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THE $\mathbf{n^{th}}$ ITERATE OF A FORMAL POWER SERIES WITH LINEAR TERM A PRIMITIVE $\mathbf{n^{th}}$ ROOT OF UNITY

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Abstract. We give a very short proof of the Theorem: *Suppose that* $f(x) = a_1x + a_2x^2 + \cdots$ *is a formal power series with coefficients in an integral domain, and* a_1 *is a primitive* n^{th} *root of unity* $(n \in \mathbb{N})$. *If the* n^{th} *iterate* $f^{(n)}(x) \equiv f\big(f(\cdots f\big(f(x)\big)\cdots\big)$ *satisfies* $f^{(n)}(x) = x+b_mx^m+b_{m+1}x^{m+1}+\cdots$, *with* $b_m \neq 0$ *and* $m > 1$, *then* $m \equiv 1 \pmod n$.

The Theorem. *Suppose that* $n \geq 2$ *is an integer and*

(1) $$f(x) = a_1x + a_2x^2 + \cdots, \text{ with } a_1 \text{ a primitive } n^{\text{th}} \text{ root of unity,}$$

is a formal power series, with coefficients in an integral domain, which has n^{th} *iterate* (n^{th} *compositional power),*

(2) $$f^{(n)}(x) \equiv f\big(f(\cdots(f(x))\cdots\big) = x + b_mx^m + b_{m+1}x^{m+1} + \cdots, b_m \neq 0, m > 1.$$

Then $m \equiv 1 \pmod n$.
Thus, there exists $q \in \mathbb{N}$ *such that* $b_k = 0$ *if* $1 < k < qn + 1$ *and* $b_{qn+1} \neq 0$.

We will give a very short proof of this theorem.

Recall that a_1 is a *primitive* n^{th} *root of unity* iff $a_1^n = 1$ and n is the smallest such positive integer.

The background. Iteration of formal power series is the combinatorial face of the iteration of complex analytic functions, the study of which goes back at least to the 1870's ([7], [3]). A series f with $a_0 = 0$ and $a_1 \neq 0$ has a compositional inverse, and will be called *invertible*. Formal series f satisfying (1) and (2) are the invertible series which have linear term $a_1 \neq 1$ of finite multiplicative order, but for which f is of infinite compositional order. We call these *hybrid series*. They have been the hardest (see below) to compute or characterize among invertible series.

Induction shows that $b_1 = a_1^n$ and b_k is a polynomial in $a_1, a_2, \ldots, a_k$. In general , if $j \geq 2$ and $f^{(j-1)}(x) = \sum_{q=1}^{\infty} c_qx^q$ then $f^{(j)}(x) = f^{(j-1)}\big(f(x)\big) = \sum_{q=1}^{\infty} c_q\big(f(x)\big)^q$. Each coefficient of x^k in $\big(f(x)\big)^q (q \leq k)$ is a *multinomial polynomial*, the sum of a large sum of terms involving multinomial coefficients $\binom{k}{r_1, r_2, \cdots, r_q} a_1^{r_1} \cdots a_q^{r_q}$ (see [10], p.17). The final formula for the coefficient b_k in $f^{(n)}$ builds up inductively.

For $a_1 = 1$, Schröder [7] (1871) gave an explicit formula for each b_k – *i.e.*, gave *A Multinomial Theorem for Formal Power Series Under Composition*. But he explicitly passed on the case $0 \neq a_1 \neq 1$. Instead he introduced the problem of

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conjugating, when possible, $f(x)$ to the linear function $\ell(x) = a_1x$, in order to get information about $f^{(n)}(x)$ without using an explicit formula. Later mathematicians – most notably Scheinberg [6] (1970) – generalized this problem to finding normal forms under conjugation for formal power series. It turns out [6, Propostion 8] that an invertible series $f(x)$ is conjugate to $\ell(x) = a_1x$ – written $f(x) \sim \ell(x)$ – if and only if $f(x)$ is not a hybrid series.[1] When f is a hybrid series then Scheinberg's conjugacy classification [6, Table 1] yields $f^{(n)}(x) \sim z + z^m + cz^{2m-1}$ where $m \equiv 1 \mod n$. But the equality proved in the Theorem of this paper was not proved until the 1990s,

We note that there has been much effort to find the coefficients b_k in (2) above. Using the Riordan matrix $(R_f)_{k\geq 0, q\geq 0} = (1, f)$ [8] (this is an infinite lower triangular matrix of multinomial polynomials), one sees that b_k is the $(k, 1)$ element of the matrix $(R_f)^n$. Closely related, one can identify $(k!)b_k$ as the $((k, 1))$ element of the the matrix $(B(f))^n$, where $B(f)$ is the matrix of Bell polynomials of f ([2], p.145). Only in 2020 did Monkam [4] generalize Schröder's result to an explicit closed formula for every coefficient of $f^{(n)}(x)$ when $a_1 \neq 0$. This gave much information about the form of the polynomial $b_k = b_k(a_1, \ldots, a_k)$. But after all is said and done, it is very hard to look at these explicit formulas and decide when, indeed, a particular $b_k = 0$, or that one has a consecutive string of zero coefficients.

Reich [5] (1992) gave a proof of the Theorem of this paper using the theory of normal forms under conjugation. Bogatyi [1] (1998) gave a proof of this Theorem involving the theory of the index of an isolated fixed point of a holomorphic map. He also sketched a proof using Scheinberg's normal forms. We give a proof which uses only the definition and associativity of composition of formal power series.

The proof.

Proof. We use a clever trick from line 11 of the proof of Theorem 9 of [6]: Equate the coefficients of x^m in $\big(f^{(n)} \circ f\big)(x)$ and in $\big(f \circ f^{(n)}\big)(x)$.

(a) $\big(f^{(n)} \circ f\big)(x)$

$$\begin{aligned} &= f(x) + b_m \cdot \big(f(x)\big)^m + b_{m+1} \cdot \big(f(x)\big)^{m+1} + \cdots \\ &= (a_1x + \cdots \underline{a_m x^m} + \cdots) + \underline{b_m\big(a_1^m x^m} + \cdots) + \text{(higher powers)} \end{aligned}$$

(b) $\big(f \circ f^{(n)}\big)(x)$

$$\begin{aligned} &= a_1 \cdot f^{(n)}(x) + a_2\big(f^{(n)}(x)\big)^2 + \cdots + a_m\big(f^{(n)}(x)\big)^m + \cdots \\ &= a_1 \cdot (x + b_m x^m + b_{m+1}x^{m+1} + \cdots) + + \cdots + a_m(x + b_m x^m + \cdots)^m + \cdots \\ &= \underline{a_1} \cdot (x + \underline{b_m x^m} + b_{m+1}x^{m+1} + \cdots) + + \cdots + \underline{a_m(x^m} + \cdots) + \text{ higher powers} \end{aligned}$$

(c) Thus, equating coefficients of x^m, we have (since $b_m \neq 0$ and our coefficients lie in an integral domain):

$$a_m + b_m a_1^m = a_1 b_m + a_m \implies b_m a_1^m = a_1 b_m \implies a_1^{m-1} = 1.$$

[1] Scheinberg also showed [6, Theorem 11], building on Sternberg [11, Section 2], that an invertible series belongs to a one-parameter subgroup of the group of all invertible series if and only if it is not hybrid. Further, Siegel [9] notes that an analytic function $f(z)$ with a_1 a root of unity is stable at 0 iff it is not given by a hybrid series.

Since the order of a_1 equals n, we see that n divides $m - 1$. Thus, $m \equiv 1 \pmod{n}$. $\square$

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