PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 10, Pages 182–194 (May 3, 2023)
https://doi.org/10.1090/bproc/152

THE KAC FORMULA AND POINCARÉ RECURRENCE THEOREM IN RIESZ SPACES

YOUSSEF AZOUZI, MOHAMED AMINE BEN AMOR, JONATHAN HOMANN, MARWA MASMOUDI, AND BRUCE A. WATSON

(Communicated by Stephen Dilworth)

Abstract. Riesz space (non-pointwise) generalizations for iterative processes are given for the concepts of recurrence, first recurrence and conditional ergodicity. Riesz space conditional versions of the Poincaré Recurrence Theorem and the Kac formula are developed. Under mild assumptions, it is shown that every conditional expectation preserving process is conditionally ergodic with respect to the conditional expectation generated by the Cesàro mean associated with the iterates of the process. Applied to processes in $L^1(\Omega, \mathcal{A}, \mu)$, where μ is a probability measure, new conditional versions of the above theorems are obtained.

1. Introduction

Riesz space generalizations of stochastic processes have been well studied, see for example [10, 17, 18] for some of the earliest and recently [7]. In this setting analogues of the Hopf-Garsia, Birkhoff and Wiener-Kakutani-Yoshida ergodic theorems were given. More recently mixing processes were considered, see [14] and [8], which revisited the concept of ergodicity in Riesz space, see [9]. The current work builds on this foundation to consider, in the Riesz space setting, a Poincaré Recurrence Theorem and Kac formula for the (conditional) mean of the recurrence time of a conditionally ergodic process. We refer the reader to [6, pages 67-103] and [16, pages 33-48], for the measure theoretic (non-conditional) versions of these results. Fundamental to our considerations is the Riesz space analogue of the L^p spaces (in particular L^1), introduced in [11] and further studied in [2] and [14]. The measure theoretic version of this spatial extension with respect to a conditional expectation operator was considered in [3–5]. It is also shown, under mild assumptions, that every conditional expectation preserving process is conditionally ergodic with respect to the conditional expectation generated by the Cesàro mean associated with the iterates of the process. When applied to processes in $L^1(\Omega, \mathcal{A}, \mu)$, where μ is a probability measure, new conditional versions of the above theorems are obtained. In particular, the Riesz space version of the Kac formula yields a conditional Kac formula for measure preserving system.

Received by the editors July 22, 2022, and, in revised form, November 24, 2022, November 26, 2022, and November 29, 2022.

2020 *Mathematics Subject Classification.* Primary 47B60, 37A30, 47A35, 60A10.

This research was funded in part by the joint South Africa - Tunisia Grant (SA-NRF: SATN180717350298, Grant number 120112.)

The fifth author is the corresponding author.

The starting point of working with stochastic processes in a Riesz space is the definition of conditional expectation operators on Riesz spaces. We use the definition of [10], which when restricted to a probability space with the preferred weak order unit the a.e. constant 1 function yields the classical definition of a conditional expectation.

Definition 1.1. Let E be a Riesz space with weak order unit. A positive order continuous projection $T\colon E \to E$, with range, $R(T)$, a Dedekind complete Riesz subspace of E, is called a conditional expectation if Te is a weak order unit of E for each weak order unit e of E.

We recall, see [16], that $(\Omega, \mathcal{A}, \mu, \tau)$ is called a measure preserving system if $(\Omega, \mathcal{A}, \mu)$ is a probability space and $\tau : \Omega \to \Omega$ is a mapping with $\mu\left(\tau^{-1}(A)\right) = \mu(A)$ for each $A \in \mathcal{A}$. A Riesz space generalization of the concept of a measure preserving system was introduced in [8] as a conditional expectation preserving system, see below.

Definition 1.2. The 4-tuple, (E, T, S, e), is called a conditional expectation preserving system if E is a Dedekind complete Riesz space, e is a weak order unit for E, T is a conditional expectation operator in E with $Te = e$ and S is an order continuous Riesz homomorphism on E with $Se = e$ and $TS = T$.

By Freudenthal's theorem the condition $TSf = Tf$ for all $f \in E$ in Definition 1.2 is equivalent to $TSPe = TPe$ for all band projections P on E.

Let (E, T, S, e) be a conditional expectation preserving system. For $f \in E$ and $n \in \mathbb{N}$ we denote

$$S_n f := \frac{1}{n}\sum_{k=0}^{n-1} S^k f. \tag{1.1}$$

We set

$$L_S f := \lim_{n\to\infty} S_n f, \tag{1.2}$$

where the above limit is the order limit and $f \in \mathcal{E}_S$ where $\mathcal{E}_S$ is the set of $f \in E$ for which the above order limit exists. Thus $L_S : \mathcal{E}_S \to E$. The set of S-invariant $f \in E$ will be denoted $\mathcal{I}_S := \{f \in E \mid Sf = f\}$. By [9, Lemma 2.3], $\mathcal{I}_S \subset \mathcal{E}_S$ and $L_S f = f$ for all $f \in \mathcal{I}_S$. Further, it is easily seen that if $f \in \mathcal{E}_S$ and $L_S f = f$ then $f \in \mathcal{I}_S$, see [9, Theorem 2.4]. These ideas are further expanded in Section 2.

In a measure preserving system $(\Omega, \mathcal{A}, \mu, \tau)$, a set $B \in \mathcal{A}$ is said to be τ-invariant if $\mu(\tau^{-1}(B)\Delta B) = 0$, i.e. $\chi_B = \chi_{\tau^{-1}(B)}$ a.e. If we take $Sf := f \circ \tau$ then $\chi_{\tau^{-1}(B)} = S\chi_B$ and, in the $L^1(\Omega, \mathcal{A}, \mu)$ sense, τ invariance of B is equivalent to S invariance of χ_B. The measure preserving system $(\Omega, \mathcal{A}, \mu, \tau)$ said to be ergodic, see [16, page 42], if every τ-invariant set has measure 0 or 1. This is equivalent to χ_B being the zero or 1 constant function in the $L^1(\Omega, \mathcal{A}, \mu)$ sense. This can be equivalently stated as χ_B being a constant function in the $L^1(\Omega, \mathcal{A}, \mu)$ sense, since the 0 and the 1 functions are the only available possibilities in the a.e. sense. Taking T as the expectation operator on $L^1(\Omega, \mathcal{A}, \mu)$, with range the a.e. constant functions, we have that $(\Omega, \mathcal{A}, \mu, \tau)$ is ergodic if and only if $\chi_B \in R(T)$ for each S invariant χ_B. This equates in the Riesz space sense to each S invariant component of the weak order unit e being in $R(T)$. However, as S is a Riesz homomorphism, the S invariant elements form a Riesz subspace generated by the S invariant components of e. Applying Freudenthal's Theorem and using that e is a weak order unit, we have that $\mathcal{I}_S \subset R(T)$ if and only of each S invariant component of e is in $R(T)$. Thus we have Definition 1.3 of conditional ergodicity in a Riesz space, which is equivalent to that given in [9], since $L_S f = f$ if and only if $f \in \mathcal{I}_S$.

Definition 1.3 (Ergodicity)**.** The conditional expectation preserving system (E, T, S, e) is said to be T-conditionally ergodic if $\mathcal{I}_S \subset R(T)$.

The notion of recurrence appears in ergodic theory, but only a measure preserving system is required to define recurrence. In particular, see [16, Definition 3.1, page 34], for $(\Omega, \mathcal{A}, \mu, \tau)$ a measure preserving system and $B \in \mathcal{A}$, a point $x \in B$ is said to be recurrent with respect to B if there is a $k \in \mathbb{N}$ for which $\tau^k x \in B$. In this setting, see [16, Theorem 3.2, page 34], the Poincaré Recurrence Theorem gives that for each $B \in \mathcal{A}$ almost every point of B is recurrent with respect to B.

The concept of recurrence in a Riesz space will be given in terms of components of a chosen weak order unit. A component p of q which is a component of e will be said to be recurrent with respect to q if p can be decomposed into a countable sequence of components, each one of which maps to a component of q under some iterate of the map S. This is formally defined below.

Definition 1.4 (Recurrence)**.** Let (E, T, S, e) be a conditional expectation preserving system and p, q be components of e with $p \le q$. We say that p is recurrent with respect to q if there are components $p_n, n \in \mathbb{N}$, of p so that $\bigvee_{n\in\mathbb{N}} p_n = p$ and $S^n p_n \le q$ for each $n \in \mathbb{N}$.

To understand the connection with the classical measure theoretic definitions we consider the simplest case, that of $(\Omega, \mathcal{A}, \mu)$ a probability space with a set $B \in \mathcal{A}$ having $\mu(B) > 0$. We take as the Riesz space $E = L^1(\Omega, A, \mu)$, e as the function on Ω which takes the value 1 a.e. on Ω. If τ is a measure preserving transformation on Ω we take $Sf(x) = f(\tau(x))$ for each $x \in \Omega$ and $f \in E$. For $A, B \in \mathcal{A}$ with $B \subset A$ let $p = \chi_B$ and $q = \chi_A$ then we have that B is recurrent with respect to A if and only if B can be decomposed into a countable union of measurable sets B_n so that $\tau^n(B_n) \subset A$, which is equivalent to

$$B \subset \bigcup_{n=1}^{\infty} \tau^{-n}(A).$$

Here the Poincaré Recurrence Theorem gives that if $x \in A \in \mathcal{A}$ then $\tau^k x \in A$ for some $k \in \mathbb{N}$, and k is a time taken to recur. These ideas are taken to the Riesz space setting in Section 3 where a Riesz space version of the Poincarè Recurrence Theorem is presented in Theorem 3.2.

If x is recurrent with respect to A there will be multiple recurrence times, but what is of interest to us is the first recurrence time. The time of first entry of x into A is given by

$$n_A(x) := \inf\{n \in \mathbb{N} \,|\, \tau^n x \in A\},$$

for $x \in A$ and for $x \notin A$ we set $n_A(x) = 0$. The function n_A is called the first recurrence time with respect to A.

If we take

$$N_A(n) = \bigcap_{j=1}^{n} (A \setminus \tau^{-j}(A)) = \bigcup \{B \in \mathcal{A} \,|\, B \subset A,\, A \cap \tau^j(B) = \emptyset \,\forall j = 1, \ldots, n\},$$

then $N_A(n)$ is the set of points of A not recurrent in the first n iterates. Thus for $x \in \Omega$, the first recurrence time of x with respect to A is

$$n_A(x) = \sum_{\substack{x \in N_A(k-1) \setminus N_A(k) \\ k \in \mathbb{N}}} k,$$

where we have set $N_A(0) = A$. The Kac Formula [16, Theorem 4.6, page 46] gives that the spatial average over Ω of the first recurrence time with respect to A, with $\mu(A) > 0$, is 1, i.e. $\int_\Omega n_A \, d\mu = 1$.

We now extend these concepts to the Riesz space setting. For S a bijective Riesz homomorphism and p a component of e in E, we set

$$N_p(n) = \bigvee \{q \,|\, q \text{ a component of } p \text{ in } E, p \wedge S^j q = 0 \text{ for all } j = 1, \dots n\} \tag{1.3}$$

and

$$N_p(0) = p. \tag{1.4}$$

Here $N_p(n)$ is the maximal component of p to have no component recurrent with respect to p in under (less than) $n+1$ iterates of S. Further $N_p(n-1) - N_p(n)$ is the maximal component of p to be recurrent with respect to p in exactly n iterates of S. We thus define first recurrence time as

$$n_p = \sum_{k=1}^{\infty} k(N_p(k-1) - N_p(k)).$$

Here n_p exists in E_u^+, the positive cone of the universal completion of E, as $N_p(k-1) - N_p(k), k \in \mathbb{N}$, are disjoint components of e in E, see [15, page 363]. Having given a form for n_p which exists in E_u^+ we can conveniently rewrite n_p as

$$n_p = \sum_{k=0}^{\infty} N_p(k). \tag{1.5}$$

These concepts form the focus of Section 4 which culminates in Theorem 4.4, a conditional Riesz space version of Kac's Lemma which does not require the concept of ergodicity. For conditionally ergodic processes Theorem 4.4 yields Corollary 4.5, which is a conditional Riesz space analogue of the classical Kac Lemma.

We complete the work with an application of these results to probabilistic systems in Section 5. Of particular interest here is the extension of Kac's Lemma to conditionally ergodic processes.

2. Preliminaries

Background material on Riesz spaces can be found in [1]. We say that a positive operator U on the Riesz space, E, is strictly positive if $Uf = 0$ for $f \in E^+$ implies $f = 0$. We denote by $C_p(E)$ the set of components of p in E. We note that if e is a weak order unit for E, then E_e, the ideal in E generated by e, has an f-algebra structure in which e is the algebraic unit and for $p, q \in C_e(E)$ we have that $p \wedge q = pq$.

Lemma 2.1. *Let (E, T, S, e) be a conditional expectation preserving system with T strictly positive, then S is injective.*

Proof. As S is a Riesz homomorphism $S(|f|) = |Sf|$. Thus if $f \in E$ with $Sf = 0$, then $S(|f|) = |Sf| = 0$. Hence $T|f| = TS|f| = 0$, but T is strictly positive so $|f| = 0$ giving $f = 0$. □

For (E,T,S,e) a conditional expectation preserving system, if S is bijective then the map S^{-1} is a bijective lattice homomorphism which has $S^{-1}e=e$ and $TS^{-1}=T$ making $\left(E,T,S^{-1},e\right)$ a conditional expectation preserving system. Further, if $Sa\wedge Sb=0$ then $a\wedge b=0$ and Su is a component of Sx if and only if u is a component of x. In particular, S and S^{-1} map components of e to components of e.

Lemma 2.2. *Let S be a bijective Riesz homomorphism, p be a component of e and let a be a component of p. Then a has a unique decomposition $a=b+c$ in two disjoint components of a (and hence also of p and of e) with $p\wedge Sb=Sb$ and $p\wedge Sc=0$.*

Proof. Let $b=a\wedge S^{-1}p$ and $c=a\wedge S^{-1}(e-p)$. As $S^{-1}e=e$, and $p,e-p$ are components of e, we have that $S^{-1}p$ and $S^{-1}(e-p)$ are components of e. Thus b and c are components of a with $b\wedge c=a\wedge S^{-1}(p\wedge(e-p))=0$. Here $b+c=a\wedge S^{-1}(p+(e-p))=a\wedge S^{-1}e=a$ with $Sc=Sa\wedge(e-p)$ making $Sc\le e-p$ so $p\wedge Sc=0$ and $S^{-1}(p\wedge Sb)=S^{-1}p\wedge a\wedge S^{-1}p=a\wedge S^{-1}p=b$ giving $p\wedge Sb=Sb$.

As for uniqueness, if $a=b'+c'$ with $b'\wedge c'=0$, where b' and c' are components of a, with $Sb\wedge Sc=0$, $Sb'\wedge Sc'=0$, $p\wedge Sb'=Sb'$ and $p\wedge Sc'=0$ then $Sb+Sc=Sb'+Sc'$ and as

$$Sb=p\wedge(Sb+Sc)=p\wedge(Sb'+Sc')=Sb',$$

making $b=b'$ and thus $c=c'$. □

Note 2.3. If $Sa\le a$ or $a\le Sa$ for some component a, then $a=Sa$. To see this, observe that $T(a-Sa)=Ta-TSa=0$ but as $a-Sa\ge 0$ or $Sa-a\ge 0$ from the strict positivity of T this gives $Sa-a=0$.

The following result from [12, Corollary 2.3] is needed in the current work.

Lemma 2.4. *If T is a strictly positive conditional expectation operator on a Dedekind complete Riesz space with weak order unit $e=Te$, then for each $g\in E_+$ we have that $P_{Tg}\ge P_g$ where P_{Tg} and P_g denote the band projections onto the bands generated by Tg and g, respectively.*

The Dedekind complete Riesz space E is said to be universally complete with respect to T (T-universally complete) if, for each increasing net (f_α) in E_+ with (Tf_α) order bounded, we have that (f_α) is order convergent in E. In this case E is an $R(T)$ module and $R(T)$ is an f-algebra with the multiplication discussed earlier, see [14].

We recall Birkhoff's ergodic theorem for a T-universally complete Riesz space from [13, Theorem 3.9].

Theorem 2.5 (Birkhoff's (complete) ergodic theorem)**.** *Let (E,T,S,e) be a conditional expectation preserving system with T strictly positive and E T-universally complete then $E=\mathcal{E}_S$ and hence $L_S=SL_S$. In addition, $TL_S=T$ and L_S is a conditional expectation operator on E.*

Applying Theorem 2.5 to the concept of conditional ergodicity as defined in Definition 1.3 we have that (E,T,S,e) is conditionally ergodic if and only if $L_Sf=Tf$ for all $f\in\mathcal{I}_S$. This leads to the following characterization of conditional ergodicity.

Corollary 2.6. *Let (E,T,S,e) be a conditional expectation preserving system with T strictly positive and E T-universally complete. The conditional expectation preserving system (E,T,S,e) is conditionally ergodic if and only if $T = L_S$.*

Combining Theorem 2.5 and Corollary 2.6 we obtain Corollary 2.7 which is fundamental to the proof of a conditional Riesz space version of the Kac formula, Theorem 4.4.

Corollary 2.7. *If (E,T,S,e) is a conditional expectation preserving system with T strictly positive, E T-universally complete and S surjective then (E,T,S^{-1},e) is a conditional expectation preserving system, $E = \mathcal{E}_S = \mathcal{E}_{S^{-1}}$, $R(T) \subset R(L_S) = \mathcal{I}_S = \mathcal{I}_{S^{-1}} = R(L_{S^{-1}})$ and $L_{S^{-1}} = L_S$.*

Proof. As E is T-universally complete, by Theorem 2.5, $E = \mathcal{E}_S$.

As T is strictly positive, S is injective, so by the surjectivity assumption, S is bijective. Thus S^{-1} exists and is a Riesz homomorphism. Further $Se = e$ gives $e = S^{-1}e$ while from $TS = T$ we have $T = TSS^{-1} = TS^{-1}$ making (E,T,S^{-1},e) a conditional expectation preserving system and again $E = \mathcal{E}_{S^{-1}}$.

Finally $f \in \mathcal{I}_S$ if and only if $Sf = f$ which is equivalent to (after applying S^{-1}) $f = S^{-1}f$. Thus $\mathcal{I}_S = \mathcal{I}_{S^{-1}}$. For $f \in \mathcal{I}_S$ and $L_S f = f$ which gives $f = L_S f \in R(L_S)$ so $\mathcal{I}_S \subset R(L_S)$. However, from Theorem 2.5 $SL_S = L_S$ giving that $R(L_S) \subset \mathcal{I}_S$ so $R(L_S) = \mathcal{I}_S$. Hence $R(L_S) = \mathcal{I}_S = \mathcal{I}_{S^{-1}} = R(L_{S^{-1}})$.

For each component p of e with $p \in R(T)$ we have that $0 \le L_S p \le L_S e = e$ giving that $L_S p \in E_e$. Thus $(p - L_S p)^2 \in E_e$ and by the averaging property of L_S, along with $p^2 = p$, we have

$$(p - L_S p)^2 = p^2 - 2p \cdot L_S p + (L_S p)^2 = p - 2p \cdot L_S p + L_S(p \cdot L_S p)$$

which after applying T and using that $TL_S = T$, $Tp = p$ and the averaging property of T gives

$$\begin{aligned} T(p - L_S p)^2 &= Tp - 2T(p \cdot L_S p) + TL_S(p \cdot L_S p) \\ &= p - T(p \cdot L_S p) \\ &= p - p \cdot TL_S p \\ &= p - p \cdot Tp = p - p^2 = 0. \end{aligned}$$

From strict positivity of T, $(p - L_S p)^2 = 0$ giving $p = L_S p$. Hence each component of e which is in $R(T)$ is also in $R(L_S)$. But every element of $R(T)$ can be expressed as an order limit of a net of linear combinations of components of e which are in $R(T)$ and $R(L_S)$ is a Dedekind complete Riesz subspace of E. Hence $R(T) \subset R(L_S) = \mathcal{I}_S$. Now from [19], $L_S = L_{S^{-1}}$. □

3. Poincaré's recurrence theorem in Riesz spaces

We begin by characterizing recurrence, in Definition 1.4, for the case of S bijective.

Lemma 3.1. *Let (E,T,S,e) be a conditional expectation preserving system with S bijective, then a component p of q is recurrent with respect to q a component of e if and only if*

$$p \le \bigvee_{n\in\mathbb{N}} S^{-n}q.$$

Proof. For $n \in \mathbb{N}$, let

$$m_n = \bigvee \{g \in C_p(E) \,|\, S^n g \leq q\}.$$

Here m_n is the maximal component of p with $S^n m_n \leq q$ and hence the maximal component of p to recur at n iterates of S. So in terms of m_n, p is recurrent with respect to q if and only if $\bigvee_{n\in\mathbb{N}} m_n \geq p$. As S is bijective, $m_n = p \wedge S^{-n}q$, making p recurrent with respect to q if and only if $p \leq \bigvee_{n\in\mathbb{N}} p \wedge S^{-n}q$ from which the result follows. □

Theorem 3.2 (Poincaré). *Let (E, T, S, e) be a conditional expectation preserving system with T strictly positive and S surjective then each component p of q where q is a component of e is recurrent with respect to q.*

Proof. By Lemma 2.1 and the surjectivity of S, S is bijective. From Lemma 3.1, it suffices to prove that

$$p \leq \bigvee_{n\in\mathbb{N}} S^{-n}q.$$

This however follows from $q \leq \bigvee_{n=1}^{\infty} S^{-n}q$, which we prove below.

Here $\bigvee_{n=1}^{\infty} S^{-n}q$ exists in E as E is Dedekind complete and $S^{-n}q \leq S^{-n}e = e$ for all $n \in \mathbb{N}$. Further $\bigvee_{n=1}^{\infty} S^{-n}q$ is a component of e, giving that

$$r := q \wedge \left(e - \bigvee_{n=1}^{\infty} S^{-n}q\right) = q \wedge \bigwedge_{-\infty}^{j=-1} (e - S^j q) \geq 0$$

is a component of q. Thus $S^n r$ is a component of e (as S maps components of e to components of e). Since $Se = e$ we have

$$0 \leq q \wedge S^n r = q \wedge S^n q \wedge \bigwedge_{-\infty}^{j=n-1} \left(e - S^j q\right) \leq q \wedge (e - q) = 0$$

which, after application of S^{-n}, gives $r \wedge S^{-n}q = 0$ for all $n \geq 1$. However $0 \leq r \leq q$ so $r \wedge S^{-n}r = 0$ for all $n \geq 1$. Applying S^{n+k} to the above gives $S^{n+k}r \wedge S^k r = 0$ for all $n \in \mathbb{N}$, $k \geq 0$, making $(S^n r)_{n\geq 0}$ a sequence of disjoint components of e. Thus

$$\sum_{n=1}^{N} S^n r = \bigvee_{n=1}^{N} S^n r \leq e$$

for all $N \in \mathbb{N}$. Now, applying T to the above equation gives

$$NTr = \sum_{n=1}^{N} Tr = \sum_{n=1}^{N} TS^n r \leq Te = e$$

for all $N \in \mathbb{N}$. Since E is Archimedean, this gives $Tr = 0$. As T is strictly positive and $r \geq 0$ it follows that $r = 0$. □

4. Kac's formula in Riesz spaces

Lemma 4.1. *Let (E, T, S, e) be a conditional expectation preserving system with T strictly positive and S surjective then*

$$N_p(n) = p \bigwedge_{j=1}^{n} (e - S^{-j}p) \tag{4.1}$$

and

$$n_p = \sum_{k=1}^{\infty} kS^{-k}p \wedge N_p(k-1). \tag{4.2}$$

Proof. From (1.3) we have

$$\begin{aligned} N_p(n) &= \bigvee \{q \in C_p(E) \,|\, S^{-j}p \wedge q = 0 \forall j = 1, \dots n\} \\ &= \bigvee \{q \in C_p(E) \,|\, (e - S^{-j}p) \geq q \forall j = 1, \dots n\} \\ &= \bigwedge_{j=1}^{n} p \wedge (e - S^{-j}p) \end{aligned}$$

and hence (4.1). Further

$$\begin{aligned} N_p(k-1) - N_p(k) &= p \wedge (e - (e - S^{-k}p)) \wedge \bigwedge_{j=1}^{k-1} (e - S^{-j}p) \\ &= S^{-k}p \wedge N_p(k-1) \end{aligned}$$

from which (4.2) follows. □

Lemma 4.2. *Let (E, T, S, e) be a conditional expectation preserving system where T is strictly positive and S is surjective. Let L_S be as defined in Theorem 2.5. If E is T-universally complete then for each component p of e we have that $n_p \in E$ and*

$$L_S n_p = L_S \left(\bigvee_{k=0}^{\infty} S^{-k}p \right).$$

Proof. Let $r \in C_e(E)$. For convenience we set $\bigwedge_{j=1}^{0} S^{-j}(e - r) = e$. We begin by proving by induction on $n \in \mathbb{N}$ that

$$L_S r = \sum_{k=1}^{n} L_S \left((e - r) \wedge \bigwedge_{j=1}^{k} S^{-j}r \right) + L_S \left(\bigwedge_{k=0}^{n} S^{-k}r \right). \tag{4.3}$$

Since $L_S S = S L_S = L_S$ we have $L_S = L_S S S^{-1} = L_S S^{-1}$ and thus

$$L_S r = L_S S^{-1} r = L_S((e - r) \wedge S^{-1}r) + L_S(r \wedge S^{-1}r),$$

from which it follows that (4.3) holds for $n = 1$. Further,

$$\begin{aligned}
L_S\left(\bigwedge_{k=0}^{n} S^{-k}r\right) &= L_S S\left(\bigwedge_{k=0}^{n} S^{-k}r\right) \\
&= L_S\left(\bigwedge_{k=1}^{n+1} S^{-k}r\right) \\
&= L_S\left((e-r)\wedge\bigwedge_{k=1}^{n+1} S^{-k}r\right) + L_S\left(r\wedge\bigwedge_{k=1}^{n+1} S^{-k}r\right) \\
&= L_S\left((e-r)\wedge\bigwedge_{k=1}^{n+1} S^{-k}r\right) + L_S\left(\bigwedge_{k=0}^{n+1} S^{-k}r\right)
\end{aligned}$$

which if (4.3) holds for n gives that (4.3) holds for $n+1$. Hence giving that (4.3) holds by induction for all $n \in \mathbb{N}$.

For $p \in C_e(E)$ we set $r = e - p$ in (4.3) to give

$$\begin{aligned}
L_S(e-p) &= \sum_{k=1}^{n} L_S\left(p\wedge\bigwedge_{j=1}^{k} S^{-j}(e-p)\right) + L_S\left(\bigwedge_{k=0}^{n} S^{-k}(e-p)\right) \\
&= \sum_{k=1}^{n} L_S(N_p(k)) + L_S\left(\bigwedge_{k=0}^{n} S^{-k}(e-p)\right).
\end{aligned}$$

So by (1.4),

$$e = L_S(e-p) + L_S p = \sum_{k=0}^{n} L_S(N_p(k)) + L_S\left(\bigwedge_{k=0}^{n} S^{-k}(e-p)\right). \tag{4.4}$$

Here we note that $\left(\sum_{k=0}^{n} N_p(k)\right)_{n\in\mathbb{N}_0}$ is an increasing sequence in E_u^+ and by (4.4), $T\left(\sum_{k=0}^{n} N_p(k)\right) \le e$, so from the T-universal completeness of E, $\left(\sum_{k=0}^{n} N_p(k)\right)_{n\in\mathbb{N}_0}$ converges in order in E, i.e. $n_p = \sum_{k=0}^{\infty} N_p(k) \in E$. Now taking the order limit as $n \to \infty$ in (4.4) gives

$$e = L_S n_p + L_S\left(\bigwedge_{k=0}^{\infty} S^{-k}(e-p)\right). \tag{4.5}$$

However $Se = e$ so $e = S^{-k}e$ giving $S^{-k}(e-p) = e - S^{-k}p$ and thus

$$\bigwedge_{k=0}^{\infty} S^{-k}(e-p) = \bigwedge_{k=0}^{\infty} (e - S^{-k}p) = e - \bigvee_{k=0}^{\infty} S^{-k}p. \tag{4.6}$$

Combining (4.5) and (4.6), and using that $L_S e = e$ we have

$$e = L_S n_p + e - L_S\left(\bigvee_{k=0}^{\infty} S^{-k}p\right)$$

from which the lemma follows. □

As $TL_S = T$, applying T to

$$L_S n_p = L_S\left(\bigvee_{k=0}^{\infty} S^{-k}p\right)$$

in Lemma 4.2 gives Corollary 4.3.

Corollary 4.3. *Let (E,T,S,e) be a conditional expectation preserving system where T is strictly positive and S is surjective. If E is T-universally complete then for each component p of e we have that $n_p \in E$ and*

$$Tn_p = T\left(\bigvee_{k=0}^{\infty} S^{-k}p\right).$$

Example. Take $E = \ell^1(4)$ with $e(x) = 1$ for all $x \in \{1,2,3,4\}$, $\tau(x) = \begin{cases} 2, & x = 1 \\ 1, & x = 2, \\ 4, & x = 3, \\ 3, & x = 4 \end{cases}$ with components of e, $p_i(x) = \delta_i(x)$ where δ is the Kronecker symbol. The surjective Riesz homomorphism S on E is given by $Sf(x) = f(\tau(x))$. Set $Tp_i = \frac{1}{2}(p_1 + p_2)$ if $i \in \{1,2\}$ and $Tp_i = \frac{1}{2}(p_3 + p_4)$ if $i \in \{3,4\}$. Take $q_1 = p_1 + p_2$ and $q_2 = p_1 + p_3$.

(a) A direct computation gives $n_{q_1} = q_1$ with $Tq_1 = Tn_{q_1}$ here $\bigvee_{k=0}^{\infty} S^{-k}q_1 = q_1$ so indeed verifying Kac.

(b) Now consider $n_{q_2} = 2q_2$ and $Tq_2 = \frac{1}{2}e$ so $Tn_{q_2} = 2Tq_2$ but here $\bigvee_{k=0}^{\infty} S^{-k}q_2 = e$ so Kac gives $Tn_{q_2} = Te = 2Tq_2$ indeed verifying Kac.

This example shows that the conditional version of the Kac formula is not as simple as that for expectations (where only case (b) appears).

Theorem 4.4 (Kac)**.** *Let (E,T,S,e) be a conditional expectation preserving system, where T is strictly positive, E is T-universally complete and S is surjective. Let L_S be as defined in Theorem 2.5. For each p a component of e we have*

$$L_S n_p = P_{L_S p} e.$$

Proof. Let

$$w = \bigvee_{k=0}^{\infty} S^{-k}p.$$

From Lemma 4.2, we have $L_S n_p = L_S w$, so it remains to show that $L_S w = P_{L_S p} e$. We recall that $R(L_S) = \mathcal{I}_S = \mathcal{I}_{S^{-1}} = R(L_{S^{-1}})$. Here $w \le e$ is a component of e and

$$S^{-1}w = \bigvee_{k=1}^{\infty} S^{-k}p \le \bigvee_{k=0}^{\infty} S^{-k}p = w.$$

Thus $w \le Sw$, so, by Note 2.3, $Sw = w$ giving $w \in \mathcal{I}_S = R(L_S)$ and so $L_S w = w$.

From the definition of w we have $w \ge p$ so $w = L_S w \ge L_S p$ and thus $w = P_w e \ge P_{L_S p} e$. But $P_{L_S p} e \ge P_p e = p$, Lemma 2.4. Here $P_{L_S p} e \in R(L_S) = \mathcal{I}_S = \mathcal{I}_{S^{-1}}$, so $P_{L_S p} e$ is S^{-1} invariant. Hence

$$P_{L_S p} e = S^{-k} P_{L_S p} e \ge S^{-k} p$$

for each $k = 0,1,2,\dots$. Taking suprema over $k = 0,1,2,\dots$ gives

$$P_{L_S p} e \ge w.$$

Hence $P_{L_S p} e = w$. □

Applying T to the above result gives

$$Tn_p = TP_{L_S p} e$$

while in the case of (E, T, S, e) being conditionally ergodic $L_S = T$ so Theorem 4.4 gives Corollary 4.5.

Corollary 4.5 (Kac)**.** *Let (E, T, S, e) be a conditionally ergodic conditional expectation preserving system, where T is strictly positive, E is T-universally complete and S is surjective. For each p a component of e we have*

$$Tn_p = P_{Tp} e.$$

5. Application to probabilistic processes

Consider a probability space $(\Omega, \mathcal{A}, \mu)$ where μ is a complete measure (i.e. all subsets of sets of measure zero are measurable). Take Σ a sub-σ-algebra of $\mathcal{A}$. Let E denote the space of a.e. equivalence classes of measurable functions $f : \Omega \to \mathbb{R}$ for which the sequence $(\mathbb{E}[\min(|f(x)|, \mathbf{n})|\Sigma])_{n\in\mathbb{N}}$ is bounded above by an a.e. finite valued measurable function. Here $\mathbf{n}$ is the (equivalence class of the) function with value n a.e. Then E is the natural domain of the conditional expectation operator $T = \mathbb{E}[\cdot|\Sigma]$. Here $L^1(\Omega, \mathcal{A}, \mu) \subset E$ and the extension of $\mathbb{E}[\cdot|\Sigma]$ to E is given by the a.e. pointwise limits

$$Tf = \lim_{n\to\infty} \mathbb{E}[\min(f^+, \mathbf{n})|\Sigma] - \lim_{n\to\infty} \mathbb{E}[\min(f^-, \mathbf{n})|\Sigma], \quad f \in E.$$

The space E is a T-universally complete Riesz space with weak order unit $\mathbf{1}$ and T is a strictly positive Riesz space conditional expectation operator on E having $T\mathbf{1} = \mathbf{1}$. If $\Sigma = \{A \subset \Omega \,|\, \mu(A) = 0 \text{ or } \mu(A) = 1\}$ then T is the expectation operator and $E = L^1(\Omega, \mathcal{A}, \mu)$.

Let $\tau : \Omega \to \Omega$ be a map with $\tau^{-1}(A) \in \mathcal{A}$ (i.e. τ is μ-measurable) and $\mathbb{E}[\chi_{\tau^{-1}(A)}|\Sigma] = \mathbb{E}[\chi_A|\Sigma]$, for all $A \in \mathcal{A}$. Further we require that for each $A \in \mathcal{A}$ there is $B_A \in \mathcal{A}$ so that $\mu(A \Delta \tau^{-1}(B_A)) = 0$. Now $Sf := f \circ \tau$ is a surjective Riesz homomorphism on E with $S\mathbf{1} = \mathbf{1}$ and $TS = T$. The system (E, T, S, e) is a conditional expectation preserving system, with S surjective. Theorem 2.5 gives that

$$L_S f = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} S^k f = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ \tau^k$$

converges a.e. pointwise to a conditional expectation operator on E (which when restricted to $L^1(\Omega, \mathcal{A}, \mu)$ is a classical conditional expectation operator). The system (E, T, S, e) is conditionally ergodic if and only if $L_S = T$ which is equivalent to $\tau^{-1}(A) = A$ with $A \in \mathcal{A}$, if and only if $A \in \Sigma$. Here $n_A = n_{\chi_A}$ in the a.e. sense on A, and applying Corollary 4.5 we obtain that

$$\mathbb{E}[n_A|\Sigma] = \chi_{\{x\in\Omega|\mathbb{P}[A|\Sigma](x)>0\}},$$

where $\mathbb{P}[A|\Sigma] = \mathbb{E}[\chi_A|\Sigma]$ is the conditional probability of A given Σ.

References

[1] Charalambos D. Aliprantis and Kim C. Border, *Infinite-dimensional analysis*, 2nd ed., Springer-Verlag, Berlin, 1999. A hitchhiker's guide, DOI 10.1007/978-3-662-03961-8. MR1717083

[2] Youssef Azouzi and Mejdi Trabelsi, *L^p-spaces with respect to conditional expectation on Riesz spaces*, J. Math. Anal. Appl. **447** (2017), no. 2, 798–816, DOI 10.1016/j.jmaa.2016.10.013. MR3573115

[3] J. J. Grobler and B. de Pagter, *Operators representable as multiplication-conditional expectation operators*, J. Operator Theory **48** (2002), no. 1, 15–40. MR1926042

[4] C. B. Huijsmans and B. de Pagter, *Averaging operators and positive contractive projections*, J. Math. Anal. Appl. **113** (1986), no. 1, 163–184, DOI 10.1016/0022-247X(86)90340-9. MR826666

[5] P. G. Dodds, C. B. Huijsmans, and B. de Pagter, *Characterizations of conditional expectation-type operators*, Pacific J. Math. **141** (1990), no. 1, 55–77. MR1028265

[6] Tanja Eisner, Bálint Farkas, Markus Haase, and Rainer Nagel, *Operator theoretic aspects of ergodic theory*, Graduate Texts in Mathematics, vol. 272, Springer, Cham, 2015, DOI 10.1007/978-3-319-16898-2. MR3410920

[7] Jacobus J. Grobler, *Stopped processes and Doob's optional sampling theorem*, J. Math. Anal. Appl. **497** (2021), no. 1, Paper No. 124875, 13, DOI 10.1016/j.jmaa.2020.124875. MR4192217

[8] Jonathan Homann, Wen-Chi Kuo, and Bruce A. Watson, *A Koopman–von Neumann type theorem on the convergence of Cesàro means in Riesz spaces*, Proc. Amer. Math. Soc. Ser. B **8** (2021), 75–85, DOI 10.1090/bproc/75. MR4214338

[9] Jonathan Homann, Wen-Chi Kuo, and Bruce A. Watson, *Ergodicity in Riesz spaces*, Positivity and its applications, Trends Math., Birkhäuser/Springer, Cham, [2021] ©2021, pp. 193–201, DOI 10.1007/978-3-030-70974-7_9. MR4383625

[10] Wen-Chi Kuo, Coenraad C. A. Labuschagne, and Bruce A. Watson, *Discrete-time stochastic processes on Riesz spaces*, Indag. Math. (N.S.) **15** (2004), no. 3, 435–451, DOI 10.1016/S0019-3577(04)80010-7. MR2093170

[11] Wen-Chi Kuo, Coenraad C. A. Labuschagne, and Bruce A. Watson, *Conditional expectations on Riesz spaces*, J. Math. Anal. Appl. **303** (2005), no. 2, 509–521, DOI 10.1016/j.jmaa.2004.08.050. MR2122234

[12] W.-C. Kuo, C. C. A. Labuschagne, and B. A. Watson, *A zero-one law for Riesz space and fuzzy processes*, Fuzzy Logic, Soft Computing and Computational Intelligence, vol. I, pp. 393–397, Eleventh International Fuzzy Systems Association World Congress, July 28–31, 2005, Tsinghua University Press/Springer, Beijing, 2005.

[13] Wen-Chi Kuo, Coenraad C. A. Labuschagne, and Bruce A. Watson, *Ergodic theory and the strong law of large numbers on Riesz spaces*, J. Math. Anal. Appl. **325** (2007), no. 1, 422–437, DOI 10.1016/j.jmaa.2006.01.056. MR2273535

[14] Wen-Chi Kuo, Michael J. Rogans, and Bruce A. Watson, *Mixing inequalities in Riesz spaces*, J. Math. Anal. Appl. **456** (2017), no. 2, 992–1004, DOI 10.1016/j.jmaa.2017.07.035. MR3688462

[15] W. A. J. Luxemburg and A. C. Zaanen, *Riesz spaces. Vol. I*, North-Holland Mathematical Library, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1971. MR0511676

[16] Karl Petersen, *Ergodic theory*, Cambridge Studies in Advanced Mathematics, vol. 2, Cambridge University Press, Cambridge, 1983, DOI 10.1017/CBO9780511608728. MR833286

[17] Gh. Stoica, *Martingales in vector lattices*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **34(82)** (1990), no. 4, 357–362. MR1309536

[18] Vladimir G. Troitsky, *Martingales in Banach lattices*, Positivity **9** (2005), no. 3, 437–456, DOI 10.1007/s11117-004-2769-1. MR2188530

[19] Bruce A. Watson, *An Andô-Douglas type theorem in Riesz spaces with a conditional expectation*, Positivity **13** (2009), no. 3, 543–558, DOI 10.1007/s11117-008-2239-2. MR2511240

LATAO Laboratory, Faculty of Mathematical, Physical and Natural Sciences of Tunis, Tunis-El Manar University, 2092 El Manar, Tunisia
Email address: josefazouzi@gmail.com

LATAO Laboratory, Faculty of Mathematical, Physical and Natural Sciences of Tunis, Tunis-El Manar University, 2092 El Manar, Tunisia
Email address: mohamedamine.benamor@ipest.rnu.tn

School of Mathematics, University of the Witwatersrand, Private Bag 3, P.O. WITS 2050, South Africa
Email address: jmhomann@gmail.com

LATAO Laboratory, Faculty of Mathematical, Physical and Natural Sciences of Tunis, Tunis-El Manar University, 2092 El Manar, Tunisia
Email address: marwa_masmoudi@hotmail.com

School of Mathematics, University of the Witwatersrand, Private Bag 3, P.O. WITS 2050, South Africa
Email address: b.alastair.watson@gmail.com