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EVOLUTION OF SUPEROSCILLATIONS FOR SPINNING PARTICLES

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ABSTRACT. Superoscillating functions are band-limited functions that can oscillate faster than their fastest Fourier component. These functions appear in various fields of science and technology, in particular they were discovered in quantum mechanics in the context of weak values introduced by Y. Aharonov and collaborators. The evolution problem of superoscillatory functions as initial conditions for the Schrödinger equation is intensively studied nowadays and the supershift property of the solution of Schrödinger equation encodes the persistence of superoscillatory phenomenon during the evolution. In this paper, we prove that the evolution of a superoscillatory initial datum for spinning particles in a magnetic field has the supershift property. Our techniques are based on the exact propagator of spinning particles, the associated infinite order differential operators and their continuity on suitable spaces of entire functions with growth conditions.

1. Introduction

In the last decade there has been increasing interest in the theory of superoscillatory functions both from mathematical and physical points of view. In quantum mechanics these functions originated in context of weak values, see [1], while in antenna theory they first appeared in [36]; for developments in optics and other applications see the recent overview paper [17] titled *Roadmap on Superoscillations*, where some of the leading experts illustrated the various features of superoscillations and applications. From the mathematical point of view, an introduction to superoscillatory functions in one variable and some investigations of the Schrödinger evolution of superoscillatory initial data can be found in [7].

The literature on superoscillations is quite large, and without claiming completeness some of the most relevant and recent results are contained in the papers [2]-[7], [12], [15], [25], [31] and [32] where the issue of persistence of superoscillatory behavior when evolved under the Schrödinger equation is considered. The papers [18]-[20], [26]-[29] and [35] are mostly concerned with the physical nature of superoscillations, while papers [10], [11], [13]-[14], [21]-[24] develop in depth the mathematical theory of superoscillations. Recently, [8] introduced a new method to generate superoscillating functions.

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The results in this paper are addressed to a general audience of mathematicians, but also physicists and engineers, and our main tools are the theory of infinite order differential operators acting on spaces of holomorphic functions and the knowledge of the Green's functions.

The prototypical superoscillating function, which appears in the theory of weak values, is

$$F_n(x,a)=\sum_{j=0}^{n}C_j(n,a)e^{i(1-\frac{2j}{n})x},\quad x\in\mathbb{R}, \tag{1.1}$$

where $a>1$ and the coefficients $C_j(n,a)$ are given by

$$C_j(n,a)=\binom{n}{j}\left(\frac{1+a}{2}\right)^{n-j}\left(\frac{1-a}{2}\right)^{j}. \tag{1.2}$$

If we fix $x\in\mathbb{R}$ and we let n go to infinity, we obtain that

$$\lim_{n\to\infty}F_n(x,a)=e^{iax},$$

and the limit is uniform on compact subsets of the real line. The term superoscillations comes from the fact that in the Fourier representation of the function (1.1) the frequencies $1-\frac{2j}{n}$ are bounded by 1, but the limit function e^{iax} has a frequency a that can be arbitrarily larger than 1.

Inspired by this example we define a *generalized Fourier sequence*. These are sequences of the form

$$f_n(x):=\sum_{j=0}^{n}Z_j(n,a)e^{ih_j(n)x},\quad n\in\mathbb{N},\quad x\in\mathbb{R}, \tag{1.3}$$

where $a\in\mathbb{R}$, $Z_j(n,a)$ and $h_j(n)$ are complex and real valued functions of the variables n,a and n, respectively. The sequence (1.3) is said to be *a superoscillating sequence* if $\sup_{j,n}|h_j(n)|\leq 1$ and there exists a compact subset of $\mathbb{R}$, which is called *a superoscillation set*, on which $f_n(x)$ converges uniformly to $e^{ig(a)x}$, where g is a continuous real valued function such that $|g(a)|>1$. Superoscillatory functions in several variables have been rigorously defined and studied in [6] and in [9] where the theory of supershifts in more than one variable was initiated.

The crucial concept that encodes the persistence of the superoscillatory behaviour of the solution of the Schrödinger equation with superoscillatory initial condition is called the supershift of the solution and is stated in Definition 2.4. This concept is motivated by the fact that the only known case where the Cauchy problem for the Schrödinger equation with initial datum given by (1.1) has a solution that is a superoscillatory function in two variables is the case of the free particle, i.e.,

$$i\frac{\partial\psi(t,x)}{\partial t}=-\frac{\partial^2\psi(t,x)}{\partial x^2}\psi(t,x),\qquad \psi(0,x)=\sum_{j=0}^{n}C_j(n,a)e^{ix(1-\frac{2j}{n})}.$$

In this case the solution is

$$\psi_n(t,x)=\sum_{j=0}^{n}C_j(n,a)e^{i(1-\frac{2j}{n})x}e^{-it(1-\frac{2j}{n})^2}, \tag{1.4}$$

and for all $t \in [-T, T]$, where T is any real positive number, we have

$$\lim_{n\to\infty} \psi_n(t,x) = e^{iax-ia^2t},$$

when x belongs to the compact sets of $\mathbb{R}$. Also in the case of nonconstant potentials we do not obtain a superoscillatory function in two or in more variables. This fact can be seen already with the well known quantum harmonic oscillator where the Cauchy problem

$$i\frac{\partial\psi(t,x)}{\partial t} = \frac{1}{2}\Big(- \frac{\partial^2}{\partial x^2} + x^2\Big)\psi(t,x), \quad \psi(0,x) = \sum_{j=0}^{n} C_j(n,a)e^{ix(1-\frac{2j}{n})}$$

has the solution

$$\begin{aligned}\psi_n(t,x) &= (\cos t)^{-1/2}\exp\Big(- \frac{i}{2}x^2\tan t\Big)\\ &\times \sum_{j=0}^{n} C_j(n,a)\exp\left(\frac{ix(1-\frac{2j}{n}))}{\cos t} - \frac{i}{2}\left(1-\frac{2j}{n}\right)^2 \tan t\right),\end{aligned} \tag{1.5}$$

which clearly is not of the form (1.4). When we take the limit for $n \to \infty$ we get

$$\lim_{n\to\infty} \psi_n(t,x) = (\cos t)^{-1/2}\exp\left(-\frac{i}{2}(x^2+a^2)\tan t + i\frac{ax}{\cos t}\right).$$

What is preserved in the two cases is the fact that in the limit process the instances $1-\frac{2j}{n} \in [-1,1]$ in the expressions (1.4) and (1.5) tend to $a > 1$, for more details see [7]. This leads to the notion of supershift in Definition 2.4 which includes as a particular case the notion of superoscillation.

The main result of this paper demonstrates the supershift property for the solution of the Schrödinger equation for spinning particles of any spin, subjected to a magnetic field. To prove our main results we take advantage of the exact form of the propagator, see [28] and also (2.2) in this paper. The precise formulation of the problem is as follows although more details are explained in Section 2.

Problem 1.1. Determine if the solution of the Schrödinger equation with the propagator of the spinning particle given by

$$\begin{aligned}K(\theta'',\phi'',\theta',\phi';T) = \Big(&\cos\left(\frac{\theta''}{2}\right)\,\cos\left(\frac{\theta'}{2}\right)\, e^{i(\phi''-BT-\phi')/2}\\ &+\sin\left(\frac{\theta''}{2}\right)\,\sin\left(\frac{\theta'}{2}\right)\, e^{-i(\phi''-BT-\phi')/2}\Big)^{2s}\end{aligned}$$

and superoscillatory initial datum in two variables

$$\psi_n(\theta,\phi) = \sum_{j=0}^{n} C_j(n,a)e^{i\theta(1-\frac{2j}{n})^p}e^{i\phi(1-\frac{2j}{n})^q} \tag{1.6}$$

has the supershift property, where $a > 1$, $C_j(n,a)$ are given by (1.2), $\theta \in [0,\pi]$, $\phi \in [0,2\pi]$ and p, $q \in \mathbb{N}$.

The strategy to solve Problem 1.1 and prove the supershift property is split into Steps (I)–(II):

Step (I). We explicitly determine the solution $\Phi_a(\theta,\phi,t)$ of the Schrödinger equation using the propagator of the spinning particle given by (2.2) with the initial condition $\psi_0(\theta,\phi) = e^{ia^p\theta+ia^q\phi}$, for $\theta \in [0,\pi]$, $\phi \in [0,2\pi]$, p, $q \in \mathbb{N}$ and $a > 1$.

Step (II)*.* Given the explicit solution $\Phi_a(\theta,\phi,t)$ obtained in the previous point, we identify suitable infinite order differential operators associated with the solution.

Step (III)*.* Thanks to the continuity property of our infinite order differential operators on the space of entire functions with exponential bound A_1 we then prove the supershift property.

As we shall see in Section 4, for spinning particles of spin $s=\frac{m}{2}$, $m\in\mathbb{N}$, the infinite order differential operators $\mathcal{A}_{m,k}(\mathcal{D}_\xi)$ and $\mathcal{B}_{m,k}(\mathcal{D}_\xi)$ defined in (4.3) and (4.4), respectively, act continuously from A_1 to A_1 for every m, q and $p\in\mathbb{N}$, see Proposition 4.5.

Thus, according to Step (II), we can deduce the supershift property of the solution. Precisely, in Theorem 4.7 we will prove that for $a>1$ the solution of the Schrödinger equation, with the propagator of the spinning particle given by (2.2) and superoscillatory initial datum (1.6), can be written as

$$\Psi_n(\theta,\phi,t)=\sum_{j=0}^{n}C_j(n,a)\Phi_{1-\frac{2j}{n}}(\theta,\phi,t),$$

where

$$\Phi_a(\theta,\phi,t)=\sum_{k=0}^{m}\binom{m}{k}e^{i\left(-\frac{m}{2}+k\right)Bt}\cos^{m-k}\left(\frac{\theta}{2}\right)\ \sin^{k}\left(\frac{\theta}{2}\right)\ e^{i\left(\frac{m}{2}-k\right)\phi}A_{m,k}(a)B_{m,k}(a)$$

for explicit functions $A_{m,k}(a)$, $B_{m,k}(a)$, given in (3.3) and (3.4) depending upon the p and q from the initial datum, and $C_j(n,a)$ are given by (1.2). Moreover, $\Phi_a(\theta,\phi,t)$ has the supershift property, that is,

$$\lim_{n\to\infty}\sum_{j=0}^{n}C_j(n,a)\Phi_{1-\frac{2j}{n}}(\theta,\phi,t)=\Phi_a(\theta,\phi,t),$$

for all $\theta\in[0,\pi]$, $\phi\in[0,2\pi]$, $t\geq 0$. The results above hold for spin $s=\frac{m}{2}$, for every $m\in\mathbb{N}$.

The plan of the paper is as follows: after this introduction, in Section 2 we first review some basic notions on superoscillating sequences in two variables and the property of the supershift. Then we recall the propagator for spinning particles and the procedure to determine it. Section 3 is devoted to determining the explicit solution of a Cauchy problem for the Schrödinger equation with the propagator for the spinning particle which is crucial to perform the evolution of a superoscillatory initial datum done in Section 4.

2. Superoscillations, supershifts and the propagator for spinning particles

We first recall some preliminary definitions related to superoscillatory functions in two variables even if they can be extended to the general case of $d\geq 2$ variables. We restrict to two dimensions since our specific evolution problem for spinning particles is two dimensional, for more details see [6, 9].

Definition 2.1 (Generalized Fourier sequence in two variables)**.** Let $(x_1,x_2)\in\mathbb{R}^2$. For $\ell=1,2$, let $(h_{j,\ell}(n))$, $j=0,\dots,n$ for $n\in\mathbb{N}_0$, be real-valued sequences. A sequence of the form

$$F_n(x_1, x_2) = \sum_{j=0}^{n} c_j(n) e^{ix_1 h_{j,1}(n)} e^{ix_2 h_{j,2}(n)},$$

where $(c_j(n))_{j,n}$, $j = 0, \ldots, n$, for $n \in \mathbb{N}_0$ is a complex-valued sequence will be called a *generalized Fourier sequence in two variables.*

Definition 2.2 (Superoscillating sequence)**.** A generalized Fourier sequence in two variables $F_n(x_1, x_2)$ is said to be *a superoscillating sequence* if

$$\sup_{j=0,\ldots,n,\ n\in\mathbb{N}_0} |h_{j,\ell}(n)| \leq 1, \quad \text{for } \ell = 1, 2,$$

and there exists a compact subset of $\mathbb{R}^2$, which will be called *a superoscillation set*, on which $F_n(x_1, x_2)$ converges uniformly to $e^{ix_1 g_1} e^{ix_2 g_2}$, where $|g_\ell| > 1$ for $\ell = 1, 2$.

An important example of a generalized Fourier sequence in two variables that will be used in the sequel is the following:

Theorem 2.3 (The case of two variables)**.** *For p and $q \in \mathbb{N}$ we define*

$$F_n(x, y) = \sum_{j=0}^{n} C_j(n, a) e^{ix_1 (h_j(n))^p} e^{ix_2 (h_j(n))^q},$$

where $C_j(n, a)$ are given by (1.2) *and $h_j(n) = 1 - \frac{2j}{n}$. Then, we have*

$$\lim_{n\to\infty} F_n(x, y) = e^{ix_1 a^p} e^{ix_2 a^q},$$

and, in particular, $F_n(x_1, x_2)$ is superoscillating when $|a| > 1$.

In the case of nonconstant potentials we have to replace the notion of superoscillations with the notion of supershift.

Definition 2.4 (Supershift property)**.** Let $\lambda \mapsto \varphi_\lambda(X)$ be a continuous complex-valued function in the variable $\lambda \in \mathcal{I}$, where $\mathcal{I} \subseteq \mathbb{R}$ is an interval, and $X \in \Omega$, where Ω is a domain. We consider $X \in \Omega$ as a parameter for the function $\lambda \mapsto \varphi_\lambda(X)$ where $\lambda \in \mathcal{I}$. When $[-1, 1]$ is contained into $\mathcal{I}$ and $a \in \mathcal{I}$, we define the sequence

$$\psi_n(X) = \sum_{j=0}^{n} C_j(n, a) \varphi_{1-\frac{2j}{n}}(X),$$

in which φ_λ is computed just at the points $1 - \frac{2j}{n}$ which belong to the interval $[-1, 1]$ and $C_j(n, a)$ are suitable coefficients, for $j = 0, \ldots, n$ and $n \in \mathbb{N}$. If

$$\lim_{n\to\infty} \psi_n(X) = \varphi_a(X)$$

for $|a| > 1$ arbitrary large (but belonging to $\mathcal{I}$), we say that the function $\lambda \mapsto \varphi_\lambda(X)$, for X fixed, admits the supershift property.

Remark 2.5. If we set $\varphi_\lambda(x) = e^{i\lambda x}$ and $X = x \in \mathbb{R}$, we obtain the superoscillating sequence described above as a particular case of the supershift. In fact, in this case, we have $\psi_n(x) = F_n(x, a)$, where $F_n(x, a)$ is defined in (1.1). The name supershift is due to the fact that we are able to obtain φ_a, for $|a| > 1$ arbitrarily large, by simply calculating the function $\lambda \mapsto \varphi_\lambda$ at infinitely many points in the neighborhood $[-1, 1]$ of the origin.

The propagator for the spinning particle. The first attempt to determine the propagator for the spinning particles was done by L. S. Schulman, see [33] and [34], but the precise propagator was found by E. Ercolessi and co-authors, in [28]. We briefly describe the ingredients to obtain the propagator because of the particular structure of the Hamiltonian for spinning particles. In fact, to obtain the exact expression of the propagator requires a regularization term in the path integral. In this part of the paper, we will resort to some physics notation to match the presentation from [28]. We will translate this notation into a more mathematical one when it is used later to be more consistent with the rest of the paper.

The Hamiltonian for the quantum mechanics of a spinning particle is described by

$$\hat{H} = \hat{B} \cdot \hat{S},$$

where $\hat{S} = [\hat{S}_1, \hat{S}_2, \hat{S}_3]$ and the spin operators $\hat{S}_j$, for $j = 1, 2, 3$, satisfy the usual commutation relations (normalizing $\hbar = 1$):

$$[\hat{S}_\ell, \hat{S}_j] = i\epsilon_{\ell jk}\hat{S}_k,$$

where $\epsilon_{\ell jk}$ is Levi-Civita symbol. In order to give the precise interpretation of the Green's function we summarize a few facts and we refer the reader to the paper [28] for more details.

We consider the group $SU(2)$ and we recall that the coherent states for a spin s ($2s$ being an integer) can be constructed as follows

$$u(\theta, \phi) = e^{-i\phi\hat{S}_3} e^{-i\theta\hat{S}_2} u_0,$$

where u_0 denotes the highest-weight state of the spin-s representation of $SU(2)$ and moreover we have that s is an eigenvalue of the operator $\hat{S}_3$ associated with the eigenfunction u_0:

$$\hat{S}_3 u_0 = s u_0.$$

We assume that θ and ϕ are the two angular coordinates parametrizing $\mathbb{S}^2$ where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. In [30], for example, one can find the relation

$$\langle u(\theta', \phi'), u(\theta, \phi)\rangle = \Big(\cos\Big(\frac{\theta'}{2}\Big) \cos\Big(\frac{\theta}{2}\Big)\, e^{i(\phi'-\phi)/2} + \sin\Big(\frac{\theta'}{2}\Big)\, \sin\Big(\frac{\theta}{2}\Big)\, e^{-i(\phi'-\phi)/2} \Big)^{2s} \tag{2.1}$$

which plays an important role in what follows. The quantum-mechanical propagator

$$K(x'', t''; x', t') = \langle u(x''), e^{-\frac{i}{\hbar}(t''-t')H} u(x')\rangle$$

may be computed using a path integral:

$$K(x'', t''; x', t') = \int_{x(t')=x'}^{x(t'')=x''} \exp\Big(\frac{i}{\hbar} \int_{t'}^{t''} \mathcal{L}(x(t), \dot{x}(t)) dt\Big) \mathcal{D}x(t),$$

where $\mathcal{L}(x(t), \dot{x}(t))$ is the Lagrangian of the system and the boundary conditions of the path integral are given by $x(t') = x'$, $x(t'') = x''$. The paths that are summed over move only forward in time and are integrated with respect to the differential $\mathcal{D}x(t)$. The kernel $K(x'', t''; x', t')$ solves the Cauchy problem for the Schrödinger equation with initial datum $\psi_0(x', t')$:

$$\psi(x'', t'') = \int_{-\infty}^{\infty} \psi_0(x', t') K(x'', t''; x', t')\, dx', \quad t'' > t'.$$

We note that formula (2.1) is of crucial importance to deduce the propagator that is obtained by a regularization procedure in the Feynman integral. Precisely, based on solid physical motivations, see [28], since the Lagrangian (or the Hamiltonian) for a quantum spinning particle does not have an explicit kinetic term that acts as a regulator, concentrating the functional measure on the continuous path, the term $\frac{1}{4}\delta((\dot\theta)^2+\sin^2\theta(\dot\phi^2))$ has been added for the regularization, where $\delta>0$ is a parameter. Then the limit, as $\delta\to 0$, has to be taken after evaluating the path integral with the saddle point approximation. For the case of a spin in a magnetic field the propagator, denoted in this case by K_{2s}, becomes:

(2.2)

$$\begin{aligned} &K_{2s}(\theta'',\phi'',\theta',\phi';T)\\ &=\lim_{\delta\to 0}\int_{\Omega(t')=\Omega'}^{\Omega(t'')=\Omega''}\exp\Big(is\int_0^T\cos\theta\dot\phi+\frac{1}{4}\delta((\dot\theta)^2+\sin^2\theta(\dot\phi^2))-B\cos\theta\Big)\mathcal{D}\Omega(t)dt\\ &=\Big(\cos\Big(\frac{\theta''}{2}\Big)\ \cos\Big(\frac{\theta'}{2}\Big)\ e^{i(\phi''-BT-\phi')/2}+\sin\Big(\frac{\theta''}{2}\Big)\ \sin\Big(\frac{\theta'}{2}\Big)\ e^{-i(\phi''-BT-\phi')/2}\Big)^{2s}. \end{aligned}$$

Remark 2.6. Observe that the propagator $K_{2s}(\theta'',\phi'',\theta',\phi';T)$ given in (2.2) can be rewritten using the relation (2.1) as

$$K_{2s}(\theta'',\phi'',\theta',\phi';T)=\langle u(\theta'',\phi''-\mu BT),u(\theta',\phi')\rangle,$$

where $T=t''-t'$, μ is the Bohr magneton that is assumed equal to 1, and B is the constant magnetic field in the direction of the z-axis.

Remark 2.7. The nontrivial procedure illustrated above that gives the correct propagator for the spinning particle has the following interpretation: the action in (2.2) becomes essentially that of a particle of charge s and mass $1/(2s\delta)$ moving on the two-sphere $\mathbb{S}^2$, coupled both with a magnetic monopole of unit strength located at the centre of the sphere and with a constant electric-type field directed along the z-axis. In this context, Dirac's quantization condition, see [27], becomes identical with the spin quantization condition, i.e., $2s$ equals an integer. The interested reader who wants to know more about the nontrivial physical aspects of this propagator can look at [28] and the references therein.

3. Evolution of oscillating functions

This section is preliminary to study the evolution of the superoscillatory initial datum (1.6) done in Section 4. In order to simplify the notation for the computations in the propagator $K_{2s}(\theta'',\phi'',\theta',\phi';T)$ we change notation as follows: the final values are denoted by

$$t''=t,\quad \theta''=\theta_F=\theta\quad\text{and}\quad \phi''=\phi_F=\phi$$

and for the initial values we set

$$t'=0,\quad \theta'=\theta_I=u\quad\text{and}\ \phi'=\phi_I=v.$$

With the above notation the solution of the Cauchy problem for initial datum $\psi_0(\theta,\phi)$ is given by

$$\psi(\theta,\phi,t)=\int_{u=0}^{\pi}\int_{v=0}^{2\pi}\psi_0(u,v)\ K_{2s}(\theta,\phi,u,v,t)dvdu. \tag{3.1}$$

Observe that for the Cauchy problem that we study, the integral exists in the Lebesgue sense.

We now study the evolution of the oscillating initfial condition $\psi_0(\theta,\phi)=e^{ia^p\theta+ia^q\phi}$ using the propagator of the spinning particle given by (2.2).

Theorem 3.1. *Let $2s := m \in \mathbb{N}$ and assume $p,\, q \in \mathbb{N}$. The Cauchy problem for the Schrödinger equation with the propagator of the spinning particle given by* (2.2) *and with initial condition*

$$\psi_0(\theta,\phi) = e^{ia^p\theta + ia^q\phi}$$

has the explicit solution:

(3.2)

$$\Phi_a(\theta,\phi,t) = \sum_{k=0}^{m} \binom{m}{k} e^{i\left(-\frac{m}{2}+k\right)Bt} \cos^{m-k}\left(\frac{\theta}{2}\right)\ \sin^{k}\left(\frac{\theta}{2}\right)\ e^{i\left(\frac{m}{2}-k\right)\phi} A_{m,k}(a) B_{m,k}(a)$$

for all $\theta \in [0,\pi]$, $\phi \in [0,2\pi]$, $t \geq 0$, where

$$A_{m,k}(a) := \frac{\pi}{2^m i^k} \sum_{r=0}^{m} c_{r,m,k} \sum_{\alpha=0}^{\infty} \frac{\left(i\pi\left(\frac{m}{2} - r + a^p\right)\right)^{\alpha}}{(\alpha+1)!} \tag{3.3}$$

and

$$B_{m,k}(a) := 2\pi \sum_{\beta=0}^{\infty} \frac{(2\pi i)^{\imath}}{(\beta+1)!} \left(-\frac{m}{2} + k + a^q\right)^{\beta} \tag{3.4}$$

with

$$c_{r,m,k} := \sum_{l=0}^{r} (-1)^l \binom{k}{l} \binom{m-k}{r-l}. \tag{3.5}$$

Moreover, the solution $\Phi_a = \Phi_a(\theta,\phi,t)$ is an entire function with respect to the variable $a \in \mathbb{C}$, for all $\theta \in [0,\pi]$, $\phi \in [0,2\pi]$, $t \geq 0$.

Proof. Keeping in mind the above positions for the variables and setting $2s = m$, for the sake of simplicity, we write the propagator as

$$K_m(\theta,\phi,u,v,t) = \left(\cos\left(\frac{\theta}{2}\right)\ \cos\left(\frac{u}{2}\right)\ e^{i\frac{(\phi-Bt-v)}{2}} + \sin\left(\frac{\theta}{2}\right)\ \sin\left(\frac{u}{2}\right)\ e^{-i\frac{(\phi-Bt-v)}{2}}\right)^m.$$

Now expand the kernel using the Binomial Theorem to find

$$\begin{aligned} &K_m(\theta,\phi,u,v,t)\\ &\quad= \sum_{k=0}^{m} \binom{m}{k} \left(\cos\left(\frac{\theta}{2}\right)\ \cos\left(\frac{u}{2}\right)\ e^{i(\phi-Bt-v)/2}\right)^{m-k}\\ &\qquad\times \left(\sin\left(\frac{\theta}{2}\right)\ \sin\left(\frac{u}{2}\right)\ e^{-i(\phi-Bt-v)/2}\right)^{k}\\ &\quad= \sum_{k=0}^{m} \binom{m}{k} \left(\cos^{m-k}\left(\frac{\theta}{2}\right)\ \cos^{m-k}\left(\frac{u}{2}\right)\ e^{i(m-k)\frac{(\phi-Bt-v)}{2}}\right)\\ &\qquad\times \left(\sin^{k}\left(\frac{\theta}{2}\right)\ \sin^{k}\left(\frac{u}{2}\right)\ e^{-ik\frac{(\phi-Bt-v)}{2}}\right)\\ &\quad= \sum_{k=0}^{m} \binom{m}{k} \cos^{m-k}\left(\frac{u}{2}\right)\ \sin^{k}\left(\frac{u}{2}\right)\ e^{i(m-k)\frac{(-Bt-v)}{2}} e^{ik\frac{(Bt+v)}{2}}\\ &\qquad\times \cos^{m-k}\left(\frac{\theta}{2}\right)\ \sin^{k}\left(\frac{\theta}{2}\right)\ e^{i\frac{(m-k)\phi}{2}}\ e^{-i\frac{k\phi}{2}}. \end{aligned}$$

Observe that

$$\frac{(m-k)(-Bt-v)}{2} + \frac{k(Bt+v)}{2} = \left(-\frac{m}{2}+k\right)Bt + \left(-\frac{m}{2}+k\right)v$$

and

$$\frac{(m-k)\phi}{2}-\frac{k\phi}{2}=\frac{m}{2}-k$$

and so we can simplify the formula for the kernel $K_m(\theta,\phi,u,v,t)$ to be:

$$K_m(\theta,\phi,u,v,t):=\sum_{k=0}^{m}\binom{m}{k}\cos^{m-k}\left(\frac{u}{2}\right)\ \sin^k\left(\frac{u}{2}\right)$$
$$\times\ e^{i\left(-\frac{m}{2}+k\right)Bt}e^{i\left(-\frac{m}{2}+k\right)v}\cos^{m-k}\left(\frac{\theta}{2}\right)\ \sin^k\left(\frac{\theta}{2}\right)\ e^{i\left(\frac{m}{2}-k\right)\phi}.$$

Hence, using (3.1), the solution to the Schrödinger equation we are after is given by

$$\begin{aligned}\Phi_a(\theta,\phi,t)&=\int_0^{\pi}\int_0^{2\pi}e^{ia^pu}e^{ia^qv}K_m(\theta,\phi,u,v,t)\,dudv\\&=\sum_{k=0}^{m}\binom{m}{k}\cos^{m-k}\left(\frac{\theta}{2}\right)\ \sin^k\left(\frac{\theta}{2}\right)e^{i\left(-\frac{m}{2}+k\right)Bt}e^{i\left(\frac{m}{2}-k\right)\phi}A_{m,k}(a)B_{m,k}(a),\end{aligned}$$

where

$$A_{m,k}(a):=\int_0^{\pi}e^{ia^pu}\cos^{m-k}\left(\frac{u}{2}\right)\sin^k\left(\frac{u}{2}\right)du,\quad B_{m,k}(a):=\int_0^{2\pi}e^{i\left(-\frac{m}{2}+k+a^q\right)v}\ dv.$$

This second term can be computed directly:

$$\begin{aligned}B_{m,k}(a)&:=\int_0^{2\pi}e^{i\left(-\frac{m}{2}+k+a^q\right)v}\ dv=\frac{e^{i2\pi\left(-\frac{m}{2}+k+a^q\right)}-1}{i\left(-\frac{m}{2}+k+a^q\right)}\\&=2\pi\sum_{\beta=0}^{\infty}\frac{(2\pi i)^\beta}{(\beta+1)!}\left(-\frac{m}{2}+k+a^q\right)^\beta.\end{aligned}$$

The computation of $A_{m,k}(a)$ is slightly more involved. Using Euler's identities we have that

$$\begin{aligned}\cos^{m-k}(t)\sin^k(t)&=\left(\frac{e^{it}+e^{-it}}{2}\right)^{m-k}\left(\frac{e^{it}-e^{-it}}{2i}\right)^k\\&=\frac{e^{imt}}{2^mi^k}(1+e^{-2it})^{m-k}(1-e^{-2it})^k.\end{aligned}$$

Substitution of this into the integral in question and an application of the Binomial Theorem gives:

$$\begin{aligned}A_{m,k}(a)&:=\int_0^{\pi}e^{ia^pu}\cos^{m-k}\left(\frac{u}{2}\right)\ \sin^k\left(\frac{u}{2}\right)du\\&=\frac{1}{2^mi^k}\int_0^{\pi}e^{ia^pu}e^{im\frac{u}{2}}(1+e^{-iu})^{m-k}(1-e^{-iu})^k\,du\\&=\frac{\pi}{2^mi^k}\sum_{j_1=0}^{m-k}\sum_{j_2=0}^{k}\binom{m-k}{j_1}\binom{k}{j_2}(-1)^{j_2}\frac{1}{\pi}\int_0^{\pi}e^{iu(a^p+\frac{m}{2}-j_1-j_2)}\,du\\&=\frac{\pi}{2^mi^k}\sum_{j_1=0}^{m-k}\sum_{j_2=0}^{k}\binom{m-k}{j_1}\binom{k}{j_2}(-1)^{j_2}\frac{e^{i\pi(a^p+\frac{m}{2}-j_1-j_2)}-1}{i\pi(a^p+\frac{m}{2}-j_1-j_2)}.\end{aligned}$$

We will sum this expression in a different order to arrive at the answer. Let $\mathcal{J}_r$ be the collection of integers (j_1, j_2) such that $j_1 + j_2 = r$ where $r = 0, \ldots, m$. Then continuing from above we have:

$$\begin{aligned} A_{m,k}(a) &= \frac{\pi}{2^m i^k} \sum_{j_1=0}^{m-k} \sum_{j_2=0}^{k} \binom{m-k}{j_1} \binom{k}{j_2} (-1)^{j_2} \frac{e^{i\pi(a^p + \frac{m}{2} - j_1 - j_2)} - 1}{i\pi(a^p + \frac{m}{2} - j_1 - j_2)} \\ &= \frac{\pi}{2^m i^k} \sum_{r=0}^{m} \left(\sum_{(j_1,j_2)\in\mathcal{J}_r} \binom{m-k}{j_1} \binom{k}{j_2} (-1)^{j_2} \right) \frac{e^{i\pi(a^p + \frac{m}{2} - r)} - 1}{i\pi(a^p + \frac{m}{2} - r)} \\ &= \frac{\pi}{2^m i^k} \sum_{r=0}^{m} \left(\sum_{(j_1,j_2)\in\mathcal{J}_r} \binom{m-k}{j_1} \binom{k}{j_2} (-1)^{j_2} \right) \left(\sum_{\alpha=0}^{\infty} \frac{\left(i\pi\left(\frac{m}{2} - r + a^p\right)\right)^{\alpha}}{(\alpha+1)!} \right) \\ &= \frac{\pi}{2^m i^k} \sum_{r=0}^{m} c_{r,m,k} \left(\sum_{\alpha=0}^{\infty} \frac{\left(i\pi\left(\frac{m}{2} - r + a^p\right)\right)^{\alpha}}{(\alpha+1)!} \right). \end{aligned}$$

Utilize that $(j_1, j_2) \in \mathcal{J}_r$ to see that

$$c_{r,m,k} := \sum_{(j_1,j_2)\in\mathcal{J}_r} \binom{m-k}{j_1} \binom{k}{j_2} (-1)^{j_2} = \sum_{l=0}^{r} (-1)^l \binom{k}{l} \binom{m-k}{r-l}.$$

This is the expression claimed in the statement of the theorem, i.e., formula (3.2) for $\Phi_a(\theta, \phi, t)$ with coefficients given by (3.3) and (3.4). The fact that the map $a \mapsto \Phi_a$ is holomorphic is direct by inspection and the proof of the theorem is complete. □

Remark 3.2. One can also show that:

$$c_{r,m,k} = \frac{1}{r!} \frac{d^r}{dt^r} \left((1-t)^k (1+t)^{m-k} \right) \Big|_{t=0} = \sum_{l=0}^{r} (-1)^l \binom{k}{l} \binom{m-k}{r-l}.$$

The first formula was discovered by computing the examples when $m = 1, 2, 3$ and then seeing what the sequence this might be in the Online Encyclopedia of Integer Sequences. It is a known sequence, Sequence A268533.

4. Infinite order differential operators and the supershift property

For the sequences of entire functions we shall consider a natural notion of convergence is the convergence in the space A_1.

Definition 4.1. A_1 is the complex algebra of entire functions such that there exists $B > 0$ with

$$\sup_{\xi\in\mathbb{C}} \left(|f(\xi)| \exp(-B|\xi|) \right) < +\infty. \tag{4.1}$$

The space A_1 has a rather complicated topology since it is a linear space obtained via an inductive limit, see for example [16]. For our purposes, it is enough to consider, for any fixed $B > 0$, the set $A_{1,B}$ of functions f satisfying (4.1), and to observe that

$$\|f\|_B := \sup_{\xi\in\mathbb{C}} \left(|f(\xi)| \exp(-B|\xi|) \right)$$

defines a norm on $A_{1,B}$, called the B-norm. One can prove that $A_{1,B}$ is a Banach space with respect to this norm.

Moreover, for f and a sequence $(f_n)_n$ of elements in A_1, we say that f_n converges to f in A_1 if and only if there exists B such that $f, f_n \in A_{1,B}$ and

$$\lim_{n\to\infty} \sup_{\xi\in\mathbb{C}} \big|f_n(\xi) - f(\xi)\big| \exp(-C|\xi|) = 0$$

for some $C \geq 0$. With these notations and definitions we can make the notion of continuity explicit (see [14] for more details). A linear operator $\mathcal{U} : A_1 \to A_1$ is continuous if and only if for any $B > 0$ there exists $B' > 0$ and $C > 0$ such that

$$\mathcal{U}(A_{1,B}) \subset A_{1,B'} \quad \text{and} \quad \|\mathcal{U}(f)\|_{B'} \leq C\|f\|_B, \qquad \text{for any } f \in A_{1,B}. \tag{4.2}$$

The following result gives a characterization of the functions in A_1 via the coefficients appearing in their Taylor series expansion.

Lemma 4.2 ([13, Lemma 4.2]). *The entire function*

$$f(\xi) = \sum_{j=0}^{\infty} f_j \xi^j$$

belongs to A_1 *if and only if there exists* $C_f > 0$ *and* $b > 0$ *such that*

$$|f_j| \leq C_f \frac{b^j}{j!}.$$

Remark 4.3. We write $f \in A_1$ to mean that $f \in A_{1,B}$ for some $B > 0$. The proof of [13, Lemma 4.2] shows that $b = 2eB$ and $C_f = \|f\|_B$.

A crucial fact in the proof of [13, Lemma 4.2] depends upon the following. We define two infinite order differential operators that will be used to study superoscillatory functions and supershifts in two variables. Observe that using the auxiliary complex variable ξ we have

$$\lambda^\ell = \frac{1}{i^\ell} \mathcal{D}_\xi^\ell e^{i\xi\lambda}\Big|_{\xi=0} \quad \text{for} \quad \lambda \in \mathbb{C}, \quad \ell \in \mathbb{N},$$

where $\mathcal{D}_\xi$ is the derivative with respect to ξ and $|_{\xi=0}$ denotes evaluation at $\xi = 0$. Let us consider the infinite order differential operators that will be associated with the two functions $A_{m,k}(a)$ and $B_{m,k}(a)$ defined in (3.3) and (3.4), respectively, where the coefficients $c_{r,m,k}$ are given in (3.5).

Definition 4.4 (Infinite order differential operators for spin $s = \frac{m}{2}$). Let m, q, $p \in \mathbb{N}$ and assume that $\xi \in \mathbb{C}$ is an auxiliary variable. Denote by $\mathcal{D}_\xi$ the complex derivative with respect to the variable ξ and define the formal infinite order differential operators

$$\mathcal{A}_{m,k}(\mathcal{D}_\xi) := \frac{\pi}{2^m i^k} \sum_{r=0}^{m} c_{r,m,k} \sum_{\alpha=0}^{\infty} \frac{\Big(i\pi\Big((\frac{m}{2} - r)\mathcal{I} + i^{-p}\mathcal{D}_\xi^p\Big)\Big)^\alpha}{(\alpha+1)!} \tag{4.3}$$

and

$$\mathcal{B}_{m,k}(\mathcal{D}_\xi) := 2\pi \sum_{\beta=0}^{\infty} \frac{(2\pi i)^\beta}{(\beta+1)!} \Big((-\frac{m}{2} + k)\mathcal{I} + i^{-q}\mathcal{D}_\xi^q\Big)^\beta, \tag{4.4}$$

where the coefficients $c_{r,m,k}$ are given in (3.5) and $\mathcal{I}$ is the identity operator.

Proposition 4.5. *Let* A_1 *be the space of entire functions as in Definition* 4.4. *Then for every* m, q *and* $p \in \mathbb{N}$ *the operators* $\mathcal{A}_{m,k}(\mathcal{D}_\xi)$ *and* $\mathcal{B}_{m,k}(\mathcal{D}_\xi)$ *defined in* (4.3) *and* (4.4), *respectively, act continuously from* A_1 *to* A_1.

Proof. It follows from the fact that the functions $A_{m,k}(a)$ and $B_{m,k}(a)$ defined in (3.3) and (3.4), respectively, are entire and from the use of Lemma 4.2. The computations are a bit involved, but follow the same strategy used in [13] to obtain the continuity estimate (4.2). Exact details are suppressed. □

Remark 4.6. One may wonder under which conditions one can guarantee that a function has the supershift property. This is an interesting problem, and a sufficient condition for the property to hold is given by analyticity, see [9].

We now return to Problem 1.1. We state the main result and we note that, by Remark 4.6, in the proof we can rely on the fact that the function in (3.2) has the supershift property.

Theorem 4.7. *Let $a > 1$ and let $s := \frac{m}{2}$ where $m \in \mathbb{N}$. Then the solution of the Schrödinger equation, with the propagator of the spinning particle given by* (2.2) *and superoscillatory initial datum* (1.6) *can be written as*

$$\Psi_n(\theta, \phi, t) = \sum_{j=0}^{n} C_j(n, a)\Phi_{1-\frac{2j}{n}}(\theta, \phi, t)$$

for all $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$, $t \geq 0$, where $\Phi_a(\theta, \phi, t)$ is given by (3.2) *and $C_j(n, a)$ are given by* (1.2) *for p, $q \in \mathbb{N}$. Moreover, $\Phi_a(\theta, \phi, t)$ has the supershift property, that is,*

$$\lim_{n\to\infty} \sum_{j=0}^{n} C_j(n, a)\Phi_{1-\frac{2j}{n}}(\theta, \phi, t) = \Phi_a(\theta, \phi, t).$$

Proof. The solution $\Phi_a(\theta, \phi, t)$ given in (3.2) can be written in terms of the infinite order differential operators $\mathcal{A}_{m,k}(\mathcal{D}_\xi)$ and $\mathcal{B}_{m,k}(\mathcal{D}_\xi)$ defined in (4.3) and (4.4), respectively, as

$$\begin{aligned}\Phi_a(\theta, \phi, t) = \sum_{k=0}^{m} \binom{m}{k} e^{i\left(-\frac{m}{2}+k\right)Bt} \cos^{m-k}\left(\frac{\theta}{2}\right) \\ \times \sin^k\left(\frac{\theta}{2}\right) e^{i\left(\frac{m}{2}-k\right)\phi} \mathcal{A}_{m,k}(\mathcal{D}_\xi)\mathcal{B}_{m,k}(\mathcal{D}_\xi)e^{ia\xi}\Big|_{\xi=0}.\end{aligned}$$

By linearity of the Schrödinger equation, the evolution of the superoscillatory initial datum is

$$\Psi_n(\theta, \phi, t) = \sum_{j=0}^{n} C_j(n, a)\Phi_{1-\frac{2j}{n}}(\theta, \phi, t).$$

If we set for simplicity

$$\Lambda_{m,k}(\theta, \phi, t) := e^{i\left(-\frac{m}{2}+k\right)Bt} \cos^{m-k}\left(\frac{\theta}{2}\right) \sin^k\left(\frac{\theta}{2}\right) e^{i\left(\frac{m}{2}-k\right)\phi},$$

we can write

$$\Phi_a(\theta, \phi, t) = \sum_{k=0}^{m} \binom{m}{k} \Lambda_{m,k}(\theta, \phi, t)\mathcal{A}_{m,k}(\mathcal{D}_\xi)\mathcal{B}_{m,k}(\mathcal{D}_\xi)e^{ia\xi}\Big|_{\xi=0},$$

so we have

$$\begin{aligned}\Psi_n(\theta,\phi,t) &= \sum_{j=0}^{n} C_j(n,a)\Phi_{1-\frac{2j}{n}}(\theta,\phi,t)\\ &= \sum_{j=0}^{n} C_j(n,a)\sum_{k=0}^{m}\binom{m}{k}\Lambda_{m,k}(\theta,\phi,t)\mathcal{A}_{m,k}(\mathcal{D}_\xi)\mathcal{B}_{m,k}(\mathcal{D}_\xi)e^{i(1-\frac{2j}{n})\xi}\Big|_{\xi=0}\\ &= \sum_{k=0}^{m}\binom{m}{k}\Lambda_{m,k}(\theta,\phi,t)\mathcal{A}_{m,k}(\mathcal{D}_\xi)\mathcal{B}_{m,k}(\mathcal{D}_\xi)\sum_{j=0}^{n} C_j(n,a)e^{i(1-\frac{2j}{n})\xi}\Big|_{\xi=0}.\end{aligned}$$

By the continuity results, in particular Proposition 4.5, it is immediate that

$$\begin{aligned}&\lim_{n\to\infty}\Psi_n(\theta,\phi,t) = \lim_{n\to\infty}\sum_{j=0}^{n} C_j(n,a)\Phi_{1-\frac{2j}{n}}(\theta,\phi,t)\\ &= \sum_{k=0}^{m}\binom{m}{k}\Lambda_{m,k}(\theta,\phi,t)\mathcal{A}_{m,k}(\mathcal{D}_\xi)\mathcal{B}_{m,k}(\mathcal{D}_\xi)\lim_{n\to\infty}\sum_{j=0}^{n} C_j(n,a)e^{i(1-\frac{2j}{n})\xi}\Big|_{\xi=0}\\ &= \sum_{k=0}^{m}\binom{m}{k}\Lambda_{m,k}(\theta,\phi,t)\mathcal{A}_{m,k}(\mathcal{D}_\xi)\mathcal{B}_{m,k}(\mathcal{D}_\xi)e^{ia\xi}\Big|_{\xi=0}\end{aligned}$$

since the sequence $\sum_{j=0}^{n} C_j(n,a)e^{i(1-\frac{2j}{n})\xi}$ converges to $e^{ia\xi}$ in A_1. One is left to simply apply the operators $\mathcal{A}_{m,k}(\mathcal{D}_\xi)\mathcal{B}_{m,k}(\mathcal{D}_\xi)$ to $e^{ia\xi}$ and take the restriction to $\xi = 0$ to yield the claimed supershift property. □

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