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# ALTERNATIVE PROOFS OF MANDREKAR'S THEOREM

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Abstract. We present two alternative proofs of Mandrekar's theorem, which states that an invariant subspace of the Hardy space on the bidisc is of Beurling type precisely when the shifts satisfy a doubly commuting condition [Proc. Amer. Math. Soc. 103 (1988), pp. 145–148]. The first proof uses properties of Toeplitz operators to derive a formula for the reproducing kernel of certain shift invariant subspaces, which can then be used to characterize them. The second proof relies on the reproducing property in order to show that the reproducing kernel at the origin must generate the entire shift invariant subspace.

## 1. Introduction

In this note we provide new proofs of Mandrekar's theorem on shift invariant subspaces of the Hardy space on the bidisc through methods mainly using reproducing kernels.

As usual, we say that a closed subspace $\mathcal{M} \subset H^2(\mathbb{D}^2)$ is *shift invariant* if $S_j\mathcal{M} \subset \mathcal{M}$ for $j = 1, 2$, where

$$S_j : H^2(\mathbb{D}^2) \mapsto H^2(\mathbb{D}^2), \quad f(z) \mapsto z_j f(z), \quad \text{for } j = 1, 2.$$

We want to prove the following.

**Theorem 1** (Theorem 2 from [4]). *An invariant subspace $\mathcal{M} \neq \{0\}$ of $H^2(\mathbb{D}^2)$ is of the form $\varphi H^2$ with $\varphi$ an inner function if and only if the shift operators $S_1$ and $S_2$ are* doubly commuting *on $\mathcal{M}$.*

That $\{S_j\}_{j=1,2}$ is doubly commuting means that the operators commute with each other, and with each other's adjoints: that is, $S_iS_j = S_jS_i$ and $S_iS_j^* = S_j^*S_i$ for $i \neq j$. A function $\varphi$ is inner if $\varphi \in H^\infty(\mathbb{D}^2)$ and $|\varphi(z)| = 1$ almost everywhere on $\mathbb{T}^2$.

### 1.1. Overview.

Mandrekar's theorem consists of two implications: the first and more straightforward implication is that if $\mathcal{M}$ is a closed invariant subspace of the form $\varphi H^2(\mathbb{D}^2)$ with $\varphi$ inner, then the shift operators are doubly commuting on $\mathcal{M}$; and the second is that if $\mathcal{M}$ is a closed shift invariant subspace on which the shift operators are doubly commuting, then $\mathcal{M} = \varphi H^2(\mathbb{D}^2)$ for some inner function $\varphi$.

We will begin by going through some preliminaries that are necessary for the new proofs of Mandrekar's theorem provided in this article. After this, the article consists of 3 additional sections. In the first section we give a new proof of the first implication by using how Toeplitz operators act on the reproducing kernel.

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In the second we give a somewhat new proof of the second implication, i.e. we prove that an invariant subspace $\mathcal{M}$ on which the shift operators are doubly commuting must be of the form $\varphi H^2$ for some inner function $\varphi$. Mandrekar's original proof is done in two steps: first he shows that a certain subspace of $\mathcal{M}$ has dimension 1 and is generated by an inner function, and then he uses a wandering subspace theorem from [9] to show that this subspace generates $\mathcal{M}$. The proof given in this section still largely uses Mandrekar's argument for showing that this subspace has dimension 1 and is generated by an inner function, but then instead uses the main idea from the alternative proof of Beurling's theorem given by Karaev in [3] in order to characterize the reproducing kernel of $\mathcal{M}$, and thus show that $\mathcal{M}$ has the desired form.

In the third and final section we give a completely new proof of the second implication by using a modified version of the classical proof of Beurling's theorem given in [2]. Namely, we show that if the shift operators are doubly commuting on $\mathcal{M}$, then either the reproducing kernel at the origin or a function which reproduces the value of a suitable derivative at the origin (if all functions in the invariant subspace vanish at the origin) must generate the entire invariant subspace $\mathcal{M}$. This is done by using the defining property of the reproducing kernel at the origin to show that no function in $\mathcal{M}$ can be orthogonal to the invariant subspace generated by this function.

1.2. **Preliminaries.** We begin with some general theory for the reproducing kernels of operator range spaces and how Toeplitz operators act on such spaces. For more details see [1] and chapter 1 of [6].

A Hilbert space $H$ with inner product $\langle \cdot, \cdot \rangle_H$ consisting of functions defined on a domain $D$ is called a reproducing kernel Hilbert space if point evaluations are bounded linear functionals. By Riesz representation theorem, this means that for every point $z_0 \in D$ there is a function $k_{z_0} \in H$, called the *reproducing kernel at* $z_0$, such that

$$f(z_0) = \langle f, k_{z_0} \rangle_H$$

for all $f \in H$. The reproducing kernel of $H^2(\mathbb{D}^n)$ is the Cauchy kernel

$$C_\lambda(z) = \prod_{j=1}^{n} \frac{1}{(1 - \overline{\lambda_j} z_j)},$$

and throughout this note, we will denote by $k_\lambda^{\mathcal{M}}$ the reproducing kernel of a closed subspace $\mathcal{M} \subset H^2(\mathbb{D}^n)$.

If $T : H \mapsto H$ is a partial isometry on a reproducing kernel Hilbert space $H$ with reproducing kernel $k_\lambda(z)$, then $T(H)$ and $T(H)^\perp$ are closed subspaces of $H$, and their reproducing kernels are given by $TT^* k_\lambda(z)$ and $(1 - TT^*)k_\lambda(z)$ respectively. In particular, we are interested in Toeplitz operators

$$T_\varphi : H \mapsto H, \quad f \mapsto \varphi f,$$

with symbol $\varphi$ belonging to the multiplier algebra of $H$. In the case of $H^2(\mathbb{D}^n)$, the multiplier algebra is $H^\infty(\mathbb{D}^n)$, the space of bounded analytic functions on $\mathbb{D}^n$.

If $T_\varphi$ is an isometry, then the above means that $\varphi H$ and $(\varphi H)^\perp$ will have reproducing kernels

$$\varphi(z)\overline{\varphi(\lambda)}k_\lambda(z) \quad \text{and} \quad (1 - \varphi(z)\overline{\varphi(\lambda)})k_\lambda(z), \tag{1}$$

respectively. In order to arrive at (1) we have used that

$$T_\varphi^* = T_{\overline{\varphi}} \quad \text{and} \quad T_{\overline{\varphi}} k_\lambda(z) = \overline{\varphi(\lambda)} k_\lambda(z). \tag{2}$$

These two equalities always hold for Toeplitz operators on reproducing kernel Hilbert spaces, and will be used frequently in this note.

Note that $T_\varphi$ is an isometry whenever $\varphi$ is an inner function and $H$ is the Hardy space or a closed subspace of the Hardy space, and thus (1) holds in these cases.

## 2. Alternate proof of the first implication

*Proof of the first implication of Theorem* 1. Let $\varphi$ be an inner function and consider the closed invariant subspace $\mathcal{M} = \varphi H^2$. We want to show that $S_1$ and $S_2$ are doubly commuting on $\mathcal{M}$.

First of all $S_1$ and $S_2$ are commuting on $\mathcal{M}$ since $z_1 z_2 \varphi(z) f(z) = z_2 z_1 \varphi(z) f(z)$ for all $f \in H^2$. In [4] Mandrekar just refers to Theorem 1(i) $\Rightarrow$ (ii) in [9] to show that $S_1$ and $S_2$ are in fact doubly commuting on such a subspace $\mathcal{M}$ if $\varphi$ is inner (or, more generally, of constant modulus on $\mathbb{T}^2$).

We will instead show that $S_1$ and $S_2$ are doubly commuting on such a subspace $\mathcal{M}$ by considering their action on the reproducing kernel. From the first formula of (1) we know that the reproducing kernel of $\mathcal{M}$ is

$$k_\lambda^{\mathcal{M}}(z) = \frac{\overline{\varphi(\lambda)}\varphi(z)}{(1-\overline{\lambda_1}z_1)(1-\overline{\lambda_2}z_2)}.$$

Since the reproducing kernels are dense in $\mathcal{M}$, we only need to show that

$$S_1 S_2^* k_\lambda^{\mathcal{M}}(z) = S_2^* S_1 k_\lambda^{\mathcal{M}}(z)$$

for the kernels. By using equation (2), we immediately see that the left hand side is $z_1 \overline{\lambda_2} k_\lambda^{\mathcal{M}}(z)$. To see that the same holds for the right hand side we argue as follows.

Since $k_\lambda^{\mathcal{M}}(z) = \varphi(z)\overline{\varphi(\lambda)} C_{\lambda_1}(z_1) C_{\lambda_2}(z_2)$, we see that

$$\begin{aligned} S_2^*(z_1 k_\lambda^{\mathcal{M}}(z)) &= \left\langle w_1 \overline{w_2} k_\lambda^{\mathcal{M}}(w), k_z^{\mathcal{M}}(w) \right\rangle_{H^2(\mathbb{D}^2)} \\ &= \left\langle w_1 \overline{w_2} \varphi(w) \overline{\varphi(\lambda)} C_{\lambda_1}(w_1) C_{\lambda_2}(w_2), \varphi(w)\overline{\varphi(z)} C_{z_1}(w_1) C_{z_2}(w_2) \right\rangle_{H^2(\mathbb{D}^2)}. \end{aligned}$$

Note that the integration defining the inner product above is with respect to $w$. Since $|\varphi(w)|^2 = 1$ almost everywhere on $\mathbb{T}^2$, we see that

$$\begin{aligned} &\left\langle w_1 \overline{w_2} \varphi(w) \overline{\varphi(\lambda)} C_{\lambda_1}(w_1) C_{\lambda_2}(w_2), \varphi(w)\overline{\varphi(z)} C_{z_1}(w_1) C_{z_2}(w_2) \right\rangle_{H^2(\mathbb{D}^2)} \\ &\quad = \left\langle w_1 \overline{w_2} \overline{\varphi(\lambda)} C_{\lambda_1}(w_1) C_{\lambda_2}(w_2), \overline{\varphi(z)} C_{z_1}(w_1) C_{z_2}(w_2) \right\rangle_{H^2(\mathbb{D}^2)} \\ &\qquad = \overline{\varphi(\lambda)}\varphi(z) \left\langle w_1 \overline{w_2} C_{\lambda_1}(w_1) C_{\lambda_2}(w_2), C_{z_1}(w_1) C_{z_2}(w_2) \right\rangle_{H^2(\mathbb{D}^2)}. \end{aligned}$$

Because of the product structure of $H^2(\mathbb{D}^2)$ we have that

$$\begin{aligned} &\left\langle w_1 \overline{w_2} C_{\lambda_1}(w_1) C_{\lambda_2}(w_2), C_{z_1}(w_1) C_{z_2}(w_2) \right\rangle_{H^2(\mathbb{D}^2)} \\ &\qquad = \left\langle w_1 C_{\lambda_1}(w_1), C_{z_1}(w_1) \right\rangle_{H^2(\mathbb{D})} \left\langle \overline{w_2} C_{\lambda_2}(w_2), C_{z_2}(w_2) \right\rangle_{H^2(\mathbb{D})}, \end{aligned}$$

and by using (2) on $H^2(\mathbb{D})$ we see that

$$\begin{aligned}\langle w_1 C_{\lambda_1}(w_1), C_{z_1}(w_1)\rangle_{H^2(\mathbb{D})} \langle \overline{w_2} C_{\lambda_2}(w_2), C_{z_2}(w_2)\rangle_{H^2(\mathbb{D})}\\ = z_1 C_{\lambda_1}(z_1) T_{\overline{z_2}}(C_{\lambda_2}(z_2)) = z_1 C_{\lambda_1}(z_1)\overline{\lambda_2} C_{\lambda_2}(z_2).\end{aligned}$$

By putting all of this together we see that

$$S_2^*(z_1 k_\lambda^{\mathcal{M}}(z)) = \overline{\varphi(\lambda)}\varphi(z) z_1 C_{\lambda_1}(z_1)\overline{\lambda_2} C_{\lambda_2}(z_2) = z_1 \overline{\lambda_2} k_\lambda^{\mathcal{M}}(z),$$

which is what we wanted to show.

This finishes the proof. □

## 3. First alternate proof of the second implication

In Mandrekar's original proof of this direction, he first proved that a certain subspace of $\mathcal{M}$ has dimension one and contains an inner function $\varphi$, and he then used a wandering subspace theorem from [9] to prove that this subspace generates $\mathcal{M}$. This implies that $\mathcal{M} = \varphi H^2$.

In this proof, we will still largely use Mandrekar's original proof to show that this subspace of $\mathcal{M}$ has dimension one and is generated by an inner function $\varphi$. But instead of using the wandering subspace theorem from [9], we will use properties of reproducing kernels to describe the reproducing kernel of $\mathcal{M}$. This will show that $\mathcal{M} = \varphi H^2$.

For completeness and to get a self-contained proof, we will present the proofs from Mandrekar's article [4] of the statements that we need to use, and furthermore we will add proofs of statements from [9] that are used in both this proof and in Mandrekar's original proof.

*First proof of the second implication of Theorem* 1. Consider the subspaces

$$O_j(\mathcal{M}) := \mathcal{M} \ominus S_j\mathcal{M},$$

and their intersection $O_1(\mathcal{M}) \cap O_2(\mathcal{M})$. This is the subspace with which Mandrekar applies Słociński's wandering subspace theorem for commuting isometries from [9].

We will essentially use Mandrekar's original proof to show that if $\mathcal{M}$ is an invariant subspace on which $\{S_j\}_{j=1,2}$ is doubly commuting, then the closed subspace $O_1(\mathcal{M}) \cap O_2(\mathcal{M})$ either has dimension 1 and contains an inner function, or it only contains the zero function. In fact, Mandrekar excludes the possibility that $O_1(\mathcal{M}) \cap O_2(\mathcal{M}) = \{0\}$ by another application of the wandering subspace theorem from [9], but we will deal with the possibility that the intersection is trivial at a later stage.

In order to show that the dimension of the intersection is smaller than or equal to 1, we must use the fact that $O_1$ is invariant under $S_2$ and $O_2$ is invariant under $S_1$. That is

$$S_2(O_1) \subset O_1 \text{ and } S_1(O_2) \subset O_2. \tag{3}$$

Mandrekar proves this by simply referring to Theorem 1(iii) in [9]. However, this can be shown with elementary arguments using that $\{S_j\}_{j=1,2}$ are doubly commuting on $\mathcal{M}$. Namely, since $S_1$ and $S_2^*$ are commuting, we have that for every $g \in \mathcal{M} \ominus S_1\mathcal{M}$

$$0 = \langle g, S_1(S_2^* f)\rangle = \langle g, S_2^*(S_1 f)\rangle = \langle S_2 g, S_1 f\rangle$$

for all $f \in \mathcal{M}$, and so $S_2 g \in \mathcal{M} \ominus S_1\mathcal{M}$. Again, here we regard $S_j$ and $S_j^*$ as operators on $\mathcal{M}$.

With this result at hand, we can go on with presenting the way Mandrekar proves that the dimension of $O_1(\mathcal{M}) \cap O_2(\mathcal{M})$ is 1 unless the intersection only contains the zero function.

Let $g_1, g_2 \in O_1(\mathcal{M}) \cap O_2(\mathcal{M})$. Then for all $m, n > 0$

$$\int_{\mathbb{T}^2} z_1^m z_2^n g_1(z)\overline{g_2(z)}|dz| = 0,$$

and by (3) we also have that

$$\int_{\mathbb{T}^2} z_2^n g_1(z)\overline{z_1^m g_2(z)}|dz| = 0$$

for all $m, n > 0$. By symmetry and since $\overline{z_1}^m = z_1^{-m}$, this means that

$$\int_{\mathbb{T}^2} z_1^m z_2^n g_1(z)\overline{g_2(z)}|dz| = 0, \tag{4}$$

for all $(m, n) \neq (0, 0)$, which means that $g_1(z)\overline{g_2(z)} = c$ a.e. on $\mathbb{T}^2$.

Now suppose that there are $g_1, g_2 \in O_1(\mathcal{M}) \cap O_2(\mathcal{M}) \setminus \{0\}$ with $g_1 \perp g_2$. In this case $|g_1|^2 = c_1 \neq 0$ and $|g_2|^2 = c_2 \neq 0$ a.e. on $\mathbb{T}^2$ by (4), but $g_1\overline{g_2} = 0$ a.e. on $\mathbb{T}^2$ since $g_1\overline{g_2}$ is a.e. constant and $g_1 \perp g_2$.

This is impossible, and hence $O_1(\mathcal{M}) \cap O_2(\mathcal{M})$ is one-dimensional. Furthermore, from the above arguments it follows that $O_1(\mathcal{M}) \cap O_2(\mathcal{M})$ contains, and hence is generated by an inner function, which we denote by $\varphi$. All that remains is to show that this implies that $\mathcal{M} = \varphi H^2(\mathbb{D}^2)$ for this inner function $\varphi$.

The next step in Mandrekar's proof relies on a wandering subspace theorem for commuting isometries due to Słociński from [9]. Here, we instead use an argument with reproducing kernels similar to that in [3].

Since $S_1$ and $S_2$ are partial isometries on any closed invariant subspace of $H^2(\mathbb{D}^2)$, applying the second formula of (1) shows that the reproducing kernel of $O_2(\mathcal{M})$ is given by

$$(1 - \overline{\lambda_2}z_2)k_\lambda^{\mathcal{M}}(z),$$

and since $O_2(\mathcal{M})$ is invariant under $S_1$ by (3), applying the second formula of (1) again shows that the reproducing kernel for $O_1(\mathcal{M}) \cap O_2(\mathcal{M})$ is given by

$$(1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)k_\lambda^{\mathcal{M}}(z). \tag{5}$$

If $O_1(\mathcal{M}) \cap O_2(\mathcal{M}) = \{0\}$, then

$$(1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)k_\lambda^{\mathcal{M}}(z) = 0 \Rightarrow k_\lambda^{\mathcal{M}}(z) = 0,$$

and so $\mathcal{M}$ is trivial.

If $O_1(\mathcal{M}) \cap O_2(\mathcal{M}) \neq \{0\}$, then we know that $O_1(\mathcal{M}) \cap O_2(\mathcal{M})$ is one-dimensional, and so its reproducing kernel will be given by $\overline{\varphi(\lambda)}\varphi(z)$ for some $\varphi$ with $H^2$-norm equal to 1. From the previous arguments, we know that this $\varphi$ will in fact be inner. Put together, this means that

$$(1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)k_\lambda^{\mathcal{M}}(z) = \overline{\varphi(\lambda)}\varphi(z)$$

$$\iff k_\lambda^{\mathcal{M}}(z) = \frac{\overline{\varphi(\lambda)}\varphi(z)}{(1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)}.$$

By (1) we recognize the right hand side of the last equation as the reproducing kernel of $\varphi H^2(\mathbb{D}^2)$. Since a Hilbert space is uniquely determined by its reproducing kernel, this finishes the proof. $\square$

A somewhat informal, but perhaps helpful way to think about the derivation of formula (5) for the reproducing kernel of $O_1(\mathcal{M}) \cap O_2(\mathcal{M})$ is to note that the orthogonal projection onto $O_j(\mathcal{M}) \subset \mathcal{M}$ is given by $P_j := (I - S_j S_j^*)$, and using that since $\{S_j\}_{j=1,2}$ are doubly commuting on $\mathcal{M}$, the projections $P_1$ and $P_2$ commute. As a consequence, the orthogonal projection onto $O_1(\mathcal{M}) \cap O_2(\mathcal{M})$ is given by $P_1 P_2$, and the reproducing kernel of $O_1(\mathcal{M}) \cap O_2(\mathcal{M})$ is

$$P_1 P_2 k_\lambda^{\mathcal{M}}(z).$$

Furthermore, it is worth noting that our argument for why

$$O_1(\mathcal{M}) \cap O_2(\mathcal{M}) = \{0\} \Rightarrow \mathcal{M} = \{0\}$$

is actually not very different from the argument given in [4]. For this Mandrekar essentially refers to the wandering subspace theorem from [9], and concludes that if $O_1(\mathcal{M}) \cap O_2(\mathcal{M}) = \{0\}$ then $\mathcal{M} = \{0\}$, since clearly the wandering subspace $O_1(\mathcal{M}) \cap O_2(\mathcal{M})$ can't generate anything else.

*Remark.* The wandering subspace argument using reproducing kernels given above works exactly the same way in $n$ variables, and Mandrekar's argument for showing that the wandering subspace is one-dimensional also works in $n$ variables, although in that case the argument will become slightly more technical. But since [9] only deals with a pair of doubly commuting isometries, Mandrekar's original theorem only concerns functions of two complex variables. However, more recently the main results in [9] have been generalized by Sarkar in [7] to deal with an $n$-tuple of pairwise doubly commuting isometries, and thus Mandrekar's original argument can be applied directly to show the corresponding theorem for $H^2(\mathbb{D}^n)$. Though, it is worth pointing out that Mandrekar's theorem on $H^2(\mathbb{D}^n)$ has already been proved by Seto in [8] by using the Wold decompositions from [9] in a slightly different way.

## 4. Second alternate proof of the second implication

In this section we give another proof of the second direction of Mandrekar's theorem, which is instead based on the proof idea of Beurling's theorem provided in [2].

We begin by proving a weaker version of the second direction of Mandrekar's theorem, where we assume that the origin is not a common zero for all functions in $\mathcal{M}$. The exact statement we prove is as follows.

**Theorem 2.** *If the shift operators $S_1$ and $S_2$ are* doubly commuting *on an invariant subspace $\mathcal{M} \neq \{0\}$ of $H^2(\mathbb{T}^2)$, which has the additional property that it contains an element which does not vanish at the origin, then $\mathcal{M}$ is of the form $\varphi H^2$ for an inner function $\varphi$.*

*Proof.* We will show that $\mathcal{M} = \varphi H^2$ for some function $\varphi \in H^2$ with constant modulus $c \neq 0$ a.e. on the boundary. By simply normalizing $\varphi$ one then obtains the desired inner function.

As always we denote by $k_\lambda^{\mathcal{M}}(z)$ the reproducing kernel of $\mathcal{M}$. Now, denote by $\varphi(z)$ the reproducing kernel at the origin, $k_0^{\mathcal{M}}(z)$. Note that $k_0(z) \neq 0$ since we assume that some function in $\mathcal{M}$ does not vanish at the origin. We will show that

$\varphi(z)$ has constant modulus on the boundary by showing that all Fourier coefficients of $|\varphi(z)|^2$ apart from the constant term are zero.

By the reproducing property of $\varphi$

$$\int_{\mathbb{T}^2} |\varphi(z)|^2 z_1^k z_2^n |dz| = \langle z_1^k z_2^n \varphi(z), \varphi(z)\rangle = 0 \tag{6}$$

for all $(k,n) \neq (0,0)$ with $k,n \geq 0$, and by taking complex conjugates, we also see that

$$\int_{\mathbb{T}^2} |\varphi(z)|^2 z_1^k z_2^n |dz| = 0$$

when $(k,n) \neq (0,0)$, and $k,n \leq 0$.

If $(k,n) \neq (0,0)$ with $k,n \geq 0$ then

$$\int_{\mathbb{T}^2} |\varphi(z)|^2 z_1^k z_2^{-n} |dz| = \int_{\mathbb{T}^2} (\varphi(z) z_1^k)\overline{(\varphi(z) z_2^n)} |dz| = \langle S_1^k \varphi(z), S_2^n \varphi(z)\rangle,$$

and since $S_1$ and $S_2$ are assumed to be doubly commuting

$$\langle S_1^k \varphi(z), S_2^n \varphi(z)\rangle = \langle \varphi(z), (S_1^*)^k S_2^n \varphi(z)\rangle = \langle \varphi(z), S_2^n (S_1^*)^k \varphi(z)\rangle = 0,$$

where the last equality is again a consequence of the reproducing property of $\varphi$.

By using the same argument with $(-k,n)$ instead of $(k,-n)$, we finally see that

$$\int_{\mathbb{T}^2} |\varphi(z)|^2 z_1^k z_2^n |dz| = 0$$

for all $(k,n) \neq (0,0)$, which shows that $|\varphi(z)|$ is constant almost everywhere on $\mathbb{T}^2$.

It remains to show that $\mathcal{M} = \varphi H^2(\mathbb{D}^2)$.

Since $\varphi$ has constant modulus on the boundary and since the polynomials are dense in $H^2(\mathbb{D}^2)$, we have that

$$\varphi H^2(\mathbb{D}^2) = \overline{\{\varphi(z)p(z) : p(z) \in \mathbb{C}[z_1, z_2]\}} \subset \mathcal{M}.$$

Now let $f \in \mathcal{M}$ be orthogonal to all elements in $\{\varphi(z)p(z) : p(z) \in \mathbb{C}[z_1, z_2]\}$. We will show that $f = 0$ by showing that all the Fourier coefficients of $\varphi \overline{f}$ are zero.

By the orthogonality assumption on $f$

$$\int_{\mathbb{T}^2} \varphi(z)\overline{f(z)} z_1^k z_2^n |dz| = 0,$$

for all $n,k \geq 0$. Furthermore, by again using the reproducing property of $\varphi$, and since $S_1$ and $S_2$ are doubly commuting on $\mathcal{M}$, we have that

$$\begin{aligned}\int_{\mathbb{T}^2} \varphi(z)\overline{f(z)} z_1^k z_2^{-n} |dz| &= \int_{\mathbb{T}^2} \varphi(z) z_1^k \overline{f(z) z_2^n} |dz| \\ &= \langle z_1^k \varphi(z), z_2^n f(z)\rangle = \langle \varphi(z), S_2^n (S_1^*)^k f(z)\rangle = 0,\end{aligned}$$

for $n \geq 1$ and $k \geq 0$. The same argument for $(-k,n)$ instead of $(k,-n)$ shows that

$$\int_{\mathbb{T}^2} \varphi(z)\overline{f(z)} z_1^{-k} z_2^n |dz| = 0$$

when $n \geq 0$ and $k \geq 1$.

It remains to show that

$$\langle \varphi, f(z) z_1^k z_2^n\rangle = \int_{\mathbb{T}^2} \varphi(z)\overline{f(z)} z_1^{-k} z_2^{-n} |dz| = 0 \tag{7}$$

for $k,n \geq 1$.

But this is an immediate consequence of the reproducing property of $\varphi$ since $f(z)z_1^k z_2^n$ belong to $\mathcal{M}$ and vanish at the origin for all $k, n \geq 1$.

It follows that all Fourier coefficients of $\varphi\overline{f}$ vanish, and so $\varphi\overline{f} = 0$. Since $|\varphi| = c \neq 0$ a.e. on $\mathbb{T}^2$, this means that $f = 0$ a.e. on $\mathbb{T}^2$, and thus $f$ is identically equal to zero. □

For functions of one variable, the general case — in which all functions in $\mathcal{M}$ might have a common zero at the origin — is easily reduced to the case above by, if necessary, simply factoring out $z^k$ for some $k \geq 1$. For functions of several variables this is no longer possible since there is no canonical factor corresponding to zeros at the origin. Instead, we modify the argument given above as follows.

*Second proof of the second implication of Theorem* 1. As above we will show that if $S_1$ and $S_2$ are doubly commuting on $\mathcal{M}$, then $\mathcal{M} = \varphi H^2(\mathbb{D}^2)$ for some function $\varphi$ with constant modulus $c \neq 0$ a.e. on the boundary. By normalizing $\varphi$ one obtains the desired inner function.

If there is some $f \in \mathcal{M}$ which does not vanish at the origin, we can just apply Theorem 2, and then there is nothing more to be done.

Now suppose $d \geq 1$ is the smallest integer such that all partial derivatives of total degree less than $d$ of all functions $f \in \mathcal{M}$ vanish at the origin. That is, for all $j_1, j_2 \in \mathbb{N}$ with $j_1 + j_2 < d$

$$\left(\frac{\partial^{j_1+j_2} f}{\partial z_1^{j_1} \partial z_2^{j_2}}\right)(0,0) = 0 \tag{8}$$

for all $f \in \mathcal{M}$. We may assume that $d < \infty$ since otherwise all functions in $\mathcal{M}$ are identically zero, i.e. $\mathcal{M} = \{0\}$.

Let $(d_1, d_2) \in \mathbb{N}^2$ with $d_1 + d_2 = d$ be any pair of integers such that

$$\left(\frac{\partial^{d_1+d_2} f}{\partial z_1^{d_1} \partial z_2^{d_2}}\right)(0,0) \neq 0$$

for some $f \in \mathcal{M}$. Consider the bounded linear functional $E_0^{(d_1,d_2)}$ on $H^2(\mathbb{D}^2)$ defined by

$$E_0^{(d_1,d_2)} : f \mapsto \left(\frac{\partial^{d_1+d_2} f}{\partial z_1^{d_1} \partial z_2^{d_2}}\right)(0,0).$$

That this functional is indeed bounded on $H^2(\mathbb{D}^2)$ is clear since it maps a function $f$ to a fixed constant multiple of its Fourier coefficient with index $(d_1, d_2)$.

By the Riesz representation theorem there exists a unique function, which we will denote by $k_0^{(d_1,d_2)}$, such that

$$E_0^{(d_1,d_2)}(f) = \left\langle f, k_0^{(d_1,d_2)} \right\rangle$$

for all $f \in H^2(\mathbb{D}^2)$. This is kind of a reproducing kernel at the origin, only it gives the value for the $(d_1, d_2)$th partial derivative instead of for the function.

Just as for the ordinary reproducing kernel, we have that $P^{\mathcal{M}} k_0^{(d_1,d_2)} \in \mathcal{M}$ is the unique function in $\mathcal{M}$ such that

$$E_0^{(d_1,d_2)}(f) = \left\langle f, P^{\mathcal{M}} k_0^{(d_1,d_2)} \right\rangle$$

for all $f \in \mathcal{M}$. From now on we will denote the function $P^{\mathcal{M}} k_0^{(d_1,d_2)}$ by $\varphi$.

Since $\varphi$ reproduces the $(d_1, d_2)$th partial derivative at the origin and since equation (8) holds for all $f \in \mathcal{M}$, we have that

$$\int_{\mathbb{T}^2} |\varphi(z)|^2 z_1^k z_2^n |dz| = \langle z_1^k z_2^n \varphi(z), \varphi(z) \rangle = 0,$$

for all $(n, k) \neq (0, 0), n, k \in \mathbb{N}$. That is $\varphi$ satisfies equation (6) from the proof of Theorem 2.

In fact, as a consequence of the reproducing property of $\varphi$ and the fact that (8) holds for all $f \in \mathcal{M}$, we have that

$$\langle \varphi, f(z) z_1^n z_2^k \rangle = 0 \tag{9}$$

for all $f \in \mathcal{M}$ and all $n, k \in \mathbb{N}$ with $(n, k) \neq (0, 0)$. To see that (9) holds, note that when we evaluate the terms of the partial derivatives of $f(z) z_1^n z_2^k$ at the origin, either they will vanish because of a monomial factor $z_1^l z_2^m$ still being left after differentiation, or they will vanish because of (8).

Now in order to finish the proof we can just use the proof of Theorem 2 from equation (6) verbatim, if we just replace any reference to "the reproducing property of $\varphi$" with a reference to equation (9). $\square$

The function $\varphi$ used above can be obtained as the unique minimizer of a suitable extremal problem. When obtained in this way, one instead shows that equation (6) holds through a variational argument. This is of course more complicated than the argument given above, but in the one variable setting this approach has been successfully applied to extend results whose usual proofs rely heavily on Hilbert space techniques to Banach spaces like $H^p(\mathbb{D})$ and $A^p(\mathbb{D})$. In this specific context though it might be worth to point out that Mandrekar's theorem for $H^p(\mathbb{D}^2)$ for $p \geq 1$ has already been proved by Redett in [5]. The argument used in that article is a modification of the idea of considering the intersection of the invariant subspace with $H^2(\mathbb{D}^2)$, applying Mandrekar's theorem, and using a density argument.

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