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A BANACH SPACE THEORETICAL CHARACTERIZATION OF ABELIAN C^*-ALGEBRAS

RYOTARO TANAKA

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ABSTRACT. A Banach space theoretical characterization of abelian C^*-algebras among all C^*-algebras is given. As an application, it is shown that if A and B are C^*-algebras (nonlinearly) isomorphic to each other with respect to the structure of Birkhoff-James orthogonality, and if either A or B is abelian, then they are $*$-isomorphic. Moreover, it is pointed out that the same kind of characterization is not valid for preduals of abelian von Neumann algebras.

1. Introduction

The study on characterizations of abelian C^*-algebras among all C^*-algebras has a long history. Various kinds of characterizations have been obtained up to the present. For example, Ogasawara [18] showed that a C^*-algebra A is abelian if and only if $a^2 \geq b^2$ whenever $a, b \in A$ and $a \geq b \geq 0$ (that is, if $t \mapsto t^2$ is operator monotone on A). In this direction, it was also shown by Wu [24] that the operator monotonicity of $t \mapsto e^t$ characterizes abelian C^*-algebras. Moreover, as a corollary to this result, it turns out that A is abelian if and only if $e^{a+b} = e^a e^b$ for each a, b in (the unitization of) A. Later, this result was generalized by Jeang and Ko [10]. Meanwhile, there exists a characterization of the Stinespring type. Namely, abelian C^*-algebras A are characterized by the property that every positive linear map from A into another C^*-algebra B becomes completely positive; actually, according to [23, Theorem 1.2], the complete positivity can be replaced with the two-positivity. The above mentioned results by Ogasawara and Wu, and of the Stinespring type, were put together and further improved by Ji and Tomiyama [11]. In addition, Kato [14] and Nakamoto [17] gave characterizations in terms of spectrum. Another very simple characterization of abelian C^*-algebras is based on the existence of nonzero nilpotent, that is, A is abelian if and only if there exists no nonzero $a \in A$ with $a^2 = 0$; see, for example, [5, Proposition II.6.4.14]. The readers interested in this topic are referred to a brief survey by Pinter-Lucke [19] and references therein; see also [1, 15] for recent developments.

As is natural, basically, the existing characterizations of abelian C^*-algebras are based on (a part of) the algebraic structure of them. At least, they required the existence of natural multiplication operations on objects. Under this circumstance,

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we try to remove this structural requirement in the present paper, and give a purely Banach space theoretical characterization of abelian C^*-algebras among all C^*-algebras, where "purely Banach space theoretical" means that the machinery for characterization can be defined in general Banach spaces. Practically, we refer to the topologizability of geometric structure spaces of C^*-algebras. The notion of geometric structure spaces was first introduced in [21] for classifying Banach spaces with respect to their structure of Birkhoff-James orthogonality. Although its original definition was given in terms of the Birkhoff-James orthogonality, it turned out from [21, Theorem 4.15] that geometric structure spaces of Banach spaces are determined by their facial structure of unit balls; see also [22] for further developments on geometric structure spaces. As an application of the results in this paper, it is shown that if A and B are C^*-algebras (nonlinearly) isomorphic to each other with respect to the structure of Birkhoff-James orthogonality, and if either A or B is abelian, then they are $*$-isomorphic. Meanwhile, it is pointed out in the last section that the same kind of characterization is not valid for preduals of abelian von Neumann algebras. More precisely, it is shown that the predual of a von Neumann algebra A has the topologizable geometric structure space if and only if $A = \mathbb{C}$.

2. Notation and preliminaries

For a Banach space X, let B_X and S_X denote the unit ball and unit sphere of X, respectively. The symbol X^* represents the (continuous) dual of X. For a subset S of X, let $\overline{S}$ denote the closure of S with respect to the norm topology. The linear span of S is denoted by $\langle S \rangle$.

Let C be a nonempty convex subset of a vector space X, and let D be a convex subset of C. Then, D is called a *face* of C if, whenever $x, y \in C$ and $tx+(1-t)y \in D$ for some $t \in (0,1)$, we obtain $x, y \in D$. Further, if $D \neq C$, the face D is *proper*. An element x of C is called an *extreme point* of C if the set $\{x\}$ is a face of C. The set of all extreme points of C is denoted by $\mathrm{ext}(C)$. In particular, we have $\mathrm{ext}([-1,1]) = \{-1,1\}$ and $\mathrm{ext}(\{c \in \mathbb{C} : |c| \leq 1\}) = \{c \in \mathbb{C} : |c| = 1\}$. For a Banach space X, let

$$\mathcal{F}(X) = \{F \subset B_X : F \text{ is a closed proper face of } B_X\},$$
$$\mathcal{F}^*(X^*) = \{G \subset B_{X^*} : G \text{ is a weakly}^* \text{ closed proper face of } B_{X^*}\}.$$

If $F \in \mathcal{F}(X)$ then the *facear* $\Phi^*(F)$ of F is defined as

$$\Phi^*(F) = \{f \in B_{X^*} : f(x) = 1 \text{ for each } x \in F\},$$

while the *prefacear* $\Phi_*(G)$ of G is defined as

$$\Phi_*(G) = \{x \in B_X : f(x) = 1 \text{ for each } f \in G\}$$

for each $G \in \mathcal{F}^*(X^*)$. We note that $\Phi^*(F) \in \mathcal{F}^*(X^*)$ and $\Phi_*(G) \in \mathcal{F}(X)$ whenever $F \in \mathcal{F}(X)$ and $G \in \mathcal{F}^*(X^*)$.

A *C^*-algebra* A is a complex Banach algebra with involution $x \mapsto x^*$ satisfying the Gelfand-Naimark axiom $\|x^*x\| = \|x\|^2$ for each $x \in A$. Let A be a C^*-algebra. An element x of A is said to be *self-adjoint* if $x^* = x$, and is said to be *positive*, denoted by $x \geq 0$, if $x = y^2$ for some self-adjoint element y of A. Let $\rho \in A^*$. Then, the formula $\rho^*(x) = \overline{\rho(x^*)}$ defines an element of A^*. We say that ρ is *hermitian* if $\rho^* = \rho$, and is *positive* if $\rho(x) \geq 0$ whenever $x \geq 0$. In particular, a positive linear functional ρ on A with $\|\rho\| = 1$ is called a *state* of A. Let $\mathcal{S}(A)$ denote the family of all states of A, and let $\mathcal{P}(A) = \mathrm{ext}(\mathcal{S}(A))$. An element of $\mathcal{P}(A)$ is called a *pure state* of A. If A is unital, then $\mathcal{S}(A) = \{\rho \in A^* : \rho(1) = \|\rho\| = 1\}$ [12, Theorem 4.3.2]. We also note that the sets $A_{\mathrm{sa}} = \{x \in A : x^* = x\}$ and $(A^*)_{\mathrm{sa}} = \{\rho \in A^* : \rho^* = \rho\}$ form real Banach spaces. Moreover, since $A = \langle A_{\mathrm{sa}} \rangle$ and $\|\rho\| = \sup\{|\rho(x)| : x \in B_{A_{\mathrm{sa}}}\}$ for each $\rho \in (A^*)_{\mathrm{sa}}$, we obtain $(A^*)_{\mathrm{sa}} = A_{\mathrm{sa}}^*$ under the (real-linear) isometric isomorphism $\rho \mapsto \rho|A_{\mathrm{sa}}$.

Let A, B be C^*-algebras. A mapping $\varphi : A \to B$ is called a $*$-*homomorphism* if it is an algebra homomorphism and $\varphi(x^*) = \varphi(x)^*$ for each $x \in A$. A bijective $*$-homomorphism is called a $*$-*isomorphism.* If B is the C^*-algebra $B(H)$ of all bounded linear operators on a complex Hilbert space H, then a $*$-homomorphism $\varphi : A \to B(H)$ is called a $*$-*representation* of A on H. An injective $*$-representation is said to be *faithful.* If $\rho \in \mathcal{S}(A)$, then we can generate a $*$-representation π_ρ of A by the Gelfand-Naimark-Segal construction [6, Theorem I.9.6]. More precisely, for each $\rho \in \mathcal{S}(A)$, there exists a $*$-representation π_ρ of A on a Hilbert space H_ρ with a unit cyclic vector ξ_ρ for π_ρ (that is, $\|\xi_\rho\| = 1$ and $\overline{\pi_\rho(A)\xi_\rho} = H_\rho$) such that $\rho(x) = \langle \pi_\rho(x)\xi_\rho, \xi_\rho \rangle$ for each $x \in A$, where $\pi_\rho(A)\xi_\rho = \{\pi_\rho(x)\xi_\rho : x \in A\}$. It is known that the *universal representation* $\sum_{\rho \in \mathcal{S}(A)} \oplus \pi_\rho$ of A on the Hilbert space $\sum_{\rho \in \mathcal{S}(A)} \oplus H_\rho$ is faithful; hence, each C^*-algebra can be considered as a C^*-subalgebra of $B(H)$ for some complex Hilbert space H [6, Theorem I.9.12].

Let H be a complex Hilbert space. If $\xi, \zeta \in H$, then the formulas $p_\xi(T) = \|T\xi\|$ and $q_{\xi,\zeta}(T) = |\langle T\xi, \zeta \rangle|$ define seminorms on $B(H)$. The topology on $B(H)$ induced by the (separating) family of seminorms $\{p_\xi : \xi \in H\}$ is called the *strong-operator topology*, while the family $\{q_{\xi,\zeta} : \xi, \zeta \in H\}$ induces the *weak-operator topology* on $B(H)$. A *von Neumann algebra* is a weak-operator closed C^*-subalgebra of $B(H)$ containing the unit of $B(H)$. Let A be a von Neumann algebra acting on H. Then, a functional ρ on A is said to be *normal* if it is weak-operator continuous on B_A. In particular, each functional on $B(H)$ of the form $T \mapsto \langle T\xi, \zeta \rangle$, where $\xi, \zeta \in H$, is restricted to a normal functional on A. The set A_* of all normal functionals on A is a norm-closed subspace of A^*. For each $x \in A$ and each $\rho \in A_*$, let $\varphi_x(\rho) = \rho(x)$. Then, the mapping $x \mapsto \varphi_x$ is an isometric isomorphism from A onto $(A_*)^*$ whose restriction to B_A is a weak-operator to weak* homeomorphism from B_A onto $B_{(A_*)^*}$ [13, Theorem 7.4.2]. The Banach space A_* is called the *predual* of A. Now, let $(A_*)_{\mathrm{sa}} = A_* \cap (A^*)_{\mathrm{sa}}$. We can see that the mapping $x \mapsto \varphi_x|(A_*)_{\mathrm{sa}}$ is an isometric isomorphism from A_{sa} onto $(A_*)_{\mathrm{sa}}^*$ whose restriction to $B_{A_{\mathrm{sa}}}$ is a weak-operator to weak* homeomorphism from $B_{A_{\mathrm{sa}}}$ onto $B_{(A_*)_{\mathrm{sa}}^*}$. Based on this identification, we represent $(A_*)_{\mathrm{sa}}$ by the symbol $(A_{\mathrm{sa}})_*$.

The notion of geometric structure spaces of Banach spaces plays central roles in this paper. Let X be a Banach space. For each maximal face F of B_X, let $I_F = \bigcup_{f \in \Phi^*(F)} \ker f$. The *geometric structure space* $\mathfrak{S}(X)$ of X is defined as the set

$$\mathfrak{S}(X) = \{I_F : F \text{ is a maximal face of } B_X\}$$

equipped with the closure operator $S \mapsto S^=$ given by

$$S^= = \{I \in \mathfrak{S}(X) : \textstyle\bigcap_{J \in S} J \subset I\}.$$

We remark that the closure operator on $\mathfrak{S}(X)$ does not induce a topology in general. The geometric space $\mathfrak{S}(X)$ of X is said to be *topologizable* if the closure operator $S \mapsto S^=$ satisfies the Kuratowski closure axioms, or equivalently, if the set $\mathfrak{C}(X) = \{S \subset \mathfrak{S}(X) : S^= = S\}$ satisfies the axioms of closed sets. The following two results on topologizability of geometric structure spaces will be useful in the rest of this paper; see [21, Corollary 4.12, and Theorem 5.2].

Theorem 2.1. *Let X be a Banach space. If $\mathfrak{S}(X)$ is topologizable, then $S \in \mathfrak{C}(X)$ for each finite subset S of $\mathfrak{S}(X)$.*

Theorem 2.2. *Let K be a locally compact Hausdorff space. Then, $\mathfrak{S}(C_0(K))$ is topologizable, and is homeomorphic to K.*

Remark 2.3. Theorem 2.2 is valid for both the real and complex cases.

The readers interested in geometric structure spaces of Banach spaces are referred to [21, 22].

3. A characterization of abelian C^*-algebras

The aim of this section is to obtain a purely Banach space theoretical characterization of abelian C^*-algebras. To this end, we make use of geometric structure spaces of Banach spaces. We begin with identifying the set $\mathfrak{S}(A)$ for a C^*-algebra A with the help of the detailed study on facial structure in operator algebras that was conducted by Edwards and Rüttimann [7] and Akemann and Pedersen [2].

Theorem 3.1 ([2, 1992])**.** *Let A be a C^*-algebra. Then, the facear mapping $F \mapsto \Phi^*(F)$ is an order reversing isomorphism from $\mathcal{F}(A)$ onto $\mathcal{F}^*(A^*)$ (or $\mathcal{F}(A_{\mathrm{sa}})$ onto $\mathcal{F}^*(A_{\mathrm{sa}}^*)$) with its inverse $G \mapsto \Phi_*(G)$.*

We note that if C is a compact subset of a locally convex space X and D is a closed minimal face of D, then D is a singleton of an extreme point of C. Indeed, a compact convex set D has an extreme point x, and $\{x\}$ is a (closed) face of C since D is a face of C. Hence, $x \in \mathrm{ext}(C)$ and $D = \{x\}$ holds by the minimality of D. This fact is used in the proof of Proposition 3.2.

Proposition 3.2. *Let A be a C^*-algebra, and let $X \in \{A, A_{\mathrm{sa}}\}$. Then, $\mathfrak{S}(X) = \{\ker \rho : \rho \in \mathrm{ext}(B_{X^*})\}$.*

Proof. By Theorem 3.1, a subset F of B_X is a maximal face of B_X if and only if $\Phi^*(F) = \{\rho\}$ for some $\rho \in \mathrm{ext}(B_{X^*})$. Hence, we have

$$\begin{aligned}\mathfrak{S}(X) &= \{I_F : F \text{ is a maximal face of } B_X\}\\ &= \{\ker \rho : \rho \in \mathrm{ext}(B_{X^*})\},\end{aligned}$$

as desired. □

To obtain a characterization of abelian C^*-algebras in terms of their geometric structure spaces, some auxiliary results will be needed. First, we recall that if $\rho \in A^*$ is hermitian, then $\|\rho\| = \|\rho|A_{\mathrm{sa}}\|$. Indeed, for any $\varepsilon > 0$, there exists an $x \in B_A$ such that $\|\rho\| - \varepsilon < \rho(x)$. Since ρ is hermitian, setting $y = 2^{-1}(x + x^*)$ yields that $y \in B_{A_{\mathrm{sa}}}$, and $\rho(y) = 2^{-1}(\rho(x) + \rho(x^*)) = \mathrm{Re}\,\rho(x) = \rho(x)$, that is, $\|\rho\| - \varepsilon < \rho(x) = \rho(y) \leq \|\rho|A_{\mathrm{sa}}\|$. Hence, we have $\|\rho\| \leq \|\rho|A_{\mathrm{sa}}\|$. The converse inequality is obvious. This fact is used in the proof of Lemma 3.3.

Lemma 3.3. *Let A be a C^*-algebra, and let ρ be a pure state of A. Then, $\rho|A_{\mathrm{sa}} \in \mathrm{ext}(B_{A_{\mathrm{sa}}^*})$.*

Proof. Suppose that $\rho|A_{\rm sa} = t\rho_1 + (1-t)\rho_2$ for some $\rho_1, \rho_2 \in B_{A^*_{\rm sa}}$ and some $t \in (0,1)$. Let

$$\overline{\rho_j}(x) = \rho_j\left(\frac{x+x^*}{2}\right) + i\rho_j\left(\frac{x-x^*}{2i}\right)$$

for each $x \in A$ and $j = 1, 2$. Then, $\overline{\rho_1}, \overline{\rho_2} \in B_{A^*}$ and $\rho = t\overline{\rho_1} + (1-t)\overline{\rho_2}$, which implies that $\rho = \overline{\rho_1} = \overline{\rho_2}$ since ρ is pure. Thus, $\rho|A_{\rm sa} = \rho_1 = \rho_2$. This proves that $\rho|A_{\rm sa} \in \mathrm{ext}(B_{A^*_{\rm sa}})$. □

It is known that an element ρ of A^* is a state of A if and only if $\lim_\lambda \rho(e_\lambda) = 1 = \|\rho\|$, where $(e_\lambda)_\lambda$ is any approximate unit for A; see [6, Lemmas I.9.5 and I.9.9]. From this, we obtain another auxiliary result.

Lemma 3.4. *Let A be a C^*-algebra, and let ρ, τ be states of A. Then, the following are equivalent:*

(i) $\rho = \tau$.
(ii) $\ker \rho = \ker \tau$.
(iii) $\ker(\rho|A_{\rm sa}) = \ker(\tau|A_{\rm sa})$.

Proof. It is sufficient to show that (iii) ⇒ (i). If $\ker(\rho|A_{\rm sa}) = \ker(\tau|A_{\rm sa})$ holds, then $\rho|A_{\rm sa} = \lambda\tau|A_{\rm sa}$ for some λ. Moreover, since ρ, τ are states of A, we obtain $\lim_a \rho(e_a)_a = \lim_a \tau(e_a) = 1$ for an approximate unit $(e_a)_a$ for A. In particular, $(e_a)_a \subset A_{\rm sa}$ implies that $\lambda = 1$. From this, $\rho|A_{\rm sa} = \tau|A_{\rm sa}$, which together with $A = \langle A_{\rm sa}\rangle$ implies that $\rho = \tau$. □

Now, we prove the main theorem in this paper.

Theorem 3.5. *Let A be a C^*-algebra. Then, the following are equivalent:*

(i) *A is abelian.*
(ii) *$A_{\rm sa}$ is abelian.*
(iii) *$\mathfrak{S}(A)$ is topologizable.*
(iv) *$\mathfrak{S}(A_{\rm sa})$ is topologizable.*

Proof. It is obvious that (i) ⇔ (ii) since $A = \langle A_{\rm sa}\rangle$. Moreover, if A is abelian, then the Gelfand-Naimark theorem ensures that A is $*$-isomorphic to $C_0(K)$ for some locally compact Hausdorff space K, in which case $A_{\rm sa}$ corresponds to the real subspace of $C_0(K)$ consisting of all real-valued continuous functions on K that vanish at infinity. Combining this with Theorem 2.2, we can conclude that $\mathfrak{S}(A)$ and $\mathfrak{S}(A_{\rm sa})$ are both topologizable. Hence, the implications (i) ⇒ (iii) and (i) ⇒ (iv) hold true.

To prove the implications (iii) ⇒ (i) and (iv) ⇒ (i), suppose that A is not abelian. Since the $*$-representation $\sum_{\rho\in\mathcal{P}(A)} \oplus\pi_\rho$ of A is faithful [6, Proof of Corollary I.9.13], there exists a pure state ρ of A such that $\pi_\rho(A)$ is not abelian, where π_ρ is the $*$-representation of A obtained from ρ by the Gelfand-Naimark-Segal construction. Let H_ρ be a complex Hilbert space on which $\pi_\rho(A)$ acts, and let ξ_ρ be a cyclic vector for π_ρ such that $\rho(a) = \langle \pi_\rho(a)\xi_\rho, \xi_\rho\rangle$ for each $a \in A$. Since ρ is pure, the representation π_ρ is irreducible [16, Theorem 5.1.6 (1)], and hence, $\pi_\rho(A)$ is strong-operator dense in $B(H_\rho)$ [16, Theorems 4.1.12 and 5.1.5]. In particular, $\dim H_\rho \geq 2$ because $\pi_\rho(A)$ is not abelian. Moreover, each nonzero vector $\xi \in H_\rho$ is cyclic for π_ρ. Indeed, since $\pi_\rho(A)$ acts irreducibly on H_ρ, we have a $u \in A$ such that $\pi_\rho(u)\xi = \xi_\rho$. This shows that $\pi_\rho(A)\xi_\rho \subset \pi_\rho(A)\xi$, which implies that $\overline{\pi_\rho(A)\xi} = H_\rho$. Therefore, by [16, Theorem 5.1.7], the formula $\rho_\xi(x) = \langle \pi_\rho(x)\xi, \xi\rangle$

defines a pure state of A for each $\xi \in S_{H_\rho}$. We note that, for $\xi, \zeta \in S_{H_\rho}$, the following are equivalent:

(a) $\rho_\xi | A_{\mathrm{sa}} = \rho_\zeta | A_{\mathrm{sa}}$.
(b) $\rho_\xi = \rho_\zeta$.
(c) $\langle T\xi, \xi \rangle = \langle T\zeta, \zeta \rangle$ for each $T \in B(H_\rho)$.
(d) $\{\xi, \zeta\}$ is linearly dependent.

Here, (b) $\Rightarrow$ (c) follows from the fact that $\pi_\rho(A)$ is strong-operator dense in $B(H_\rho)$. If (c) holds, the rank one projection E_ζ onto $\langle \{\zeta\} \rangle$ satisfies $\|E_\zeta \xi\|^2 = \langle E_\zeta \xi, \xi \rangle = \langle E_\zeta \zeta, \zeta \rangle = 1$. Hence, it follows that $\xi = E_\zeta \xi \in \langle \{\zeta\} \rangle$, that is, (c) $\Rightarrow$ (d). The other implications are obvious.

Suppose that ξ_0 is a unit vector in H_ρ such that $\xi_\rho \perp \xi_0$. For each $(s, \lambda) \in [0,1] \times \mathbb{C}$ with $s^2 + |\lambda|^2 = 1$, let $\xi_{(s,\lambda)} = s\xi_\rho + \lambda \xi_0$. If $(t, \mu) \in [0,1] \times \mathbb{C}$, and if $t^2 + |\mu|^2 = 1$, then $\{\xi_{(s,\lambda)}, \xi_{(t,\mu)}\}$ is linearly dependent if and only if $(s, \lambda) = (t, \mu)$. Set $p_1 = (1,0)$, $p_2 = (0,1)$, $p_3 = 2^{-1/2}(1,1)$ and $p_4 = 2^{-1/2}(1,i)$. Then, there are infinitely many $(s,\lambda) \in [0,1] \times \mathbb{C}$ with $s^2 + |\lambda|^2 = 1$ such that $\rho_{\xi_{(s,\lambda)}} \neq \rho_{\xi_{p_j}}$ for all $j \in \{1,2,3,4\}$.

Now, let $X \in \{A, A_{\mathrm{sa}}\}$. To show that $\mathfrak{S}(X)$ is not topologizable, suppose to the contrary that the closure operator $S \mapsto S^=$ satisfies the Kuratowski closure axioms. Then, each finite subset S of $\mathfrak{S}(X)$ satisfies $S^= = S$ by Theorem 2.1. Set $S = \{\ker(\rho_{\xi_{p_j}} | X) : j = 1,2,3,4\}$. We note that $S \subset \mathfrak{S}(X)$ by Lemma 3.3. Take an arbitrary $(s,\lambda) \in [0,1] \times \mathbb{C}$ with $s^2 + |\lambda|^2 = 1$. If $x \in \bigcap_{j=1}^4 (\ker \rho_{\xi_{p_j}} | X)$, then we get

$$\begin{aligned}
(\rho_{\xi_{p_1}} | X)(x) &= \langle \pi_\rho(x) \xi_\rho, \xi_\rho \rangle = 0, \\
(\rho_{\xi_{p_2}} | X)(x) &= \langle \pi_\rho(x) \xi_0, \xi_0 \rangle = 0, \\
(\rho_{\xi_{p_3}} | X)(x) &= 2^{-1} \langle \pi_\rho(x)(\xi_\rho + \xi_0), \xi_\rho + \xi_0 \rangle = 0, \text{ and} \\
(\rho_{\xi_{p_4}} | X)(x) &= 2^{-1} \langle \pi_\rho(x)(\xi_\rho + i\xi_0), \xi_\rho + i\xi_0 \rangle = 0.
\end{aligned}$$

It follows that

$$\langle \pi_\rho(x) \xi_\rho, \xi_\rho \rangle = \langle \pi_\rho(x) \xi_0, \xi_0 \rangle = \langle \pi_\rho(x) \xi_\rho, \xi_0 \rangle = \langle \pi_\rho(x) \xi_0, \xi_\rho \rangle = 0,$$

and hence, $(\rho_{\xi_{(s,\lambda)}} | X)(x) = 0$. This shows that $\ker(\rho_{\xi_{(s,\lambda)}} | X) \in S^= = S$, that is, by Lemma 3.4, $\rho_{\xi_{(s,\lambda)}} | X = \rho_{\xi_{p_j}} | X$ for some $j \in \{1,2,3,4\}$. However, as was noted in the preceding paragraph, there are (infinitely many) $(s,\lambda) \in [0,1] \times \mathbb{C}$ with $s^2 + |\lambda|^2 = 1$ such that $\rho_{\xi_{(s,\lambda)}} \neq \rho_{\xi_{p_j}}$ for all $j \in \{1,2,3,4\}$, which is a contradiction. Thus, $\mathfrak{S}(X)$ cannot be topologizable. This proves the implications (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (i). □

We conclude this section with an application of Theorem 3.5 to the theory of geometric nonlinear classification of Banach spaces. Recall that an element x of a Banach space X over $\mathbb{K}$ is said to be *Birkhoff-James orthogonal* to another $y \in X$, denoted by $x \perp_{BJ} y$, if $\|x + \lambda y\| \geq \|x\|$ for each $\lambda \in \mathbb{K}$. The Birkhoff-James orthogonality is a generalization of orthogonality in Hilbert spaces from the viewpoint of best approximation that was first introduced by Birkhoff [4] and was studied in detail by James [8, 9]; see [3] for a comprehensive survey of generalized orthogonality types in Banach spaces. The geometric structure of Banach spaces is closely related to the behavior of Birkhoff-James orthogonality in them. Recently, in [20, 21], some results on the classification of Banach spaces based on the structure of Birkhoff-James orthogonality were given. More precisely, the classes of classical sequence spaces and spaces of continuous functions were classified by using a nonlinear equivalence "$\sim_{BJ}$," where we declare that $X \sim_{BJ} Y$ for Banach spaces X and Y if there exists a Birkhoff-James orthogonality preserver $T : X \to Y$, that is, if there exists a bijection $T : X \to Y$ such that $x \perp_{BJ} y$ if and only if $Tx \perp_{BJ} Ty$. A key ingredient of the theory is the fact that each Birkhoff-James orthogonality preserver induces a homeomorphism between geometric structure spaces. Recall that a mapping $\Phi : \mathfrak{S}(X) \to \mathfrak{S}(Y)$ is called a *homeomorphism* if $\Phi(S^{=}) = \Phi(S)^{=}$ for each $S \subset \mathfrak{S}(X)$, and that X and Y are said to be isomorphic with respect to geometric structure spaces, denoted by $X \sim_{\mathfrak{S}} Y$, if there exists a homeomorphism $\Phi : \mathfrak{S}(X) \to \mathfrak{S}(Y)$. With this notation, $X \sim_{BJ} Y$ implies that $X \sim_{\mathfrak{S}} Y$ [21, Theorem 3.10]. Moreover, it is also known that the topologizability of a geometric structure space is preserved under homeomorphisms [21, Theorem 3.11].

Now, we present an application of Theorem 3.5.

Corollary 3.6. *Let A, B be C^*-algebras. Suppose that either A or B is abelian. Then, the following are equivalent:*

(i) *A and B are $*$-isomorphic.*
(ii) *$A \sim_{BJ} B$.*
(iii) *$A \sim_{\mathfrak{S}} B$.*
(iv) *$A_{\rm sa}$ and $B_{\rm sa}$ are isometrically isomorphic.*
(v) *$A_{\rm sa} \sim_{BJ} B_{\rm sa}$.*
(vi) *$A_{\rm sa} \sim_{\mathfrak{S}} B_{\rm sa}$.*

Proof. Since every isometric isomorphism is a Birkhoff-James orthogonality preserver, the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) $\Rightarrow$ (iv) hold true. Moreover, it is clear that (i) $\Rightarrow$ (iv). Hence, to complete the proof, it is sufficient to show that (iii) $\Rightarrow$ (i) and (vi) $\Rightarrow$ (i). We may assume that A is abelian. By the Gelfand-Naimark theorem, A is $*$-isomorphic to $C_0(K)$ for some locally compact Hausdorff space K. In this case, by Theorem 2.2, both $\mathfrak{S}(A)$ and $\mathfrak{S}(A_{\rm sa})$ are topologizable, and homeomorphic to K. In particular, if either (iii) or (vi) holds, then $\mathfrak{S}(B)$ or $\mathfrak{S}(B_{\rm sa})$ is topologizable. Combining this with Theorem 3.5, we can conclude that B is also abelian. Now, let L be a locally compact Hausdorff space such that B is $*$-isomorphic to $C_0(L)$. Then, again by Theorem 2.2, both $\mathfrak{S}(B)$ and $\mathfrak{S}(B_{\rm sa})$ are homeomorphic to L, which together with $A \sim_{\mathfrak{S}} B$ (or $A_{\rm sa} \sim_{\mathfrak{S}} B_{\rm sa}$) implies that K and L are homeomorphic. Thus, the implications (iii) $\Rightarrow$ (i) and (vi) $\Rightarrow$ (i) hold true. This completes the proof. □

Remark 3.7. Corollary 3.6 does not extend to general C^*-algebras. Indeed, we know that every Jordan $*$-isomorphism between C^*-algebras is an isometric isomorphism preserving the self-adjoint parts. Hence, if A and B are Jordan $*$-isomorphic but not $*$-isomorphic C^*-algebras, then (ii) to (vi) are true but (i) is false.

4. The case of preduals

This section is devoted to showing that the predual version of Theorem 3.5 does not hold. In fact, the criterion for topologizability of geometric structure spaces of preduals of von Neumann algebras is much stricter than that of C^*-algebras.

As in the case of C^*-algebras, we begin with identifying the sets $\mathfrak{S}(A_*)$ and $\mathfrak{S}((A_{\rm sa})_*)$ for a von Neumann algebra A. The main tool used here is the following result by Edwards and Rüttimann [7]; see also Akemann and Pedersen [2].

Theorem 4.1 ([7, 1988]; [2, 1992])**.** *Let A be a von Neumann algebra. Then, the facear mapping $G \mapsto \Phi^*(G)$ is an order reversing isomorphism from $\mathcal{F}(A_*)$ onto $\mathcal{F}^*(A)$ (or $\mathcal{F}((A_*)_{\mathrm{sa}})$ onto $\mathcal{F}^*(A_{\mathrm{sa}})$) with its inverse $F \mapsto \Phi_*(F)$.*

By the exact same argument as that in the proof of Proposition 3.2, we obtain the following result.

Lemma 4.2. *Let A be a von Neumann algebra, and let $X \in \{A, A_{\mathrm{sa}}\}$. Then, $\mathfrak{S}(X_*) = \{\ker \varphi_x \cap X_* : x \in \mathrm{ext}(B_X)\}$, where φ_x is an element of $(A_*)^*$ given by $\varphi_x(\rho) = \rho(x)$ for each $\rho \in A_*$.*

We also make use of the following characterization of extreme points of the unit ball of (the real part of) a C^*-algebra; see, for example, [13, Theorem 7.3.1 and Proposition 7.4.6].

Theorem 4.3. *Let A be a unital C^*-algebra, and let $v \in B_A$. Then, the following hold:*

(i) *$v \in \mathrm{ext}(B_A)$ if and only if v is a partial isometry such that*

$$(1 - vv^*)A(1 - v^*v) = \{0\}.$$

(ii) *$v \in \mathrm{ext}(B_{A_{\mathrm{sa}}})$ if and only if $v = 2e - 1$ for some projection $e \in A$.*

Remark 4.4. If $v = 2e - 1$ for some projection e, then v is self-adjoint and unitary. Hence, it follows from Theorem 4.3 (i) that $v \in \mathrm{ext}(B_A)$. Namely, $\mathrm{ext}(B_{A_{\mathrm{sa}}}) \subset \mathrm{ext}(B_A)$.

Let A be a von Neumann algebra acting on a Hilbert space H, and let $\xi \in H$. Then, the functional ω_ξ on A defined by $\omega_\xi(x) = \langle x\xi, \xi\rangle$ belongs to $(A_{\mathrm{sa}})_*$. We note that if $x, y \in B(H)$ satisfies $\omega_\xi(x) = \omega_\xi(y)$ for each $\xi \in H$, then $x = y$ by the polarization identity.

In contrast to the case of C^*-algebras, there are few examples of von Neumann algebras A such that $\mathfrak{S}(A)$ or $\mathfrak{S}(A_{\mathrm{sa}})$ is topologizable. First, we consider the predual of a von Neumann algebra.

Theorem 4.5. *Let A be a von Neumann algebra. Then, $\mathfrak{S}(A_*)$ is topologizable if and only if $A = \mathbb{C}$.*

Proof. Suppose that A acts on a Hilbert space H. If $A = \mathbb{C}$, then $A_* = A^* = \mathbb{C}$; in this case, $\mathfrak{C}(A_*)$ satisfies the axioms of closed sets by Theorem 3.5. Conversely, if $A \neq \mathbb{C}$, then there exists a projection $e \in A$ such that $0 < e < 1$. Let $u = 2e - 1$ and $v = (1 - i)e - 1$. We note that $1, u, v \in \mathrm{ext}(B_A)$ by Theorem 4.3 since they are unitary elements of A. Moreover, for each $\rho \in \ker \varphi_1 \cap \ker \varphi_u$, we have $\rho(1) = \rho(u) = 0$, which implies that $\rho(e) = 0$. Hence, it follows that $\rho(v) = 0$; that is, $\ker \varphi_v \in \{\ker \varphi_1, \ker \varphi_u\}^=$. Meanwhile, if $\xi \in S_{e(H)}$ and $\zeta \in S_{(1-e)(H)}$, then $\omega_\xi - \omega_\zeta \in \ker \varphi_1 \setminus \ker \varphi_v$, while $\omega_\xi + \omega_\zeta \in \ker \varphi_u \setminus \ker \varphi_v$. This means that $\ker \varphi_v \notin \{\ker \varphi_1, \ker \varphi_u\}$. Therefore, $\{\ker \varphi_1, \ker \varphi_u\} \notin \mathfrak{C}(A_*)$. Combining this with Theorem 2.1, we see that $\mathfrak{S}(A_*)$ is not topologizable. □

Next, we consider the case of real parts. Then, an auxiliary result will be needed.

Lemma 4.6. *Let A be a von Neumann algebra. If A contains three mutually orthogonal nonzero projections e_1, e_2, e_3, then $\mathfrak{S}((A_{\mathrm{sa}})_*)$ is not topologizable.*

Proof. Suppose that A acts on a Hilbert space H. Replacing e_3 with $1-e_1-e_2$ if necessary, we may assume that $e_1+e_2+e_3=1$. Let $v_j=2e_j-1$ for $j=1,2,3$. Then, $v_1,v_2,v_3\in \mathrm{ext}(B_{A_{\mathrm{sa}}})$ by Theorem 4.3 (ii). If $\rho\in\ker\varphi_1\cap\ker\varphi_{v_1}\cap\ker\varphi_{v_2}$, we have $\rho(1)=\rho(v_1)=\rho(v_2)=0$. It follows that $\rho(e_1)=\rho(e_2)=0$, and hence,

$$\rho(v_3)=\rho(2(1-e_1-e_2)-1)=\rho(1)-2\rho(e_1)-2\rho(e_2)=0.$$

This shows that $\ker\varphi_{v_3}\cap(A_{\mathrm{sa}})_*\in\{\ker\varphi_1\cap(A_{\mathrm{sa}})_*,\ker\varphi_{v_1}\cap(A_{\mathrm{sa}})_*,\ker\varphi_{v_2}\cap(A_{\mathrm{sa}})_*\}^=$. Meanwhile, if $\xi_j\in S_{e_j(H)}$ for $j=1,2,3$, then $\omega_{\xi_1}-\omega_{\xi_3}\in\ker\varphi_1\setminus\ker\varphi_{v_3}$, while $\omega_{\xi_1}+\omega_{\xi_2}\in(\ker\varphi_{v_1}\cap\ker\varphi_{v_2})\setminus\ker\varphi_{v_3}$. Therefore, $\ker\varphi_{v_3}\cap(A_{\mathrm{sa}})_*\notin\{\ker\varphi_1\cap(A_{\mathrm{sa}})_*,\ker\varphi_{v_1}\cap(A_{\mathrm{sa}})_*,\ker\varphi_{v_2}\cap(A_{\mathrm{sa}})_*\}$, which proves that

$$\{\ker\varphi_1\cap(A_{\mathrm{sa}})_*,\ker\varphi_{v_1}\cap(A_{\mathrm{sa}})_*,\ker\varphi_{v_2}\cap(A_{\mathrm{sa}})_*\}\notin\mathfrak{C}((A_{\mathrm{sa}})_*).$$

Combining this with Theorem 2.1, we see that $\mathfrak{C}((A_{\mathrm{sa}})_*)$ does not satisfy the axioms of closed sets. □

We conclude this paper with Theorem 4.7.

Theorem 4.7. *Let A be a von Neumann algebra. Then, $\mathfrak{S}((A_{\mathrm{sa}})_*)$ is topologizable if and only if $A=\mathbb{C}$ or $A=\ell^2_\infty$.*

Proof. Suppose that $\mathfrak{S}((A_{\mathrm{sa}})_*)$ is topologizable. Then, by Lemma 4.6, A does not contain three or more orthogonal nonzero projections. Hence, we have $A\in\{\mathbb{C},\ell^2_\infty,M_2(\mathbb{C})\}$ by considering the type decomposition for A; see [13, Section 6.5].

Suppose that $A=M_2(\mathbb{C})$. For each $x\in M_2(\mathbb{C})$, let $\|x\|_1=\sigma_1(x)+\sigma_2(x)$, where $\sigma_1(x),\sigma_2(x)$ are the singular values of x. Set $\psi_y(x)=\mathrm{tr}(xy)$ for each $x,y\in M_2(\mathbb{C})$. Then, it is known that the mapping $y\mapsto\psi_y$ is an isometric isomorphism from $(M_2(\mathbb{C}),\|\cdot\|_1)$ onto $A^*=A_*$. Moreover, we have $\psi_y^*=\psi_{y^*}$ for each $y\in M_2(\mathbb{C})$, which implies that $(A_{\mathrm{sa}})_*=\{\psi_y:y\in M_2(\mathbb{C})_{\mathrm{sa}}\}$. Now, let

$$e_1=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad e_2=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad e_3=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad e_4=\begin{bmatrix}0&i\\-i&0\end{bmatrix}.$$

Then, e_1,e_2,e_3,e_4 are all self-adjoint unitaries in A; hence, by Theorem 4.3, they are extreme points of $B_{A_{\mathrm{sa}}}$. Moreover, if $\psi_y\in(A_{\mathrm{sa}})_*$ satisfies $y\in\bigcap_{j=1}^4\ker\varphi_{e_j}$, then $y=0$. This shows that $\{\ker\varphi_{e_j}\cap(A_{\mathrm{sa}})_*:j=1,2,3,4\}^==\mathfrak{S}((A_{\mathrm{sa}})_*)$. Meanwhile, setting

$$u=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix},$$

we have $u\in\mathrm{ext}(B_{A_{\mathrm{sa}}})$, $\psi_{e_2}\in(\ker\varphi_{e_1}\cap\ker\varphi_{e_3}\cap\ker\varphi_{e_4})\setminus\ker\varphi_u$, and $\psi_{e_3}\in\ker\varphi_{e_2}\setminus\ker\varphi_u$. Since $\psi_{e_2},\psi_{e_3}\in(A_{\mathrm{sa}})_*$, we derive that

$$\ker\varphi_u\cap(A_{\mathrm{sa}})_*\notin\{\ker\varphi_{e_j}\cap(A_{\mathrm{sa}})_*:j=1,2,3,4\}.$$

Thus, $\{\ker\varphi_{e_j}\cap(A_{\mathrm{sa}})_*:j=1,2,3,4\}\notin\mathfrak{C}((A_{\mathrm{sa}})_*)$. From this and Theorem 2.1, it turns out that $\mathfrak{S}((A_{\mathrm{sa}})_*)$ is not topologizable. This proves that $A\neq M_2(\mathbb{C})$.

For the converse, if $A=\mathbb{C}$ then $A_*=A^*=\mathbb{C}$, in which case we have $(A_{\mathrm{sa}})_*=\mathbb{R}=A_{\mathrm{sa}}$. Hence, by Theorem 3.5, $\mathfrak{S}((A_{\mathrm{sa}})_*)$ is topologizable. Next, suppose that $A=\ell^2_\infty$. Then, $A_*=A^*=\ell^2_1$ and $(A_{\mathrm{sa}})_*=\ell^2_1\cap\mathbb{R}^2$. Since $T(a,b)=(a+b,a-b)$ defines an isometric isomorphism $T:\ell^2_1\cap\mathbb{R}^2\to\ell^2_\infty\cap\mathbb{R}^2$, it follows that $(A_{\mathrm{sa}})_*=\ell^2_\infty\cap\mathbb{R}^2=(\ell^2_\infty)_{\mathrm{sa}}$. Thus, again by Theorem 3.5, $\mathfrak{S}((A_{\mathrm{sa}})_*)$ is topologizable. This completes the proof. □

References

[1] Toshikazu Abe and Osamu Hatori, *On a characterization of commutativity for C^*-algebras via gyrogroup operations*, Period. Math. Hungar. **72** (2016), no. 2, 248–251, DOI 10.1007/s10998-016-0126-3. MR3498587

[2] Charles A. Akemann and Gert K. Pedersen, *Facial structure in operator algebra theory*, Proc. London Math. Soc. (3) **64** (1992), no. 2, 418–448, DOI 10.1112/plms/s3-64.2.418. MR1143231

[3] Javier Alonso, Horst Martini, and Senlin Wu, *Orthogonality types in normed linear spaces*, Surveys in geometry I, Springer, Cham, [2022] ©2022, pp. 97–170, DOI 10.1007/978-3-030-86695-2_4. MR4404395

[4] Garrett Birkhoff, *Orthogonality in linear metric spaces*, Duke Math. J. **1** (1935), no. 2, 169–172, DOI 10.1215/S0012-7094-35-00115-6. MR1545873

[5] B. Blackadar, *Operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006. Theory of C^*-algebras and von Neumann algebras; Operator Algebras and Non-commutative Geometry, III, DOI 10.1007/3-540-28517-2. MR2188261

[6] Kenneth R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996, DOI 10.1090/fim/006. MR1402012

[7] C. M. Edwards and G. T. Rüttimann, *On the facial structure of the unit balls in a JBW*-triple and its predual*, J. London Math. Soc. (2) **38** (1988), no. 2, 317–332, DOI 10.1112/jlms/s2-38.2.317. MR966303

[8] Robert C. James, *Inner product in normed linear spaces*, Bull. Amer. Math. Soc. **53** (1947), 559–566, DOI 10.1090/S0002-9904-1947-08831-5. MR21242

[9] Robert C. James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc. **61** (1947), 265–292, DOI 10.2307/1990220. MR21241

[10] Jyh-Shyang Jeang and Chun-Chieh Ko, *On the commutativity of C^*-algebras*, Manuscripta Math. **115** (2004), no. 2, 195–198, DOI 10.1007/s00229-004-0488-3. MR2098469

[11] Guoxing Ji and Jun Tomiyama, *On characterizations of commutativity of C^*-algebras*, Proc. Amer. Math. Soc. **131** (2003), no. 12, 3845–3849, DOI 10.1090/S0002-9939-03-06991-0. MR1999933

[12] Richard V. Kadison and John R. Ringrose, *Fundamentals of the theory of operator algebras. Vol. I*, Graduate Studies in Mathematics, vol. 15, American Mathematical Society, Providence, RI, 1997. Elementary theory; Reprint of the 1983 original, DOI 10.1090/gsm/015. MR1468229

[13] Richard V. Kadison and John R. Ringrose, *Fundamentals of the theory of operator algebras. Vol. II*, Graduate Studies in Mathematics, vol. 16, American Mathematical Society, Providence, RI, 1997. Advanced theory; Corrected reprint of the 1986 original, DOI 10.1090/gsm/016. MR1468230

[14] Yoshinobu Kato, *A characterization of commutative C^*-algebras by normal approximate spectra*, Math. Japon. **24** (1979/80), no. 2, 209–210. MR549997

[15] Lajos Molnár, *A few conditions for a C^*-algebra to be commutative*, Abstr. Appl. Anal., posted on 2014, Art. ID 705836, 4, DOI 10.1155/2014/705836. MR3212443

[16] Gerard J. Murphy, *C^*-algebras and operator theory*, Academic Press, Inc., Boston, MA, 1990. MR1074574

[17] Ritsuo Nakamoto, *A spectral characterization of commutative C^*-algebras*, Math. Japon. **24** (1979/80), no. 4, 399–400. MR557471

[18] Tôzirô Ogasawara, *A theorem on operator algebras*, J. Sci. Hiroshima Univ. Ser. A **18** (1955), 307–309. MR73955

[19] James Pinter-Lucke, *Commutativity conditions for rings: 1950–2005*, Expo. Math. **25** (2007), no. 2, 165–174, DOI 10.1016/j.exmath.2006.07.001. MR2319575

[20] Ryotaro Tanaka, *Nonlinear equivalence of Banach spaces based on Birkhoff-James orthogonality*, J. Math. Anal. Appl. **505** (2022), no. 1, Paper No. 125444, 12, DOI 10.1016/j.jmaa.2021.125444. MR4280839

[21] Ryotaro Tanaka, *Nonlinear equivalence of Banach spaces based on Birkhoff-James orthogonality, II*, J. Math. Anal. Appl. **514** (2022), no. 1, Paper No. 126307, 19, DOI 10.1016/j.jmaa.2022.126307. MR4419543

[22] Ryotaro Tanaka, *Nonlinear classification of Banach spaces based on geometric structure spaces*, J. Math. Anal. Appl. **521** (2023), no. 2, Paper No. 126944, 17, DOI 10.1016/j.jmaa.2022.126944. MR4527441

[23] Jun Tomiyama, *On the difference of n-positivity and complete positivity in C^*-algebras*, J. Functional Analysis **49** (1982), no. 1, 1–9, DOI 10.1016/0022-1236(82)90083-0. MR680854
[24] Wei Wu, *An order characterization of commutativity for C^*-algebras*, Proc. Amer. Math. Soc. **129** (2001), no. 4, 983–987, DOI 10.1090/S0002-9939-00-05724-5. MR1814137

Katsushika Division, Institute of Arts and Sciences, Tokyo University of Science, Tokyo 125-8585, Japan
Email address: r-tanaka@rs.tus.ac.jp