rangle dV_g &= \int_a^s \int_{\partial D(t)} \frac{\langle \nabla S, \nabla f \rangle}{|\nabla f|} d\sigma dt \\ &= \int_a^s \int_{\partial D(t)} \langle \nabla S, \nu \rangle d\sigma dt \\ &= \int_a^s \left(\int_{D(t)} \Delta S dV_g \right) dt \\ &= \int_a^s \left(\int_{D(t)} (S + \langle \nabla S, \nabla f \rangle - 2|\text{Ric}|^2) dV_g \right) dt. \end{aligned}$$

In the second equality of (3.2), ν is the outward unit normal vector of $\partial D(t)$ and in the third and fourth equalities, we have used the divergence theorem and (2.10), respectively.

Define the functions $\Phi(s)$ and $\Psi(s)$ by

$$\Phi(s) = \int_a^s \left(\int_{D(t)} \langle \nabla S, \nabla f \rangle dV_g \right) dt$$

and

$$\Psi(s) = \int_a^s \left(\int_{D(t)} (S - 2|Ric|^2) dV_g \right) dt.$$

Hence, (3.2) becomes

$$(3.3) \quad \Phi'(s) = \Phi(s) + \Psi(s).$$

Differentiating (3.3), we get that

$$(3.4) \quad \Phi''(s) = \Phi'(s) + \Psi'(s).$$

Noting $\Phi'(a) = 0$, by (3.4), we obtain that

$$\Phi'(s) = e^s \int_a^s \Psi'(t) e^{-t} dt.$$

Consequently,

$$\begin{aligned} \Phi'(b) &= e^b \int_a^b \Psi'(t) e^{-t} dt \\ &= e^b \int_a^b \left(\int_{D(t)} (S - 2|Ric|^2) dV_g \right) e^{-t} dt \\ &\leq e^b \int_a^b \left(\int_{D(t)} (S - \frac{1}{2}S^2) dV_g \right) e^{-t} dt \\ &= e^b \int_a^b \left(\int_{D(t)} \left(\frac{1}{2} - \frac{1}{2}(S-1)^2 \right) dV_g \right) e^{-t} dt \\ &\leq \frac{1}{2} e^b \int_a^b \text{Vol}(D(t)) e^{-t} dt \\ (3.5) \quad &\leq \frac{1}{2} \text{Vol}(M) (e^{b-a} - 1). \end{aligned}$$

Since $\Phi'(b) = \int_M \langle \nabla S, \nabla f \rangle dV_g$, we get

$$(3.6) \quad \int_M \langle \nabla S, \nabla f \rangle dV_g \leq \frac{1}{2} \text{Vol}(M) (e^{b-a} - 1).$$

Finally, plugging (3.6) into (2.25) we conclude

$$\begin{aligned} 8\pi^2 \chi(M) &= \int_M |W|^2 dV_g + \frac{1}{6} \text{Vol}(M) - \frac{1}{12} \int_M \langle \nabla S, \nabla f \rangle dV_g \\ (3.7) \quad &\geq \int_M |W|^2 dV_g + \frac{1}{24} \text{Vol}(M) (5 - e^{b-a}), \end{aligned}$$

which is (1.4). Now consider the case of the equality in (1.4). Suppose, by contrary, that f is not constant. Then, $\text{Vol}(D(t)) < \text{Vol}(M)$ for any $t < b$. Therefore the

strict inequalities in (3.5) and thus in (3.7) must hold. This is a contradiction. So f must be constant and M^4 is Einstein. This finishes the proof of the theorem. \square

3.3. Proof of Corollary 1.

Proof. Using (1.4) and (2.18), we get

$$(3.8) \quad 4\pi^2 (2\chi(M) \pm 3\tau(M)) \geq 2 \int_M |W^\pm|^2 dV_g + \frac{1}{24} \text{Vol}(M) (5 - e^{f_{\max} - f_{\min}}).$$

Hence, the assumption $f_{\max} - f_{\min} \leq \log 5$ implies that

$$\chi(M) \geq \frac{3}{2} |\tau(M)|,$$

as asserted. \square

3.4. Proof of Theorem 2.

Proof. Since M^4 has positive scalar curvature, using a result on a compact oriented Riemannian 4-manifold with positive scalar curvature, which was proved by Gursky in [19] (see [37] also), we know that M^4 must satisfy

$$(3.9) \quad 8\pi^2 (\chi(M) - 2) \leq \int_M |W|^2 dV_g.$$

Moreover, equality holds if and only if M^4 is conformally equivalent to a sphere \mathbb{S}^4 . By (3.9) and (1.4), we get

$$(3.10) \quad \frac{1}{24} \text{Vol}(M) (5 - e^{f_{\max} - f_{\min}}) \leq 16\pi^2,$$

which proves (1.6).

If the equality in (3.10) holds, then M must be Einstein and conformally equivalent to a sphere \mathbb{S}^4 . Hence, f is constant, $\mathring{Ric} = 0$, $S = 2$, and $W = 0$. The decomposition of the curvature tensor implies that the sectional curvature of M must be constant $\frac{1}{6}$. Thus M must be a standard sphere \mathbb{S}^4 with the radius $\sqrt{6}$. In particular, such a sphere has volume $\text{Vol}(M) = 96\pi^2$.

Next, we deal with the second assertion in the theorem, i.e., the estimate on the Yamabe invariant. Since the L^2 -norm of the Weyl tensor is conformally invariant in dimension 4, one sees that

$$\int_M \left(\frac{S^2}{24} - \frac{1}{2} |\mathring{Ric}|^2 \right) dV_g$$

is conformally invariant as well. Let $\mathcal{Y}(M, [g])^2$ be the Yamabe invariant associated to (M^4, g) . Then, given the Yamabe metric $\bar{g} \in [g]$, we obtain

$$\begin{aligned}
 \mathcal{Y}(M, [g])^2 &= \frac{1}{\text{Vol}_{\bar{g}}(M)} \left(\int_M \bar{S} dV_{\bar{g}} \right)^2 \\
 &= \int_M \bar{S}^2 dV_{\bar{g}} \\
 &\geq \int_M \left(\bar{S}^2 - 12|\overset{\circ}{Ric}|^2 \right) dV_{\bar{g}} \\
 (3.11) \qquad &= \int_M \left(S^2 - 12|Ric|^2 \right) dV_g,
 \end{aligned}$$

where we have used the conformal invariance in the last equality. Moreover, equality holds if and only if (M^4, g) is conformally Einstein.

On the other hand, it is easy to check from (2.23) and (2.24) that

$$\begin{aligned}
 \int_M |\overset{\circ}{Ric}|^2 dV_g &= \int_M \left(|Ric|^2 - \frac{S^2}{4} \right) dV_g \\
 &= \frac{1}{2} \int_M S^2 dV_g - \text{Vol}(M) - \int_M \frac{S^2}{4} dV_g \\
 (3.12) \qquad &= \frac{1}{4} \int_M S^2 dV_g - \text{Vol}(M).
 \end{aligned}$$

Plugging (3.12) into (3.11), one obtains that

$$(3.13) \qquad \mathcal{Y}(M, [g])^2 \geq -2 \int_M S^2 dV_g + 12 \text{Vol}(M).$$

Hence, we may use (2.20) to get

$$(3.14) \qquad 8\pi^2 \chi(M) \leq \int_M |W|^2 dV_g + \frac{1}{24} \mathcal{Y}(M, [g])^2$$

and then it suffices to use (1.4) to see that

$$(3.15) \qquad \text{Vol}(M) (5 - e^{f_{\max} - f_{\min}}) \leq \mathcal{Y}(M, [g])^2,$$

as asserted.

Finally, if the equality in (3.15) holds, then the equality in (1.4) must hold. Hence, Theorem 1 implies that (M, g) must be Einstein with $4 \text{Vol}(M) = \mathcal{Y}(M, [g])^2$. To prove the inverse, one only needs to note that the equalities in (1.4) and (3.11) hold if (M, g) is Einstein. So, the proof is finished. \square

Remark 3. We point out that (1.6) can be alternatively obtained by using (1.7) as follows. For any compact 4-dimensional manifold M^4 , due to Aubin and Schoen, one has the following inequality $\mathcal{Y}(M^4, [g])^2 \leq \mathcal{Y}(\mathbb{S}^4, [g_0])^2$, where g_0 denotes the metric of standard sphere \mathbb{S}^4 . Moreover, equality holds if and only if (M, g) is conformally equivalent to a sphere \mathbb{S}^4 . At the same time, we have $\mathcal{Y}(\mathbb{S}^4, [g_0])^2 = 384\pi^2$. Hence, (1.6) holds. For the equality case, the same argument as in the proof of Theorem 2 can be applied.

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