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ON THE BOOLEAN ALGEBRA TENSOR PRODUCT VIA CARATHÉODORY SPACES OF PLACE FUNCTIONS

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ABSTRACT. We show that the Carathéodory space of place functions on the free product of two Boolean algebras is Riesz isomorphic with Fremlin's Archimedean Riesz space tensor product of their respective Carathéodory spaces of place functions. We provide a solution to Fremlin's problem 315Y(f) [*Measure Theory*, Torres Fremlin, Colchester, 2004] concerning completeness in the free product of Boolean algebras by applying our results on the Archimedean Riesz space tensor product to Carathéodory spaces of place functions.

1. Introduction and preliminary material

Fremlin asserts in problem 315Y(f) of [6] that the Boolean algebra tensor product of two nontrivial Boolean algebras is complete if and only if one is finite and the other is complete. In Theorem 4.6 of [4], we prove that the Fremlin tensor product of two Dedekind complete Riesz spaces rarely is Dedekind complete. In fact, if the tensor product is Dedekind complete, then one of the two spaces is Riesz isomorphic to the set of all finite-valued functions on a subset of that space. To connect 315Y(f) of [6] with Theorem 4.6 of [4], we employ Carathéodory's Riesz space of place functions on a Boolean algebra. The main result is Theorem 2.1 with applications given in Section 3.

The necessary terms for Boolean algebras, the free product, Riesz spaces, and Carathéodory spaces of place functions are provided in this section. We reserve $\mathcal{A}$, $\mathcal{B}$ for Boolean algebras and E, F, G for Archimedean Riesz spaces.

Boolean algebras and their free product. For Boolean algebras, see chapter 31 of [6]. Two elements x and y of a Boolean algebra are called *disjoint* if $x \wedge y = 0$, in which case we write $x \perp y$. Two subsets A and B of a Boolean algebra are called disjoint if $x \perp y$ for every $x \in A$ and $y \in B$, in which case we write $A \perp B$. We define the *disjoint sum* of two elements x and y in a Boolean algebra by

$$x \oplus y = (x \wedge y') \vee (x' \wedge y).$$

A Boolean algebra is *complete* if every nonempty subset has a supremum.

Definition 1.1 (312F of [6])**.** Let $\mathcal{A}$ and $\mathcal{B}$ be Boolean algebras. A function $\chi\colon \mathcal{A} \to \mathcal{B}$ is said to be a *Boolean homomorphism* if for all x, $y \in \mathcal{A}$,

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(i) $\chi(x \wedge y) = \chi(x) \wedge \chi(y)$;
(ii) $\chi(x \oplus y) = \chi(x) \oplus \chi(y)$;
(iii) $\chi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$.

A bijective Boolean homomorphism is called a *Boolean isomorphism.* If there exists an isomorphism $\chi\colon \mathcal{A} \to \mathcal{B}$, then the Boolean algebras $\mathcal{A}$ and $\mathcal{B}$ are said to be *Boolean isomorphic.*

Proposition 312H of [6] proves additionally that every Boolean homomorphism preserves finite suprema, that is $\chi(x \vee y) = \chi(x) \vee \chi(y)$ for every x, $y \in \mathcal{A}$.

The *Stone space* of a Boolean algebra $\mathcal{A}$ is the set Z of nonzero ring homomorphisms from $\mathcal{A}$ to $\mathbb{Z}_2$. Set

$$\hat{a} = \{z : z \in Z, z(a) = 1\}.$$

By Stone's Theorem (see, for instance, 311E of [6]), the canonical map

$$a \mapsto \hat{a} : \mathcal{A} \to \mathcal{P}(Z)$$

is an injective ring homomorphism which we call the *Stone representation.* For more on Stone spaces, see [6] where Fremlin defines and utilizes Stone spaces to define the Boolean algebra tensor product, called the *free product.*

Definition 1.2 (Fremlin, 315I of [6])**.**

(i) Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of Boolean algebras. For each $i \in I$, let Z_i be the Stone space of $\mathcal{A}_i$. Set $Z = \prod_{i\in I} Z_i$, with the product topology. Then the *free product* of $\{\mathcal{A}_i\}_{i\in I}$ is the algebra of open-and-closed sets in Z, denoted $\bigotimes_{i\in I} \mathcal{A}_i$.
(ii) For $i \in I$ and $a \in \mathcal{A}_i$, the set $\hat{a} \subseteq Z_i$ representing a is an open-and-closed subset of Z_i; because $z \mapsto z(i)\colon Z \to Z_i$ is continuous,

$$\epsilon_i(a) = \{z : z(i) \in \hat{a}\}$$

is open-and-closed, so belongs to $\mathcal{A}$. In this context, $\epsilon_i\colon \mathcal{A}_i \to \mathcal{A}$ is called the *canonical map.*

In the following theorem, we list the necessary material from 315J and 315K of [6] in the language of Fremlin.

Theorem 1.3. *Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of Boolean algebras, with free product $\mathcal{A}$.*

(i) *The canonical map $\epsilon_i\colon \mathcal{A}_i \to \mathcal{A}$ is a Boolean homomorphism for every $i \in I$.*
(ii) *For any Boolean algebra $\mathcal{B}$ and any family $\{\varphi_i\}_{i\in I}$ such that φ_i is a Boolean homomorphism from $\mathcal{A}_i$ to $\mathcal{B}$ for every i, there is a unique Boolean homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ such that $\varphi_i = \varphi \circ \epsilon_i$ for each i.*
(iii) *Write C for the set of those members of $\mathcal{A}$ expressible in the form $\inf_{j\in J} \epsilon_j(a_j)$, where $J \subseteq I$ is finite and $a_j \in \mathcal{A}_j$ for every j. Then every member of $\mathcal{A}$ is expressible as the supremum of a disjoint finite subset of C.*
(iv) *$\mathcal{A} = \{0_{\mathcal{A}}\}$ if and only if there is some $i \in I$ such that $\mathcal{A}_i = \{0_{\mathcal{A}_i}\}$.*
(v) *If $\mathcal{A}_i \neq \{0_{\mathcal{A}}\}$ for every $i \in I$, then ϵ_i is injective for every $i \in I$.*
(vi) *Let $\mathcal{A}_i \neq \{0_{\mathcal{A}}\}$ for every $i \in I$. If $J \subseteq I$ is finite and a_j is a nonzero member of $\mathcal{A}_j$ for each $j \in J$, then $\inf_{j\in J} \epsilon_j(a_j) \neq 0$.*

Archimedean Riesz spaces and their tensor product. See [11] for Archimedean Riesz spaces and [5] for Riesz bimorphisms.

Theorem 1.4 (4.2 of [5])**.** *Let E and F be Archimedean Riesz spaces. There exist an Archimedean Riesz space G and a Riesz bimorphism $\varphi\colon E\times F\to G$ such that whenever H is an Archimedean Riesz space and $\psi\colon E\times F\to H$ is a Riesz bimorphism, there is a unique Riesz homomorphism $T\colon G\to H$ such that $T\circ\varphi=\psi$.*

G of Theorem 1.4 is the *Archimedean Riesz space tensor product of E and F*, denoted by $E\bar{\otimes}F$. The "universal property of $E\bar{\otimes}F$" refers to the implication that any Archimedean Riesz space paired with a Riesz bimorphism satisfying Theorem 1.4 is Riesz isomorphic to G. The Riesz bimorphism $\otimes\colon E\times F\to E\bar{\otimes}F$ embeds the algebraic tensor product $E\otimes F$ into $E\bar{\otimes}F$ via $\otimes(e,f)=e\otimes f$ for all $e\in E$ and $f\in F$.

We define a few terms needed for the statement of Theorem 1.6.

Definition 1.5. Let E be an Archimedean Riesz space and let I be a nonempty set. $c_{00}(I,E)$ is the set of all maps $f\colon I\to E$ for which

$$S(f)=\{x\in I: f(x)\neq 0\}$$

is finite. If $E=\mathbb{R}$, then $c_{00}(I,E)$ is written $c_{00}(I)$.

Let f and g be elements of a Riesz space E. The ideals generated by f and g are denoted by E_f and E_g respectively. We denote the principal bands generated by f and g with $[f]$ and $[g]$ respectively (see, for instance, pg. 30 of [11] for definitions).

A Riesz space is *Dedekind complete* if every bounded subset of E has a supremum. Every Dedekind complete Riesz space is Archimedean. In [4], we characterized when the tensor product of two Dedekind complete Riesz spaces is Dedekind complete.

Theorem 1.6 (4.6 of [4])**.** *Suppose E and F are Dedekind complete Riesz spaces. The following are equivalent.*

1. *$E_x\bar{\otimes}F_y$ is Dedekind complete for every $x\in E^+$ and $y\in F^+$.*
2. *[E_x is finite dimensional for every $x\in E^+$] or [F_y is finite dimensional for every $y\in F^+$].*
3. *$E\cong c_{00}(I)$ for a set $I\subseteq E$ or $F\cong c_{00}(J)$ for a set $J\subseteq F$.*
4. *$E\bar{\otimes}F\cong c_{00}(I,F)$ for a set $I\subseteq E$ or $E\bar{\otimes}F\cong c_{00}(J,E)$ for a set $J\subseteq F$.*
5. *$E\bar{\otimes}F$ is Dedekind complete.*

As an intermediary between Archimedean Riesz spaces and Boolean algebras, we consider Boolean algebras of bands. The following three statements are used in Section 3 and are given for the reader's convenience.

Theorem 1.7 (22.6, 22.8 of [10])**.** *Let E be a Riesz space and define*

$$\mathcal{B}(E)=\{B\subseteq E: B \textit{ is a band}\}.$$

$\mathcal{B}(E)$ is an order complete distributive lattice. $\mathcal{B}(E)$, partially ordered by inclusion, is a Boolean algebra if and only if E is Archimedean.

Lemma 1.8. *Let E be a Riesz space and $f,\ g\in E$. Then $|f|\wedge|g|=0$ implies $[f]\perp[g]$.*

Proof. Let $|f|\wedge|g|=0$. Certainly, $E_f\perp E_g$. Suppose $h_1\in[f]$ and $h_2\in[g]$. Using the fact that $E_f\perp E_g$, it is straightforward to show that $|h_1|\wedge|h_2|=0$ via 6.1 and 7.8 of [11]. Thus, $[g_1]\perp[g_2]$. ☐

Lemma 1.9. *If E is an infinite dimensional Archimedean Riesz space, then $\mathcal{B}(E)$ is not finite.*

Proof. By the contrapositive of Theorem 26.10 in [10], there is an infinite subset of mutually disjoint nonzero elements in E. Thus, there are an infinite number of mutually disjoint bands in E by Lemma 1.8. □

Carathéodory spaces of place functions.

Definition 1.10 (pg. 40 of [1])**.** Let E be a Riesz space and $e \in E^+$. Then $x \in E^+$ is said to be a *component* of e whenever $x \wedge (e - x) = 0$.

The collection of all components of e, denoted $\mathcal{C}(e)$, is a Boolean algebra under the partial ordering induced by E (pg. 40 of [1]). With e as a strong order unit (pg. 51 of [11]), a connection between Archimedean Riesz spaces and Boolean algebras is described explicitly in the following theorem.

Theorem 1.11 (4.1 of [3])**.** *Let $\mathcal{A}$ be a Boolean algebra. There exists an Archimedean Riesz space E with a strong unit e with the following properties.*

(i) *There exists a Boolean isomorphism $\chi\colon \mathcal{A} \to \mathcal{C}(e)$.*
(ii) *E is the linear span of $\mathcal{C}(e)$.*

(E, χ) is unique up to isomorphism. It is denoted by $\mathcal{C}(\mathcal{A})$ and is called the *space of place functions on $\mathcal{A}$* in the sense of Carathéodory.

Let λ_i, $\gamma_j \in \mathbb{R}$ be nonzero; n, $m \in \mathbb{N}$; $x_i \in \mathcal{A}$ be pairwise disjoint; and $y_j \in \mathcal{A}$ be pairwise disjoint. Two elements

$$f = \sum_{i=1}^{n} \lambda_i \chi(x_i) \quad \text{and} \quad g = \sum_{j=1}^{m} \gamma_j \chi(y_j)$$

are equivalent if $\bigvee_{i=1}^{n} x_i = \bigvee_{j=1}^{m} y_j$ and if $\lambda_i = \gamma_j$ whenever $x_i \wedge y_j \neq 0$. $\mathcal{C}(\mathcal{A})$ is the set of all such equivalence classes. Henceforth, we take $f = \sum_{i=1}^{n} \lambda_i \chi(x_i)$ to represent all elements of $\mathcal{C}(\mathcal{A})$ that are equivalent to f.

We define addition in $C(\mathcal{A})$ in the style of Goffman in [7] and Jakubik in [9]. For a different approach, see [3]. For x, $y \in \mathcal{A}$, let $x -_1 y$ be the complement of $x \wedge y$ relative to x, that is, $x \wedge (x \wedge y)'$. Then addition in $C(\mathcal{A})$ is defined by

$$f + g = \sum_{i=1}^{n} \sum_{j=1}^{m} (\lambda_i + \gamma_j) \chi(x_i \wedge y_j) + \sum_{i=1}^{n} \lambda_i \chi(x_i -_1 \bigvee_{j=1}^{m} y_j) + \sum_{j=1}^{m} \gamma_j \chi(y_j -_1 \bigvee_{i=1}^{n} x_i)$$

where in the summation only those terms are taken into account in which $\lambda_i + \gamma_j \neq 0$ and the elements $x_i \wedge y_j$, $x_i -_1 \bigvee y_j$, and $y_j -_1 \bigvee x_i$ are nonzero. It is routine to verify that addition is well-defined in $\mathcal{C}(\mathcal{A})$.

Jakubik proves in [9] that the completeness of a Boolean algebra is equivalent to the Dedekind completeness of its Carathéodory space of place functions. However, his propositions assume complete distributivity. Since this work has no need for a Boolean algebra to be completely distributive, Theorem 1.13 is proven with credit to Propositions 5.2(a) and 5.6 of [9] for its similarity.

Definition 1.12 (pg. 231 of [9])**.** Let Y be a sublattice of a lattice X. Y is said to be a *regular sublattice* of X if:

(i) whenever $x_0 \in Y$ and $\emptyset \neq X \subseteq Y$ such that $x_0 = \sup_Y X$, then $x_0 = \sup_X X$; and

(ii) whenever $x_1 \in Y$ and $\emptyset \neq X \subseteq Y$ such that $x_1 = \inf_Y X$, then $x_1 = \inf_X X$.

Theorem 1.13. *Let $\mathcal{A}$ be a Boolean algebra. $\mathcal{A}$ is complete if and only if $\mathcal{C}(\mathcal{A})$ is Dedekind complete.*

Proof. Assume that $\mathcal{A}$ is complete. Let D be a bounded subset of $\mathcal{C}(\mathcal{A})$. Then there exists $g \in \mathcal{C}(\mathcal{A})$ such that $g \geq f$ for every $f \in \mathcal{C}(\mathcal{A})$. Find $\lambda_i \in \mathbb{R}$, $n \in \mathbb{N}$, and $x_i \in \mathcal{A}$ such that $g = \sum_{i=1}^{n} \lambda_i \chi(x_i)$. Set

$$x = x_1 \vee \cdots \vee x_n \qquad \text{and} \qquad \lambda = max\{\lambda_1, \cdots, \lambda_n\}.$$

Then $D \subseteq [0, \lambda\chi(x)]$. By assumption, the interval $[0, x]$ is complete in $\mathcal{A}$. It follows from Corollary 4.4 of [9] that $\mathcal{A}$ is a regular subset of $\mathcal{C}(\mathcal{A})$. Then the interval $[0, \chi(x)]$ is complete as a subset of $\mathcal{C}(\mathcal{A})$. In particular, $[0, \lambda\chi(x)]$ is complete, so $\sup(D)$ exists in $\mathcal{C}(\mathcal{A})$.

To prove sufficiency, assume that $\mathcal{C}(\mathcal{A})$ is complete. Let $\chi \colon \mathcal{A} \to \mathcal{C}(e)$ be the Boolean isomorphism from Theorem 1.11. Note that $e = \chi(1_\mathcal{A})$. Let D be a subset of $\mathcal{A}$. Since $\mathcal{C}(\mathcal{A})$ is Dedekind complete, $\sup \chi(D)$ exists in $\mathcal{C}(\mathcal{A})^+$. For every $x \in D$, $\chi(x)$ is a component of $\chi(1_\mathcal{A})$. Thus, $\sup \chi(D) = 2 \sup \chi(D) \wedge \chi(1_\mathcal{A})$ so that $0 = \sup \chi(D) \wedge (\chi(1_\mathcal{A}) - \sup \chi(D))$. By definition, $\sup \chi(D)$ is a component of e.

Let $y = \chi^{-1}(\sup \chi(D))$. Since χ is a Boolean isomorphism, y is an upper bound for D. Suppose there exists y' such that $x \leq y' < y$ for every $x \in D$. Then $\chi(y') \geq \sup_{x \in D} \chi(x) = \chi(y)$. Thus, $\chi(y') = \chi(y)$. Since χ is one-to-one, $y' = y$. Therefore, $y = \sup(D)$ exists in $\mathcal{A}$. □

2. The Fremlin tensor product of Carathéodory spaces of place functions

In this section, we relate Boolean algebras $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{A} \otimes \mathcal{B}$ to their Carathéodory spaces of place functions $\mathcal{C}(\mathcal{A})$, $\mathcal{C}(\mathcal{B})$, and $\mathcal{C}(\mathcal{A} \otimes \mathcal{B})$. The notation of Theorem 1.11 is used with the addition of subscripts to indicate which Boolean algebra is at work. The symbols in (1), (2), and (3) will be used freely.

(1) $\chi_A \colon \mathcal{A} \to C(\mathcal{A})$, $\chi_B \colon \mathcal{B} \to C(\mathcal{B})$, and $\hat{\chi} \colon \mathcal{A} \otimes \mathcal{B} \to C(\mathcal{A} \otimes \mathcal{B})$ are the Boolean isomorphisms from Theorem 1.11.

(2) $C(\mathcal{A})$, $C(\mathcal{B})$, and $C(\mathcal{A} \otimes \mathcal{B})$ have units $\chi_A(1_\mathcal{A})$, $\chi_B(1_\mathcal{B})$, and $\hat{\chi}(1_{\mathcal{A} \otimes \mathcal{B}})$ respectively.

(3) $\epsilon_A \colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$ and $\epsilon_B \colon \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$ are the canonical Boolean homomorphisms in Definition 1.2.

Theorem 2.1. *$C(\mathcal{A}) \bar{\otimes} C(\mathcal{B})$ and $C(\mathcal{A} \otimes \mathcal{B})$ are Riesz isomorphic.*

Proof. Assume that $\mathcal{A}$ and $\mathcal{B}$ are nontrivial Boolean algebras. For $f \in \mathcal{C}(\mathcal{A})$, there exist $n \in \mathbb{N}$, pairwise disjoint $x_i \in \mathcal{A}$, and nonzero $\lambda_i \in \mathbb{R}$ such that $f = \sum_{i=1}^{n} \lambda_i \chi_A(x_i)$. For $g \in \mathcal{C}(\mathcal{B})$, there exist $m \in \mathbb{N}$, pairwise disjoint $u_j \in \mathcal{B}$, and nonzero $\gamma_j \in \mathbb{R}$ so $g = \sum_{j=1}^{m} \gamma_j \chi_B(u_j)$. Define $\psi \colon \mathcal{C}(\mathcal{A}) \times \mathcal{C}(\mathcal{B}) \to \mathcal{C}(\mathcal{A} \otimes \mathcal{B})$ by

$$\begin{aligned} \psi(f, g) &= \psi\left(\sum_{i=1}^{n} \lambda_i \chi_A(x_i), \sum_{j=1}^{m} \gamma_j \chi_B(u_j)\right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} (\lambda_i \gamma_j) \hat{\chi}(\epsilon_A(x_i) \wedge \epsilon_B(u_j)). \end{aligned}$$

It follows from Theorem 1.3 (iv) and (vi) that the definition of ψ is independent of the representations chosen for f and g.

Let $f_1 = f$ and $f_2 = \sum_{k=1}^{p} \delta_k \chi_A(y_k)$ for nonzero $\delta_k \in \mathbb{R}$, $p \in \mathbb{N}$ and pairwise disjoint $y_k \in \mathcal{A}$. Recall that $f_1 + f_2$ is defined to be

$$\sum_i \sum_k (\lambda_i + \delta_k)\chi_A(x_i \wedge y_k) + \sum_i \lambda_i \chi_A(x_{i-1} \bigvee_k y_k) + \sum_k \delta_k \chi_A(y_{k-1} \bigvee_i x_i).$$

Claim 1. ψ is bilinear.

$$\begin{aligned}
&\psi(f_1 + f_2, g)\\
&= \psi\left(\sum_i \lambda_i \chi_A(x_i) + \sum_k \delta_k \chi_A(y_k), \sum_j \gamma_j \chi_B(u_j)\right)\\
&= \sum_{i,k,j} (\lambda_i + \delta_k)\gamma_j \hat{\chi}\left(\epsilon_A(x_i \wedge y_k) \wedge \epsilon_B(u_j)\right) + \sum_{i,j} (\lambda_i\gamma_j)\hat{\chi}\left(\epsilon_A(x_{i-1} \bigvee_k y_k) \wedge \epsilon_B(u_j)\right)\\
&+ \sum_{k,j} (\delta_k\gamma_j)\hat{\chi}\left(\epsilon_A(y_{k-1} \bigvee_i x_i) \wedge \epsilon_B(u_j)\right)\\
&= \sum_{i,j}\left[\sum_k (\lambda_i\gamma_j)\hat{\chi}\left(\epsilon_A(x_i \wedge y_k) \wedge \epsilon_B(u_j)\right) + (\lambda_i\gamma_j)\hat{\chi}\left(\epsilon_A(x_{i-1} \bigvee_k y_k) \wedge \epsilon_B(u_j)\right)\right]\\
&+ \sum_{k,j}\left[\sum_i (\delta_k\gamma_j)\hat{\chi}\left(\epsilon_A(x_i \wedge y_k) \wedge \epsilon_B(u_j)\right) + (\delta_k\gamma_j)\hat{\chi}\left(\epsilon_A(y_{k-1} \bigvee_i x_i) \wedge \epsilon_B(u_j)\right)\right]\\
&= \sum_{i,j} (\lambda_i\gamma_j)\left[\hat{\chi}\left(\bigvee_k \epsilon_A(x_i \wedge y_k) \wedge \epsilon_B(u_j)\right) + \hat{\chi}\left(\epsilon_A(x_{i-1} \bigvee_k y_k) \wedge \epsilon_B(u_j)\right)\right]\\
&+ \sum_{k,j} (\delta_k\gamma_j)\left[\hat{\chi}\left(\bigvee_i \epsilon_A(x_i \wedge y_k) \wedge \epsilon_B(u_j)\right) + \hat{\chi}\left(\epsilon_A(y_{k-1} \bigvee_i x_i) \wedge \epsilon_B(u_j)\right)\right] \quad (*)\\
&= \sum_{i,j} (\lambda_i\gamma_j)\hat{\chi}(\epsilon_A(x_i) \wedge \epsilon_B(u_j)) + \sum_{k,j} (\delta_k\gamma_j)\hat{\chi}(\epsilon_A(y_k) \wedge \epsilon_B(u_j))\\
&= \psi(f_1, g) + \psi(f_2, g).
\end{aligned}$$

$(*)$ is justified because $y_k \perp y_{k'}$ for all $k \neq k'$ and $x_i \perp x_{i'}$ for all $i \neq i'$. Symmetrically, $\psi(f, g_1 + g_2) = \psi(f, g_1) + \psi(f, g_2)$ for $f \in \mathcal{C}(\mathcal{A})$ and $g_1,\ g_2 \in \mathcal{C}(\mathcal{B})$. It follows from the definition of ψ that $\psi(\lambda f, g) = \psi(f, \lambda g) = \lambda\psi(f, g)$ for every $\lambda \in \mathbb{R}$.

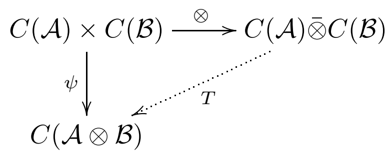
Claim 2. ψ is a Riesz bimorphism.

Assume $f_1 \wedge f_2 = 0$ and $g \in \mathcal{C}(\mathcal{B})^+$. Using the same representations for f_1, f_2, and g as above, it follows that $x_i \perp y_k$ for all i and k. Then since the maps $\hat{\chi}$ and ϵ_A are Boolean homomorphisms and $\{x_i\}_{i=1}^n$, $\{y_k\}_{k=1}^p$ are each pairwise disjoint,

$$\begin{aligned}
\psi(f_1, g) \wedge \psi(f_2, g) &= \psi\left(\sum_i \lambda_i \chi_A(x_i), \sum_j \gamma_j \chi_B(u_j)\right) \wedge \psi\left(\sum_k \delta_k \chi_A(y_k), \sum_j \gamma_j \chi_B(u_j)\right)\\
&= \left(\sum_{i,j} (\lambda_i\gamma_j)\hat{\chi}(\epsilon_A(x_i) \wedge \epsilon_B(u_j))\right) \wedge \left(\sum_{k,j} (\delta_k\gamma_j)\hat{\chi}(\epsilon_A(y_k) \wedge \epsilon_B(u_j))\right)\\
&= 0.
\end{aligned}$$

Likewise if $f \in \mathcal{C}(\mathcal{A})^+$ and $g_1 \wedge g_2 = 0$ in $\mathcal{C}(\mathcal{B})$, then $\psi(f, g_1) \wedge \psi(f, g_2) = 0$. By Theorem 19.1 of [11], ψ is a Riesz bimorphism.

It follows from the universal property of the Fremlin tensor product that there is a unique Riesz homomorphism $T\colon C(\mathcal{A})\bar{\otimes}C(\mathcal{B}) \to C(\mathcal{A} \otimes \mathcal{B})$ such that $\psi = T \circ \otimes$.



Claim 3. T is a Riesz isomorphism.

Step 1 (T is onto). . Let $h \in C(\mathcal{A} \otimes \mathcal{B})$. Then $h = \sum_{i=1}^{n} \lambda_i \hat{\chi}(e_i)$ for some pairwise disjoint $e_i \in \mathcal{A} \otimes \mathcal{B}$, $n \in \mathbb{N}$, and nonzero $\lambda_i \in \mathbb{R}$. Fix $i \in \{1, \cdots, n\}$. By Theorem 1.3(iii), there exists a finite disjoint subset $\{\epsilon_A(a_k) \wedge \epsilon_B(b_k)\}_{k=1}^{m}$ ($m \in \mathbb{N}$) of $\mathcal{A} \otimes \mathcal{B}$ such that

$$e_i = \bigvee_{k=1}^{m} \epsilon_A(a_k) \wedge \epsilon_B(b_k).$$

Then it follows from the definition of ψ that

$$\begin{aligned} \hat{\chi}(e_i) &= \hat{\chi}\left(\bigvee_{k=1}^{m} \epsilon_A(a_k) \wedge \epsilon_B(b_k)\right) \\ &= \bigvee_{k=1}^{m} \hat{\chi}(\epsilon_A(a_k) \wedge \epsilon_B(b_k)) \\ &= \bigvee_{k=1}^{m} \psi(\chi_A(a_k), \chi_B(b_k)) \\ &= \bigvee_{k=1}^{m} T \circ \otimes(\chi_A(a_k), \chi_B(b_k)). \end{aligned}$$

Since T preserves finite suprema, $\hat{\chi}(e_i)$ is in the image of T for every i. It follows from the linearity of T that h is in the image of T.

Step 2 (T is one-to-one). Suppose $f \in C(\mathcal{A}) \otimes C(\mathcal{B})$, the algebraic tensor product of $C(\mathcal{A})$ and $C(\mathcal{B})$, such that f is nonzero. Then for some $n \in \mathbb{N}$, nonzero $\lambda_k \in \mathbb{R}$, and nontrivial $x_k \in \mathcal{A}$, $u_k \in \mathcal{B}$ such that

$$f = \sum_{k=1}^{n} \lambda_k \chi_A(x_k) \otimes \chi_B(u_k).$$

Since ϵ_A, ϵ_B, and $\hat{\chi}$ are injective Boolean isomorphisms,

$$\begin{aligned} T(f) &= T\left(\sum_{k=1}^{n} \lambda_k \chi_A(x_k) \otimes \chi_B(u_k)\right) \\ &= \sum_{k=1}^{n} \lambda_k \psi\left(\chi_A(x_k), \chi_B(u_k)\right) \\ &= \sum_{k=1}^{n} \lambda_k \hat{\chi}(\epsilon_A(x_k) \wedge \epsilon_B(u_j)) \\ &\neq 0. \end{aligned}$$

Let g be a nonzero element of the Riesz space tensor product of $C(\mathcal{A})$ and $C(\mathcal{B})$, i.e. $g \in C(\mathcal{A})\bar{\otimes}C(\mathcal{B})$. By Theorem 2.2 of [2], for all $\delta > 0$ there exists $f \in \mathcal{C}(\mathcal{A})^+ \otimes \mathcal{C}(\mathcal{B})^+$ such that $0 \leq |g| - f \leq \delta\hat{\chi}(1_{\mathcal{A}\otimes\mathcal{B}})$. Since $C(\mathcal{A})\bar{\otimes}C(\mathcal{B})$ is Archimedean, choose $\delta > 0$ such that $|g| \wedge \delta\hat{\chi}(1_{\mathcal{A}\otimes\mathcal{B}}) \neq |g|$. Then f is nonzero. We have shown that $T(f) \neq 0$ when $0 \neq f \in \mathcal{C}(\mathcal{A}) \otimes \mathcal{C}(\mathcal{B})$. Since T is a Riesz homomorphism, $0 < T(f) \leq |T(g)|$. Therefore, $T(g) \neq 0$, and T is a Riesz isomorphism. Consequently, $C(\mathcal{A})\bar{\otimes}C(\mathcal{B})$ is Riesz isomorphic to $C(\mathcal{A} \otimes \mathcal{B})$.

□

3. Applications

In this section, we use Theorem 2.1 to provide a solution for Fremlin's problem 315Y(f) in [6]. The statement leads to an observation on Dedekind completeness in the Fremlin tensor product of place functions and a statement on bands in the Fremlin tensor product of infinite dimensional Archimedean Riesz spaces.

Problem 3.1 (Fremlin, 315Y(f) of [6])**.** Let $\mathcal{A}$ and $\mathcal{B}$ be Boolean algebras. $\mathcal{A}\otimes\mathcal{B}$ is complete if and only if *either* $\mathcal{A} = \{0\}$ *or* $\mathcal{B} = \{0\}$ *or* $\mathcal{A}$ is finite and $\mathcal{B}$ is complete *or* $\mathcal{B}$ is finite and $\mathcal{A}$ is complete.

Proof. If $\mathcal{A} = \{0\}$ or $\mathcal{B} = \{0\}$, the result is trivial. Assume $\mathcal{A}$ and $\mathcal{B}$ are nontrivial Boolean algebras.

Suppose $\mathcal{A}\otimes\mathcal{B}$ is complete. It follows from Theorems 2.1 and 1.13 that $\mathcal{C}(\mathcal{A}\otimes\mathcal{B}) \cong \mathcal{C}(\mathcal{A})\bar{\otimes}\mathcal{C}(\mathcal{B})$ is Dedekind complete. By Proposition 3.6 of [8], $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\mathcal{B})$ are Dedekind complete. From Theorem 1.13, $\mathcal{A}$ and $\mathcal{B}$ are complete. It remains to show that one of the Boolean algebras is finite. However, the Dedekind completeness of $\mathcal{C}(\mathcal{A})\bar{\otimes}\mathcal{C}(\mathcal{B})$ implies that $\mathcal{C}(\mathcal{A}) \cong c_{00}(I)$ for a set $I \subseteq \mathcal{C}(\mathcal{A})$ or $\mathcal{C}(\mathcal{B}) \cong c_{00}(J)$ for a set $J \subseteq \mathcal{C}(\mathcal{B})$ (see Theorem 1.6). Since each Carathéodory space of place functions contains a unit, $\mathcal{C}(\mathcal{A})$ or $\mathcal{C}(\mathcal{B})$ is finite dimensional. Thus, $\mathcal{A}$ is finite or $\mathcal{B}$ is finite.

The sufficiency is proven analogously via Theorem 1.6. □

Corollary 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be nontrivial Boolean algebras. $\mathcal{C}(\mathcal{A})\bar{\otimes}\mathcal{C}(\mathcal{B})$ is Dedekind complete if and only if one of $\mathcal{A}$ or $\mathcal{B}$ is finite and the other is complete.*

Recall that for an Archimedean Riesz space E, its collection of bands, denoted $\mathcal{B}(E)$, forms a complete Boolean algebra. Our last application shows that for Archimedean Riesz spaces E and F, the set of bands in $E\bar{\otimes}F$ is rarely Boolean isomorphic to $\mathcal{B}(E) \otimes \mathcal{B}(F)$. That is, if E and F are infinite dimensional, not every band B of $E\bar{\otimes}F$ can be "decomposed" into the Fremlin tensor product of a band in E and a band in F.

Corollary 3.3. *Let E and F be infinite dimensional Archimedean Riesz spaces. Then $\mathcal{B}(E) \otimes \mathcal{B}(F)$ is not Boolean isomorphic to $\mathcal{B}(E\bar{\otimes}F)$.*

Proof. By Lemma 1.9, neither $\mathcal{B}(E)$ nor $\mathcal{B}(F)$ is finite. Then $\mathcal{B}(E) \otimes \mathcal{B}(F)$ is not complete by Problem 3.1. However, Theorem 1.7 states that the Boolean algebra of bands is complete for any Archimedean Riesz space, so $\mathcal{B}(E\bar{\otimes}F)$ is complete. ☐

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