PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 10, Pages 195–207 (May 12, 2023)
https://doi.org/10.1090/bproc/168

A SHORT NOTE ON SIMPLICIAL STRATIFICATIONS

DOMINIK J. WRAZIDLO

(Communicated by Julie Bergner)

Abstract. We show that the simplicial stratification associated to a triangulation of a PL pseudomanifold possesses a canonical system of trivializations of link bundles that satisfies a natural compatibility condition over nested singular strata. Consequently, Agustín Vicente and Fernández de Bobadilla's generalization of Banagl's intersection space construction is applicable to all PL pseudomanifolds (and in particular, to all complex algebraic varieties).

1. Introduction

The purpose of this paper is to show that Agustín Vicente and Fernández de Bobadilla's generalization [1] of Banagl's intersection space construction applies to every triangulated PL pseudomanifold which is equipped with its simplicial stratification and a canonical system of trivializations of link bundles that we construct. Since any complex algebraic variety can be canonically Whitney stratified (and hence triangulated, see [12]), we can therefore assign to it the family of intersection spaces associated to all possible triangulations. The informational content of this family, seen as an invariant of the complex algebraic variety, can be subject to future study.

Many spaces arising in topology and geometry are not manifolds, but have singularities, so that essential tools like classical Poincaré duality on (co)homology are no longer available. To overcome this issue for the important class of stratified pseudomanifolds, Goresky and MacPherson introduced intersection (co)homology groups that depend on an additional parameter called a perversity (see [13, 14]). As an important feature of intersection homology theory, they showed that intersection (co)homology groups of stratified pseudomanifolds satisfy a generalized form of Poincaré duality with respect to complementary perversities. More recently, intersection spaces were introduced by Banagl in [3, 4] as a spatial imitation of the intersection chain complex that underlies the definition of intersection homology groups. While the singular homology of intersection spaces is not isomorphic to intersection homology of singular spaces, Banagl showed that both theories are related by mirror symmetry on Calabi-Yau 3-folds.

Received by the editors October 18, 2022, and, in revised form, March 22, 2023, March 23, 2023, and March 23, 2023.

2020 *Mathematics Subject Classification.* Primary 55N33, 57N80, 32S60, 32S20, 57P10.

Key words and phrases. Singularities, stratified spaces, intersection homology, Poincaré duality, intersection spaces, triangulations.

The author was supported by a Feodor Lynen Return Fellowship of the Alexander von Humboldt Foundation.

The method of intersection spaces assigns generalized reduced Poincaré duality spaces depending on a perversity to certain stratified pseudomanifolds by performing a homotopy theoretic modification in a neighborhood of the singular set. The missing fundamental class of these generalized reduced Poincaré duality spaces has been constructed for certain depth one stratifications (see [17] and [20]). By construction, intersection spaces of stratified spaces depend on the choice of homology truncations (Moore approximations) of (recursively modified) links. Over the rationals, results in the isolated singularities case (see [19]) suggest nevertheless that different choices of homology truncations might still result in intersection spaces with weakly equivalent differential graded algebras. In [14], Goresky and MacPherson show that intersection homology of a singular space is independent of the underlying stratification. On the other hand, the Betti numbers of intersection spaces depend in general heavily on the choice of the stratification.

Intersection spaces are rich homotopy theoretic invariants of stratified spaces, and their rational cohomology has been studied from the perspectives of linear algebra [11], sheaf theory [1, 6, 9, 10, 18], Sullivan's PL polynomial differential forms [19], smooth differential forms [5], and L^2 harmonic forms [7]. However, the existence of intersection spaces is generally obstructed, and the obstruction vanishes for example for spaces having one manifold stratum with trivial link budle (see e.g. Banagl-Chriestenson [8]). In [1], Agustín Vicente and Fernández de Bobadilla extended Banagl's construction to make it applicable to stratifications of arbitrary depth with compatibly trivializable link bundles. They adopt a relative viewpoint by providing a recursive construction of *intersection space pairs*, which are space pairs whose relative cohomology generalizes the cohomology of intersection spaces. Their construction requires as input a stratified space with a *trivial conical structure* (see Definition 2.8), that is, a choice of a system of trivializations of link bundles satisfying a natural compatibility condition over nested singular strata that is shown in diagram (2.12). So far, only specific classes of stratified pseudomanifolds with trivial link bundles are known to satisfy this compatibility condition, and can hence serve as possible input for the intersection space pair construction of [1]. For instance, toric varieties are known to admit a stratification together with an appropriate system of trivializations of link bundles, which are both naturally induced by the torus action (see [1, Corollary 3.32]). We also point out that any stratified pseudomanifold with trivial link bundles (including the case that all singular strata are contractible, see e.g. Corollary 9.9 in [1]) possesses an *intersection space complex*, which is constructed in [1] as a sheaf theoretic counterpart of the intersection space pair without requiring the compatibility condition for trivializations of link bundles. However, the intersection space complex is a much weaker invariant than the intersection space pair itself. (For example, it does not capture the rational homotopy type or even the cohomological cup product structure of the intersection space pair.)

In this paper, we show that the intersection space pair construction of [1] applies to every triangulated PL pseudomanifold which is equipped with its simplicial stratification and the canonical system of trivializations of link bundles that we construct. Recall that an n-dimensional PL pseudomanifold X is by definition a PL-space such that for some (and hence every) triangulation of X by a simplicial complex, every simplex is contained in an n-simplex, and every $(n-1)$-simplex is a face of exactly two n-simplices. Moreover, the *simplicial stratification* associated to a fixed triangulation of X is given by the filtration

$$X = X_n \supset X_{n-2} \supset \cdots \supset X_0 \tag{1.1}$$

whose ith part X_i is the union of all closed i-simplices of the triangulation. Thus, the singular strata of dimension i are the open i-simplices of the triangulation, and the regular part $X \setminus X_{n-2}$ is the union of all open n- and $(n-1)$-simplices. Since the singular strata of a simplicial triangulation are contractible, all link bundles are automatically trivial. In our main result below, we employ the first barycentric subdivision of the given triangulation to construct in a canonical way a system of link bundles and trivializations that give rise to a trivial conical structure. The compatibility condition for the system of trivializations of link bundles will follow essentially from associativity of the join operation.

The main result of this paper, which we prove in Section 4, is the following.

Theorem 1.1. *Let X be a PL pseudomanifold. Fix a triangulation T of X, and let $\mathcal{X}_T$ be the simplicial stratification* (1.1) *of X associated to T. Then, the stratified space $(X, \mathcal{X}_T)$ possesses a canonical trivial conical structure in the sense of Definition* 2.8.

As an immediate consequence of Theorem 1.1, we obtain

Corollary 1.2. *The intersection space pair construction of Agustín Vicente and Fernández de Bobadilla* [1] *is applicable to all PL pseudomanifolds by choosing the simplicial stratification associated to a triangulation and our system of canonical trivializations of link bundles.*

2. Compatibility for trivial link bundles of stratified spaces

2.1. **Stratified spaces.** In this section, we recall the notion of a topologically stratified space to fix terminology.

Definition 2.1 (Topologically stratified spaces)**.** A 0-*dimensional topologically stratified space* X is a countable set with the discrete topology. For $d > 0$, a d-*dimensional topologically stratified space* is a paracompact Hausdorff topological space X equipped with a filtration

$$X = X_d \supset X_{d-1} \supset \cdots \supset X_0 \supset X_{-1} = \varnothing \tag{2.1}$$

of X by closed subsets X_j such that $X_d \setminus X_{d-1} \neq \varnothing$, for every $0 \leq j \leq d$ the complement $X_j \setminus X_{j-1}$ is a j-dimensional topological manifold (that is, a second countable Hausdorff topological space that is locally Euclidean), and for each point $x \in X_j \setminus X_{j-1}$ there exist an open neighborhood U of x in X, a compact $(d-j-1)$-dimensional topologically stratified space L (called a *link* of $X_j \setminus X_{j-1}$ at x in X) with filtration

$$L = L_{d-j-1} \supset L_{d-j-2} \supset \cdots \supset L_0 \supset L_{-1} = \varnothing, \tag{2.2}$$

and a homeomorphism $\phi\colon U \xrightarrow{\cong} \mathbb{R}^j \times c^\circ(L)$, where $c^\circ(Z) = (Z \times [0,1))/(Z \times \{0\})$ is the open cone on a topological space Z (with the convention that $c^\circ(\varnothing)$ consists only of the vertex point), such that ϕ takes $U \cap X_{j+i+1}$ homeomorphically onto $\mathbb{R}^j \times c^\circ(L_i)$ $(\subset \mathbb{R}^j \times c^\circ(L))$ for $-1 \leq i \leq d-j-1$.

Remark 2.2 (Topological stratified pseudomanifolds). A d-dimensional topologically stratified space X with filtration (2.1) is called a *d-dimensional topological stratified pseudomanifold* if $X_{d-1} = X_{d-2}$ and $X \setminus X_{d-1}$ is dense in X (see e.g.

Definition 4.1.1 in [2]). If X is a topological stratified pseudomanifold, then it follows that the links with filtration (2.2) are topological stratified pseudomanifolds as well.

Remark 2.3. Note that the notion of pseudomanifold introduced in Definition 3.4 in [1] differs in the following two ways. First, it contains the additional assumption that the pairs (X_j, X_{j-1}) are locally finite relative CW-complexes. Second, as remarked right before [1, Definition 3.4], the singular set of a pseudomanifold is not assumed to have codimension at least 2.

From now on, we reserve the terminology "stratified space" to refer to a topologically stratified space. For a d-dimensional stratified space X with filtration (2.1), the connected components of the j-dimensional manifold $X_j \setminus X_{j-1}$ are called the *strata of dimension j (or of codimension $d-j$)*. The strata contained in $X_d \setminus X_{d-1}$ are called *regular*, and the other strata are called *singular*.

If V is a union of regular strata of X, then it is easy to see that the closure $\overline{V}^X$ of V in X is a d-dimensional stratified space with filtration

$$X_d \cap \overline{V}^X \supset X_{d-1} \cap \overline{V}^X \supset \cdots \supset X_0 \cap \overline{V}^X \supset X_{-1} \cap \overline{V}^X = \varnothing.$$

Moreover, by induction on the dimension of X, we can show that if X has singular strata of dimension j, then X_j is a j-dimensional stratified space with filtration

$$X_j \supset X_{j-1} \supset \cdots \supset X_0 \supset X_{-1} = \varnothing.$$

We write $\mathcal{S}(X)$ for the set of singular strata of a stratified space X with fixed filtration (2.1). For $S, S' \in \mathcal{S}(X)$, we write $S \prec S'$ if $S \neq S'$ and S is contained in (or, equivalently, has nonempty intersection with) the closure of S' in X. Then, note that we have $\dim(S) < \dim(S')$. We write $S \preceq S'$ if $S = S'$ or $S \prec S'$. Then, $(\mathcal{S}(X), \preceq)$ is a partially ordered set.

2.2. **Conical structures and trivializations.** The notion of conical structures was introduced by Agustín Vicente and Fernández de Bobadilla for pairs of topological spaces (X, Y) with respect to a stratified space X_{d-k} contained in Y (see Definition 3.5 of [1]). In Definition 2.4, we restate the definition of conical structures by using a different terminology, and only for the special case that X is a d-dimensional stratified space with subspace the stratified space $Y = X_{d-1}$ $(= X_{d-k})$ (compare Remark 3.14 in [1]). In the present paper, we show that triangulated PL pseudomanifolds give rise to conical structures that are *trivial* in the sense of Definition 2.8, and can hence be used as input for the intersection space pair construction of [1].

The following notation will be used in Definition 2.4. For a fiber bundle $\pi \colon E \to B$ with fiber F, we denote by $c(\pi) \colon c_\pi(E) \to B$ the associated cone bundle. That is, $c(\pi)$ is the fiber bundle over B with total space $c_\pi(E) = (E \times [0,1])/\sim_\pi$, where $(x,t) \sim_\pi (x',t')$ if and only if $\pi(x) = \pi(x')$ and $t = 0 = t'$, projection map $c(\pi)$ induced by the composition $E \times [0,1] \xrightarrow{\mathrm{pr}_E} E \xrightarrow{\pi} B$, and fiber $c(F) = (F \times [0,1])/(F \times \{0\})$ the closed cone on F. For subsets $A \subset E$, we obtain subspaces $c_\pi(A) \subset c_\pi(E)$ of the form $c_\pi(A) = (A \times [0,1])/\sim_\pi$. In particular, we can identify $c_\pi(\varnothing) = B$, which is the subspace of the cone points of the fibers because the cone on the empty set $\varnothing$ consists by convention only of the cone point. In the following, we consider B naturally as a subspace of $c_\pi(E)$ via $B = c_\pi(\varnothing) \subset c_\pi(E)$. Moreover, we consider E naturally as a subspace of $c_\pi(E)$ via $E = E \times \{1\} \subset c_\pi(E)$. Thus, we have $c(\pi)|_E = \pi$.

Definition 2.4 (Conical structures; compare Definition 3.5 in [1]). A *conical structure* for a d-dimensional stratified space X with stratification (2.1) is a family $\{(C_S, \partial\pi_S, \theta_S)\}_{S\in\mathcal{S}(X)}$ of triples consisting for every $S \in \mathcal{S}(X)$ of

- a closed neighborhood C_S of S in $X \setminus X_{\dim(S)-1}$ such that
  - for all $S, T \in \mathcal{S}(X)$ we have[1] $C_S \cap C_T \neq \varnothing$ if and only if $S \preceq T$ or $T \preceq S$,
- a fiber bundle $\partial\pi_S\colon \partial C_S \to S$ with total space ∂C_S the boundary of C_S in $X \setminus X_{\dim(S)-1}$, with structure group denoted by G_S, and with fiber X^S a compact stratified space of dimension $\operatorname{codim}(S) - 1$ with stratification

  $$X^S = X^S_{\operatorname{codim}(S)-1} \supset X^S_{\operatorname{codim}(S)-2} \supset \cdots \supset X^S_0 \supset X^S_{-1} = \varnothing,$$

  such that
  - for all $S, T \in \mathcal{S}(X)$ with $T \prec S$ there exists a pair (C^T_S, Z^T_S), where Z^T_S is a union of singular strata of X^T of codimension $\operatorname{codim}_X(S)$ in X^T, and C^T_S is a closed neighborhood of Z^T_S in $X^T \setminus X^T_{\dim(Z^T_S)-1}$ with boundary ∂C^T_S, such that the structure group G_T of $\partial\pi_T$ consists of homeomorphisms $\alpha\colon X^T \xrightarrow{\cong} X^T$ that satisfy $\alpha(Z^T_S) = Z^T_S$, $\alpha(C^T_S) = C^T_S$, and $\alpha(\partial C^T_S) = \partial C^T_S$, and the subbundles of $\partial\pi_T$ induced by the inclusions $Z^T_S \subset X^T$, $C^T_S \subset X^T$, and $\partial C^T_S \subset X^T$ have total spaces $\partial C_T \cap S$, $\partial C_T \cap C_S$, and $\partial C_T \cap \partial C_S$, respectively, and
- a homeomorphism $\theta_S\colon C_S \xrightarrow{\cong} c_{\partial\pi_S}(\partial C_S)$ that extends the identity maps on ∂C_S and S,

such that for all $T \prec S$ in $\mathcal{S}(X)$, we have

$$\theta_T(C_T \cap S) = c_{\partial\pi_T}(\partial C_T \cap S) \setminus T \tag{2.3}$$

and

$$\theta_T(C_T \cap C_S) = c_{\partial\pi_T}(\partial C_T \cap C_S) \setminus T, \tag{2.4}$$

the composition

$$\pi_S = c(\partial\pi_S) \circ \theta_S\colon C_S \to S$$

satisfies

$$(\pi_S)^{-1}(C_T \cap S) = C_T \cap C_S \tag{2.5}$$

and

$$(\pi_S)^{-1}(\partial C_T \cap S) = \partial C_T \cap C_S, \tag{2.6}$$

and the diagrams

$$\begin{tikzcd} C_T \cap C_S \arrow[r, "\pi_S|"] \arrow[d, "\cong"', "\theta_T|"] & C_T \cap S \arrow[d, "\cong"', "\theta_T|"] \\ (\partial C_T \cap C_S) \times (0,1] \arrow[r, "\pi_S| \times \mathrm{id}_{(0,1]}"] & (\partial C_T \cap S) \times (0,1] \end{tikzcd} \tag{2.7}$$

[1]This assumption implies that for $S, T \in \mathcal{S}(X)$ we have $C_T \cap S \neq \varnothing$ if and only if $T \preceq S$.

and

$$\begin{tikzcd} \partial C_T \cap C_S \arrow[r, "\pi_S|"] \arrow[dr, "\partial\pi_T|"'] & \partial C_T \cap S \arrow[d, "\partial\pi_T|"] \\ & T \end{tikzcd} \tag{2.8}$$

commute.

Remark 2.5. We point out that our diagrams (2.7) and (2.8) correspond to the properties (3) and (4) of Definition 3.5 in [1], respectively.

Remark 2.6. For $S \in \mathcal{S}(X)$, it follows from the definition of π_S that $\pi_S|_{\partial C_S} = \partial\pi_S$ because θ_S extends the identity map on ∂C_S. Hence, for all $T \prec S$ in $\mathcal{S}(X)$, (2.5) and (2.6) imply that

$$(\partial\pi_S)^{-1}(C_T \cap S) = C_T \cap \partial C_S \tag{2.9}$$

and

$$(\partial\pi_S)^{-1}(\partial C_T \cap S) = \partial C_T \cap \partial C_S, \tag{2.10}$$

respectively.

Remark 2.6 implies that, for all $T \prec S$ in $\mathcal{S}(X)$, diagram (2.8) restricts to a commutative diagram

$$\begin{tikzcd} \partial C_T \cap \partial C_S \arrow[r, "\partial\pi_S|"] \arrow[dr, "\partial\pi_T|"'] & \partial C_T \cap S \arrow[d, "\partial\pi_T|"] \\ & T. \end{tikzcd} \tag{2.11}$$

We note that the map $\partial\pi_S|$ in diagram (2.11) is a fiber bundle with fiber X^S.

Definition 2.7 (Trivializations of conical structures)**.** Let X be a d-dimensional stratified space with conical structure $\{(C_S, \partial\pi_S, \theta_S)\}_{S\in\mathcal{S}(X)}$. A *trivialization* of the conical structure $\{(C_S, \partial\pi_S, \theta_S)\}_{S\in\mathcal{S}(X)}$ is a family $\{\psi_S\}_{S\in\mathcal{S}(X)}$ of trivializations $\psi_S \colon \partial C_S \xrightarrow{\cong} X^S \times S$ of the fiber bundles $\partial\pi_S \colon \partial C_S \to S$ such that for all $T \prec S$ in $\mathcal{S}(X)$, there exists a homeomorphism $\beta_S^T \colon \partial C_S^T \xrightarrow{\cong} X^S \times Z_S^T$ such that the diagram

$$\begin{tikzcd} \partial C_T \cap \partial C_S \arrow[r, "\psi_S|", "\cong"'] \arrow[d, "\psi_T|", "\cong"'] & X^S \times (\partial C_T \cap S) \arrow[d, "\mathrm{id}_{X^S} \times \psi_T|", "\cong"'] \\ \partial C_S^T \times T \arrow[r, "\beta_S^T \times \mathrm{id}_T", "\cong"'] & X^S \times Z_S^T \times T \end{tikzcd} \tag{2.12}$$

commutes[2].

Definition 2.8. Let X be a d-dimensional stratified space. A *trivial conical structure* for X is a family $\{(C_S, \partial\pi_S, \theta_S, \psi_S)\}_{S\in\mathcal{S}(X)}$, where $\{(C_S, \partial\pi_S, \theta_S)\}_{S\in\mathcal{S}(X)}$ is a conical structure for X with trivialization $\{\psi_S\}_{S\in\mathcal{S}(X)}$.

[2] There exists a unique homeomorphism $\partial C_S^T \times T \xrightarrow{\cong} X^S \times Z_S^T \times T$ such that diagram (2.12) commutes, and by commutativity of diagram (2.11), this homeomorphism is of the form $(x, y) \mapsto (\beta_S^T(x, y), y)$. Thus, the claim means that $\beta_S^T(x, y)$ does not depend on y.

3. Link bundles in simplicial stratifications

For a background on locally finite simplicial complexes in $\mathbb{R}^\infty$, see Hudson [16, Chapter III]. We will not distinguish formally between a simplicial complex and its geometric realization as a subset of $\mathbb{R}^\infty$, which should not lead to any confusion. The join of two compact simplicial complexes A and B in $\mathbb{R}^\infty$ is denoted by $A * B$, where the notation implicitly means that A and B are joinable (see [16, p. 6]). Note that $A, B \subset A * B$, and the join operator is commutative and associative.

Let T be a locally finite simplicial complex. We introduce some terminology that is also used by Haefliger [15, Proposition 1.4]. We denote the barycenter of a closed simplex σ of T by $\widehat{\sigma}$. Note that we have $\sigma = \widehat{\sigma} * \partial\sigma$. Let T' denote the first barycentric subdivision of T. For a closed simplex σ of T, we define with respect to T'

- the *dual complex* $D'(\sigma)$ *of* σ to be the full simplicial subcomplex of T' consisting of those simplices whose vertices are the barycenters $\widehat{\tau}$ of the simplices τ of T containing σ,
- the *link* $L'(\sigma)$ *of* σ to be the full simplicial subcomplex of T' consisting of those simplices whose vertices are the barycenters $\widehat{\tau}$ of the simplices τ of T containing σ and different from σ, and
- the *star* $St'(\sigma)$ *of* σ to be the full[3] simplicial subcomplex of T' consisting of those simplices that are a face of a simplex of T' containing the barycenter $\widehat{\sigma}$ of σ.

Then, we have $D'(\sigma) = \widehat{\sigma} * L'(\sigma)$, $\partial St'(\sigma) = \partial\sigma * L'(\sigma)$, and $St'(\sigma) = \partial\sigma * D'(\sigma) = \widehat{\sigma} * \partial St'(\sigma)$.

Let σ be a closed simplex of T. For closed simplices α and β of T' with $\alpha \subset L'(\sigma)$ and $\beta \subset \partial\sigma$, we define the continuous map

$$\Phi_{\alpha,\beta} \colon \alpha \times (\widehat{\sigma} * \beta) \longrightarrow \alpha * \beta, \qquad \Phi_{\alpha,\beta}(a, t\widehat{\sigma} + (1-t)b) = ta + (1-t)b.$$

Remark 3.1. If $b_1, \dots, b_p$ are the vertices of β, then $\widehat{\sigma}, b_1, \dots, b_p$ are the vertices of the simplex $\widehat{\sigma} * \beta$ of T'. Hence, using barycentric coordinates, every point $x \in \widehat{\sigma} * \beta$ can be written uniquely as $x = x_0\widehat{\sigma} + \sum_{i=1}^p x_i b_i$ with $x_0, \dots, x_p \geq 0$ and $x_0 + \dots + x_p = 1$. Note that for $x_0 \neq 1$, we may write $x = t\widehat{\sigma} + (1-t)b$ with $t = x_0$ and $b = \sum_{i=1}^p \frac{x_i}{1-x_0} b_i \in \beta$. Thus, it follows that $\Phi_{\alpha,\beta}(a, x) = x_0 a + \sum_{i=1}^p x_i b_i$ for $a \in \alpha$ and $x \in \widehat{\sigma} * \beta$.

Note that the map $\Phi_{\alpha,\beta}$ restricts to a homeomorphism

$$\phi_{\alpha,\beta} \colon \alpha \times ((\widehat{\sigma} * \beta) \setminus \beta) \xrightarrow{\cong} (\alpha * \beta) \setminus \beta, \qquad \phi_{\alpha,\beta}(a, t\widehat{\sigma} + (1-t)b) = ta + (1-t)b.$$

By gluing the maps $\Phi_{\alpha,\beta}$ for $\alpha \subset L'(\sigma)$ and $\beta \subset \partial\sigma$, we obtain a continuous map

$$\Phi_\sigma \colon L'(\sigma) \times \sigma = L'(\sigma) \times (\widehat{\sigma} * \partial\sigma) \longrightarrow L'(\sigma) * \partial\sigma = \partial St'(\sigma).$$

Similarly, by gluing the homeomorphisms $\phi_{\alpha,\beta}$ for all closed simplices $\alpha \subset L'(\sigma)$ and $\beta \subset \partial\sigma$ of T', we obtain a homeomorphism

$$\phi_\sigma \colon L'(\sigma) \times \sigma^\circ = L'(\sigma) \times ((\widehat{\sigma} * \partial\sigma) \setminus \partial\sigma) \xrightarrow{\cong} (L'(\sigma) * \partial\sigma) \setminus \partial\sigma =: E'(\sigma),$$

where $\sigma^\circ := \sigma \setminus \partial\sigma$ and $E'(\sigma) := \partial St'(\sigma) \setminus \partial\sigma$. By construction, ϕ_σ is the restriction of Φ_σ to $L'(\sigma) \times (\sigma \setminus \partial\sigma)$.

[3] Indeed, let σ' be a simplex of T' whose vertices are all contained in $St'(\sigma)$. Then, we have $\sigma' \subset \tau$ for some simplex τ of T, where $\widehat{\tau}$ is a vertex of σ'. Since $\widehat{\tau} \in St'(\sigma)$, $\widehat{\tau}$ is a face (vertex) of a simplex ρ' of T' containing the barycenter $\widehat{\sigma}$ of σ.

Since the maps $\Phi_{\alpha,\beta}$ are simplicial, it follows that the map Φ_σ is a PL map. Consequently, ϕ_σ is a PL homeomorphism. (Recall that the property of being a PL map is stable under restriction to open subsets, and that a homeomorphism is a PL map if and only if its inverse is a PL map.)

Lemma 3.2. *Let $K \subset T'$ be a full simplicial subcomplex, that is, a simplicial subcomplex such that every simplex of T' whose vertices lie in K is contained in K. Then, for any closed simplices α and β of T' such that $\alpha * \beta$ is a simplex of T', we have $\alpha * \beta \subset K$ if and only if $\alpha \subset K$ and $\beta \subset K$.*

Proof. Since $\alpha, \beta \subset \alpha * \beta$, it is clear that $\alpha * \beta \subset K$ implies $\alpha, \beta \subset K$. Conversely, suppose that $\alpha, \beta \subset K$. Then, $\alpha * \beta$ is a simplex of T' whose vertices are contained in K, and it follows that $\alpha * \beta \subset K$. $\square$

Proposition 3.3. *Let $K \subset T'$ be a full simplicial subcomplex such that $\partial\sigma \subset K$. Then, the map Φ_σ restricts to a map*

$$\Phi_\sigma| \colon (L'(\sigma) \cap K) \times \sigma \longrightarrow \partial St'(\sigma) \cap K,$$

and the homeomorphism ϕ_σ restricts to a homeomorphism

$$\phi_\sigma| \colon (L'(\sigma) \cap K) \times \sigma^\circ \xrightarrow{\cong} E'(\sigma) \cap K.$$

Proof. Let α and β be closed simplices of T' with $\alpha \subset L'(\sigma)$ and $\beta \subset \partial\sigma$. Since $\beta \subset \partial\sigma \subset K$, Lemma 3.2 implies that we have $\alpha \in K$ if and only if $\alpha * \beta \in K$. Consequently, by gluing the maps $\Phi_{\alpha,\beta}$ for $\alpha \subset L'(\sigma) \cap K$ and $\beta \subset \partial\sigma$, we obtain a continuous map

$$(L'(\sigma) \cap K) \times \sigma = (L'(\sigma) \cap K) \times (\widehat{\sigma} * \partial\sigma) \longrightarrow \partial St'(\sigma) \cap K.$$

Similarly, by gluing the homeomorphisms $\phi_{\alpha,\beta}$ for $\alpha \subset L'(\sigma) \cap K$ and $\beta \subset \partial\sigma$, we obtain a homeomorphism

$$(L'(\sigma) \cap K) \times (\sigma \setminus \partial\sigma) = (L'(\sigma) \cap K) \times ((\widehat{\sigma} * \partial\sigma) \setminus \partial\sigma) \xrightarrow{\cong} (\partial St'(\sigma) \cap K) \setminus \partial\sigma.$$

$\square$

Proposition 3.4. *Let $K \subset T'$ be a full simplicial subcomplex such that $\widehat{\sigma} \in K$ and $L'(\sigma) \subset K$. Then, the map Φ_σ restricts to a map*

$$\Phi_\sigma| \colon L'(\sigma) \times (\sigma \cap K) \longrightarrow \partial St'(\sigma) \cap K,$$

and the homeomorphism ϕ_σ restricts to a homeomorphism

$$\phi_\sigma| \colon L'(\sigma) \times (\sigma^\circ \cap K) \xrightarrow{\cong} E'(\sigma) \cap K.$$

Proof. Let α and β be closed simplices of T' with $\alpha \subset L'(\sigma)$ and $\beta \subset \partial\sigma$. Since $\widehat{\sigma} \in K$, Lemma 3.2 implies that we have $\beta \in K$ if and only if $\widehat{\sigma} * \beta \in K$. Hence,

$$\sigma \cap K = \widehat{\sigma} * (\partial\sigma \cap K). \tag{3.1}$$

Moreover, since $\alpha \subset L'(\sigma) \subset K$, Lemma 3.2 implies that we have $\beta \in K$ if and only if $\alpha * \beta \in K$. Altogether, we have $\widehat{\sigma} * \beta \in K$ if and only if $\alpha * \beta \in K$. Consequently, by gluing the maps $\Phi_{\alpha,\beta}$ for $\alpha \subset L'(\sigma)$ and $\beta \subset \partial\sigma \cap K$, we obtain a continuous map

$$L'(\sigma) \times (\sigma \cap K) \overset{(3.1)}{=} L'(\sigma) \times (\widehat{\sigma} * (\partial\sigma \cap K)) \longrightarrow \partial St'(\sigma) \cap K.$$

Similarly, by gluing the homeomorphisms $\phi_{\alpha,\beta}$ for $\alpha \subset L'(\sigma)$ and $\beta \subset \partial\sigma \cap K$, we obtain a homeomorphism

$$L'(\sigma) \times ((\sigma \setminus \partial\sigma) \cap K) \overset{(3.1)}{=} L'(\sigma) \times ((\widehat{\sigma} * (\partial\sigma \cap K)) \setminus \partial\sigma) \xrightarrow{\cong} (\partial St'(\sigma) \cap K) \setminus \partial\sigma.$$

□

Theorem 3.5. *Let σ and τ be simplices of T such that τ is a proper face of σ. Then, the homeomorphisms ϕ_σ and ϕ_τ induce a commutative diagram*

$$\begin{tikzcd} L'(\sigma) \times (\sigma^\circ \cap L'(\tau)) \times \tau^\circ \arrow[r, "\cong", "\phi_\sigma| \times \mathrm{id}_{\tau^\circ}"'] \arrow[d, "\mathrm{id}_{L'(\sigma)} \times \phi_\tau|"', "\cong"] & (E'(\sigma) \cap L'(\tau)) \times \tau^\circ \arrow[d, "\phi_\tau|"', "\cong"] \\ L'(\sigma) \times (\sigma^\circ \cap E'(\tau)) \arrow[r, "\cong", "\phi_\sigma|"'] & E'(\sigma) \cap E'(\tau). \end{tikzcd} \tag{3.2}$$

Proof. First, let us show that the homeomorphisms ϕ_σ and ϕ_τ restrict in the desired ways:

- To produce the vertical maps in diagram (3.2), we have to show that ϕ_τ restricts to homeomorphisms

$$(\sigma^\circ \cap L'(\tau)) \times \tau^\circ \xrightarrow{\cong} \sigma^\circ \cap E'(\tau), \tag{3.3}$$

$$(E'(\sigma) \cap L'(\tau)) \times \tau^\circ \xrightarrow{\cong} E'(\sigma) \cap E'(\tau). \tag{3.4}$$

  In fact, since $\sigma^\circ = \sigma \setminus \partial\sigma$ and $E'(\sigma) = \partial St'(\sigma) \setminus \partial\sigma$, it suffices to show that ϕ_τ restricts to homeomorphisms

$$(\sigma \cap L'(\tau)) \times \tau^\circ \xrightarrow{\cong} \sigma \cap E'(\tau),$$
$$(\partial St'(\sigma) \cap L'(\tau)) \times \tau^\circ \xrightarrow{\cong} \partial St'(\sigma) \cap E'(\tau),$$
$$(\partial\sigma \cap L'(\tau)) \times \tau^\circ \xrightarrow{\cong} \partial\sigma \cap E'(\tau).$$

  We show that these restrictions exist by applying Proposition 3.3 for $K = \sigma$, $K = \partial St'(\sigma) = L'(\sigma) * \partial\sigma$, and $K = \partial\sigma$, respectively. (Note that these choices of K are full simplicial subcomplexes of T' such that $\partial\tau \subset K$ since $\tau \subset \sigma$.)
- To produce the horizontal maps in diagram (3.2), we have to show that ϕ_σ restricts to homeomorphisms

$$L'(\sigma) \times (\sigma^\circ \cap L'(\tau)) \xrightarrow{\cong} E'(\sigma) \cap L'(\tau),$$
$$L'(\sigma) \times (\sigma^\circ \cap \partial St'(\tau)) \xrightarrow{\cong} E'(\sigma) \cap \partial St'(\tau),$$

  where note that the second map coincides with the lower horizontal map in diagram (3.2) because we have $\sigma^\circ = \sigma \setminus \partial\sigma$ and $E'(\sigma) = \partial St'(\sigma) \setminus \partial\sigma$, so that we may replace $\partial St'(\tau)$ by $\partial St'(\tau) \setminus \partial\tau = E'(\tau)$ since $\partial\tau \subset \partial\sigma$. We show that these restrictions exist by applying Proposition 3.4 for $K = L'(\tau)$ and $K = \partial St'(\tau) = L'(\tau) * \partial\tau$, respectively. (Note that these choices of K are full simplicial subcomplexes of T', and we conclude from the definition of $L'(-)$ that $L'(\sigma) \subset L'(\tau) \subset K$ since $\tau \subset \sigma$, and $\widehat{\sigma} \in L'(\tau) \subset K$.)

To show that diagram (3.2) commutes, we fix a point $(x, y, z) \in L'(\sigma) \times (\sigma^\circ \cap L'(\tau)) \times \tau^\circ$, and have to show that

$$\phi_\tau(\phi_\sigma(x, y), z) = \phi_\sigma(x, \phi_\tau(y, z)). \tag{3.5}$$

For this purpose, we fix simplices $\alpha \in L'(\sigma)$, $\beta \in \partial\sigma \cap L'(\tau)$, and $\gamma \in \partial\tau$ of T' such that $x \in \alpha$, $y \in \widehat{\sigma} * \beta$, and $z \in \widehat{\tau} * \gamma$. (We note that $y \in \sigma^\circ$ and $z \in \tau^\circ$ imply $y \in (\widehat{\sigma} * \beta) \setminus \beta$ and $z \in (\widehat{\tau} * \gamma) \setminus \gamma$, respectively.) Let $a_1, \dots, a_p$, $b_1, \dots, b_q$, and $c_1, \dots, c_r$ be the vertices of α, β, and γ, respectively. Then, using barycentric coordinates, we can uniquely write

$$x = \sum_{i=1}^{p} x_i a_i, \qquad x_1, \dots, x_p \geq 0, \quad x_1 + \cdots + x_p = 1,$$

$$y = y_0 \widehat{\sigma} + \sum_{j=1}^{q} y_j b_j, \qquad y_0, y_1, \dots, y_q \geq 0, \quad y_0 + y_1 + \cdots + y_q = 1,$$

$$z = z_0 \widehat{\tau} + \sum_{k=1}^{r} z_k c_k, \qquad z_0, z_1, \dots, z_r \geq 0, \quad z_0 + z_1 + \cdots + z_r = 1.$$

We show (3.5) by computing both sides separately:

- To evaluate the left hand side of (3.5), we note that ϕ_σ restricts to the homeomorphism

$$\phi_{\alpha,\beta} \colon \alpha \times ((\widehat{\sigma} * \beta) \setminus \beta) \xrightarrow{\cong} (\alpha * \beta) \setminus \beta,$$

  and ϕ_τ restricts to the homeomorphism

$$\phi_{\alpha*\beta,\gamma} \colon (\alpha * \beta) \times ((\widehat{\tau} * \gamma) \setminus \gamma) \xrightarrow{\cong} (\alpha * \beta * \gamma) \setminus \gamma.$$

  Since $x \in \alpha$ and $y \in (\widehat{\sigma} * \beta) \setminus \beta$, we obtain

$$\phi_\sigma(x, y) = \phi_{\alpha,\beta}(x, y) = \phi_{\alpha,\beta}(\sum_{i=1}^{p} x_i a_i, y_0 \widehat{\sigma} + \sum_{j=1}^{q} y_j b_j) = \sum_{i=1}^{p} x_i y_0 a_i + \sum_{j=1}^{q} y_j b_j.$$

  As $\alpha \in L'(\sigma)$ and $\beta \in \partial\sigma$ are independent, $\alpha * \beta$ is a simplex of T' with vertices $a_1, \dots, a_p, b_1, \dots, b_q$. Since $\phi_\sigma(x, y) \in \alpha * \beta$ and $z \in (\widehat{\tau} * \gamma) \setminus \gamma$, we conclude that

$$\begin{aligned} \phi_\tau(\phi_\sigma(x, y), z) &= \phi_{\alpha*\beta,\gamma}(\phi_\sigma(x, y), z) \\ &= \phi_{\alpha*\beta,\gamma}(\sum_{i=1}^{p} x_i y_0 a_i + \sum_{j=1}^{q} y_j b_j, z_0 \widehat{\tau} + \sum_{k=1}^{r} z_k c_k) \\ &= \sum_{i=1}^{p} x_i y_0 z_0 a_i + \sum_{j=1}^{q} y_j z_0 b_j + \sum_{k=1}^{r} z_k c_k. \end{aligned}$$

- To evaluate the right hand side of (3.5), we note that ϕ_τ restricts to the homeomorphism

$$\phi_{\widehat{\sigma}*\beta,\gamma} \colon (\widehat{\sigma} * \beta) \times ((\widehat{\tau} * \gamma) \setminus \gamma) \xrightarrow{\cong} (\widehat{\sigma} * \beta * \gamma) \setminus \gamma,$$

  and ϕ_σ restricts to the homeomorphism

$$\phi_{\alpha,\beta*\gamma} \colon \alpha \times ((\widehat{\sigma} * \beta * \gamma) \setminus (\beta * \gamma)) \xrightarrow{\cong} (\alpha * \beta * \gamma) \setminus (\beta * \gamma).$$

Since $y \in \widehat{\sigma} * \beta$, and $z \in (\widehat{\tau} * \gamma) \setminus \gamma$, we obtain

$$\begin{aligned}
\phi_\tau(y, z) = \phi_{\widehat{\sigma}*\beta,\gamma}(y, z) &= \phi_{\widehat{\sigma}*\beta,\gamma}(y_0\widehat{\sigma} + \sum_{j=1}^{q} y_j b_j, z_0\widehat{\tau} + \sum_{k=1}^{r} z_k c_k) \\
&= y_0 z_0 \widehat{\sigma} + \sum_{j=1}^{q} y_j z_0 b_j + \sum_{k=1}^{r} z_k c_k.
\end{aligned}$$

As $\beta \in L'(\tau)$ and $\gamma \in \partial\tau$ are independent, $\beta * \gamma$ is a simplex of T' with vertices $b_1, \dots, b_q, c_1, \dots, c_r$. Since $x \in \alpha$ and $\phi_\tau(y, z) \in (\widehat{\sigma} * \beta * \gamma) \setminus (\beta * \gamma)$ (where note that $\phi_\tau(y, z) \notin \beta * \gamma$ because $y \in \sigma^\circ$ and $z \in \tau^\circ$ imply that $y_0 z_0 > 0$), we conclude that

$$\begin{aligned}
\phi_\sigma(x, \phi_\tau(y, z)) &= \phi_{\alpha,\beta*\gamma}(x, \phi_\tau(y, z)) \\
&= \phi_{\alpha,\beta*\gamma}(\sum_{i=1}^{p} x_i a_i, y_0 z_0 \widehat{\sigma} + \sum_{j=1}^{q} y_j z_0 b_j + \sum_{k=1}^{r} z_k c_k) \\
&= \sum_{i=1}^{p} x_i y_0 z_0 a_i + \sum_{j=1}^{q} y_j z_0 b_j + \sum_{k=1}^{r} z_k c_k.
\end{aligned}$$

This completes the proof of Theorem 3.5. $\square$

4. Proof of Theorem 1.1

Let X be a PL pseudomanifold. Fix a triangulation T of X, and let $\mathcal{X}_T$ be the simplicial stratification (1.1) of X associated to T. Recall that $\mathcal{S}(X)$ denotes the set of singular strata of the stratified space X. Using the notation of Section 3, the following table provides the data that define a canonical trivial conical structure (in the sense of Definition 2.8) on the stratified space $(X, \mathcal{X}_T)$.

$S \in \mathcal{S}(X)$	open simplices σ° of T of codimension ≥ 2
C_S	$\widehat{E}'(\sigma) := \mathrm{St}'(\sigma) \setminus \partial\sigma = (D'(\sigma) * \partial\sigma) \setminus \partial\sigma$
∂C_S	$E'(\sigma) = \partial\,\mathrm{St}'(\sigma) \setminus \partial\sigma = (L'(\sigma) * \partial\sigma) \setminus \partial\sigma$
trivial fiber bundle $\begin{tikzcd} \partial C_S \arrow[r, "\partial\pi_S"] \arrow[dr, "\cong", "\psi_S"'] & S \\ & X^S \times S \arrow[u, "\mathrm{pr}_2"'] \end{tikzcd}$ with fiber X^S and structure group $G_S = \{\mathrm{id}_{X^S}\}$	$\begin{tikzcd} E'(\sigma) \arrow[r, "=:\partial\rho_\sigma"] \arrow[dr, "\cong", "\phi_\sigma^{-1}"'] & \sigma^\circ \\ & L'(\sigma) \times \sigma^\circ \arrow[u, "\mathrm{pr}_2"'] \end{tikzcd}$ (the full simplicial subcomplex $L'(\sigma) \subset T'$ is stratified by its simplicial stratification)
homeomorphism $\theta_S \colon C_S \xrightarrow{\cong} c_{\partial\pi_S}(\partial C_S)$	$\eta_\sigma \colon \widehat{E}'(\sigma) = ((\widehat{\sigma} * L'(\sigma)) * \partial\sigma) \setminus \partial\sigma \cong c_{\partial\rho_\sigma}(E'(\sigma))$
Z_S^T	$\sigma^\circ \cap L'(\tau)$
C_S^T	$\widehat{E}'(\sigma) \cap L'(\tau)$
∂C_S^T	$E'(\sigma) \cap L'(\tau)$

Using the data from the above table, it is now straightforward to check that

$$\{(C_S, \partial\pi_S, \theta_S)\}_{S \in \mathcal{S}(X)} := \{(\widehat{E}'(\sigma), \partial\rho_\sigma, \eta_\sigma)\}_{\sigma^\circ \in T}$$

is a canonical conical structure (in the sense of Definition 2.4) on the stratified space $(X, \mathcal{X}_T)$. For example, for $S, T \in \mathcal{S}(X)$ with $T \prec S$, the subbundles of $\partial\pi_T$ induced by the inclusions $Z_S^T \subset X^T$ and $\partial C_S^T \subset X^T$ have total spaces $\partial C_T \cap S$ and

$\partial C_T \cap \partial C_S$ by (3.3) and (3.4), respectively. (Similarly, by applying Proposition 3.3 to $K = \mathrm{St}'(\sigma)$ and $K = \partial\sigma$, it can be shown that the subbundle of $\partial\pi_T$ induced by the inclusion $C_S^T \subset X^T$ has total space $\partial C_T \cap C_S$.) Furthermore, we see that

$$\{\psi_S\}_{S\in\mathcal{S}(X)} := \{\phi_\sigma^{-1}\}_{\sigma^\circ\in T}$$

is a trivialization (in the sense of Definition 2.7) of the above conical structure, where the commutative diagram (2.12) is canonically provided by Theorem 3.5.

This completes the proof of Theorem 1.1.

Acknowledgment

The author would like to thank Markus Banagl for fruitful discussions.

References

[1] Marta Agustín Vicente and Javier Fernández de Bobadilla, *Intersection space constructible complexes*, Doc. Math. **25** (2020), 1653–1725. MR4170059

[2] M. Banagl, *Topological invariants of stratified spaces*, Springer Monographs in Mathematics, Springer, Berlin, 2007. MR2286904

[3] Markus Banagl, *Singular spaces and generalized Poincaré complexes*, Electron. Res. Announc. Math. Sci. **16** (2009), 63–73, DOI 10.3934/era.2009.16.63. MR2577502

[4] Markus Banagl, *Intersection spaces, spatial homology truncation, and string theory*, Lecture Notes in Mathematics, vol. 1997, Springer-Verlag, Berlin, 2010, DOI 10.1007/978-3-642-12589-8. MR2662593

[5] Markus Banagl, *Foliated stratified spaces and a de Rham complex describing intersection space cohomology*, J. Differential Geom. **104** (2016), no. 1, 1–58. MR3544285

[6] Markus Banagl, Nero Budur, and Laurenţu Maxim, *Intersection spaces, perverse sheaves and type IIB string theory*, Adv. Theor. Math. Phys. **18** (2014), no. 2, 363–399. MR3273317

[7] Markus Banagl and Eugénie Hunsicker, *Hodge theory for intersection space cohomology*, Geom. Topol. **23** (2019), no. 5, 2165–2225, DOI 10.2140/gt.2019.23.2165. MR4019892

[8] Markus Banagl and Bryce Chriestenson, *Intersection spaces, equivariant Moore approximation and the signature*, J. Singul. **16** (2017), 141–179, DOI 10.5427/jsing.2017.16g. MR3687584

[9] Markus Banagl and Laurentiu Maxim, *Deformation of singularities and the homology of intersection spaces*, J. Topol. Anal. **4** (2012), no. 4, 413–448, DOI 10.1142/S1793525312500185. MR3021771

[10] Markus Banagl and Laurentiu Maxim, *Intersection spaces and hypersurface singularities*, J. Singul. **5** (2012), 48–56, DOI 10.5427/jsing.2012.5d. MR2928933

[11] Christian Geske, *Algebraic intersection spaces*, J. Topol. Anal. **12** (2020), no. 4, 1157–1194, DOI 10.1142/S1793525319500778. MR4146576

[12] R. Mark Goresky, *Triangulation of stratified objects*, Proc. Amer. Math. Soc. **72** (1978), no. 1, 193–200, DOI 10.2307/2042563. MR500991

[13] Mark Goresky and Robert MacPherson, *Intersection homology theory*, Topology **19** (1980), no. 2, 135–162, DOI 10.1016/0040-9383(80)90003-8. MR572580

[14] Mark Goresky and Robert MacPherson, *Intersection homology. II*, Invent. Math. **72** (1983), no. 1, 77–129, DOI 10.1007/BF01389130. MR696691

[15] A. Haefliger, *Introduction to piecewise linear intersection homology*, Intersection cohomology (Bern, 1983), Progr. Math., vol. 50, Birkhäuser Boston, Boston, MA, 1984, pp. 1–21, DOI 10.1007/978-0-8176-4765-0_1. MR788172

[16] J. F. P. Hudson, *Piecewise linear topology*, W. A. Benjamin, Inc., New York-Amsterdam, 1969. University of Chicago Lecture Notes prepared with the assistance of J. L. Shaneson and J. Lees. MR0248844

[17] Mathieu Klimczak, *Poincaré duality for spaces with isolated singularities*, Manuscripta Math. **153** (2017), no. 1-2, 231–262, DOI 10.1007/s00229-016-0884-5. MR3635981

[18] Laurentiu Maxim, *Intersection spaces, perverse sheaves and string theory*, J. Singul. **15** (2016), 118–125, DOI 10.5427/jsing.2016.15f. MR3562857

[19] Dominik J. Wrazidlo, *On the rational homotopy type of intersection spaces*, J. Singul. **20** (2020), 251–273, DOI 10.5427/jsing.2020.20k. MR4141082

[20] Dominik J. Wrazidlo, *A fundamental class for intersection spaces of depth one Witt spaces*, Manuscripta Math. **166** (2021), no. 1-2, 199–236, DOI 10.1007/s00229-020-01238-7. MR4296376

Mathematisches Institut, Universität Heidelberg, Im Neuenheimer Feld 205, 69120 Heidelberg, Germany

Email address: `dwrazidlo@mathi.uni-heidelberg.de`