PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 10, Pages 91–100 (March 31, 2023)
https://doi.org/10.1090/bproc/151

A VERSION OF KRUST'S THEOREM FOR ANISOTROPIC MINIMAL SURFACES

BENNETT PALMER

(Communicated by Lu Wang)

ABSTRACT. We generalize Krust's theorem to an anisotropic setting by showing the following. If Σ is an anisotropic minimal surface in an axially symmetric normed linear space which is a graph over a convex domain contained in a plane orthogonal to the axis of symmetry, then its conjugate anisotropic minimal surface must also be a graph.

We also generalize a reflection principle of Lawson relating symmetries of an anisotropic minimal surface with symmetries of its conjugate surface.

1. Introduction

Most of the surfaces we see in the real world occur as interfaces. They serve as boundaries between immiscible materials or between distinct phases of one material. The geometry of the interface forms so as to attempt to minimize an appropriate surface energy subject to whatever constraints are imposed by the environment of the interface. When one of the materials is in an ordered phase (i.e crystalline or liquid crystalline), the surface energy is anisotropic, i.e. it depends on the direction of the surface at each point. A particular type of anisotropic surface energy, which we will use here, is a homogeneous one that is independent of position. Its equilibria are called anisotropic minimal surfaces.

Krust's theorem states that if a minimal surface can be represented as a graph over a convex domain, its conjugate minimal surface is also a graph [3]. The significance of the conclusion of this theorem comes from the fact that any minimal graph over a convex domain minimizes area with respect to its boundary values. Recently, Krust's theorem was generalized to maximal surfaces in Lorentz-Minkowski space by R. López [7], see also the paper by Akamine and Fujino [8].

In this note, we will give a generalization of Krust's theorem to the anisotropic setting. Although an anisotropic minimal surface cannot, in general, be embedded in a one-parameter 'associated family' of such surfaces, there is still a notion of conjugate anisotropic minimal surface. However this conjugate anisotropic minimal surface is an equilibrium for a distinct energy functional that is dual to the original one. Under the assumption that the original anisotropic functional is axially symmetric, we show that Krust's theorem still holds, the conjugate surface is a graph if the original anisotropic minimal surface is a graph over a convex domain.

Received by the editors September 13, 2022, and, in revised form, November 8, 2022.
2020 *Mathematics Subject Classification.* Primary 49Q10; Secondary 49N15, 53A05.

2. Preliminaries

Let $\|\cdot\|_*$ be a norm that is smooth on $\mathbf{R}^3\backslash\{0\}$. For a sufficiently smooth, oriented immersed surface $X : \Sigma \to \mathbf{R}^3$ with unit normal field ν, we have the corresponding anisotropic energy functional

$$\mathcal{F}[X] = \int_\Sigma \|\nu\|_* \, d\Sigma.$$

Such a functional is sometimes referred to as an even constant coefficient parametric elliptic functional.

Let $\|\cdot\|$ denote the dual norm to $\|\cdot\|_*$. The sphere of this norm $W := \{\|y\| = 1\}$ will be called the Wulff shape. We assume that W is smooth and uniformly convex so that the Gauss map of W is a bijection from W to S^2. If γ denotes the support function of W pulled back to S^2, then the inverse of the Gauss map of W is given by $\chi : S^2 \to \mathbf{R}^3$, $\nu \mapsto D\gamma|_\nu + \gamma(\nu)\nu \in W \subset \mathbf{R}^3$. If γ is a positive function on S^2, the Wulff shape W can be recovered from the formula

$$W = \partial \bigcap_{\nu\in S^2} \{Y \in \mathbf{R}^3 \mid Y \cdot \nu \leq \gamma(\nu)\}. \tag{1}$$

For an oriented surface with Gauss map ν, we define the Cahn-Hoffman field by $\xi = \chi \circ \nu$. This is a type of anisotropic normal field that was introduced in [1]. Note that $\gamma = \xi \cdot \nu$.

It is then easy to see that

$$\mathcal{F}[X] = \int_\Sigma \gamma(\nu) \, d\Sigma.$$

For any relatively compact $\Omega \subset \Sigma$ with sufficiently smooth boundary, the first variation of $\mathcal{F}$ in the direction δX is given by

$$\delta\mathcal{F}[X] = -\int_\Omega \Lambda\psi \, d\Sigma - \oint_{\partial\Omega} \xi \times dX \cdot \delta X \, ds, \tag{2}$$

where

$$\Lambda = 2H\gamma - \nabla \cdot D\gamma \tag{3}$$

is the anisotropic mean curvature. Here $D\gamma$ is the gradient of γ on S^2 evaluated at ν. The surfaces with $\Lambda \equiv 0$ are exactly the anisotropic minimal surfaces for the given functional.

If a surface is represented as a graph over a plane, the hypothesis that the Wulff shape W is smooth and convex implies the absolute ellipticity of the operator expressing the anisotropic mean curvature in terms of the height function over the plane. Here, absolute ellipticity means that the linearization of this operator is elliptic for any sufficiently smooth surface.

Proposition 2.1. *$\Lambda \equiv 0$ if and only if the vector valued 1-form $\xi \times dX$ is closed on Σ.*

Proof. Take $\delta X = E_i =$ usual basis vectors. Since all translations preserve the value of the functional on any relatively compact $\Omega \subset \Sigma$, we get

$$0 = -\int_\Omega \Lambda\nu_i \, d\Sigma - \oint_{\partial\Omega} \xi \times dX \cdot E_i \, ds,$$

so if $\Lambda \equiv 0$, the integral of the vector valued 1-form $\xi \times dX$ vanishes over any closed curve.

Conversely, if $\xi \times dX$ is closed and p is a point with $\Lambda(p) \neq 0$, we choose a small neighborhood U of p in Σ with $\nu_i \neq 0$ and $\Lambda \neq 0$ in U. Then the first variation formula gives

$$0 = -\int_U \Lambda \nu_i \, d\Sigma \neq 0,$$

which is a contradiction. □

If Σ is simply connected, then we define the *conjugate surface* $\hat{X}$ by the formula

$$d\hat{X} = -\xi \times dX.$$

Note that $\hat{X} : \Sigma \to (\mathbf{R}^3)^*$, since both ξ and dX are $\mathbf{R}^3$ valued. The cross product of vectors $a, b \in \mathbf{R}^3$ defines $a \times b \in (\mathbf{R}^3)^*$ by $\langle c, a \times b\rangle = \det(a, b, c)$. Note also that ξ^* is the Cahn-Hoffman field of $\hat{X}$, since $\xi \cdot d\hat{X} = 0$ and $\|\xi^*\|_* = 1$.

For surfaces in $\tilde{Y} : \Sigma \to (\mathbf{R}^3)^*$, we have the dual functional

$$\mathcal{F}^*[Y] = \int_\Sigma \|\nu_Y\| \; d\Sigma.$$

Proposition 2.2. *If X is defined on a simply connected surface and $\Lambda = 0$ holds, then $\hat{X}$ is an equilibrium surface for $\mathcal{F}^*$.*

In the case where W is smooth and convex, there is a bijection $W \to W^* := \{Y \in \mathbf{R}^3 | \, \|Y\|_* = 1\}$, given by $\xi \to \xi^*$, where ξ^* is the unique element of W^* satisfying

$$\langle \xi, \xi^* \rangle = 1. \tag{4}$$

By the previous proposition, the surface $\hat{X}$ will be an equilibrium for $\mathcal{F}^*$ exactly when $d(\xi^* \times d\tilde{X}) = 0$ holds. However using (4) and the fact that $d\xi$ and dX have parallel images, we get

$$\begin{aligned} \xi^* \times d\hat{X} &= \xi^* \times d(-\xi \times dX) \\ &= \langle \xi, \xi^* \rangle dX - \langle dX, \xi^* \rangle \xi \\ &= dX, \end{aligned}$$

so $d(\xi^* \times d\hat{X}) = ddX = 0$ holds. □

An interesting feature of the duality, which mimics the classical case of the area functions, is that for any anisotropic minimal surface Σ for which the dual surface is defined we have

$$\mathcal{F}[\Sigma] = \mathcal{F}^*[\Sigma^*].$$

To see this, we note the following formulas

$$\gamma = \xi \cdot \nu, \quad \xi^* = \frac{\nu}{\gamma}, \quad \hat{\nu} = \frac{\xi}{|\xi|},$$

where $\hat{\nu}$ is the normal to W^* at ξ^*.

Let e_i, $i = 1, 2$, be any locally defined orthonormal frame on Σ with dual frame ω_i. Using $|\cdot|$ for the Euclidean norm and the formula $(u_1 \times u_2) \times y = (u_1 \cdot y)u_2 -$

$(u_2 \cdot y)u_1$, we get

$$
\begin{aligned}
\xi^* \cdot \hat{\nu}\, d\hat{\Sigma} &= \frac{1}{|\xi|} |d\hat{X}(e_1) \times d\hat{X}(e_2)|\, d\Sigma \\
&= \frac{1}{|\xi|} |(\xi \times dX(e_1)) \times (\xi \times dX(e_2))|\, d\Sigma \\
&= \frac{1}{|\xi|} |\ (\xi \times dX(e_1)) \cdot \xi dX(e_2) - (\xi \times dX(e_1)) \cdot dX(e_2)\xi\ |\, d\Sigma \\
&= \frac{1}{|\xi|} |\ -(\xi \times dX(e_1)) \cdot dX(e_2)\xi\ |\, d\Sigma \\
&= \frac{1}{|\xi|} |\ (\xi \cdot \nu)\xi\ |\, d\Sigma \\
&= \gamma\, d\Sigma.
\end{aligned}
$$

2.1. **The axially symmetric case.** In the case where the functional $\mathcal{F}$ is axially symmetric, we can assume $\gamma = \gamma(\nu_3)$. In this case, the Wulff shape will be axially symmetric and, with respect to the inward pointing normal, the principal radius of curvature of a parallel is $\mu_2^{-1} := \gamma - \nu_3 \gamma'(\nu_3)$, while the principal radius of curvature of a meridian is $\mu_1^{-1} = (1 - \nu_3^2)\gamma''(\nu_3) + \mu_2^{-1}$. These radii are positive due to the convexity of W. The inverse of the Gauss map of W can then be expressed

$$
\chi = \frac{1}{\mu_2}\nu + \gamma'(\nu_3)E_3. \tag{5}
$$

We refer the reader to [4] for details. A consequence of the previous formula, which we will use later, is that the third coordinate of χ is

$$
\chi \cdot E_3 = \frac{\nu_3}{\mu_2}\nu + \gamma'(\nu_3). \tag{6}
$$

Since $\|\nu\|_* = \gamma(\nu_3)$, $\gamma(\nu_3)$ must be even so $\gamma'(0) = 0$ and $\chi \cdot E_3 \equiv 0$ along the arc in W where $\nu_3 = 0$. This forces χ_3 to be positive for ν_3 positive. Up to rescaling and translation, there is a unique non planar axially symmetric anisotropic, minimal surface called the *anisotropic catenoid*. Since the Wulff shape W is axially symmetric, it can be parameterized $\chi = \chi(V, \theta) = (U(V)\cos\theta, U(V)\sin\theta, V)$. Then, the anisotropic catenoid can be expressed

$$
X(V, \theta) = (r(U(V))\cos\theta, r(U(V))\sin\theta, z(V)),
$$

where

$$
r(V) = \frac{1}{2U(V)}, \quad dz = -\frac{1}{2U(V)^2}\, dV.
$$

Straightforward calculations show that the conjugate surface is the helicoidal surface given by

$$
\hat{X}(V, \theta) = (\frac{V}{U}\sin\theta, -\frac{V}{U}\cos\theta, -\theta).
$$

Since $V \mapsto (V/U(V), V/U(V))$, $V > 0$ is just a parameterized ray, this shows the remarkable fact that for any axially symmetric anisotropic energy, the conjugate of its anisotropic catenoid is the usual helicoid. (See Figure 1.)

The following result, which is probably well known, is supplied for motivation but will not be essential for the remainder of the paper.

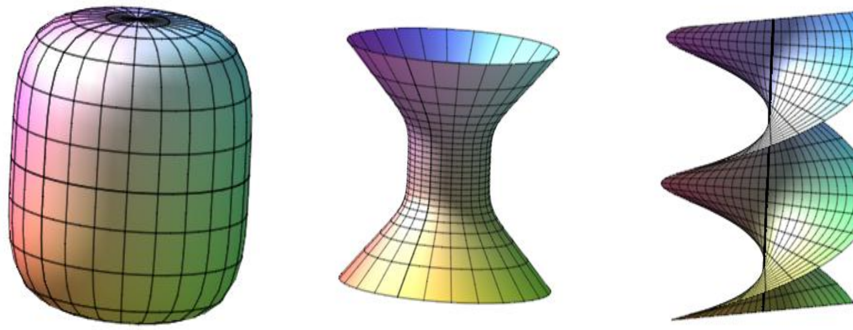

FIGURE 1. left: A Wulff shape center: its anisotropic catenoid; right: helicoid. In Proposition 4.1, the waist of the catenoid is a curve of planar reflection that corresponds to the vertical line of the helicoid which is an arc of geodesic reflection.

Theorem 2.3. *Let $\Sigma \to \mathbf{R}^3$ be a sufficiently smooth compact orientable surface with boundary which is an equilibrium for the axially symmetric functional $\mathcal{F}$. If Σ can be represented as a graph over a convex planar domain Ω contained in a plane perpendicular to the axis of symmetry of $\mathcal{F}$, then Σ minimizes $\mathcal{F}$ with respect to its boundary values, i.e.*

$$\mathcal{F}[\Sigma] \leq \mathcal{F}[S]$$

holds for any orientable surface S with $\partial S = \partial\Sigma$.

The proof will be presented in the Appendix.

3. Krust's theorem

The following result generalizes Krust's theorem to axially symmetric anisotropic surface energies. Our proof is a modification of a proof given by López to generalize Krust's theorem to maximal surface in three-dimensional Lorentz-Minkowski space $\mathbf{L}^3$. In [7], López gives three proofs for surfaces in $\mathbf{L}^3$. Ideas from his second are employed here since they do not rely on complex analytic tools which are unavailable in the anisotropic case.

Theorem 3.1. *Let Σ be an anisotropic minimal surface for an axially symmetric functional $\mathcal{F}$ that can be represented as a graph over a convex domain Ω in a plane orthogonal to the axis of symmetry of the functional. Then its conjugate anisotropic minimal surface $\hat{X}$ is also a graph.*

Proof. We may assume that the axis of symmetry is the vertical axis. We express Σ as the image of the embedding

$$X : \Omega \to \mathbf{R}^3, \quad (x, y) \mapsto (x, y, z(x, y)).$$

Then the conjugate surface $\hat{\Sigma}$ is the image of the immersion $\hat{X} : \Omega \to \mathbf{R}^3$, with $d\hat{X} = -\xi \times dX$ on Ω. For distinct points $p_i \in \Omega$, we show that their images $\hat{X}(p_i)$ have distinct orthogonal projections to the plane $z = 0$.

Let w denote the unit vector in the direction of the line segment from p_1 to p_2 and define $a := w \times E_3$ which is a unit normal to the plane span$\{w, E_3\}$. Note that $dX(w) = w + z'(t)E_3$, where $z'(t) = \partial_t z(p_1 + tw)$. By (5)

$$\begin{aligned}\partial_t \hat{X}(p_1 + tw) &= -\xi \times dX(w)\\ &= -\xi \times w - z'(t)\xi \times E_3\\ &= -(\frac{1}{\mu_2}\nu \times w + \gamma'(\nu_3)E_3 \times w + \frac{z'(t)}{\mu_2}\nu \times w)\end{aligned}$$
,

Using the formula $u_1 \times v_1 \cdot u_1 \times v_2 = (u_1 \cdot v_1)(u_2 \cdot v_2) - (u_1 \cdot v_2)(u_2 \cdot v_1)$, we then obtain

$$\begin{aligned}\partial_t \hat{X}(p_1 + tw) \cdot a &= \partial_t \hat{X}(p_1 + tw) \cdot w \times E_3\\ &= -(\frac{1}{\mu_2}\nu \times w \cdot w \times E_3 + \gamma'(\nu_3)E_3 \times w \cdot w \times E_3 + \frac{z'(t)}{\mu_2}\nu \times w \cdot w \times E_3)\\ &= |w|^2(\frac{\nu_3}{\mu_2} + \gamma'(\nu_3)) + \frac{(z'(t))^2\nu_3}{\mu_2}.\end{aligned}$$

The second term above is clearly positive and by (6), the first term equates to $|w|^2\xi_3$ which is also positive. Therefore $\partial_t \hat{X}(p_1 + tw) \cdot a > 0$ holds. Therefore

$$0 < \int_0^{|p_1 - p_2|} \partial_t \hat{X}(p_1 + tw) \cdot a \, dt = (\hat{X}(p_2) - \hat{X}(p_1)) \cdot a.$$

Since a is a horizontal vector, this shows that the points $\hat{X}(p_i)$, $i = 1, 2$, have distinct projections to the plane $z = 0$, proving the result. □

Another consequence of $\hat{\Sigma}$ being a graph is that it is stable as an anisotropic minimal surface. Also, it is shown in [5] that the Gauss map $\hat{\nu}$ of $\hat{\Sigma}$ is critical for an energy functional

$$E[f] = \int_{\hat{\Sigma}} \langle (D^2\hat{\gamma} + \hat{\gamma}I)\nabla f, \nabla f\rangle d\hat{\Sigma},$$

where $\hat{\gamma} := \xi^* \cdot \hat{\nu}$ is the support function of W^* regarded as a function on S^2 and $D^2\hat{\gamma}$ is its Hessian on S^2. The competing maps are those mapping $\hat{\Sigma}$ to S^2 having the same boundary values as $\hat{\nu}$. Since $\hat{\Sigma}$ is stable, $\hat{\nu}$ is in fact the absolute minimizer of this energy functional among all smooth maps $\hat{\Sigma} \to S^2$ having the same boundary values as $\hat{\Sigma}$ (see [5] for details).

4. Reflection principle

In [6], Lawson formulated a reflection principle for minimal surfaces in space forms that we will now consider in the anisotropic case. A non trivial planar arc C contained in a surface Σ is called an arc of planar symmetry if Σ is invariant with respect to reflection through the plane containing the arc. A vertical line segment $\ell \subset \Sigma$, given by $x = 0 = y$, will be called an arc of geodesic symmetry if the map $(x, y, z) \mapsto (-x, -y, z)$ restricted to Σ is a symmetry of Σ.

We will only consider rotationally symmetric energy functionals having a density $\gamma(\nu_3)$. Because of this, we may assume that any arc of planar symmetry is contained in a horizontal plane, since the functional is only invariant with respect to reflection through horizontal planes. We will show the following:

Proposition 4.1. *Let Σ be an anisotropic minimal surface with energy density $\gamma = \gamma(\nu_3)$ being real analytic. A non trivial arc $C \subset \Sigma$ is a horizontal arc of planar symmetry if and only if its corresponding arc in the conjugate surface $\hat{\Sigma}$ is a vertical arc of geodesic symmetry.*

Proof. The method of proof is similar to that used in [6] although there are significant differences with the isotropic case.

First assume C is a planar curve in Σ and that Σ is invariant under reflection across the plane. Then Σ meets the plane in a right angle and it is clear from the reflective symmetry that $\nu_3 \equiv 0$ along C. Recall that $\xi := \chi \circ \nu$ where ν is the Gauss map of Σ. If C' denotes the unit tangent vector to C, then since both ν and C' are horizontal vectors, we get

$$\begin{aligned} d\hat{X}(C') &= -\xi \times C' \\ &= -(\frac{1}{\mu_2}\nu + \gamma'(0)E_3) \times C' \\ &= \pm\gamma(0)E_3. \end{aligned}$$

It follows that the image $\hat{C}$ of C in $\hat{\Sigma}$ is a vertical line segment.

Conversely, if $\hat{C}$ is a vertical line, then it follows from the previous calculation that along C both C' and ξ are lie in a horizontal plane. Since $\chi : S^2 \to W$ maps the equator $\nu_3 = 0$ onto the circle $\chi_3 = 0$, this means that $\nu_3 \equiv 0$ along C. So Σ meets the plane containing C in a right angle.

In order to complete the proof, it remains to show that if Σ contains a vertical line (resp. if Σ meets a horizontal plane in a right angle), then Σ is invariant with respect to geodesic reflection across the horizontal line (resp. with respect to reflection across the horizontal plane).

Let D denote the unit disc in the (x_1, x_2) plane and let $D_\pm$ be the half disc with $(-1)^{\pm 1}x_1 > 0$. We may assume that a neighborhood of a point in the vertical line ℓ is the image of D under a conformal embedding $X : D \to \mathbf{R}^3$, with the points of D_+ located on one side of ℓ. Set $S_\pm = X(D_\pm)$. We claim that the reflection $\psi(x, y, z) = (-x, -y, z)$ extends S_+ to a smooth anisotropic minimal immersion $X^* : D \to R^3$.

If $\nu = (\nu_1, \nu_2, \nu_3)$ denotes the normal of Σ, then a direct calculation shows that the normal of $\psi \circ X$ is $(\nu_1, \nu_2, -\nu_3)$. Recall that $\gamma(-\nu_3) = \gamma(\nu_3)$ and note that $\nu_3 \equiv 0$ holds on $S_+ \cap \ell$ since ℓ is a vertical line tangent to Σ. Since the map ψ is an isometry of $\mathbf{R}^3$ and its effect on the normal leaves γ invariant, it follows that the surface $D_- \mapsto \mathbf{R}^3$, $(x_1, x_2) \mapsto (\psi \circ X)(-x_1, x_2)$ is an anisotropic minimal surface which extends $X|_{D_+}$ to a C^1 immersion of $X^* : D \to \mathbf{R}^3$.

We will show that X^* is, in fact, of class C^2. To see this we first claim that the mean curvature H of Σ vanishes on the line segment $\{x_1 = 0\} \cup D$. Using (3), we get

$$\begin{aligned} 0 &= \nabla \cdot D\gamma - 2H\gamma \\ &= \nabla \cdot (\gamma'(\nu_3)\nabla z) - 2H\gamma \\ &= \gamma''(\nu_3)\nabla\nu_3 \cdot \nabla z + 2H\nu_3\gamma'(\nu_3) - 2H\gamma. \end{aligned}$$

Now use that $\nu_3 \equiv 0$ when $x_1 = 0$ and as explained above $\gamma'(0) = 0$. Also, on the segment $x_1 = 0$, we have $\nu_3 \equiv 0$ and so $\nabla\nu_3 \cdot \nabla z \equiv 0$ along ℓ. Since γ is positive, it then follows that $H \equiv 0$ along ℓ.

It is clear that $(\partial^2 x/\partial x_2^2)(0, x_2) \equiv 0 \equiv (\partial^2 y/\partial x_2^2)(0, x_2)$. Since $H \equiv 0$ on ℓ, the Laplacians of all three coordinate functions vanish also and it then follows that $(\partial^2 x/\partial x_1^2)(0, x_2) \equiv 0 \equiv (\partial^2 y/\partial x_1^2)(0, x_2)$ also. This implies that these derivatives can be continuously extended to continuous functions in D which are odd with respect to x_1. It is straightforward to check that they are the derivatives of the reflected functions x and y on D_-. In a similar way, $(\partial^2 z/\partial x_i^2)$, $i = 1, 2$, can be extended as an even continuous function. A similar statement holds for the mixed second order partial derivatives of the three coordinate functions.

Since all of the coordinates are of class C^2, it follows that Σ^* is a classical solution of the variational problem. □

In Figure 1 the waist of the anisotropic catenoid is an arc of planar reflection which corresponds to the vertical line in the helicoid. The vertical line is an arc of linear reflection.

Remark. Brian White [9] has shown the solvability of the Plateau problem for smooth even parametric elliptic functionals with arbitrary prescribed simple closed boundary curve C which lies on the boundary of a convex set in $\mathbf{R}^3$. Because of this, we can find many examples of anisotropic minimal surfaces containing a line in their boundaries to which the previous result can be applied.

5. Appendix

We present here the proof of Theorem 2.3. The idea of approximating Ω by a polygonal domain and then applying the compactness theorem was provided to us by Frank Morgan. We wish to express our gratitude for his help.

Proof of Theorem 2.3. First assume that a comparison surface S is contained in the solid cylinder $\overline{\Omega \times \mathbf{R}}$. We can assume that S is oriented in such a way so that the induced orientation of the boundary agrees with the induced orientation of $\partial\Sigma$. Assume S satisfies the conditions stated above. Since Σ is a graph $(x, y) \mapsto (x, y, z(x, y))$, we can extend its Cahn-Hoffman field to a vector field $\tilde{\xi}$ on $\overline{\Omega \times \mathbf{R}}$ by $\hat{\xi}(x, y, z) = \xi(x, y)$. The vanishing of the anisotropic mean curvature on Σ implies that $\nabla \cdot \xi \equiv 0$ holds on Σ which is the same as $\nabla \cdot \tilde{\xi} \equiv 0$ on $\Omega \times \mathbf{R}$.

The two-chain $\Sigma - S$ is the oriented boundary of an oriented three-chain U. By the divergence theorem

$$0 = \int_U \nabla \cdot \tilde{\xi}\, dV = \int_\Sigma \tilde{\xi} \cdot \nu\, d\Sigma - \int_S \tilde{\xi} \cdot \nu_1\, dS.$$

Therefore, if ξ_1 denotes the Cahn-Hoffman field of S, then

$$\begin{aligned}\mathcal{F}[\Sigma] &= \int_\Sigma \xi \cdot \nu \, d\Sigma \\ &= \int_\Sigma \tilde{\xi} \cdot \nu \, d\Sigma \\ &= \int_S \tilde{\xi} \cdot \nu_1 \, dS \\ &\leq \int_S \xi_1 \cdot \nu_1 \, dS \\ &= \mathcal{F}[S].\end{aligned}$$

by (1).

Now consider the case where the comparison surface S intersects the exterior of the solid cylinder $\overline{\Omega \times \mathbf{R}}$. In order to handle this we will need to extend $\mathcal{F}$ to the category of integral currents. We refer the reader to Chapter 5 of [2] for details.

By the convexity of Ω, we can find a sequence of convex polygonal domains P_i such that $\partial\Omega$ is inscribed in ∂P_i, $P_1 \supset P_2 \supset P_3 \supset \cdots$ and

$$\text{distance}(\partial P_i, \Omega) < 1/i. \tag{7}$$

Let $\pi_i : \mathbf{R}^3 \to \overline{P_i \times \mathbf{R}}$ denote the Lipschitz map which sends each point to its nearest point in $\overline{P_i \times \mathbf{R}}$. We decompose the compliment of $\overline{P_i \times \mathbf{R}}$ in $\mathbf{R}^3$ as $E_i \cup \Phi_i$ where E_i is the set of points q with $\pi(q)$ lying on an edge of $\overline{P_i \times \mathbf{R}}$ in $\mathbf{R}^3$ and Φ_i is the set of points q with $\pi_i(q)$ lying in an open face. Let F be any face of $\partial P_i \times \mathbf{R}$. We claim that the restriction of π to $S \cup \pi_i^{-1}(F)$ is energy non increasing.

Assume first that $S \cap \pi_i^{-1}(F)$ has only one component which we'll denote by S_F. Note that F is a vertical plane having a horizontal normal vector ν_o. The Cahn-Hoffman field of F is therefore

$$\xi_o := \xi(\nu_o) = D\gamma|_{\nu_o} + \gamma(0)\nu_o = \gamma'(0)E_3 + \gamma(0)\nu_o = \gamma(0)\nu_o \tag{8}$$

since $\gamma'(0) = 0$. Note also that $S \cap \pi_i^{-1}(F)$ lies in the infinite rectangular box $F \times \mathbf{R}$.

Each point $m \in S_F$ either lies in F or it is joined by a line segment parallel to ν_o which connects it to $\pi_i(m)$. The union of all such line segments $\overline{m\pi_i(m)}$ is a three-dimensional domain $U \subset F \times \mathbf{R}$. The boundary of U consists of S_F, $\pi_i(S_F)$ and a union of planar domains having normals equal to $\pm E_3 \times \nu_o$.

We now apply the divergence theorem using the constant vector field ξ_o in U to get

$$\begin{aligned}0 &= \int_U \nabla \cdot \xi_o \, dV \\ &= \int_{\pi_i S_F} \xi_o \cdot dS - \int_{\partial U \setminus \pi_i S_F} \xi_o \cdot dS \\ &= \mathcal{F}[\pi_i S_F] - \int_{S_F} \xi_o \cdot N \, dS\end{aligned}$$

since ξ_o is perpendicular to $E_3 \times \nu_o$. Here N is the normal to S_F pointing out of U. Again, by (1) we get

$$\mathcal{F}[\pi_i S_F] = \int_{S_F} \xi_o \cdot N \, dS \leq \int_{S_F} \gamma(N) \, dS = \mathcal{F}[S_F].$$

If $S \cap \pi_i^{-1}(F)$ has multiple boundary components, we apply the same argument to each component. This proves the claim and it follows that $\pi_i|_S$ is energy non increasing since $\pi_i(S \cap \Phi_i)$ has Hausdorff 2 measure zero.

We choose a compact set $K \subset \mathbf{R}^3$ which contains $\pi_1 S$. Since π_i is Lipschitz, each $\pi_i S$ is an integral current contained in K. Further, the masses $M(\pi_i S)$ and $M(\partial \pi_i S)$ are uniformly bounded by $M(S) + M(\partial S)$. By the compactness theorem ([2] Theorem 4.2.17), we can extract a subsequence, again denoted by $\{\pi_i S\}$, which converges with respect to the weak topology on the space of currents. If π denotes projection into $\overline{\Omega \times \mathbf{R}}$, then, because of (7), it is clear that $\pi_i S \to \pi S$ in this topology.

We then obtain

$$\mathcal{F}[S] \geq \liminf \mathcal{F}[\pi_i S] \geq \mathcal{F}[\pi S] \geq \mathcal{F}[\Sigma].$$

The first inequality holds since π is energy non increasing, the second follows from the lower semicontinuity of $\mathcal{F}$ ([2] Th. 5.15), and the third follows from the first part of the proof since πS is contained in the cylinder $\overline{\Omega \times \mathbf{R}}$. $\square$

Acknowledgments

Part of this research was done while the author was a guest of Texas Tech University. We would like to extend our thanks for their hospitality.

Again, the authors would like to thank Frank Morgan for his assistance.

References

[1] J. Cahn, and D. L. Hoffman, *A vector thermodynamics for anisotropic surfaces II. Curved and faceted surfaces*, Acta Metallurgica **22** (1974), no. 10, 1205–1214.

[2] Herbert Federer, *Geometric measure theory*, Springer, 2014.

[3] Hermann Karcher, *Construction of minimal surfaces*, Surveys in geometry, 1989, 1–96.

[4] Miyuki Koiso and Bennett Palmer, *Rolling construction for anisotropic Delaunay surfaces*, Pacific J. Math. **234** (2008), no. 2, 345–378, DOI 10.2140/pjm.2008.234.345. MR2373453

[5] Miyuki Koiso and Bennett Palmer, *Geometry and stability of surfaces with constant anisotropic mean curvature*, Indiana Univ. Math. J. **54** (2005), no. 6, 1817–1852, DOI 10.1512/iumj.2005.54.2613. MR2189687

[6] H. Blaine Lawson Jr., *Complete minimal surfaces in S^3*, Ann. of Math. (2) **92** (1970), 335–374, DOI 10.2307/1970625. MR270280

[7] Rafael López, *The Lorentzian version of a theorem of Krust*, Bull. Malays. Math. Sci. Soc. **44** (2021), no. 2, 683–692, DOI 10.1007/s40840-020-00968-x. MR4217079

[8] Shintaro Akamine and Hiroki Fujino, *Extension of Krust theorem and deformations of minimal surfaces*, Ann. Mat. Pura Appl. (4) **201** (2022), no. 6, 2583–2601, DOI 10.1007/s10231-022-01211-z. MR4498400

[9] Brian White, *Existence of smooth embedded surfaces of prescribed genus that minimize parametric even elliptic functionals on 3-manifolds*, J. Differential Geom. **33** (1991), no. 2, 413–443. MR1094464

(An Anisotropic Version of Krust's Theorem) Department of Mathematics, Idaho State University, Pocatello, Idaho 83209

Email address: palmbenn@isu.edu