MILNOR OPERATIONS AND CLASSIFYING SPACES

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ABSTRACT. We give an example of a nonzero odd degree element of the classifying space of a connected Lie group such that all higher Milnor operations vanish on it. It is a counterexample of a conjecture of Kono and Yagita [Trans. Amer. Math. Soc. 339 (1993), pp. 781–798].

1. INTRODUCTION

For each prime number p, there are the mod p and Brown-Peterson cohomology. For a compact connected Lie group G, the mod p cohomology of the classifying space BG has no nonzero odd degree element if the integral cohomology of G has no p-torsion. So does the Brown-Peterson cohomology. On the one hand, if the integral homology of G has p-torsion, the mod p cohomology of BG has a nonzero odd degree element. On the other hand, for the Brown-Peterson cohomology, Kono and Yagita conjectured the following:

Conjecture 1.1 (Kono and Yagita, (1) in Conjecture 4 in [KY93]). *There is no nonzero odd degree element in the Brown-Peterson cohomology of the classifying space of a compact Lie group.*

Conjecture 1.1 is interesting in conjunction with Totaro's conjecture on the cycle map from the Chow ring of the classifying space of a complex linear algebraic group G to its Brown-Peterson cohomology. In [Tot97], Totaro showed that the cycle map from the Chow ring of a complex smooth algebraic variety to its ordinary cohomology factors through the Brown-Peterson cohomology after localized at p. In [Tot99], he defined the Chow ring $CH^*(BG)$ of a linear algebraic group G and conjectured the following.

Conjecture 1.2 (Totaro, p.250 in [Tot99]). *For a complex linear algebraic group G, if there is no nonzero odd degree element in the Brown-Peterson cohomology $BP^*(BG)$, the cycle map*

$$CH^i(BG)_{(p)} \to (\mathbb{Z}_{(p)} \otimes_{BP^*} BP^*(BG))^{2i}$$

is an isomorphism.

With Conjectures 1.1 and 1.2, we expect a close connection between the Chow ring in algebraic geometry and the Brown-Peterson cohomology in algebraic topology. In [KY93], Kono and Yagita confirmed Conjecture 1.1 for some compact

connected Lie groups with p-torsion by computing the Atiyah-Hirzebruch spectral sequences. The non-triviality of Milnor operations on odd degree elements yields non-trivial differentials sending odd degree elements to nonzero elements, so odd degree elements do not survive to the E_∞-term. With their computational results on the Brown-Peterson cohomology of classifying spaces, Kono and Yagita conjectured the following:

Conjecture 1.3 (Kono and Yagita, Conjecture 5 in [KY93]). *For each nonzero odd degree element x of the mod p cohomology of the classifying space of a compact connected Lie group, there exists an integer i such that for $m \geq i$,*

$$Q_m x \neq 0.$$

Conjecture 1.3 is interesting in the cohomology theory of classifying spaces of non-simply connected Lie groups. In [VV05], Vavpetič and Viruel showed that if p is an odd prime, Conjecture 1.3 holds for the projective unitary group $PU(p)$. Moreover, the cohomology of classifying spaces of non-simply connected Lie groups has recently enjoyed renewed interest. Many mathematicians have studied it in various contexts. Antieau, Gu and Williams ([AW14], [Gu19], [Gu20], [GZZZ22]) studied it for the topological period-index problem. Antieau, the author and Tripathy ([Ant16], [Kam15], [Kam17], [Tri16]) studied it for integral Hodge conjecture modulo torsion. Furthermore, the Atiyah-Hirzebruch spectral sequence is used in theoretical physics to study anomalies, cf. García-Etxebarria and Montero [GEM19].

In this paper, we give a counterexample for Conjecture 1.3 in the case $p = 2$. Our result is as follows: Let $\mathbb{H}$ be the quaternions. Let $Sp(1) \subset \mathbb{H}$ be the symplectic group consisting of unit quaternions. Let G be the quotient of the 3-fold product $Sp(1)^3$ of the symplectic groups $Sp(1)$ by the subgroup Γ_2 generated by $(-1, -1, 1)$ and $(-1, 1, -1)$.

Theorem 1.4. *In the mod 2 cohomology of the classifying space of the compact connected Lie group G above, there exists a nonzero element x_{13} of degree 13 such that*

$$Q_m x_{13} = 0$$

for $m \geq 1$.

This paper is organized as follows. In Section 2, we describe the action of Milnor operations on the mod 2 cohomology of $BSO(3)$. In Section 3, we prove Theorem 1.4 as Proposition 3.5.

2. Milnor operations

In this section, we recall Milnor operations

$$Q_m \colon H^i(X; \mathbb{Z}/2) \to H^{i+2^m+1}(X; \mathbb{Z}/2)$$

and the mod 2 cohomology of the classifying space $BSO(3)$. Milnor operations Q_m are defined by

$$Q_0 = \mathrm{Sq}^1, \quad Q_m = \mathrm{Sq}^{2^m} Q_{m-1} + Q_{m-1} \mathrm{Sq}^{2^m} \quad (m \geq 1).$$

They have the following properties:

$$Q_m Q_n = Q_n Q_m,$$
$$Q_m^2 = 0,$$

and

$$Q_m(x \cdot y) = (Q_m x) \cdot y + x \cdot (Q_m y).$$

These formulae are essential in our proofs Propositions 2.2 and 3.5. The mod 2 cohomology of $BSO(3)$ is a polynomial ring

$$H^*(BSO(3); \mathbb{Z}/2) = \mathbb{Z}/2[w_2, w_3]$$

generated by two elements w_2, w_3 of degree 2, 3, respectively. The action of Steenrod squares on these elements is well-known as the Wu formula. In particular, we have

$$\mathrm{Sq}^1 w_2 = w_3, \qquad\qquad \mathrm{Sq}^2 w_2 = w_2^2,$$
$$\mathrm{Sq}^1 w_3 = 0, \qquad\qquad \mathrm{Sq}^2 w_3 = w_2 w_3.$$

By the Wu formula and by the definition and elementary properties of Milnor operations stated above, it is easy to obtain

$$Q_0 w_2 = w_3, \qquad Q_1 w_2 = w_2 w_3, \qquad Q_0 Q_1 w_2 = w_3^2,$$
$$Q_0 w_3 = 0, \qquad Q_1 w_3 = w_3^2, \qquad Q_0 Q_1 w_3 = 0.$$

This section aims to prove the following lemma on the action of Milnor operations on the mod 2 cohomology of $BSO(3)$.

Lemma 2.1. *For $m \geq 2$, there exists a polynomial g_m in w_2^2 and w_3^2 such that we have*

$$Q_m Q_1 w_2 = g_m w_3^4.$$

in the mod 2 cohomology of $BSO(3)$.

To prove Lemma 2.1, we may recall the relation between Dickson invariants and Milnor operations as Proposition 2.2. The connection between Dickson invariants and Milnor operations is an exciting subject in algebraic topology. Thus, we refer the reader to the classical work of Adams and Wilkerson ([AW80], [Wil83]) for more detail on the background of this section. However, to make this paper self-contained as far as possible, we give detailed proof for Lemma 2.1 without mentioning Dickson invariants and the above background.

Let $(\mathbb{Z}/2)^2 = \mathbb{Z}/2 \times \mathbb{Z}/2$ be the elementary abelian 2-subgroup of $SO(3)$ generated by diagonal matrices

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We denote by $\iota \colon (\mathbb{Z}/2)^2 \to SO(3)$ the inclusion map. The induced homomorphism

$$B\iota^* \colon H^*(BSO(3); \mathbb{Z}/2) \to H^*(B(\mathbb{Z}/2)^2; \mathbb{Z}/2)$$

is injective, and its image is the subring generated by the following elements.

$$B\iota^*(w_2) = s_1^2 + s_1 s_2 + s_2^2,$$
$$B\iota^*(w_3) = s_1^2 s_2 + s_1 s_2^2.$$

Proposition 2.2. *Suppose that $m \geq 2$. For an element x of the mod 2 cohomology of $B(\mathbb{Z}/2)^2$, let*

$$D_m x = Q_m x + B\iota^*(w_2^{2^{m-1}}) Q_{m-1} x + B\iota^*(w_3^{2^{m-1}}) Q_{m-2} x.$$

Then, we have

$$D_m x = 0.$$

Proof. Here, in the proof of Proposition 2.2, by the mod 2 cohomology ring, we mean the mod 2 cohomology ring of $B(\mathbb{Z}/2)^2$ unless otherwise stated explicitly. Recall that

$$Q_m(x \cdot y) = (Q_m x) \cdot y + x \cdot (Q_m y),$$

for $x, y \in H^*(X; \mathbb{Z}/2)$. For $i = 1$ and 2, we have

$$
\begin{aligned}
D_m s_i &= Q_m s_i + B\iota^*(w_2^{2^{m-1}}) Q_{m-1} s_i + B\iota^*(w_3^{2^{m-1}}) Q_{m-2} s_i \\
&= s_i^{2^{m+1}} + (s_1^2 + s_1 s_2 + s_2^2)^{2^{m-1}} s_i^{2^m} + (s_1^2 s_2 + s_1 s_2^2)^{2^{m-1}} s_i^{2^{m-1}} \\
&= \left(s_i^4 + (s_1^2 + s_1 s_2 + s_2^2) s_i^2 + (s_1^2 s_2 + s_1 s_2^2) s_i \right)^{2^{m-1}} \\
&= 0.
\end{aligned}
$$

Thus, for elements x, y in the mod 2 cohomology ring, we have

$$D_m(x \cdot y) = D_m x \cdot y + x \cdot D_m y.$$

Therefore, since the mod 2 cohomology ring is generated by s_1, s_2, the fact that $D_m s_i = 0$ for $i = 1, 2$ implies that $D_m x = 0$ for each element x in the mod 2 cohomology ring. $\qquad\square$

Now, for $m \geq 2$, we describe the action of the Milnor operation Q_m in terms of certain polynomials $f_{m,0}$, $f_{m.1}$ in w_2^2 and w_3^2 and Milnor operations Q_0, Q_1. Since the induced homomorphism

$$B\iota^* \colon H^*(BSO(3); \mathbb{Z}/2) \to H^*(B(\mathbb{Z}/2)^2; \mathbb{Z}/2)$$

is injective, by Proposition 2.2, for each x in the mod 2 cohomology of $BSO(3)$, we have

$$Q_m x = w_2^{2^{m-1}} Q_{m-1} x + w_3^{2^{m-1}} Q_{m-2} x.$$

We may write it in the following form.

$$
\begin{pmatrix} Q_m x \\ Q_{m-1} x \end{pmatrix} = \begin{pmatrix} w_2^{2^{m-1}} & w_3^{2^{m-1}} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Q_{m-1} x \\ Q_{m-2} x \end{pmatrix}.
$$

Let us define a matrix A_m whose coefficients are polynomials in w_2^2, w_3^2 as follows:

$$
A_m = \begin{pmatrix} w_2^{2^{m-1}} & w_3^{2^{m-1}} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_2^{2^{m-2}} & w_3^{2^{m-2}} \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} w_2^4 & w_3^4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_2^2 & w_3^2 \\ 1 & 0 \end{pmatrix}.
$$

Furthermore, let us define polynomials $f_{m,0}$, $f_{m,1}$ by

$$
\begin{pmatrix} f_{m,1} & f_{m,0} \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} A_m.
$$

Then, for x in the mod 2 cohomology of $BSO(3)$, we have

$$
Q_m x = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} Q_m x \\ Q_{m-1} x \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} A_m \begin{pmatrix} Q_1 x \\ Q_0 x \end{pmatrix} = f_{m,1} Q_1 x + f_{m,0} Q_0 x.
$$

Proof of Lemma 2.1. We have the following congruence.

$$
A_m \equiv \begin{pmatrix} w_2^{2^{m-1}} & 0 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} w_2^2 & 0 \\ 1 & 0 \end{pmatrix} \equiv \begin{pmatrix} w_2^{2^m - 2} & 0 \\ w_2^{2^{m-1} - 2} & 0 \end{pmatrix} \quad \mod (w_3^2).
$$

Hence, we have $f_{m,0} \equiv 0 \mod (w_3^2)$. Therefore, there exists a polynomial g_m in w_2^2 and w_3^2 such that
$$f_{m,0} = g_m w_3^2.$$
Recall the fact that $Q_0 Q_1 w_2 = w_3^2$ and $Q_1 Q_1 = 0$. Then, we have
$$
\begin{aligned}
Q_m Q_1 w_2 &= f_{m,1} Q_1 Q_1 w_2 + f_{m,0} Q_0 Q_1 w_2 \\
&= f_{m,0} Q_0 Q_1 w_2 \\
&= g_m w_3^4. \hspace{3cm} \square
\end{aligned}
$$

Example 2.3. For $m = 2, 3, 4$, elements $Q_m x$ and polynomials g_m in Lemma 2.1 are as follows:
$$
\begin{aligned}
Q_2 x &= w_2^2 Q_1 x + w_3^2 Q_0 x, & g_2 &= 1, \\
Q_3 x &= (w_2^6 + w_3^4) Q_1 x + w_2^4 w_3^2 Q_0 x, & g_3 &= w_2^4, \\
Q_4 x &= (w_2^{14} + w_2^8 w_3^4 + w_2^2 w_3^8) Q_1 x + (w_2^{12} + w_3^8) w_3^2 Q_0 x, & g_4 &= w_2^{12} + w_3^8.
\end{aligned}
$$

3. The nonzero odd degree element

In this section, we prove Theorem 1.4 as Proposition 3.5.

We begin with recalling the definition of the connected Lie group G in Section 1 and set up notations. Let us consider the 3-fold product of symplectic groups $Sp(1) \subset \mathbb{H}$ consisting of unit quaternions. Let
$$\Gamma_3 = \{(\pm 1, \pm 1, \pm 1)\}$$
be the center of $Sp(1)^3$. Let Γ_2 be its subgroup generated by $(-1, 1, -1), (1, -1, -1)$ and
$$G = Sp(1)^3 / \Gamma_2.$$
Let $\mathbb{Z}/2 = \{(\pm 1, 1, 1)\} \subset \Gamma_3$. Then, $\mathbb{Z}/2$ and Γ_2 generate Γ_3. Moreover, we have
$$Sp(1)^3 / \Gamma_3 = SO(3)^3.$$
Therefore, we have the following fiber sequence:
$$B\mathbb{Z}/2 \to BG \to BSO(3)^3.$$

We denote by
$$\{E_r^{p,q}, d_r \colon E_r^{p,q} \to E_r^{p+r, q-r+1}\}$$
the Leray-Serre spectral sequence associated with this fiber sequence. Let us denote its E_r-term by
$$E_r = \bigoplus_{p,q} E_r^{p,q}.$$
We compute the mod 2 cohomology of BG using the above Leray-Serre spectral sequence. Although it is easy, we quickly review it. See [Kam19] for more detail.

We describe the E_2-term and compute the first non-trivial differential d_2. Let
$$B\pi_i \colon BSO(3)^3 \to BSO(3)$$
be the map induced by the projection to the ith factor for $i = 1, 2, 3$. We denote by w_i', w_i'', w_i''' the cohomology classes $B\pi_1^*(w_i)$, $B\pi_2^*(w_i)$, $B\pi_3^*(w_i)$, respectively. Let u_1 be the generator of the mod 2 cohomology $H^1(B\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2$ of the fiber $B\mathbb{Z}/2$. The E_2-term is given by
$$E_2 = \mathbb{Z}/2[w_2', w_3', w_2'', w_3'', w_2''', w_3'''] \otimes \mathbb{Z}/2[u_1].$$

To compute the differential d_2, we consider the Leray-Serre spectral sequence

$$\{\bar{E}_r^{p,q}, \bar{d}_r \colon \bar{E}_r^{p,q} \to \bar{E}_r^{p+r,q-r+1}\}$$

associated with the fiber sequence

$$B\mathbb{Z}/2 \to BSp(1) \to BSO(3).$$

Recall that its E_2-term is given as follows:

$$\bar{E}_2 = \mathbb{Z}/2[w_2, w_3] \otimes \mathbb{Z}/2[u_1],$$

and its first nontrivial differential $\bar{d}_2$ is given by

$$\bar{d}_2(u_1) = w_2.$$

Let

$$B\iota_i \colon BSO(3) \to BSO(3)^3$$

be the map induced by the inclusion map ι_i of $Sp(1)$ for $i = 1, 2, 3$ such that

$$\iota_1(g) = (g, 1, 1), \quad \iota_2(g) = (1, g, 1), \quad \iota_3(g) = (1, 1, g).$$

Then we have the following commutative diagram,

$$
\begin{array}{ccccc}
B\mathbb{Z}/2 & \longrightarrow & BSp(1) & \longrightarrow & BSO(3) \\
\big\Vert & & \big\downarrow & & \big\downarrow {\scriptstyle B\iota_i} \\
B\mathbb{Z}/2 & \longrightarrow & BG & \longrightarrow & BSO(3)^3.
\end{array}
$$

Furthermore, we have

$$
\begin{array}{lll}
B\iota_1^*(w_2') = w_2, & B\iota_1^*(w_2'') = 0, & B\iota_1^*(w_2''') = 0, \\
B\iota_2^*(w_2') = 0, & B\iota_2^*(w_2'') = w_2, & B\iota_2^*(w_2''') = 0, \\
B\iota_3^*(w_2') = 0, & B\iota_3^*(w_2'') = 0, & B\iota_3^*(w_2''') = w_2.
\end{array}
$$

Now, we are ready to compute the differential d_2. Suppose that the first nontrivial differential d_2 is given as follows:

$$d_2(u_1) = \alpha_1 w_2' + \alpha_2 w_2'' + \alpha_3 w_2''',$$

where $\alpha_1, \alpha_2, \alpha_3$ are in $\mathbb{Z}/2$. Since

$$B\iota_i^*(d_2(u)) = \alpha_i w_2,$$

and

$$\bar{d}_2(u_1) = w_2,$$

we obtain

$$\alpha_i = 1$$

for $i = 1, 2, 3$. Thus, the first nontrivial differential $d_2 \colon E_2^{0,1} \to E_2^{2,0}$ is given as follows:

$$d_2(u_1) = w_2' + w_2'' + w_2'''.$$

Let us recall the relation between the transgression and Steenrod squares. For $r \geq 2$, the transgression

$$d_r \colon E_r^{0,r-1} \to E_r^{r,0}$$

commutes with Steenrod squares Sq^i. In other words, if $d_r(x) = y$ then we may have an element $\mathrm{Sq}^i x \in E_s^{0,r-1+i}$ for $r \leq s$, an element $\mathrm{Sq}^i y \in E_{r+i}^{r+i,0}$ and there hold that $d_s(\mathrm{Sq}^i x) = 0$ for $r \leq s < r + i$ and that $d_{r+i}(\mathrm{Sq}^i x) = \mathrm{Sq}^i y$.

Starting with the above E_2 and d_2, since $u_1^2 = \mathrm{Sq}^1 u_1$, $u_1^4 = \mathrm{Sq}^2 u_1^2$, and $u_1^8 = \mathrm{Sq}^4 u_1^4$, we have the following E_r-tems and differentials up to $r \leq 9$.

$$E_3 = \mathbb{Z}/2[w_2', w_3', w_2'', w_3'', w_3'''] \otimes \mathbb{Z}/2[u_1^2],$$
$$d_3(u_1^2) = \mathrm{Sq}^1(w_2' + w_2'' + w_2''')$$
$$= w_3' + w_3'' + w_3''',$$
$$E_4 = \mathbb{Z}/2[w_2', w_3', w_2'', w_3''] \otimes \mathbb{Z}/2[u_1^4],$$
$$d_4(u_1^4) = 0,$$
$$E_5 = E_4,$$
$$d_5(u_1^4) = \mathrm{Sq}^2(w_3' + w_3'' + w_3''')$$
$$= w_2'w_3' + w_2''w_3'' + w_2'''w_3'''$$
$$= w_2'w_3'' + w_2''w_3',$$
$$E_6 = \mathbb{Z}/2[w_2', w_3', w_2'', w_3'']/(w_2'w_3'' + w_2''w_3') \otimes \mathbb{Z}/2[u_1^8],$$
$$d_6(u_1^8) = 0,$$
$$d_7(u_1^8) = 0,$$
$$d_8(u_1^8) = 0,$$
$$E_9 = E_6.$$

To compute higher terms and differentials, let us consider the ring homomorphism

$$\phi \colon \mathbb{Z}/2[w_2', w_3', w_2'', w_3''] \to \mathbb{Z}/2[w_2', w_2'', t_1]$$

defined by

$$\phi(w_2') = w_2',$$
$$\phi(w_3') = t_1 w_2',$$
$$\phi(w_2'') = w_2'',$$
$$\phi(w_3'') = t_1 w_2''.$$

We assign weight 0, 1, 0, 1 to w_2', w_3', w_2'', w_3'', respectively. We also assign weight 1 to t_1. Then, the ring homomorphism ϕ is weight-preserving.

Let

$$M = \mathbb{Z}/2[w_2', w_3', w_2'', w_3'']/(w_2'w_3'' + w_2''w_3').$$

It is the bottom line of the E_9-term of the spectral sequence such that

$$M = \bigoplus_p E_9^{p,0}.$$

The ring homomorphism ϕ induces the weight-preserving ring homomorphism

$$\bar{\phi} \colon M \to \mathbb{Z}/2[w_2', w_2'', t_1].$$

It is clear that the ring homomorphism $\bar{\phi}$ is injective. Thus, M is isomorphic to the subring $\bar{\phi}(M)$ of $\mathbb{Z}/2[w_2', w_2'', t_1]$. Therefore, both M and $\bar{\phi}(M)$ are integral domains.

The next nontrivial differential is d_9. It is given by

$$
\begin{aligned}
d_9(u_1^8) &= \mathrm{Sq}^4(w_2'w_3'' + w_2''w_3') \\
&= w_2'^2 w_2'' w_3'' + w_3' w_3''^2 + w_2''^2 w_2' w_3' + w_3'' w_3'^2 \\
&= w_2' w_2''(w_2' w_3'' + w_2'' w_3') + w_3' w_3''^2 + w_3'' w_3'^2 \\
&= w_3' w_3''^2 + w_3'' w_3'^2.
\end{aligned}
$$

Since

$$
\bar{\phi}(w_3' w_3''^2 + w_3'' w_3'^2) = t_1^3 w_2' w_2''(w_2' + w_2'')
$$

is nonzero in $\mathbb{Z}/2[w_2', w_2'', t_1]$, multiplication by $w_3' w_3''^2 + w_3'' w_3'^2$ is injective on M. Therefore, we have

$$
E_{10} = \mathbb{Z}/2[w_2', w_3', w_2'', w_3'']/(w_2' w_3'' + w_2'' w_3', w_3' w_3''^2 + w_3'' w_3'^2) \otimes \mathbb{Z}/2[u_1^{16}].
$$

We would like to point out that $w_2' w_3'' + w_2'' w_3'$, $w_3' w_3''^2 + w_3'' w_3'^2$ is a regular sequence in the polynomial ring $\mathbb{Z}/2[w_2', w_2'', w_3', w_3'']$.

Finally, using the commutativity between the transgression and Steenrod squares again, we have

$$
\begin{aligned}
d_r(u_1^{16}) &= 0 \quad \text{for } 10 \le r \le 16 \text{ and} \\
d_{17}(u_1^{16}) &= \mathrm{Sq}^8(w_3' w_3''^2 + w_3'^2 w_3'') \\
&= w_2' w_3' w_3''^4 + w_2'' w_3' w_3'^4 \\
&= (w_2' w_3'' + w_2'' w_3')w_3' w_3''^3 + w_2'' w_3'(w_3' + w_3'')(w_3' w_3''^2 + w_3'^2 w_3'') \\
&= 0.
\end{aligned}
$$

Hence, we have $d_r = 0$ for $r \ge 10$ and $E_\infty = E_{10}$.

To describe the E_∞-term, let

$$
N = \mathbb{Z}/2[w_2', w_3', w_2'', w_3'']/(w_2' w_3'' + w_2'' w_3', w_3' w_3''^2 + w_3'' w_3'^2).
$$

It is the bottom line of the E_∞-term of the spectral sequence such that

$$
N = \bigoplus_p E_\infty^{p,0}.
$$

It is also the subring of the mod 2 cohomology ring of BG generated by w_2', w_3', w_2'', w_3''. What we need is the fact that the induced homomorphism

$$
N \to H^*(BG; \mathbb{Z}/2)
$$

is injective, and N is closed under the action of Milnor operations Q_m for $m \ge 0$.

For a graded set $\{x_1, x_2, \dots\}$, we denote by $\mathbb{Z}/2\{x_1, x_2, \dots\}$ the graded $\mathbb{Z}/2$-module spanned by $\{x_1, x_2, \dots\}$. Recall that we defined weight of $w_2', w_3', w_2'', w_3'', t_1$ as $0, 1, 0, 1, 1$, respectively. We have direct sum decompositions of M and N with respect to weight. Namely, M_k, N_k are graded submodules of M, N spanned by monomials of weight k, respectively.

We will define the element x_{13} as an element in N_1. We also need Proposition 3.1 on the basis for N_1 to show that x_{13} is nonzero.

Proposition 3.1. *For N_0, N_1, N_2, we have*
$$N_0 = \mathbb{Z}/2\{w_2'^m w_2''^n \mid m, n \geq 0\},$$
$$N_1 = \mathbb{Z}/2\{w_2'^m w_3', w_2'^m w_2''^n w_3'' \mid m, n \geq 0\},$$
$$N_2 = \mathbb{Z}/2\{w_2'^m w_3'^2, w_2'^m w_2''^n w_3' w_3'', w_2''^n w_3''^2 \mid m, n \geq 0\}.$$

Proof. The weight of monomials in
$$\bar{\phi}(w_3' w_3''^2 + w_3'^2 w_3'') = t_1^3 w_2' w_2''^2 + t_1^3 w_2'^2 w_2''$$
is 3. Therefore, the ideal of M generated by
$$w_3' w_3''^2 + w_3'^2 w_3''$$
is spanned by monomials of weight greater than or equal to 3. Hence, we have.
$N_i = M_i$ for $i = 0, 1, 2$. It is clear that
$$\bar{\phi}(M_0) = \mathbb{Z}/2\{w_2'^m w_2''^n\},$$
$$\bar{\phi}(M_1) = \mathbb{Z}/2\{t_1 w_2'^m w_2''^n \mid m + n \geq 1\},$$
$$\bar{\phi}(M_2) = \mathbb{Z}/2\{t_1^2 w_2'^m w_2''^n \mid m + n \geq 2\}$$
and that
$$\bar{\phi}(\mathbb{Z}/2\{w_2'^m w_2''^n\}) = \mathbb{Z}/2\{w_2'^m w_2''^n\},$$
$$\bar{\phi}(\mathbb{Z}/2\{w_2'^m w_3', w_2'^m w_2''^n w_3''\}) = \mathbb{Z}/2\{t_1 w_2'^m w_2''^n \mid m + n \geq 1\},$$
$$\bar{\phi}(\mathbb{Z}/2\{w_2'^m w_3'^2, w_2'^m w_2''^n w_3' w_3'', w_2''^n w_3''^2\}) = \mathbb{Z}/2\{t_1^2 w_2'^m w_2''^n \mid m + n \geq 2\},$$
where m, n range over the set of nonnegative integers. Since the ring homomorphism $\bar{\phi}$ is injective, we obtain the desired results. $\square$

We need the following lemma on N_k ($k \geq 3$) to show that $Q_m x_{13} = 0$ for $m \geq 2$.

Proposition 3.2. *Suppose that $k \geq 3$. For $1 \leq i \leq k - 1$, $m \geq 0$, $n \geq 0$, we have*
$$w_2'^m w_2''^n w_3'^i w_3''^{k-i} = w_2''^{m+n} w_3' w_3''^{k-1}$$
in N_k.

Proof. For $i \geq 2$, we have
$$w_3'^i w_3''^{k-i} = w_3'^2 w_3'' \cdot w_3'^{i-2} w_3''^{k-i-1}$$
$$= w_3' w_3''^2 \cdot w_3'^{i-2} w_3''^{k-i-1} \quad (\because w_3'^2 w_3'' = w_3' w_3''^2)$$
$$= w_3'^i w_3''^{k-i}.$$
Iterating this process, we have $w_3'^i w_3''^{k-i} = w_3' w_3''^{k-1}$. For $m \geq 1$, we have
$$w_2'^m w_2''^n w_3' w_3''^{k-1} = w_2' w_3'' \cdot w_2'^{m-1} w_2''^n w_3' w_3''^{k-2}$$
$$= w_3' w_2'' \cdot w_2'^{m-1} w_2''^n w_3' w_3''^{k-2} \quad (\because w_2' w_3'' = w_3' w_2'')$$
$$= w_2'^{m-1} w_2''^{n+1} w_3'^2 w_3''^{k-2}$$
$$= w_2'^{m-1} w_2''^{n+1} w_3' w_3''^{k-1} \quad (\because w_3'^2 w_3^{k-2} = w_3' w_3''^{k-1}).$$
Hence, we have the desired result $w_2'^m w_2''^n w_3'^i w_3''^{k-i} = w_2''^{m+n} w_3' w_3''^{k-1}$. $\square$

Remark 3.3. With Proposition 3.2, it is easy to find a basis for N_k. And we have the following.
$$N_k = \mathbb{Z}/2\{w_2'^m w_3'^k, w_2''^n w_3' w_3''^{k-1}, w_2''^n w_3''^k \mid m, n \geq 0\}.$$

Remark 3.4. It is easy to compute the Poincaré series

$$\frac{(1 - t^5)(1 - t^9)}{(1 - t^2)^2(1 - t^3)^2}$$

of N since $w_2'w_3'' + w_2''w_3'$, $w_3'w_3''^2 + w_3''w_3'^2$ is a regular sequence. To prove the linear independence of elements in Propositions 3.1 and 3.2, one may compute the Poincaré series of each N_k and add them up to obtain the Poincaré series of N above.

Proposition 3.5. *Let us define an element x_{13} of degree* 13 *in the mod* 2 *cohomology of BG by*

$$x_{13} := B\pi_1^*(Q_1 w_2)w_2''^2(w_2'^2 + w_2''^2).$$

Then, x_{13} is nonzero and for $m \geq 1$, we have

$$Q_m x_{13} = 0.$$

Proof. First, we verify that x_{13} is nonzero. Since $B\pi_1^*(Q_1 w_2) = w_2'w_3'$, we have

$$x_{13} = w_2'^3 w_2''^2 w_3' + w_2' w_2''^4 w_3'$$
$$= w_2'^4 w_2'' w_3'' + w_2'^2 w_2''^3 w_3''$$
$$\neq 0$$

in N_1 by Proposition 3.1. Next, we compute $Q_m x_{13}$. Since Q_m acts trivially on $w_2''^2(w_2'^2 + w_2''^2)$,

$$Q_m x_{13} = B\pi_1^*(Q_m Q_1 w_2)w_2''^2(w_2'^2 + w_2''^2).$$

For $m = 1$, since $Q_1 Q_1 = 0$, we have $Q_1 x_{13} = 0$. For $m \geq 2$, by Lemma 2.1, we have

$$B\pi_1^*(Q_m Q_1 w_2)w_2''^2(w_2'^2 + w_2''^2) = B\pi_1^*(g_m w_3^4)w_2''^2(w_2'^2 + w_2''^2)$$
$$= B\pi_1^*(g_m)w_3'^4 w_2''^2(w_2'^2 + w_2''^2).$$

By Proposition 3.2, we obtain

$$w_3'^4 w_2''^2 w_2'^2 = w_2''^4 w_3' w_3''^3,$$
$$w_3'^4 w_2''^2 w_2''^2 = w_2''^4 w_3' w_3''^3,$$

hence, we have

$$w_3'^4 w_2''^2 (w_2'^2 + w_2''^2) = 0.$$

Therefore, we obtain $Q_m x_{13} = 0$.   □

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