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RANKS OF $RO(G)$-GRADED STABLE HOMOTOPY GROUPS OF SPHERES FOR FINITE GROUPS G

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ABSTRACT. We describe the distribution of infinite groups within the $RO(G)$-graded stable homotopy groups of spheres for a finite group G.

CONTENTS

1. INTRODUCTION

1.1. **Overview.** In ordinary stable homotopy theory, one of the most basic theorems is Serre's Finiteness Theorem [Ser53] stating that the n-th stable homotopy group of the sphere, $\pi_n(S^0)$, is finite for $n > 0$. Since we understand that $\pi_0(S^0) \cong \mathbb{Z}$, this means that rationally the structure of stable homotopy is very simple, and attention is quickly focused on torsion. Equivariantly, it is still true that rationalisation is a massive simplification, but the residual structure in the rationalisation is worth some attention.

Let G be a finite group. If V is a real orthogonal G-representation, its one-point compactification S^V is a sphere with G-action and one can define the V-th G-equivariant homotopy group of the sphere $\pi_V^G(X)$ by considering equivariant homotopy classes of maps out of S^V. Taking $X = S^0$ and stabilising yields the $RO(G)$-graded stable homotopy groups of the sphere [May96, Ch. IX]. The purpose of this note is to identify the crudest feature of these groups: their ranks as abelian groups. This is a straightforward deduction from well-known results, but some interesting features emerge by giving a systematic account.

Example. Let $G = C_2$ be the cyclic group of order two. Then

$$RO(C_2) \cong \mathbb{Z}\{1, \sigma\},$$

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where 1 is the one-dimensional trivial representation and σ is the one-dimensional sign representation. Computations of Araki–Iriye [AI82] show that $\pi_\alpha^{C_2}(S^0)$ is infinite if

$$\alpha \in \mathbb{Z}\{2(1-\sigma)\} \cup \mathbb{Z}\{\sigma\}.$$

Our results recover this observation, and show that these are the *only* degrees for which $\pi_\alpha^{C_2}(S^0)$ is infinite.

Using rational equivariant stable homotopy theory, we prove the following:

Theorem A (Theorem 2.3)**.** *Let G be a finite group and $\alpha \in RO(G)$. Then*

$$\pi_\alpha^G(S^0) \otimes \mathbb{Q} = [S^\alpha, S^0]^G \otimes \mathbb{Q} = \prod_{(H)} \mathrm{Hom}_{W_G(H)}(\pi_0(S^{\alpha^H}), \mathbb{Q}),$$

where the $W_G(H)$-module $\mathbb{Q}$ has trivial action, and the product is taken over conjugacy classes of subgroups $H \leq G$. Thus $\pi_\alpha^G(S^0) \otimes \mathbb{Q}$ is a rational vector space of dimension r_α, where

$$r_\alpha = |\{(H) \mid \alpha^H = 0 \textit{ and } W_G(H) \textit{ acts trivially on } \pi_0(S^{\alpha^H})\}|.$$

We lay the groundwork for applying Theorem A in Section 3. We then compute the ranks of the $RO(G)$-graded stable homotopy groups of spheres for various G in Section 5.

In Section 4, we discuss two natural variations where the same techniques give information. Since the sphere is rationally an Eilenberg-MacLane spectrum for the Burnside Mackey functor, $S^0 \simeq_{\mathbb{Q}} H\mathbb{A}$, we may view our methods as a calculation of the rationalisation of $H_G^\star(S^0; \mathbb{A})$ (where $\star$ denotes $RO(G)$-grading). The same methods apply to give a calculation of the rationalisation of $H_G^\star(S^0; M)$ for any Mackey functor M. For the second variation, we may consider the Picard-graded stable homotopy groups of spheres: invertible objects are again characterised in terms of orientations and dimension functions (see [FLM01]).

Finally, we note that our results provide a basis for understanding other large-scale phenomena in the $RO(G)$-graded stable homotopy groups of spheres. For example, Iriye [Iri83] showed that Nishida's nilpotence theorem [Nis73] holds equivariantly: an element $\pi_\star^G(S^0)$ is torsion if and only if it is nilpotent. Theorem A therefore explicitly describes the regions of $\pi_\star^G(S^0)$ in which elements can be nilpotent and non-nilpotent.

1.2. **Finite generation.** For most of the paper we will work rationally, but we would like to draw conclusions about the integral situation. For completeness we include the proofs of the basic finiteness statements that permit this deduction.

Lemma 1.1. *For any $\alpha \in RO(G)$, the sphere S^α is a finite G-cell spectrum.*

Proof. For an actual representation V, the sphere S^V is a smooth compact manifold and hence admits the structure of a finite G-CW-complex. By exactness of Spanier–Whitehead duality, $DS^V \simeq S^{-V}$ is also a finite G-CW spectrum (since by the Wirthmüller isomorphism $DG/H_+ \simeq G/H_+$). Now if $\alpha = V - W$, $S^\alpha \simeq S^V \wedge S^{-W}$, so the result follows. □

The following consequence fails for infinite compact Lie groups.

Lemma 1.2. *For any $\alpha \in RO(G)$ the abelian group $\pi_\alpha^G(S^0)$ is finitely generated. Consequently $\pi_\alpha^G(S^0)$ is finite if and only if $\pi_\alpha^G(S^0) \otimes \mathbb{Q} = 0$.*

Proof. From the Segal–tom Dieck splitting theorem [tD75], we see that $\pi_n^G(S^0)$ is a finitely generated abelian group. By Lemma 1.1, it follows that $\pi_\alpha^G(S^0)$ is finitely generated. □

Theorem A describes the $RO(G)$-graded rational homotopy groups of the sphere. By Lemma 1.2, this determines precisely those degrees $\alpha \in RO(G)$ for which $\pi_\alpha^G(S^0)$ is finite.

1.3. **Conventions.** Henceforth everything is rational. We write G for a finite group, H for a subgroup of G, and $W_G(H) = N_G(H)/H$ for the Weyl group of H. We use $*$ to denote $\mathbb{Z}$-graded groups and $\star$ to denote $RO(G)$-graded groups. If $V \in RO(G)$, then $|V|$ denotes its (virtual) dimension.

2. Rational stable homotopy

For finite groups, it is easy to give a complete model of rational G-spectra [GM95, App. A]. We do not need the full strength of this description, so we describe what we want in a convenient form.

First, note that for any X and Y, passage to geometric fixed points gives a map

$$\Phi^H : [X, Y]^G \longrightarrow [\Phi^H X, \Phi^H Y].$$

The codomain admits an action of the Weyl group $W_G(H)$ by conjugation, and Φ^H takes values in the $W_G(H)$-equivariant maps.

Theorem 2.1. *If X and Y are rational, the maps Φ^H give an isomorphism*

$$[X, Y]^G_* = \bigoplus_{(H)} H^0(W_G(H); [\Phi^H X, \Phi^H Y]_*),$$

where the sum is taken over conjugacy classes of subgroups $H \leq G$. Furthermore, passage to homotopy groups gives isomorphisms

$$H^0(W_G(H); [\Phi^H X, \Phi^H Y]_*) = \mathrm{Hom}_{W_G(H)}(\pi_*(\Phi^H X), \pi_*(\Phi^H Y)).$$

Proof. Filtering EG_+ by skeleta gives a spectral sequence

$$H^*(G; [X, Y]_*) \Rightarrow [EG_+ \wedge X, Y]^G_*$$

for (integral) stable maps. When Y is rational, this collapses to an isomorphism

$$H^0(G; [X, Y]_*) = [EG_+ \wedge X, Y]^G_*.$$

We may combine this with the splitting $S^0 \simeq \bigvee_{(H)} e_H S^0$ using the idempotents e_H of the rational Burnside ring to give the first stated isomorphism, since

$$\begin{aligned}[e_H S^0 \wedge X, Y]^G = [e_H S^0 \wedge X, Y]^{N_G(H)} &= [e_{\{e\}} S^0 \wedge \Phi^H X, \Phi^H Y]^{W_G(H)} \\ &= [EW_G(H)_+ \wedge \Phi^H X, \Phi^H Y]^{W_G(H)}.\end{aligned}$$

The second isomorphism comes from the classical version of Serre's Theorem [Ser53]. □

Remark 2.2. An alternative approach is to use [GM95]. We observe that $X \simeq \prod_n \Sigma^n H \underline{\pi}_n^G(X)$ and then use the fact that all rational Mackey functors are projective and injective to deduce

$$[X, Y]^G \xrightarrow{\cong} \prod_n \mathrm{Hom}(\underline{\pi}_n^G(X), \underline{\pi}_n^G(Y)).$$

Now we use the structure of Mackey functors to deduce

$$\mathrm{Hom}(\underline{\pi}_n^G(X), \underline{\pi}_n^G(Y)) \cong \prod_{(H)} \mathrm{Hom}_{W_G(H)}(\pi_n(\Phi^H X), \pi_n(\Phi^H Y)),$$

as claimed.

Since G acts trivially on S^0, $W_G(H)$ acts trivially on $\pi_0(S^0) = \mathbb{Q}$. We then have the following consequence of Theorem 2.1:

Theorem 2.3. *Let G be a finite group and $\alpha \in RO(G)$. Then*

$$\pi_\alpha^G(S^0) = [S^\alpha, S^0]^G = \prod_{(H)} \mathrm{Hom}_{W_G(H)}(\pi_0(S^{\alpha^H}), \mathbb{Q}),$$

where the product is taken over conjugacy classes of subgroups $H \leq G$. Thus $\pi_\alpha^G(S^0)$ is a rational vector space of dimension r_α, where

$$r_\alpha = |\{(H) \mid \alpha^H = 0 \text{ and } W_G(H) \text{ acts trivially on } \pi_0(S^{\alpha^H})\}|.$$

3. Geometry of the ranks of the $RO(G)$-graded stable stems

To make the answer in Theorem 2.3 explicit there are now two ingredients: (a) the dimension of the fixed points and (b) the orientations.

3.1. **Virtual representations of fixed point dimension zero.** If we list the simple real representations $S_1, S_2, \ldots, S_r$ of G, we may identify $RO(G) = \mathbb{Z}^r$. Now, for each subgroup $H \leq G$ we have a dimension vector

$$d_H = (\dim(S_1^H), \ldots, \dim(S_r^H)),$$

and the space of virtual representations α with $\alpha^H = 0$ is

$$N_H = \{x \mid x \cdot d_H = 0\},$$

which is isomorphic to $\mathbb{Z}^{r-1}$ as an abelian group. The only α for which $\pi_\alpha^G(S^0)$ can be infinite are those lying in some N_H, and the maximum rank of $\pi_\alpha^G(S^0)$ is the number of conjugacy classes of H with $\alpha \in N_H$.

When $H = G$, the Weyl group $W_G(H)$ is trivial, and we immediately draw a useful conclusion.

Corollary 3.1. *If V is a virtual representation with $V^G = 0$ then*

$$\mathrm{rk}\, \pi_V^G(S^0) \geq 1.$$

Remark 3.2. One special case is when V is a multiple of the reduced regular representation $\bar{\rho}$. This was observed to the second author by Bert Guillou, who noted that it follows from the fact that $\Phi^G(S^0) \simeq S^0$ and that geometric fixed points are given by inverting the Euler class of the reduced regular representation.

On this same theme, if V is a representation with $V^G = 0$, the inclusion of the origin gives a map $a_V : S^0 \longrightarrow S^V$ whose G-fixed points generate $\pi_0(S^0)$. The element a_V is thus of infinite order in $\pi_{-V}^G(S^0)$. The G-component of the map a_V will not usually be invertible integrally. However, by Theorem 2.1, there is a rational map $a'_V \in \pi_V^G(S^0)$ whose G-component is the inverse of a_V. The problem of finding the smallest positive multiple of a'_V that is integral is of considerable interest; the case of the group of order 2 was studied classically by Landweber [Lan69], but is now best treated using motivic homotopy theory [BGI21, GI20]. For the group of odd prime order p, it was studied by Iriye [Iri89].

3.2. **Orientability.** For any real representation V the group G acts on $H_{|V|}(S^V)$, giving a homomorphism

$$o_V : G \longrightarrow \mathrm{Aut}(\mathbb{Z}) = \mu_2.$$

In view of the Künneth isomorphism

$$H_{|V|}(S^V) \otimes H_{|W|}(S^W) \xrightarrow{\cong} H_{|V+W|}(S^{V\oplus W}),$$

this gives the *orientation character*, a homomorphism

$$o : RO(G) \longrightarrow \mathrm{Hom}(G, \mu_2).$$

Elements of the kernel $RO^+(G)$ of o are *orientable* virtual representations.

Example 3.3. Clearly 2α is orientable for any α. More generally, the image of any complex representation is orientable, as is any element in the image of $RSO(G) \longrightarrow RO(G)$.

Remark 3.4. It is clear that an orientable representation $\rho : G \longrightarrow O(n)$ is one that takes values in representations of determinant 1, so that it comes from $RSO(G)$. However, this is not true of virtual representations. For example, if $G = \Sigma_3$, then $V - \sigma - 1$ is orientable (where V is the reduced regular representation and σ is the sign representation). However, only even multiples of σ or V come from $RSO(\Sigma_3)$.

If $W_G(H)$ is of odd order, then all the gradings in N_H give infinite groups. In general, on each such null space N_H we have an orientation character

$$o_H : N_H \longrightarrow \mathrm{Hom}(W_G(H), \mu_2)$$

defined by considering the action of $W_G(H)$ on $H_{|(S^V)^H|}((S^V)^H)$ for $V \in N_H$. As noted above, the kernel N_H^+ contains all even vectors of N_H and the image of all complex representations.

The set of α for which $\pi_\alpha^G(S^0)$ is infinite is $\bigcup_H N_H^+$. The rank r_α of $\pi_\alpha^G(S^0)$ is the number of conjugacy classes H with $\alpha \in N_H^+$.

3.3. **Bases.** If we choose a subgroup H giving an associated fixed point vector d_H, we note that the component of the trivial representation S_1 is always 1, so that N_H has basis $S_2 - d_H(2), S_3 - d_H(3), \ldots, S_r - d_H(r)$. The orientation o_H is thus described by the homomorphisms

$$o_H(2), o_H(3), \ldots, o_H(r) : W_G(H) \longrightarrow \mu_2,$$

where $o_H(i) = o_H(S_i - d_H(i))$. Since $W_G(H)$ always acts trivially on the trivial representation, the orientation $o_H(S_i - d_H(i)) = o_H(S_i)$, and $o_H(i)$ is the determinant of S_i^H. Since o_H is a homomorphism, this determines its values throughout. All the homomorphisms factor through the largest elementary abelian 2-quotient $E_2(H)$ of $W_G(H)$ (i.e., we factor out commutators and squares).

4. The two variations

In effect, our calculation in Theorem 2.3 was of

$$\pi_\alpha^G(S^0) \otimes \mathbb{Q} = [S^\alpha, S^0]^G \otimes \mathbb{Q} = [S^\alpha, H\mathbb{A}]^G \otimes \mathbb{Q} = H_G^0(S^\alpha; \mathbb{A}) \otimes \mathbb{Q}.$$

We point out that the same methods allow us to calculate

$$[S^\alpha, HM]^G \otimes \mathbb{Q} = H_G^0(S^\alpha; M) \otimes \mathbb{Q}$$

for any invertible spectrum S^α and rational G-Mackey functor M. Indeed we still have

$$[S^\alpha, HM]^G \otimes \mathbb{Q} = \prod_{(H)} \mathrm{Hom}_{W_G(H)}(H_0(S^{\alpha^H}), M^{eH}),$$

where M corresponds to $\{M^{eH}\}_H$ under the equivalence

$$G\text{-MackeyFunctors}/\mathbb{Q} \simeq \prod_{(H)} \mathbb{Q}W_G(H)\text{-modules}.$$

More explicitly, $M^{eH} = M(G/H)/(\text{proper transfers})$. In other words,

$$\mathrm{rk}[S^\alpha, HM]^G \otimes \mathbb{Q} = \sum_{(H)} z_H \cdot m(\alpha, H),$$

where $z_H = 1$ if $\alpha^H = 0$ and $z_H = 0$ otherwise, and where $m(\alpha, H)$ is the multiplicity of the simple $\mathbb{Q}W_G(H)$-representation $H_0(S^{\alpha^H})$ in M^{eH}.

The only M^{eH} which can possibly give infinite groups are those with summands coming from a homomorphism $W_G(H) \longrightarrow \mu_2$. Since the sphere corresponds to the Burnside Mackey functor $\mathbb{A}$ with $\mathbb{A}^{eH} = \mathbb{Q}$ (with trivial action), it has almost as many $RO(G)$-gradings which are infinite as is possible.

5. Examples

We conclude by explicitly calculating the ranks of the $RO(G)$-graded stable homotopy groups of spheres for groups G with small subgroup lattices.

5.1. **Cyclic group of order two.** We have

$$RO(C_2) \cong \mathbb{Z}\{1, \sigma\},$$

where 1 is the 1-dimensional trivial representation and σ is the sign representation. Then

$$N_e \cong \mathbb{Z}\{1 - \sigma\}, \quad N_{C_2} \cong \mathbb{Z}\{\sigma\}.$$

Since $W_{C_2}(C_2) = e$, we have

$$N^+_{C_2} = N_{C_2} \cong \mathbb{Z}\{\sigma\}.$$

On the other hand, $W_{C_2}(e) \cong C_2/e \cong C_2$ acts by (-1) on $1 - \sigma$, so

$$N^+_e \cong \mathbb{Z}\{2(1 - \sigma)\}.$$

Each representation $V \in N^+_{C_2} \cup N^+_e$ satisfies $\mathrm{rk}\,\pi^{C_2}_V(S^0) \geq 1$. Since $N^+_{C_2} \cap N^+_e = \{0\}$, we also have $\mathrm{rk}\,\pi^{C_2}_0(S^0) = 2$. Altogether, we find:

Proposition 5.1. *We have*

$$\mathrm{rk}\,\pi^{C_2}_V(S^0) = \begin{cases} 2 & \text{if } V = 0, \\ 1 & \text{if } V \in (\mathbb{Z}\{\sigma\} \cup \mathbb{Z}\{2(1-\sigma)\}) \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 5.2. The fact that $\pi^{C_2}_V(S^0)$ is infinite for $V \in \mathbb{Z}\{\sigma\} \cup \mathbb{Z}\{2(1-\sigma)\}$ appears in [AI82, Thm. 7.6]. A proof that these are the only degrees for which $\pi^{C_2}_V(S^0)$ is infinite using the C_2-equivariant Adams spectral sequence was communicated to the second author by Bert Guillou and Dan Isaksen.

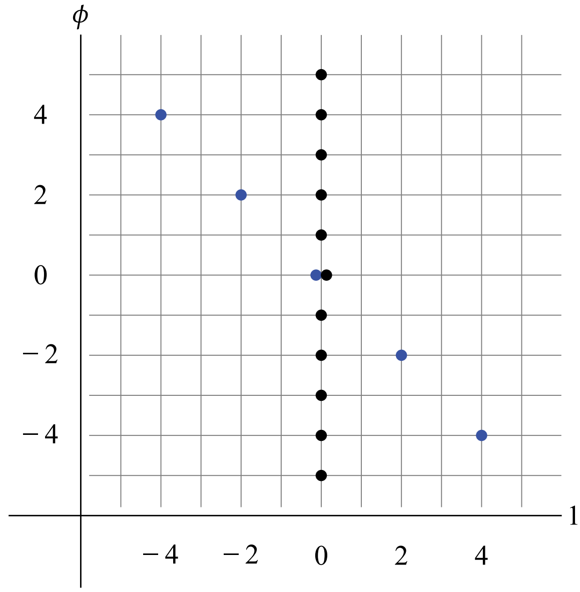

FIGURE 1. Degrees in $RO(C_2)$ where $\pi_V^{C_2}(S^0)$ has infinite rank. A black • indicates a copy of $\mathbb{Z}$ arising from $N_{C_2}^+$ and a blue • indicates a copy of $\mathbb{Z}$ arising from N_e^+.

5.2. **Cyclic group of odd prime order.** Let $q = \frac{p-1}{2}$. We have

$$RO(C_p) \cong \mathbb{Z}\{1, \phi_1, \ldots, \phi_q\},$$

where 1 is the 1-dimensional trivial representation and $\phi_t : C_p \to \mathrm{Aut}(\mathbb{R}^2) \cong \mathrm{Aut}(\mathbb{C})$ sends the generator of C_p to $\cdot e^{2\pi i t/p}$. Then

$$N_e \cong \mathbb{Z}\{2 - \phi_1, \ldots, 2 - \phi_q\}, \quad N_{C_p} \cong \mathbb{Z}\{\phi_1, \ldots, \phi_q\}.$$

Since $W_{C_p}(e) \cong C_p$ and $W_{C_p}(C_p) = e$ necessarily act trivially on $\mathbb{Z}$, we have

$$N_e^+ = N_e, \quad N_{C_p}^+ \cong N_{C_p}.$$

Finally, we have

$$N_e^+ \cap N_{C_p}^+ \cong \mathbb{Z}\{\phi_1 - \phi_2, \ldots, \phi_1 - \phi_q\}.$$

Proposition 5.3. *We have*

$$\mathrm{rk}\, \pi_V^{C_p}(S^0) = \begin{cases} 2 & \text{if } V \in \mathbb{Z}\{\phi_1 - \phi_2, \ldots, \phi_1 - \phi_q\}, \\ 1 & \text{if } V \in (\mathbb{Z}\{2 - \phi_1, \ldots, 2 - \phi_q\} \cup \mathbb{Z}\{\phi_1, \ldots, \phi_q\}) \\ & \qquad \setminus \mathbb{Z}\{\phi_1 - \phi_2, \ldots, \phi_1 - \phi_q\}, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 5.4. We note that $\phi_1, \ldots, \phi_q$ have similar behaviour. Thus we are considering $RO(C_p) = \mathbb{Z} \oplus N_{C_p}$ and the same picture as for C_2, but now the vertical line represents N_{C_p} and the antidiagonal N_e represents another subspace isomorphic to N_G.

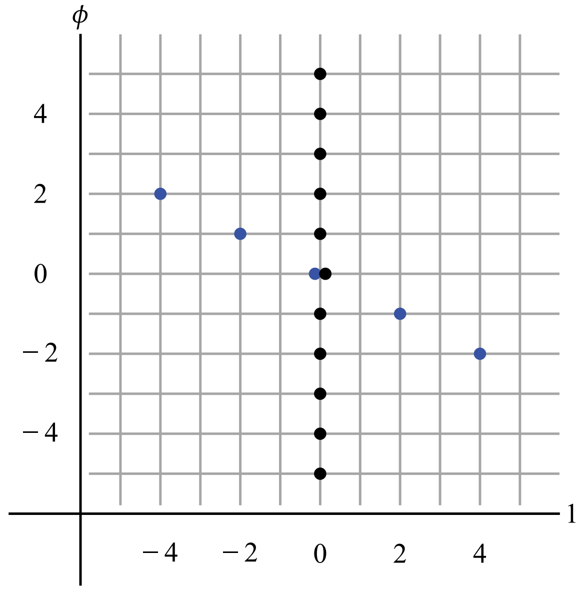

FIGURE 2. Degrees in $RO(C_3)$ where $\pi_V^{C_3}(S^0)$ has infinite rank. A black • indicates a copy of $\mathbb{Z}$ arising from $N_{C_3}^+$ and a blue • indicates a copy of $\mathbb{Z}$ arising from N_e^+.

5.3. **Cyclic groups of odd prime power order.** Let $q = \frac{p^n-1}{2}$. Then

$$RO(C_{p^n}) \cong \mathbb{Z}\{1, \phi_1, \dots, \phi_q\} \cong \mathbb{Z}^{q+1}.$$

For all $0 \leq m \leq n$, we have

$$N_{C_{p^m}}^+ = N_{C_{p^m}} \cong \mathbb{Z}\{2 - \phi_i : p^m \mid i\} \oplus \mathbb{Z}\{\phi_j : p^m \nmid j\}.$$

Indeed, let γ denote a generator of C_{p^n}, so $\gamma^{p^{n-m}}$ is a generator for C_{p^m}. Since $\phi_i : \gamma \mapsto \cdot e^{\frac{2\pi i}{p^n}}$,

$$\phi_i : \gamma^{p^{n-m}} \mapsto \cdot e^{\frac{2\pi i p^{n-m}}{p^n}} = \cdot e^{\frac{2\pi i}{p^m}}.$$

Therefore ϕ_i pulls back to a trivial C_{p^m}-representation if and only if $p^m \mid i$.

Describing the intersections of these subspaces gets complicated quickly. For example, if $0 \leq k < m \leq n$, then

$$N_{C_{p^k}}^+ \cap N_{C_{p^m}}^+ \cong \mathbb{Z}\{2-\phi_i : p^m \mid i\} \oplus \mathbb{Z}\{\phi_j : p^k \nmid j\} \oplus \mathbb{Z}\{\phi_{p^k} - \phi_\ell : \ell > p^k,\ p^k \mid \ell,\ p^m \nmid \ell\}.$$

Here, we use that $p^m \mid i$ implies $p^k \mid i$, and similarly, $p^k \nmid j$ implies $p^m \nmid j$.

5.4. **Klein four group.** Let $K = C_2 \times C_2 = \{e, i, j, k\}$. We have

$$RO(K) \cong \mathbb{Z}\{1, \sigma_i, \sigma_j, \sigma_k\},$$

where 1 is the 1-dimensional trivial representation, σ_i is the 1-dimensional representation on which e and i act trivially and j and k act by (-1), and similarly for

σ_j and σ_k. Then

$$\begin{aligned} N_e &\cong \mathbb{Z}\{1-\sigma_i, 1-\sigma_j, 1-\sigma_k\}, \\ N_{\langle i\rangle} &\cong \mathbb{Z}\{1-\sigma_i, \sigma_j, \sigma_k\}, \\ N_{\langle j\rangle} &\cong \mathbb{Z}\{\sigma_i, 1-\sigma_j, \sigma_k\}, \\ N_{\langle k\rangle} &\cong \mathbb{Z}\{\sigma_i, \sigma_j, 1-\sigma_k\}, \\ N_K &\cong \mathbb{Z}\{\sigma_i, \sigma_j, \sigma_k\}. \end{aligned}$$

The Weyl group $W_K(e) \cong K$ acts nontrivially on σ_i, σ_j, and σ_k, so we have

$$N_e^+ \cong \mathbb{Z}\{2(1-\sigma_i), \sigma_i-\sigma_j, \sigma_i-\sigma_k\}.$$

The Weyl group $W_K(\langle i\rangle) \cong K/\langle i\rangle \cong \langle j\rangle \cong \langle k\rangle$ acts nontrivially on σ_i but trivially on σ_j and σ_k, so we have

$$N_{\langle i\rangle}^+ \cong \mathbb{Z}\{2(1-\sigma_i), \sigma_j, \sigma_k\},$$

and similarly,

$$\begin{aligned} N_{\langle j\rangle}^+ &\cong \mathbb{Z}\{\sigma_i, 2(1-\sigma_j), \sigma_k\}, \\ N_{\langle k\rangle}^+ &\cong \mathbb{Z}\{\sigma_i, \sigma_j, 2(1-\sigma_k)\}. \end{aligned}$$

Finally, since $W_K(K) \cong e$ must act trivially on $\mathbb{Z}$, we have

$$N_K^+ \cong N_K \cong \mathbb{Z}\{\sigma_i, \sigma_j, \sigma_k\}.$$

To determine the ranks of $\pi_\alpha^G(S^0)$, we now compute intersections. In the following, we let $a \in \{i,j,k\}$, $a' \in \{i,j,k\} \setminus \{a\}$, and $a'' \in \{i,j,k\} \setminus \{a,a'\}$. Then we have

$$\begin{aligned} N_e^+ \cap N_{\langle a\rangle}^+ &\cong \mathbb{Z}\{2(1-\sigma_a), \sigma_{a'}-\sigma_{a''}\}, \\ N_e^+ \cap N_K^+ &\cong \mathbb{Z}\{\sigma_i-\sigma_j, \sigma_i-\sigma_k\}, \\ N_{\langle a\rangle}^+ \cap N_{\langle a'\rangle}^+ &\cong \mathbb{Z}\{2(1-\sigma_a-\sigma_{a'}), \sigma_{a''}\}, \\ N_{\langle a\rangle}^+ \cap N_K^+ &\cong \mathbb{Z}\{\sigma_{a'}, \sigma_{a''}\}, \\ N_e^+ \cap N_{\langle a\rangle}^+ \cap N_{\langle a'\rangle}^+ &\cong \mathbb{Z}\{2(1-\sigma_a-\sigma_{a'}+\sigma_{a''}\}, \\ N_{\langle a\rangle}^+ \cap N_{\langle a''\rangle}^+ \cap N_K^+ &\cong \mathbb{Z}\{\sigma_{a'}\}, \\ N_e^+ \cap N_{\langle a\rangle}^+ \cap N_K^+ &\cong \mathbb{Z}\{\sigma_{a'}-\sigma_{a''}\}, \\ N_{\langle a\rangle} \cap N_{\langle a'\rangle} \cap N_{\langle a''\rangle} &\cong \mathbb{Z}\{2(1-\sigma_a-\sigma_{a'}-\sigma_{a''})\}, \end{aligned}$$

and all 4- and 5-fold intersections are $\{0\}$.

Proposition 5.5. *With a, a', a'' as above, we have*

$$\operatorname{rk} \pi_V^K(S^0) = \begin{cases} 5 & \text{if } V=0, \\ 3 & \text{if } V \in (\mathbb{Z}\{\sigma_a-\sigma_{a'}\} \cup \mathbb{Z}\{\sigma_{a'}\} \cup \mathbb{Z}\{2(1-\sigma_a-\sigma_{a'} \pm \sigma_{a''})\}) \setminus \{0\}, \\ 2 & \text{if } V \in (\bigcup_{H\neq H'} N_H^+ \cap N_{H'}^+) \setminus (\bigcup_{H\neq H'\neq H''\neq H} N_H^+ \cap N_{H'}^+ \cap N_{H''}^+), \\ 1 & \text{if } V \in (\bigcup_H N_H^+) \setminus (\bigcup_{H\neq H'} N_H^+ \cap N_{H'}^+), \\ 0 & \text{otherwise.} \end{cases}$$

5.5. **Dihedral groups of order $2p$, p odd.** We have

$$RO(D_{2p}) \cong \mathbb{Z}\{1, \sigma, \phi_1, \dots, \phi_q\},$$

where 1 is the trivial representation, σ is the sign representation, and $\phi_t : D_{2p} \to \mathrm{Aut}(\mathbb{R}^2) \cong \mathrm{Aut}(\mathbb{C})$ sends the generator of $C_p \subseteq D_{2p}$ to $\cdot e^{2\pi i t/p}$ and the generator of $C_2 \subseteq D_{2p}$ to reflection across the real axis. Then

$$\begin{aligned} N_e &\cong \mathbb{Z}\{1-\sigma, 2-\phi_1, \dots, 2-\phi_q\}, \\ N_{C_2} &\cong \mathbb{Z}\{1-\sigma, 1-\phi_1, \dots, 1-\phi_q\}, \\ N_{C_p} &\cong \mathbb{Z}\{1-\sigma, \phi_1, \dots, \phi_q\}, \\ N_{D_{2p}} &\cong \mathbb{Z}\{\sigma, \phi_1, \dots, \phi_q\}. \end{aligned}$$

Since $W_{D_{2p}}(C_2) \cong e \cong W_{D_{2p}}(D_{2p})$, we have $N^+_{C_2} = N_{C_2}$ and $N^+_{D_{2p}} = N_{D_{2p}}$. On the other hand, $W_{D_{2p}}(e) \cong D_{2p}$ and $W_{D_{2p}}(C_p) \cong C_2$, so

$$\begin{aligned} N^+_e &\cong \mathbb{Z}\{2(1-\sigma), 1+\sigma-\phi_1, \phi_1-\phi_2, \dots, \phi_1-\phi_q\}, \\ N^+_{C_p} &\cong \mathbb{Z}\{2(1-\sigma), 2\phi_1, \phi_1-\phi_2, \dots, \phi_1-\phi_q\}. \end{aligned}$$

We now compute intersections:

$$\begin{aligned} N^+_e \cap N^+_{C_2} &\cong \mathbb{Z}\{2(1-\sigma), \phi_1-\phi_2, \dots, \phi_1-\phi_q\}, \\ N^+_e \cap N^+_{C_p} &\cong \mathbb{Z}\{2(1-\sigma), \phi_1-\phi_2, \dots, \phi_1-\phi_q\}, \\ N^+_e \cap N^+_{D_{2p}} &\cong \mathbb{Z}\{4\sigma-2\phi_1, \phi_1-\phi_2, \dots, \phi_1-\phi_q\}, \\ N^+_{C_2} \cap N^+_{C_p} &\cong \mathbb{Z}\{2(1-\sigma), \phi_1-\phi_2, \dots, \phi_1-\phi_q\}, \\ N^+_{C_2} \cap N^+_{D_{2p}} &\cong \mathbb{Z}\{\sigma-\phi_1, \phi_1-\phi_2, \dots, \phi_1-\phi_q\}, \\ N^+_{C_p} \cap N^+_{D_{2p}} &\cong \mathbb{Z}\{2\phi_1, \phi_1-\phi_2, \dots, \phi_1-\phi_q\}, \\ N^+_e \cap N^+_{C_2} \cap N^+_{C_p} &\cong \mathbb{Z}\{2(1-\sigma), \phi_1-\phi_2, \dots, \phi_1-\phi_q\}, \\ N^+_e \cap N^+_{C_2} \cap N^+_{D_{2p}} &\cong \mathbb{Z}\{\phi_1-\phi_2, \dots, \phi_1-\phi_q\}, \\ N^+_e \cap N^+_{C_p} \cap N^+_{D_{2p}} &\cong \mathbb{Z}\{\phi_1-\phi_2, \dots, \phi_1-\phi_q\}, \\ N^+_{C_2} \cap N^+_{C_p} \cap N^+_{D_{2p}} &\cong \mathbb{Z}\{\phi_1-\phi_2, \dots, \phi_1-\phi_q\}, \\ N^+_e \cap N^+_{C_2} \cap N^+_{C_p} \cap N^+_{D_{2p}} &\cong \mathbb{Z}\{\phi_1-\phi_2, \dots, \phi_1-\phi_q\}. \end{aligned}$$

Proposition 5.6. *We have*

$$\mathrm{rk}\, \pi_V^{D_{2p}}(S^0) = \begin{cases} 4 & \text{if } V \in \mathbb{Z}\{\phi_1-\phi_2, \dots, \phi_1-\phi_q\}, \\ 2 & \text{if } V \in (\bigcup_{H \neq H'} N^+_H \cap N^+_{H'}) \setminus (\bigcup_{H \neq H' \neq H'' \neq H} N^+_H \cap N^+_{H'} \cap N^+_{H''}), \\ 1 & \text{if } V \in (\bigcup_H N^+_H) \setminus (\bigcup_{H \neq H'} N^+_H \cap N^+_{H'}), \\ 0 & \text{otherwise.} \end{cases}$$

5.6. **Quaternion group.** Let $Q = Q_8$ denote the quaternion group of order 8. We have

$$RO(Q) = \mathbb{Z}\{1, \sigma_i, \sigma_j, \sigma_k, h\},$$

where $1, \sigma_i, \sigma_j, \sigma_k$ are the pullbacks of the K-representations of the same name along the quotient map $Q \to Q/C_2 \cong K$, and h is the unique irreducible 4-dimensional representation of Q. Then with a, a', and a'' as in our analysis of K,

$$\begin{aligned}
N_e &\cong \mathbb{Z}\{1-\sigma_i, 1-\sigma_j, 1-\sigma_k, 4-h\},\\
N_{C_2} &\cong \mathbb{Z}\{1-\sigma_i, 1-\sigma_j, 1-\sigma_k, h\},\\
N_{\langle a\rangle} &\cong \mathbb{Z}\{1-\sigma_a, \sigma_{a'}, \sigma_{a''}, h\},\\
N_Q &\cong \mathbb{Z}\{\sigma_i, \sigma_j, \sigma_k, h\},
\end{aligned}$$

and

$$\begin{aligned}
N_e^+ &\cong \mathbb{Z}\{2(1-\sigma_i), \sigma_i-\sigma_j, \sigma_i-\sigma_k, 4-h\},\\
N_{C_2}^+ &\cong \mathbb{Z}\{2(1-\sigma_i), \sigma_i-\sigma_j, \sigma_i-\sigma_k, h\},\\
N_{\langle a\rangle}^+ &\cong \mathbb{Z}\{2(1-\sigma_a), \sigma_{a'}, \sigma_{a''}, h\},\\
N_Q^+ &\cong \mathbb{Z}\{\sigma_i, \sigma_j, \sigma_k, h\}.
\end{aligned}$$

The 2-fold intersections are as follows:

$$\begin{aligned}
N_e^+ \cap N_{C_2}^+ &\cong \mathbb{Z}\{2(1-\sigma_i), \sigma_i-\sigma_j, \sigma_i-\sigma_k\},\\
N_e^+ \cap N_{\langle a\rangle}^+ &\cong \mathbb{Z}\{2(1-\sigma_a), \sigma_{a'}-\sigma_{a''}, h-4\sigma_{a'}\},\\
N_e^+ \cap N_Q^+ &\cong \mathbb{Z}\{\sigma_i-\sigma_j, \sigma_i-\sigma_k, 4\sigma_i-h\},\\
N_{C_2}^+ \cap N_{\langle a\rangle}^+ &\cong \mathbb{Z}\{2(1-\sigma_a), \sigma_{a'}-\sigma_{a''}, h\},\\
N_{C_2}^+ \cap N_Q^+ &\cong \mathbb{Z}\{\sigma_i-\sigma_j, \sigma_i-\sigma_k, h\},\\
N_{\langle a\rangle}^+ \cap N_{\langle a'\rangle}^+ &\cong \mathbb{Z}\{2(1-\sigma_a-\sigma_{a'}), \sigma_{a''}, h\},\\
N_{\langle a\rangle}^+ \cap N_Q^+ &\cong \mathbb{Z}\{\sigma_{a'}, \sigma_{a''}, h\}.
\end{aligned}$$

The 3-fold intersections are as follows:

$$\begin{aligned}
N_e^+ \cap N_{C_2}^+ \cap N_{\langle a\rangle}^+ &\cong \mathbb{Z}\{2(1-\sigma_a), \sigma_{a'}-\sigma_{a''}\},\\
N_e^+ \cap N_{C_2}^+ \cap N_Q^+ &\cong \mathbb{Z}\{\sigma_i-\sigma_j, \sigma_i-\sigma_k\},\\
N_e^+ \cap N_{\langle a\rangle}^+ \cap N_Q^+ &\cong \mathbb{Z}\{\sigma_{a'}-\sigma_{a''}, h-4\sigma_{a'}\},\\
N_e^+ \cap N_{\langle a\rangle}^+ \cap N_{\langle a'\rangle}^+ &\cong \mathbb{Z}\{2(1-\sigma_a-\sigma_{a'}+\sigma_{a''}), h-4\sigma_{a''}\},\\
N_{C_2}^+ \cap N_{\langle a\rangle}^+ \cap N_Q^+ &\cong \mathbb{Z}\{\sigma_{a'}-\sigma_{a''}, h\},\\
N_{C_2}^+ \cap N_{\langle a\rangle}^+ \cap N_{\langle a'\rangle}^+ &\cong \mathbb{Z}\{2(1-\sigma_a-\sigma_{a'}+\sigma_{a''}), h\},\\
N_{\langle a\rangle}^+ \cap N_{\langle a'\rangle}^+ \cap N_Q^+ &\cong \mathbb{Z}\{\sigma_{a''}, h\},\\
N_{\langle a\rangle}^+ \cap N_{\langle a'\rangle}^+ \cap N_{\langle a''\rangle}^+ &\cong \mathbb{Z}\{2(1-\sigma_a-\sigma_{a'}-\sigma_{a''}), h\}.
\end{aligned}$$

The 4-fold intersections are as follows:

$$\begin{aligned}
N_e^+ \cap N_{C_2}^+ \cap N_{\langle a\rangle}^+ \cap N_Q^+ &\cong \mathbb{Z}\{\sigma_{a'}-\sigma_{a''}\},\\
N_e^+ \cap N_{C_2}^+ \cap N_{\langle a\rangle}^+ \cap N_{\langle a'\rangle}^+ &\cong \mathbb{Z}\{2(1-\sigma_a-\sigma_{a'}+\sigma_{a''})\},\\
N_e^+ \cap N_{\langle a\rangle}^+ \cap N_{\langle a'\rangle}^+ \cap N_{\langle a''\rangle}^+ &\cong \mathbb{Z}\{2(1-\sigma_a-\sigma_{a'}-\sigma_{a''})+h\},\\
N_{C_2}^+ \cap N_{\langle a\rangle}^+ \cap N_{\langle a'\rangle}^+ \cap N_Q^+ &\cong \mathbb{Z}\{h\},\\
N_{C_2}^+ \cap N_{\langle a\rangle}^+ \cap N_{\langle a'\rangle}^+ \cap N_{\langle a''\rangle}^+ &\cong \mathbb{Z}\{h\},\\
N_{\langle a\rangle}^+ \cap N_{\langle a'\rangle}^+ \cap N_{\langle a''\rangle}^+ \cap N_Q^+ &\cong \mathbb{Z}\{h\}.
\end{aligned}$$

The 5-fold intersections are as follows:

$$N_e^+ \cap N_{C_2}^+ \cap N_{\langle a\rangle}^+ \cap N_{\langle a'\rangle}^+ \cap N_Q^+ \cong \{0\},$$
$$N_e^+ \cap N_{C_2}^+ \cap N_{\langle a\rangle}^+ \cap N_{\langle a'\rangle}^+ \cap N_{\langle a''\rangle}^+ \cong \{0\},$$
$$N_e^+ \cap N_{\langle a\rangle}^+ \cap N_{\langle a'\rangle}^+ \cap N_{\langle a''\rangle}^+ \cap N_Q^+ \cong \{0\},$$
$$N_{C_2}^+ \cap N_{\langle a\rangle}^+ \cap N_{\langle a'\rangle}^+ \cap N_{\langle a''\rangle}^+ \cap N_Q^+ \cong \mathbb{Z}\{h\}.$$

For completeness, the unique 6-fold intersection is

$$N_e^+ \cap N_{C_2}^+ \cap N_{\langle a\rangle}^+ \cap N_{\langle a'\rangle}^+ \cap N_{\langle a''\rangle}^+ \cap N_Q^+ \cong \{0\}.$$

Proposition 5.7. *With a, a', a'' as above, we have*

$$\operatorname{rk} \pi_V^K(S^0) = \begin{cases} 6 & \text{if } V = 0, \\ 5 & \text{if } V \in \mathbb{Z}\{h\} \setminus \{0\}, \\ 4 & \text{if } V \in (\mathbb{Z}\{\sigma_{a'} - \sigma_{a''}\} \cup \mathbb{Z}\{2(1 - \sigma_a - \sigma_{a'} + \sigma_{a''})\} \\ & \qquad \cup \mathbb{Z}\{2(1 - \sigma_a - \sigma_{a'} - \sigma_{a''}) + h\}) \setminus \mathbb{Z}\{h\}, \\ 3 & \text{if } V \in (\bigcup_{H,H',H''} N_H^+ \cap N_{H'}^+ \cap N_{H''}^+) \\ & \qquad \setminus (\bigcup_{H,H',H'',H'''} \bigcap_{L \in \{H,H',H'',H'''\}} N_L^+), \\ 2 & \text{if } V \in (\bigcup_{H \neq H'} N_H^+ \cap N_{H'}^+) \setminus (\bigcup_{H,H',H''} N_H^+ \cap N_{H'}^+ \cap N_{H''}^+), \\ 1 & \text{if } V \in (\bigcup_H N_H^+) \setminus (\bigcup_{H \neq H'} N_H^+ \cap N_{H'}^+), \\ 0 & \text{otherwise.} \end{cases}$$

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