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COMPARING INVARIANTS OF TORIC IDEALS OF BIPARTITE GRAPHS

KIERAN BHASKARA AND ADAM VAN TUYL

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Abstract. Let G be a finite simple graph and let I_G denote its associated toric ideal in the polynomial ring R. For each integer $n \geq 2$, we completely determine all the possible values for the tuple $(\operatorname{reg}(R/I_G), \deg(h_{R/I_G}(t)), \operatorname{pdim}(R/I_G), \operatorname{depth}(R/I_G), \dim(R/I_G))$ when G is a connected bipartite graph on n vertices.

1. Introduction

Let I be a homogeneous ideal of a polynomial ring $S = \mathbb{K}[x_1, \ldots, x_m]$ over an algebraically closed field $\mathbb{K}$ of characteristic zero. Associated with I are a number of homological invariants that are encoded in the minimal graded free resolution of S/I; some of these invariants are the (Castelnuovo–Mumford) regularity $\operatorname{reg}(S/I)$, the projective dimension $\operatorname{pdim}(S/I)$, the (Krull) dimension $\dim(S/I)$, the depth $\operatorname{depth}(S/I)$ and $\deg(h_{S/I}(t))$, the degree of the h-polynomial of S/I. Four of these invariants are related by the inequality

$$\operatorname{reg}(S/I) - \deg(h_{S/I}(t)) \leq \dim(S/I) - \operatorname{depth}(S/I)$$

(see, for example [25, Corollary B.28]), while the depth and projective dimension are related via the well-known Auslander–Buchsbaum formula.

A recent program in combinatorial commutative algebra is to understand what possible pairs (or tuples) of these invariants can be realized for specific families of ideals, most notably, ideals that are defined combinatorially (e.g., edge ideals, binomial edge ideals, and toric ideals of graphs). This circle of problems was introduced by Hibi, Higashitani, Kimura, and O'Keefe [12], who compared the depth and dimension of toric ideals of graphs, and Hibi and Matsuda [15, 16], who first showed that for any pair r, d of positive integers, there exists a (lexsegment) monomial ideal I with $(r, d) = (\operatorname{reg}(S/I), \deg(h_{S/I}(t)))$. Hibi, Matsuda, and Van Tuyl [18] later showed a similar result for the edge ideals of graphs. An investigation of the possible pairs $(\operatorname{reg}(S/I), \deg(h_{S/I}(t)))$ for other families of ideals soon followed, most notably, for binomial edge ideals [17, 21, 22] and toric ideals of graphs [6]. Other recent work has focused on determining the possible pairs $(\operatorname{reg}(S/I), \operatorname{pdim}(S/I))$ [9], the pairs $(\operatorname{depth}(S/I), \dim(S/I))$ [13, 19], and comparisons of the multiplicity of S/I to $\operatorname{reg}(S/I)$ and $\deg(h_{S/I}(t))$ [23] when I is an edge ideal.

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Hibi, Kimura, Matsuda, and Van Tuyl [14] introduced a variation of this problem by asking if we can completely describe all pairs $(\mathrm{reg}(S/I), \deg(h_{S/I}(t)))$ if we restrict to edge ideals of connected graphs on n vertices; in the case of Cameron–Walker graphs, they were able to answer this question. In a related direction, Erey and Hibi [5] completely described all the possible pairs $(\mathrm{reg}(S/I), \mathrm{pdim}(S/I))$ as I varies over all the edge ideals of connected bipartite graphs on n vertices. Recently, Ficarra and Sgroi [7] gave an almost complete description of $(\mathrm{reg}(S/I), \mathrm{pdim}(S/I))$ for the case of binomial edge ideals of graphs on n non-isolated vertices. Inspired by [5, 7], in this paper we characterize the values of the invariants that can occur if I is the toric ideal of a connected bipartite graph on n vertices.

We introduce some notation to describe our results. Let $G = (V, E)$ be a finite simple graph with edge set $E = \{e_1, \ldots, e_q\}$ and vertex set $V = \{v_1, \ldots, v_n\}$. Define a $\mathbb{K}$-algebra homomorphism $\varphi : R = \mathbb{K}[e_1, \ldots, e_q] \to \mathbb{K}[v_1, \ldots, v_n]$ by $\varphi(e_i) = v_j v_k$ if $e_i = \{v_j, v_k\} \in E$. The *toric ideal of* G, denoted I_G, is the kernel of φ. The graph G is a *bipartite graph* if there exists $V_1, V_2 \subseteq V$ such that $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$ with the property that every $e \in E$ has an endpoint in both V_1 and V_2. We define $\mathrm{CBPT}_{\mathrm{reg,deg,pdim,depth,dim}}(n)$ to be the set

$$\left\{ (\mathrm{reg}(R/I_G), \deg(h_{R/I_G}(t)), \mathrm{pdim}(R/I_G), \mathrm{depth}(R/I_G), \dim(R/I_G)) \;\middle|\; \begin{array}{c} G \text{ is a connected} \\ \text{bipartite graph} \\ \text{on } n \text{ vertices} \end{array} \right\}.$$

Our main result is a complete characterization of the elements in the above set:

Theorem 1.1. *Let $n \geq 2$. Then* $\mathrm{CBPT}_{\mathrm{reg,deg,pdim,depth,dim}}(n)$ *is given by*

$$\left\{ (r, r, p, n-1, n-1) \;\middle|\; 0 < r < \left\lfloor \frac{n}{2} \right\rfloor,\ 1 \leq p \leq r(n-2-r) \right\} \cup \{(0, 0, 0, n-1, n-1)\}.$$

Notice that Theorem 1.1 describes all five invariants; the only other result similar to Theorem 1.1 is a result of Hibi, Kanno, Kimura, Matsuda, and Van Tuyl [13] which described four of these invariants for the edge ideals of Cameron–Walker graphs.

The proof of Theorem 1.1 requires both new and old results. Recent work of Almousa, Dochtermann, and Smith [1] allows us to bound the regularity of toric ideals of bipartite graphs by looking at subgraphs. We also require an old graph theory result of Jackson [20] on the existence of cycles in bipartite graphs, which enables us to find an upper bound on the number of edges of G in terms of the regularity of R/I_G (see Lemma 3.4). Note that it is well-known that the toric ideals of bipartite graphs are Cohen–Macaualy; consequently, $\dim(R/I_G) = \mathrm{depth}(R/I_G) = n-1$ (cf. [27, Corollary 10.1.21]) and $\deg(h_{R/I_G}(t)) = \mathrm{reg}(R/I_G)$ (cf. [25, Corollary B.28]). Consequently, proving Theorem 1.1 is equivalent to proving what values the regularity and projective dimension can obtain; these details are given in Theorem 3.5.

Our paper is structured as follows. In Section 2 we recall the relevant background on graph theory, commutative algebra, and toric ideals of graphs. In Section 3, we focus on the regularity and projective dimension of toric ideals of bipartite graphs. In Section 4 we combine the previous results of the paper to prove Theorem 1.1.

2. Background

Throughout this paper $\mathbb{K}$ will denote an algebraically closed field of characteristic zero. In this section we collect the facts needed to prove our main results.

2.1. **Graph theory.** We recall the relevant graph theory terminology. A *finite simple graph* (or a *graph*) $G = (V(G), E(G))$ consists of a non-empty finite set $V(G) = \{v_1, \ldots, v_n\}$, called the vertices, and a finite set $E(G) = \{e_1, \ldots, e_q\} \subseteq \{\{u, v\} \mid u, v \in V(G), u \neq v\}$ of distinct unordered pairs of distinct elements of $V(G)$, called the edges. We sometimes write V (resp. E) for $V(G)$ (resp. $E(G)$) if G is clear from the context. A graph $H = (V(H), E(H))$ is said to be a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In this case, we say that G contains H and write $H \subseteq G$.

A *walk* of G is a sequence of edges $w = (e_1, e_2, \ldots, e_m)$, where each $e_i = \{u_{i_1}, u_{i_2}\} \in E$ and $u_{i_2} = u_{(i+1)_1}$ for each $i = 1, \ldots, m-1$. Equivalently, a walk is a sequence of vertices $(u_1, \ldots, u_m, u_{m+1})$ such that $\{u_i, u_{i+1}\} \in E$ for all $i = 1, \ldots, m$. Here, m is referred to as the *length* of the walk. A walk is *even* if m is even, and it is *closed* if $u_{m+1} = u_1$. Two vertices u and v are said to be *connected* if there is a walk between them. A graph G is said to be *connected* if every two distinct vertices of G are connected. A *connected component* of G is a maximal connected subgraph of G.

A *cycle* of a graph G is a closed walk $(u_1, \ldots, u_m, u_{m+1} = u_1)$ of vertices of G (with $m \geq 3$) such that the only vertices in the walk that are not pairwise distinct are u_1 and u_{m+1}. A cycle of length m is called an *m-cycle*. The *m-cycle graph*, denoted C_m, is the graph with the vertex set $V(C_m) = \{v_1, \ldots, v_m\}$ and edge set $E(C_m) = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{m-1}, v_m\}, \{v_m, v_1\}\}$. A *tree* is a connected graph that contains no cycles; a *forest* is a graph where each connected component is a tree.

We are primarily interested in bipartite graphs. A graph $G = (V, E)$ is *bipartite* if there exists a partition (or bipartition) $V = V_1 \cup V_2$ with $V_1, V_2 \subseteq V$ and $V_1 \cap V_2 = \emptyset$ such that every edge of E joins a vertex in V_1 and a vertex in V_2. Given $a, b \geq 1$, the *complete bipartite graph*, $K_{a,b}$, is the graph with partition $V = \{x_1, \ldots, x_a\} \cup \{y_1, \ldots, y_b\}$ and edge set $\{\{x_i, y_j\} \mid 1 \leq i \leq a, 1 \leq j \leq b\}$. Note that C_m is a bipartite graph if and only if m is even. All trees and forests are also bipartite graphs.

A *matching* of a graph G is a collection of pairwise non-adjacent edges of G. The *matching number* of G, denoted $\operatorname{mat}(G)$, is the largest size of any matching of G. Note that if G is a bipartite graph, then it is straightforward to verify that

$$\operatorname{mat}(G) \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor. \tag{2.1}$$

The following result, which determines the existence of cycles in a bipartite graph, will play a pivotal role in later results.

Theorem 2.1 ([20, Theorem 3])**.** *Let $m \geq 2$ be an integer, and let G be a bipartite graph with bipartition $V = A \cup B$, where $|A| = a$, $|B| = b$, and $2 \leq m \leq a \leq b$. If*

$$|E(G)| > \begin{cases} a + (b-1)(m-1), & a \leq 2m-2, \\ (a+b-2m+3)(m-1), & a \geq 2m-2, \end{cases}$$

then G contains a cycle of length at least $2m$.

2.2. **Commutative algebra.** Our goal is to compare a number of homological invariants. We recall their definitions and some properties. Let $S = \mathbb{K}[x_1, \ldots, x_m]$ and let I be a homogeneous ideal of S. The *minimal graded free resolution of S/I*

has the form:

$$0 \to \bigoplus_j S(-j)^{\beta_{p,j}(S/I)} \to \bigoplus_j S(-j)^{\beta_{p-1,j}(S/I)} \to \cdots$$
$$\to \bigoplus_j S(-j)^{\beta_{1,j}(S/I)} \to S \to S/I \to 0$$

where $\beta_{i,j}(S/I)$ denotes the (i,j)*-th graded Betti number* of S/I.

Two of the invariants that we will consider are encoded into this resolution. The (*Castelnuovo–Mumford*) *regularity* of S/I is defined to be

$$\operatorname{reg}(S/I) = \max\{j - i \mid \beta_{i,j}(S/I) \neq 0\}.$$

The *projective dimension* of S/I is the length of the minimal graded free resolution, i.e.,

$$\operatorname{pdim}(S/I) = \max\{i \mid \beta_{i,j}(S/I) \neq 0 \text{ for some } j\}.$$

The *Hilbert series* of S/I is the formal power series

$$\mathrm{HS}_{S/I}(t) = \sum_{i \geq 0} [\dim_{\mathbb{K}}(S/I)_i] t^i$$

where $\dim_{\mathbb{K}}(S/I)_i$ is the dimension of the ith graded piece of S/I.

The *depth* of S/I, denoted $\operatorname{depth}(S/I)$, is the length of any maximal regular sequence of S/I that is contained in the maximal ideal $\mathfrak{m} = \langle x_1, \ldots, x_m \rangle \subset S$. The (*Krull*) *dimension* of S/I, denoted $\dim(S/I)$, is the supremum of the lengths of all chains of prime ideals in S/I. It is well-known that $\operatorname{depth}(S/I) \leq \dim(S/I)$ for all homogeneous ideals I of S. We say S/I is *Cohen–Macaulay* if $\operatorname{depth}(S/I) = \dim(S/I)$.

By the Hilbert–Serre theorem [4, Corollary 4.1.8], there exists a unique polynomial $h_{S/I}(t) \in \mathbb{Z}[t]$, called the *h-polynomial* of S/I, such that $\mathrm{HS}_{S/I}(t)$ can be written as

$$\mathrm{HS}_{S/I}(t) = \frac{h_{S/I}(t)}{(1-t)^{\dim(S/I)}}$$

with $h_{S/I}(1) \neq 0$. We denote the *degree of the h-polynomial* $h_{S/I}(t)$ by $\deg(h_{S/I}(t))$.

We want to compare the invariants $\operatorname{reg}(S/I)$, $\operatorname{pdim}(S/I)$, $\operatorname{depth}(S/I)$, $\dim(S/I)$, and $\deg(h_{S/I}(t))$. The following result provides some useful relations among these invariants.

Theorem 2.2. *Let I be a proper homogeneous ideal of $S = \mathbb{K}[x_1, \ldots, x_m]$. Then*

(i) $\operatorname{pdim}(S/I) + \operatorname{depth}(S/I) = m$*;*
(ii) *if S/I is Cohen-Macaulay, then* $\operatorname{reg}(S/I) = \deg(h_{S/I}(t))$.

Proof. Statement (i) is a special case of the Auslander–Buchsbaum formula [4, Theorem 1.3.3]. Statement (ii) is [25, Corollary B.28] (or Corollary B.4.1 in earlier printings). □

2.3. **Toric ideals of graphs.** We define the family of ideals that are studied in this paper and some of their properties. Let G be a graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $E = \{e_1, \ldots, e_q\}$ with $q \geq 1$. Let $\mathbb{K}[V] = \mathbb{K}[v_1, \ldots, v_n]$ and $\mathbb{K}[E] = \mathbb{K}[e_1, \ldots, e_q]$ be polynomial rings in the vertex and edge variables, respectively. Define a $\mathbb{K}$-algebra homomorphism $\varphi \colon \mathbb{K}[E] \to \mathbb{K}[V]$ by $\varphi(e_i) = v_{i_1} v_{i_2}$ for all $e_i = \{v_{i_1}, v_{i_2}\} \in E$, $1 \leq i \leq q$. The *toric ideal of G*, denoted I_G, is defined to be the kernel of the homomorphism φ.

Remark 2.3. The ideal I_G is called a toric ideal since I_G is a prime binomial ideal. Indeed, the image of φ is an integral domain, and since $\varphi(\mathbb{K}[E])$ is isomorphic to $\mathbb{K}[E]/I_G$ by the first isomorphism theorem, it follows that I_G is a prime ideal. Theorem 2.5, shows that I_G is a binomial ideal. Note that in the definition of I_G, we avoid the case that G has no edges to ensure that $\mathbb{K}[E]$ has at least one variable.

Remark 2.4. We write $\mathbb{K}[G]$ to denote the quotient ring $\mathbb{K}[E]/I_G$. Note that in the literature (see e.g., [1, 10, 12]), $\mathbb{K}[G]$ often denotes the *edge ring of* G (i.e., the image $\mathrm{im}(\varphi)$ of φ). As mentioned in the previous remark, $\mathrm{im}(\varphi)$ and $\mathbb{K}[E]/I_G$ are isomorphic as rings; however, we must take care when stating results about gradings on these rings, as they may differ. In all subsequent appearances of the notation $\mathbb{K}[G]$, we have ensured that results from the literature concerning $\mathbb{K}[G]$ remain true under our interpretation.

While the generators of I_G are defined implicitly, there is a well-known connection between the closed even walks of a graph G and a (possibly non-minimal) set of generators for I_G. For a closed even walk $\Gamma = (e_{i_1}, \ldots, e_{i_{2m}})$ of graph G, we define a binomial

$$f_\Gamma = e_{i_1}e_{i_3}\cdots e_{i_{2m-1}} - e_{i_2}e_{i_4}\cdots e_{i_{2m}}.$$

We can now describe a set of generators of I_G.

Theorem 2.5 ([26, Proposition 3.1])**.** *If I_G is the toric ideal of a graph G, then*

$$I_G = \langle f_\Gamma \mid \Gamma \textit{ is a closed even walk of } G\rangle.$$

If G is bipartite, then $I_G = \langle f_\Gamma \mid \Gamma$ is a even cycle of $G\rangle$.

We now collect together some facts about the toric ideals of bipartite graphs. We need a result due to Almousa, Dochtermann, and Smith [1] that allows us to find bounds on the regularity using subgraphs (while [1] includes the connected hypothesis, it can be shown that this hypothesis is not required; for our purposes, we only require the original statement).

Theorem 2.6 ([1, Theorem 6.11])**.** *Suppose $G \subseteq K_{a,b}$ is a connected bipartite graph and let $G' \subseteq G$ be a connected subgraph with at least two vertices. Then $\mathrm{reg}(\mathbb{K}[G']) \leq \mathrm{reg}(\mathbb{K}[G])$.*

The next result summarizes some useful results in the literature.

Theorem 2.7. *Let G be a connected bipartite graph on $n \geq 2$ vertices.*

(i) [10, Theorem 1] *or* [27, Theorem 14.4.19] $\mathrm{reg}(\mathbb{K}[G]) \leq \mathrm{mat}(G) - 1$.
(ii) [27, Corollary 10.1.21] $\dim(\mathbb{K}[G]) = n - 1$.
(iii) [11, Corollary 5.26] $\mathbb{K}[G]$ *is Cohen–Macaulay.*

We get the following useful facts as corollaries.

Corollary 2.8. *Let G be a connected bipartite graph on $n \geq 2$ vertices with q edges.*

(i) $\mathrm{depth}(\mathbb{K}[G]) = \dim(\mathbb{K}[G]) = n - 1$.
(ii) $0 \leq \deg(h_{\mathbb{K}[G]}(t)) = \mathrm{reg}(\mathbb{K}[G]) < \left\lfloor \frac{n}{2} \right\rfloor$.
(iii) $0 \leq \mathrm{pdim}(\mathbb{K}[G]) = q - n + 1$.

Proof. By Theorem 2.7 (iii), the ring $\mathbb{K}[G]$ is Cohen–Macaulay. Statement (i) now follows from the definition of Cohen–Macaulayness and Theorem 2.7 (ii). For statement (ii), because $\mathbb{K}[G]$ is Cohen–Macaulay, Theorem 2.2 (ii) implies that $\deg(h_{\mathbb{K}[G]}(t)) = \operatorname{reg}(\mathbb{K}[G])$. The inequality follows from Theorem 2.7 (i) and inequality (2.1). For statement (iii), Theorem 2.2 (i) gives $\operatorname{pdim}(\mathbb{K}[G]) = q - \operatorname{depth}(\mathbb{K}[G]) = q - n + 1$ since $\mathbb{K}[E]$ has q variables. □

For some special families of graphs, we can give exact values for the regularity.

Lemma 2.9. *The following formulas hold:*

(i) *if* $G = K_{a,b}$, *then* $\operatorname{reg}(\mathbb{K}[G]) = \min\{a, b\} - 1$*;*
(ii) *if* $G = C_{2r}$ *with* $r \geq 2$, *then* $\operatorname{reg}(\mathbb{K}[G]) = r - 1$.

Proof. For (i), see [3, Corollary 4.11]. For (ii), Theorem 2.5 implies $I_{C_{2r}}$ is a principal ideal with a minimal generator of degree r. The conclusion follows from this fact. □

We also require a classification of the toric ideals of bipartite graphs with regularity and projective dimension equal to zero.

Lemma 2.10. *Let G be a bipartite graph on $n \geq 2$ vertices. Then the following are equivalent:*

(i) *G is a forest.*
(ii) $\operatorname{reg}(\mathbb{K}[G]) = 0$.
(iii) $\operatorname{pdim}(\mathbb{K}[G]) = 0$.

Proof. If G is a forest, then G has no even cycles, so $I_G = \langle 0 \rangle$ by Theorem 2.5. Thus $\operatorname{reg}(\mathbb{K}[G]) = \operatorname{pdim}(\mathbb{K}[G]) = 0$. Conversely, if G is not a forest, then G has at least one even cycle. So, by Theorem 2.5, $I_G \neq \langle 0 \rangle$, from which it follows that both the regularity and projective dimension of $\mathbb{K}[G]$ are non-zero. □

Remark 2.11. Lemma 2.10 is only classifying bipartite graphs with regularity and projective dimension zero. There are non-bipartite graphs whose toric ideals have regularity and projective dimension zero, e.g., the toric ideals of C_{2r+1} with $r \geq 1$.

Remark 2.12. While we have only highlighted the results about the regularity of toric ideals of bipartite graphs that we require, we point out that a number of other results are known from the perspective of the a-invariant (we thank Rafael Villarreal for pointing out this connection). Given a homogeneous ideal $I \subseteq S = \mathbb{K}[x_1, \ldots, x_m]$, the *$a$-invariant* of S/I, denoted $a(S/I)$, is the degree of $\operatorname{HS}_{S/I}(t)$ as a rational function (in lowest terms). So, $a(S/I) = \deg h_{S/I}(t) - \dim(S/I)$. When G is a connected bipartite graph, we can use Corollary 2.8 to show that $a(\mathbb{K}[G]) = \operatorname{reg}(\mathbb{K}[G]) - n + 1$. Consequently, studying the regularity of toric ideals of connected bipartite graphs is equivalent to studying their a-invariant. In [24], Valencia and Villarreal gave a combinatorial interpretation for the a-invariant, and a linear program to compute this invariant. Also see [27, Section 11.5] for other properties of $a(\mathbb{K}[G])$; for example, Lemma 2.9 (i) is equivalent to [27, Corollary 11.5.2] which computes the a-invariant for the toric ideal of $K_{a,b}$. Remark 4.3 gives another combinatorial interpretation of $\operatorname{reg}(\mathbb{K}[G])$ for bipartite graphs.

3. Comparing the regularity and projective dimension

Let $\operatorname{CBPT}(n)$ denote the set of connected bipartite graphs on n vertices. In this section, we focus on comparing the regularity and projective dimension of the toric ideals of graphs G with $G \in \operatorname{CBPT}(n)$. In particular, we describe the set

$$\operatorname{CBPT}^{\operatorname{pdim}}_{\operatorname{reg}}(n) = \{(\operatorname{reg}(\mathbb{K}[G]), \operatorname{pdim}(\mathbb{K}[G])) : G \in \operatorname{CBPT}(n)\}.$$

Understanding this set will be key to proving Theorem 1.1.

We begin with a simple inequality that we will use in subsequent lemmas.

Lemma 3.1. *Let n and r be integers. If $r \leq \lfloor \frac{n}{2} \rfloor - 1$, then $0 \leq n - 2 - 2r$.*

Proof. Observe that $r \leq \lfloor \frac{n}{2} \rfloor - 1 \leq \frac{n}{2} - 1 = \frac{n-2}{2}$. Hence $2r \leq n - 2$, so $0 \leq n - 2 - 2r$. □

Lemmas 3.2 and Lemma 3.3 give ranges of positive integers r and p that can be realized as $r = \operatorname{reg}(\mathbb{K}[G])$ and $p = \operatorname{pdim}(\mathbb{K}[G])$ for some connected bipartite graph G. More precisely, Lemma 3.2 (resp. Lemma 3.3) shows that any $n \geq 4$, $0 < r < \lfloor \frac{n}{2} \rfloor$ and $1 \leq p \leq r^2$ (resp. $r^2 \leq p \leq r(n-2-r)$) can be realized as $r = \operatorname{reg}(\mathbb{K}[G])$ and $p = \operatorname{pdim}(\mathbb{K}[G])$ for some connected bipartite graph G on n vertices. (Note that the $p = r^2$ case is covered twice.)

Lemma 3.2. *Let n, r, p be integers with $n \geq 4$, $0 < r < \lfloor \frac{n}{2} \rfloor$, and $1 \leq p \leq r^2$. Then there exists a connected bipartite graph G on n vertices with $\operatorname{reg}(\mathbb{K}[G]) = r$ and $\operatorname{pdim}(\mathbb{K}[G]) = p$.*

Proof. We describe the construction of the desired graph. Letting n, r, p be as given, we define the bipartite graph $G_{n,r,p}$ as follows. Let our vertex set V be

$$V = \{x_1, \ldots, x_{r+1}, y_1, \ldots, y_{r+1}, z_1, \ldots, z_{n-2r-2}\}$$

with bipartition $V_1 = \{x_1, \ldots, x_{r+1}\}$ and $V_2 = \{y_1, \ldots, y_{r+1}, z_1, \ldots, z_{n-2r-2}\}$. To define our edge set, let E_1 and E_2 be

$$E_1 = \{\{x_1, y_1\}, \{y_1, x_2\}, \{x_2, y_2\}, \{y_2, x_3\}, \ldots, \{x_{r+1}, y_{r+1}\}, \{y_{r+1}, x_1\}\};$$
$$E_2 = \{\{x_{r+1}, z_j\} \mid 1 \leq j \leq n - 2r - 2\}.$$

(Note that $n - 2r - 2 \geq 0$ by Lemma 3.1. If $n - 2r - 2 = 0$, then $E_2 = \emptyset$ and there are no z_i vertices.) Note E_1 is a cycle of length $2r + 2$. Let E_3 be any $p - 1 \geq 0$ edges with one vertex in $\{x_1, \ldots, x_{r+1}\}$ and the other in $\{y_1, \ldots, y_{r+1}\}$ that do not already appear in E_1. Because we can have at most $(r+1)^2 = r^2 + 2r + 1$ edges between the x_i's and y_j's, and since E_1 already used $2r + 2$ of these edges, there are only $r^2 - 1 = r^2 + 2r + 1 - (2r + 2)$ possible choices for these $p - 1$ edges. Since $p \leq r^2$, it is possible to find $p - 1$ such edges. Let $E = E_1 \cup E_2 \cup E_3$ be the edge set of $G_{n,r,p}$. See Figure 1 for an example of the graph $G_{n,r,p}$ for $n = 10, r = 3$, and $p = 2$. We claim that $G = G_{n,r,p}$ is a connected bipartite graph on n vertices with $\operatorname{reg}(\mathbb{K}[G]) = r$ and $\operatorname{pdim}(\mathbb{K}[G]) = p$.

By construction G is a connected bipartite graph on $n = (r+1) + (r+1) + (n - 2r - 2)$ vertices with $q = 2r + 2 + (n - 2r - 2) + (p - 1) = n + p - 1$ edges. Observe that G is a subgraph of the complete bipartite graph $K_{r+1,n-r-1}$. Since $r + 1 \leq n - r - 1$ by Lemma 3.1, by Theorem 2.6 and Lemma 2.9 (i), we have

$$\operatorname{reg}(\mathbb{K}[G]) \leq \operatorname{reg}(\mathbb{K}[K_{r+1,n-r-1}]) = \min\{r + 1, n - r - 1\} - 1 = r.$$

Since G contains the cycle C_{2r+2} as a subgraph (the edges of E_1 form this cycle) we have $r = \operatorname{reg}(\mathbb{K}[C_{2r+2}]) \leq \operatorname{reg}(\mathbb{K}[G])$ by Theorem 2.6 and Lemma 2.9 (ii). Thus $\operatorname{reg}(\mathbb{K}[G]) = r$. Furthermore, it follows from Corollary 2.8 (iii) that

$$\operatorname{pdim}(\mathbb{K}[G]) = q - n + 1 = (n + p - 1) - n + 1 = p,$$

as desired. □

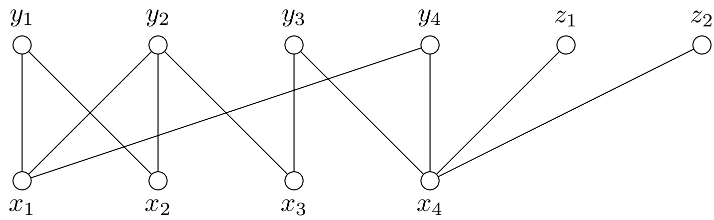

FIGURE 1. The graph $G = G_{10,3,2}$ with $\text{reg}(\mathbb{K}[G]) = 3$ and $\text{pdim}(\mathbb{K}[G]) = 2$

Lemma 3.3. *Let n, r, p be integers with $n \geq 4$, $0 < r < \lfloor \frac{n}{2} \rfloor$, and $r^2 \leq p \leq r(n-2-r)$. Then there exists a connected bipartite graph G on n vertices with $\text{reg}(\mathbb{K}[G]) = r$ and $\text{pdim}(\mathbb{K}[G]) = p$.*

Proof. We describe the construction of the desired graph. Letting n, r, p be as given, we define the bipartite graph $H_{n,r,p}$ as follows. Let our vertex set V be

$$V = \{x_1, \ldots, x_{r+1}, y_1, \ldots, y_{r+1}, z_1, \ldots, z_{n-2r-2}\}$$

with bipartition $V_1 = \{x_1, \ldots, x_{r+1}\}$ and $V_2 = \{y_1, \ldots, y_{r+1}, z_1, \ldots, z_{n-2r-2}\}$. To define our edge set, let E_1 and E_2 be

$$E_1 = \{\{x_i, y_j\} \mid 1 \leq i, j \leq r+1\};$$
$$E_2 = \{\{x_{r+1}, z_j\} \mid 1 \leq j \leq n-2r-2\}.$$

(Note that $n-2r-2 \geq 0$ by Lemma 3.1. If $n-2r-2 = 0$, then $E_2 = \emptyset$ and there are no z_i vertices.) Note that there can be at most $r(n-2r-2) = r(n-2-r) - r^2$ edges between $\{x_1, \ldots, x_r\}$ and $\{z_1, \ldots, z_{n-2r-2}\}$. Let E_3 be a set containing any $p - r^2 \geq 0$ of these edges. Observe that our hypotheses imply that $r(n-2r-2) \geq p - r^2$ so it is possible to find $p - r^2$ such edges. Let $E = E_1 \cup E_2 \cup E_3$ be the edge set of $H_{n,r,p}$. See Figure 2 for an example of the graph $H_{n,r,p}$ for $n = 10, r = 3$, and $p = 12$. We claim that $G = H_{n,r,p}$ is a connected bipartite graph on n vertices with $\text{reg}(\mathbb{K}[G]) = r$ and $\text{pdim}(\mathbb{K}[G]) = p$.

By construction G is a connected bipartite graph on $n = (r+1) + (r+1) + (n-2r-2)$ vertices with $q = (r+1)^2 + (n-2r-2) + (p-r^2) = n+p-1$ edges. Observe that G is a subgraph of the complete bipartite graph $K_{r+1,n-r-1}$. Since $r+1 \leq n-r-1$ by Lemma 3.1, by Theorem 2.6 and Lemma 2.9 (i), we have

$$\text{reg}(\mathbb{K}[G]) \leq \text{reg}(\mathbb{K}[K_{r+1,n-r-1}]) = \min\{r+1, n-r-1\} - 1 = r.$$

On the other hand, note that G contains the complete bipartite graph $K_{r+1,r+1}$ on $\{x_1, \ldots, x_{r+1}, y_1, \ldots, y_{r+1}\}$. Theorem 2.6 and Lemma 2.9 (i) imply that $r \leq \text{reg}(\mathbb{K}[G])$. Thus $\text{reg}(\mathbb{K}[G]) = r$, as desired. Furthermore, it follows from Corollary 2.8 (iii) that

$$\text{pdim}(\mathbb{K}[G]) = q - n + 1 = (n+p-1) - n + 1 = p,$$

completing the proof. □

The next result, which is of independent interest, provides an upper bound on the number of edges in a bipartite graph G in terms of the regularity.

Lemma 3.4. *Let G be a connected bipartite graph on $n \geq 2$ vertices. If $\text{reg}(\mathbb{K}[G]) = r$, then G has at most $(r+1)(n-r-1)$ edges.*

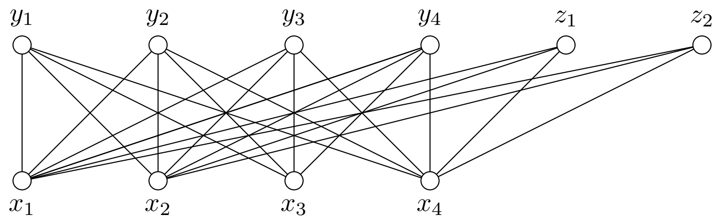

FIGURE 2. The graph $G = H_{10,3,12}$ with $\text{reg}(\mathbb{K}[G]) = 3$ and $\text{pdim}(\mathbb{K}[G]) = 12$

Proof. Suppose towards a contradiction that $|E(G)| > (r+1)(n-r-1)$. Since G is bipartite we can consider G as a subgraph of $K_{a,b}$ with $1 \le a \le b$ and $a+b=n$. Observe that if $a=1$, then G is a tree since G is connected. Hence $r=0$ by Lemma 2.10, so $|E(G)| = n-1 = (r+1)(n-r-1)$ and we obtain a contradiction.

So we may assume that $2 \le a \le b$. Since $G \subseteq K_{a,b}$ and $\text{reg}(\mathbb{K}[K_{a,b}]) = a-1$ by Lemma 2.9 (i), it follows by Theorem 2.6 that $a \ge r+1$. If $a = r+1$, then $b = n-r-1$ and so G is a subgraph of $K_{r+1,n-r-1}$ which has $(r+1)(n-r-1)$ edges, contradicting the assumption that $|E(G)| > (r+1)(n-r-1)$. So we can assume $a \ge r+2$. Hence $2 \le r+2 \le a \le b$, and so we can apply Theorem 2.1. We have two cases:

Case 1 ($a \le 2(r+2)-2$). Since $0 \le a-r-1$, we have $b \le a+b-r-1 = n-r-1$. Hence $br \le rn - r^2 - r$, so

$$br + (n-r-1) \le rn - r^2 - r + (n-r-1) = (r+1)(n-r-1).$$

It follows that $a + (b-1)(r+1) = br + (n-r-1) \le (r+1)(n-r-1) < |E(G)|$, so we conclude by Theorem 2.1 that G contains a cycle of length at least $2(r+2)$.

Case 2 ($a \ge 2(r+2)-2$). Then $(a+b-2(r+2)+3)(r+1) = (n-2r-1)(r+1) \le (n-r-1)(r+1) < |E(G)|$, so again we conclude by Theorem 2.1 that G contains a cycle of length at least $2(r+2)$.

In either case, G contains an even cycle C of length at least $2(r+2)$. But then $\text{reg}(\mathbb{K}[G]) \ge \text{reg}(\mathbb{K}[C]) \ge r+1$ by Theorem 2.6 and Lemma 2.9 (ii), contradicting the fact that $\text{reg}(\mathbb{K}[G]) = r$. This final contradiction concludes the proof. $\square$

We now arrive at the main result of this section.

Theorem 3.5. *Let $n \ge 2$ be an integer. Then*

$$\text{CBPT}^{\text{pdim}}_{\text{reg}}(n) = \left\{(r,p) \in \mathbb{Z}^2 \mid 0 < r < \left\lfloor \frac{n}{2} \right\rfloor, 1 \le p \le r(n-2-r)\right\} \cup \{(0,0)\}.$$

Proof. We show both inclusions, starting with $\supseteq$. If $n \in \{2,3\}$, then the RHS set is $\{(0,0)\}$, and by Lemma 2.10, we know that there is a tree G on n vertices with $\text{reg}(\mathbb{K}[G]) = \text{pdim}(\mathbb{K}[G]) = 0$. So suppose $n \ge 4$ and take an element (r,p) of the RHS set. If $(r,p) = (0,0)$, then again by Lemma 2.10, there is a tree G on n vertices with $\text{reg}(\mathbb{K}[G]) = \text{pdim}(\mathbb{K}[G]) = 0$. Otherwise, if $(r,p) \ne (0,0)$, we must have $0 < r < \left\lfloor \frac{n}{2} \right\rfloor$, and $1 \le p \le r(n-2-r)$. Lemma 3.2 and Lemma 3.3 imply that there is a connected bipartite graph G on n vertices with $\text{reg}(\mathbb{K}[G]) = r$ and $\text{pdim}(\mathbb{K}[G]) = p$, which verifies the first inclusion.

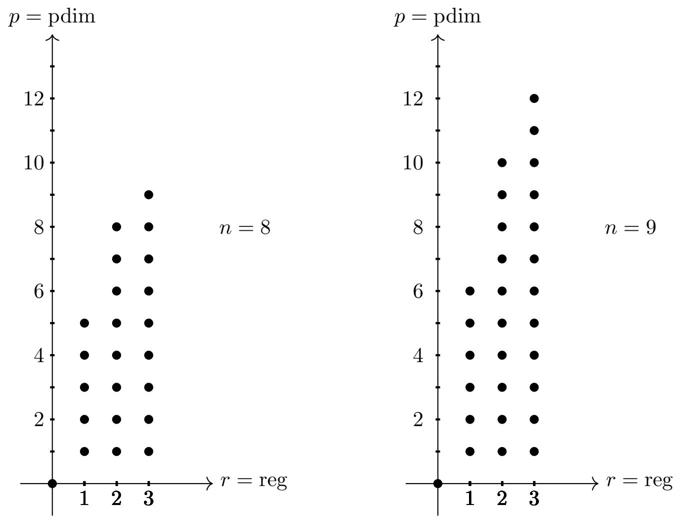

FIGURE 3. Possible $(r,p) = (\mathrm{reg}(\mathbb{K}[G]), \mathrm{pdim}(\mathbb{K}[G]))$ for all connected bipartite graphs on 8 and 9 vertices

Now, let $n \geq 2$ and $G \in \mathrm{CBPT}(n)$. By Corollary 2.8 (ii), we have $0 \leq \mathrm{reg}(\mathbb{K}[G]) < \lfloor \frac{n}{2} \rfloor$. If $\mathrm{reg}(\mathbb{K}[G]) = 0$, then $\mathrm{pdim}(\mathbb{K}[G]) = 0$ by Lemma 2.10. Hence letting $r = \mathrm{reg}(\mathbb{K}[G])$ and $p = \mathrm{pdim}(\mathbb{K}[G])$, we just need to show that if $0 < r < \lfloor \frac{n}{2} \rfloor$, then $1 \leq \mathrm{pdim}(\mathbb{K}[G]) \leq r(n-2-r)$. So suppose $0 < r < \lfloor \frac{n}{2} \rfloor$. Since $r \neq 0$, we must have $1 \leq p$ by Lemma 2.10.

It remains to show that $p \leq r(n-2-r)$. By Lemma 3.4, we have that $q \leq (r+1)(n-r-1)$, where q is the number of edges of G. Hence Corollary 2.8 (iii) gives

$$\begin{aligned} p = q - n + 1 &\leq (r+1)(n-r-1) - n + 1 \\ &= rn - r^2 - 2r \\ &= r(n-r-2) \end{aligned}$$

as desired, which concludes the proof. □

As an example of Theorem 3.5, Figure 3 shows the sets $\mathrm{CBPT}^{\mathrm{pdim}}_{\mathrm{reg}}(n)$ for $n = 8, 9$. Now that we have a description of $\mathrm{CBPT}^{\mathrm{pdim}}_{\mathrm{reg}}(n)$ for each n, we can compute its cardinality.

Corollary 3.6. *For each $n \geq 2$,*

$$|\,\mathrm{CBPT}^{\mathrm{pdim}}_{\mathrm{reg}}(n)| = 1 - \frac{1}{6}\left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\left(2\left\lfloor \frac{n}{2} \right\rfloor - 3n + 5\right).$$

Proof. This can be checked directly for $n = 2, 3$. Let $n \geq 4$. By Theorem 3.5, we have

$$\begin{aligned} |\operatorname{CBPT}^{\text{pdim}}_{\text{reg}}(n)| - 1 &= \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor - 1} r(n-2-r) = (n-2) \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor - 1} r - \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor - 1} r^2 \\ &= \frac{n-2}{2} \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) - \frac{1}{6} \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \left(2 \left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \\ &= -\frac{1}{6} \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \left(2 \left\lfloor \frac{n}{2} \right\rfloor - 3n + 5 \right). \end{aligned}$$ □

Our second corollary shows that all tuples $(r, p) \in \{(0, 0)\} \cup \mathbb{N}^2$ can be realized as $(\operatorname{reg}(\mathbb{K}[G]), \operatorname{pdim}(\mathbb{K}[G]))$ for some connected bipartite graph G. Here $\mathbb{N} = \{1, 2, 3, \ldots, \}$.

Corollary 3.7. *Let r and p be integers. Then there is a connected bipartite graph G on at least two vertices with $\operatorname{reg}(\mathbb{K}[G]) = r$ and $\operatorname{pdim}(\mathbb{K}[G]) = p$ if and only if $r = p = 0$ or $r, p \geq 1$. Equivalently,*

$$\bigcup_{n \geq 2}^{\infty} \operatorname{CBPT}^{\text{pdim}}_{\text{reg}}(n) = \{(0, 0)\} \cup \mathbb{N}^2.$$

Proof. Let G be a connected bipartite graph on $n \geq 2$ vertices with $\operatorname{reg}(\mathbb{K}[G]) = r$ and $\operatorname{pdim}(\mathbb{K}[G]) = p$. By Theorem 3.5, $(r, p) = (0, 0)$, or $r, p \geq 1$.

Conversely, suppose $r = p = 0$ or $r, p \geq 1$. If $r = p = 0$, then the unique tree G on two vertices is a connected bipartite graph with $\operatorname{reg}(\mathbb{K}[G]) = \operatorname{pdim}(\mathbb{K}[G]) = 0$, so $(0, 0) \in \operatorname{CBPT}^{\text{pdim}}_{\text{reg}}(2)$. So assume $r, p \geq 1$. Let $N = 2 + r + \max\{r, p\}$. Then $N \geq 2 + r + r = 2 + 2r$. Hence

$$0 < r < r + 1 = \left\lfloor \frac{2r+2}{2} \right\rfloor \leq \left\lfloor \frac{N}{2} \right\rfloor$$

and thus $0 < r < \lfloor \frac{N}{2} \rfloor$. Also, observe that since $r \geq 1$,

$$r(N - 2 - r) = r \max\{r, p\} \geq rp \geq p,$$

so $1 \leq p \leq r(N - 2 - r)$. We then have $(r, p) \in \operatorname{CBPT}^{\text{pdim}}_{\text{reg}}(N)$ by Theorem 3.5. □

4. Proof of main theorem

Using the previous sections, we can prove the main result of this paper, namely, a description of all the elements of $\operatorname{CBPT}_{\text{reg,deg,pdim,depth,dim}}(n)$. In particular, we now prove:

Theorem 4.1. *Let $n \geq 2$. Then $\operatorname{CBPT}_{\text{reg,deg,pdim,depth,dim}}(n)$ is given by*

$$\left\{ (r, r, p, n-1, n-1) \;\middle|\; 0 < r < \left\lfloor \frac{n}{2} \right\rfloor,\ 1 \leq p \leq r(n-2-r) \right\} \cup \{(0, 0, 0, n-1, n-1)\}.$$

Proof. Fix an $n \geq 2$ and let T denote the set in the statement. We will first show that all the elements of $\operatorname{CBPT}_{\text{reg,deg,pdim,depth,dim}}(n)$ belong to T.

Let G be any connected bipartite graph on n vertices, and set

$$(r, d_1, p, d_2, d_3) = (\operatorname{reg}(\mathbb{K}[G], \deg(h_{\mathbb{K}[G]}(t)), \operatorname{pdim}(\mathbb{K}[G]), \operatorname{depth}(\mathbb{K}[G]), \dim(\mathbb{K}[G])).$$

By Corollary 2.8 (i) and (ii), we have $0 \leq r = d_1 < \lfloor \frac{n}{2} \rfloor$ and $d_2 = d_3 = n - 1$, i.e., $(r, d_1, p, d_2, d_3) = (r, r, p, n-1, n-1)$. Since $G \in \operatorname{CBPT}(n)$, by Theorem 3.5

we have $r = p = 0$, or $0 < r < \lfloor \frac{n}{2} \rfloor$ and $1 \leq p \leq r(n-2-r)$. Consequently, $(r, d_1, p, d_2, d_3) \in T$.

For the reverse containment, note that

$$(0, 0, 0, n-1, n-1) \in \mathrm{CBPT}_{\mathrm{reg},\mathrm{deg},\mathrm{pdim},\mathrm{depth},\mathrm{dim}}(n)$$

since any connected tree G on n vertices satisfies

$$(\mathrm{reg}(\mathbb{K}[G], \deg(h_{\mathbb{K}[G]}(t)), \mathrm{pdim}(\mathbb{K}[G]), \mathrm{depth}(\mathbb{K}[G]), \dim(\mathbb{K}[G])) = (0, 0, 0, n-1, n-1)$$

by Corollary 2.8 and Lemma 2.10. So, consider any $(r, r, p, n-1, n-1) \in T$ with $0 < r$. Because the tuple (r, p) belongs to $\mathrm{CPBT}^{\mathrm{pdim}}_{\mathrm{reg}}(n)$ by Theorem 3.5, there exists a connected bipartite graph G on n vertices with $\mathrm{reg}(\mathbb{K}[G]) = r$ and $\mathrm{pdim}(\mathbb{K}[G]) = p$. But by Corollary 2.8 (i) and (ii), this graph G also has $\deg(h_{\mathbb{K}[G]}(t)) = r$ and $\dim(\mathbb{K}[G]) = \mathrm{depth}(\mathbb{K}[G]) = n-1$. Thus $(r, r, p, n-1, n-1) \in \mathrm{CBPT}_{\mathrm{reg},\mathrm{deg},\mathrm{pdim},\mathrm{depth},\mathrm{dim}}(n)$, as desired. □

Remark 4.2. Theorem 4.1 focuses on the *connected* bipartite graphs. It is possible to provide a generalization of Theorem 4.1 to describe all the possible values for these invariants for all bipartite graphs, not just connected bipartite graphs. In particular, one needs to make use of the fact that these invariants behave well over tensor products. However, additional care needs to be taken for bipartite graphs with isolated vertices. See [2] for the worked out details.

Remark 4.3. In Theorem 3.5, we completely determined the set $\mathrm{CBPT}^{\mathrm{pdim}}_{\mathrm{reg}}(n)$. Akihiro Higashitani pointed out to us that describing this set is equivalent to describing a combinatorially defined set; we quickly sketch out these details. Associated with a toric ideal of a graph G is a polytope P_G. The *codegree* of P_G is given by $\mathrm{codeg}(P_G) = \min\{k \in \mathbb{Z} \mid \mathrm{int}(kP_G) \cap \mathbb{Z}^n \neq \emptyset\}$. This invariant is measuring the smallest integer k such that the interior of the polytope kP_G has an integer lattice point. In the case G is a bipartite graph, it can be shown that $\deg(h_{\mathbb{K}[G]}(t)) + \mathrm{codeg}(P_G) = n$. Further, when G is a bipartite graph, by Corollary 2.8 (ii) and (iii) we have $\mathrm{codeg}(P_G) = n - \mathrm{reg}(\mathbb{K}[G])$ and $|E| = \mathrm{pdim}(\mathbb{K}[G]) + n - 1$. Consequently, determining the elements of $\mathrm{CBPT}^{\mathrm{pdim}}_{\mathrm{reg}}(n)$ is equivalent to determining the elements of

$$\mathrm{CBPT}^{|E|}_{\mathrm{codeg}}(n) = \{(\mathrm{codeg}(P_G), |E|) \mid G \in \mathrm{CBPT}(n)\}.$$

This observation suggests it might be interesting to consider all pairs $(\mathrm{codeg}(P_G), |E|)$ for all graphs, not just bipartite graphs.

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Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario L8S 4L8, Canada

Email address: kieran.bhaskara@mcmaster.ca

Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario L8S 4L8, Canada

Email address: vantuyl@math.mcmaster.ca