PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 10, Pages 288–295 (August 23, 2023)
https://doi.org/10.1090/bproc/184

ON THE COSMETIC CROSSING CONJECTURE FOR SPECIAL ALTERNATING LINKS

JOE BONINGER

(Communicated by David Futer)

Abstract. We prove that a family of links, which includes all special alternating knots, does not admit non-nugatory crossing changes which preserve the isotopy type of the link. Our proof incorporates a result of Lidman and Moore [Trans. Amer. Math. Soc. **369** (2017), pp. 3639–3654] on crossing changes to knots with L-space branched double-covers, as well as tools from Scharlemann and Thompson's [Comment. Math. Helv. **64** (1989), pp. 527–535] proof of the cosmetic crossing conjecture for the unknot.

1. Introduction

The cosmetic crossing conjecture, attributed to Xiao-Song Lin [12, Problem 1.58], posits that changing a nontrivial crossing in a link diagram must change the isotopy type of the link. More concretely, given an oriented link $L \subset S^3$, define a *crossing disk* to be a disk $D \subset S^3$ which intersects L transversely at two points of opposite orientation. A *crossing change* is then performed by passing a neighborhood of one point of $L \cap D$ through a neighborhood of the other, as in Figure 1. The crossing is said to be *nugatory* if ∂D bounds a disk in $S^3 - L$, and a crossing change is *cosmetic* if it preserves the isotopy type of L.

Conjecture 1.1 (Cosmetic crossing conjecture)**.** *For any knot $L \subset S^3$, only a nugatory crossing admits a cosmetic crossing change.*

Conjecture 1.1 has been affirmed for two-bridge knots [19] and fibered knots [11], and significant partial results exist for genus one knots and satellite knots [1, 2, 9, 10]. Further, Lidman and Moore have verified the conjecture for all knots $L \subset S^3$ such that the branched double-cover $\Sigma(L)$ is an L-space, and L has square-free determinant [13]; their work has been extended by Ito [8].

In this note, we prove the cosmetic crossing conjecture for all special alternating knots in S^3. (The case of special alternating knots with square-free determinant is included in [13].)

Theorem 1.2. *Let $L \subset S^3$ be a special alternating knot. Then L admits no cosmetic, non-nugatory crossing change.*

Actually, we prove Conjecture 1.1 for a family of oriented links which includes all non-split special alternating links with certain orientations, and some non-alternating links—see Theorem 3.2.

Received by the editors March 7, 2023, and, in revised form, June 14, 2023 and July 10, 2023.
2020 *Mathematics Subject Classification.* Primary 57K10.
This material is based upon work supported by the National Science Foundation under Award No. 2202704.

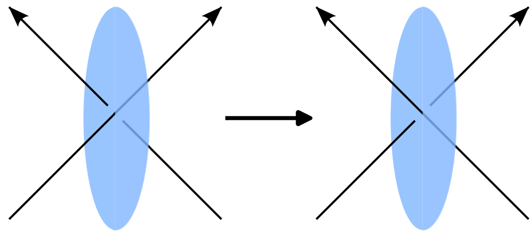


FIGURE 1. A crossing change

A diagram $D \subset S^2$ of a link $L \subset S^3$ is *alternating* if crossings alternate over-under as one traverses any link component of the diagram. The diagram is *special* if one of its checkerboard surfaces, constructed by shading the components of $S^2 - D$ in a checkerboard fashion and taking the union of the shaded regions with half-twisted bands at each crossing, is orientable. Equivalently, a diagram is special if one of its Tait graphs is bipartite. A link $L \subset S^3$ is called *special alternating* if it admits a diagram which is both alternating and special. Special alternating links include $(2, n)$-torus links, and many twist and pretzel knots. More generally, as alluded to above, a special alternating diagram can be constructed from any embedding of a bipartite planar graph in S^2.

Our proof of Theorem 1.2 incorporates a key result from Lidman and Moore [13], as well as tools from Scharlemann and Thompson's proof of Conjecture 1.1 for the unknot [18, Theorem 1.4]. As a corollary, we obtain the following:

Corollary 1.3. *Suppose a link $L \subset S^3$ admits a cosmetic, non-nugatory crossing change, and $\Sigma(L)$ is an L-space. Then L bounds two minimal-genus Seifert surfaces, with Seifert forms represented by matrices (v_{ij}) and (v'_{ij}), such that $v_{11} = v'_{11} + 1$ and $v_{ij} = v'_{ij}$ otherwise.*

Corollary 1.3 is analogous to a finding of Balm, Friedl, Kalfagianni and Powell [1, Corollary 1.3], who use a related approach to study genus one knots.

2. Background

A three-manifold Y is an *L-space* if it is a rational homology sphere with $\operatorname{rank}(\widehat{HF}(Y)) = |H_1(Y;\mathbb{Z})|$, where $\widehat{HF}$ denotes the hat flavor of Heegaard Floer homology. Of importance to us is the fact that, if $L \subset S^3$ is a non-split alternating link, then its branched double-cover, $\Sigma(L)$, is an L-space [16].

Let $L \subset S^3$, and D a crossing disk for L as above. A *crossing arc* is an embedded arc $\gamma \subset D$ connecting the two points of $L \cap D$, and we use $\tilde{\gamma}$ to denote the closed curve which is the preimage of γ in the branched covering $\Sigma(L) \to S^3$. Lidman and Moore proved the following:

Theorem 2.1 ([13, Remark 13])**.** *Let L be an oriented knot with $\Sigma(L)$ an L-space, D a crossing disk for L, and γ a crossing arc in D. If the crossing change induced by D is cosmetic, and $\tilde{\gamma}$ is nullhomologous in $\Sigma(L)$, then D is nugatory.*

Their argument uses the surgery characterization of an unknot in an L-space, due to Gainullin [4]. In the appendix, we extend Theorem 2.1 to links.

Next, we recall the *Gordon-Litherland form*. Given a surface $S \subset S^3$, this is a symmetric, bilinear form $\mathcal{G}_S : H_1(S)^2 \to \mathbb{Z}$ [5]. Briefly, let $\nu(S) \subset S^3$ denote the unit normal bundle of S, with projection $p : \nu(S) \to S$. Given homology classes $a, b \in H_1(S)$, represented by embedded multi-curves $\alpha, \beta \subset S$, we define

$$\mathcal{G}_S(a,b) = \mathrm{lk}(\alpha, p^{-1}\beta),$$

where lk is the linking number. If $L \subset S^3$ is an oriented link, and S a compatibly oriented Seifert surface for L, then $\mathcal{G}_S$ coincides with the symmetrized Seifert form of S, and the signature $\sigma(\mathcal{G}_S)$ equals the signature of L. If, in addition, S is connected, then the nullity $\eta(\mathcal{G}_S)$ is a link invariant called the *nullity* of L, $\eta(L)$. (In some literature, $\eta(L)$ is defined to be $\eta(\mathcal{G}_S) + 1$.)

Convention 2.2. All links are oriented, and we require Seifert surfaces be oriented compatibly with the link. We allow Seifert surfaces to be disconnected, but not to have closed components.

A surface in S^3 is called *definite* if its Gordon-Litherland form is positive- or negative-definite. If $D \subset S^2$ is an alternating link diagram, then the two checkerboard surfaces of D are known to be definite; conversely, definite surfaces can be used to characterize alternating links topologically [6, 7]. In particular, a suitably oriented, non-split special alternating link bounds a definite Seifert surface.

3. Proof of main result

We say a Seifert surface spanning an oriented, non-split link $L \subset S^3$ is *taut* if it has maximal Euler characteristic among all Seifert surfaces of L. (For equivalence with the standard definition of tautness, see [18, Lemma 1.2].) We have:

Lemma 3.1. *Suppose non-split $L \subset S^3$ bounds a definite Seifert surface S. Then S is taut in $S^3 - L$, and conversely every taut Seifert surface for L is definite.*

Proof. First, we argue that S has the maximal number of components of any Seifert surface for L. Suppose some Seifert surface S' has $b_0(S') > b_0(S)$. We form a connected Seifert surface $\hat{S}$ for L by joining the components of S using $b_0(S) - 1$ tubes, and likewise form a connected surface $\hat{S}'$ by adding $b_0(S') - 1$ tubes to S'. We have

$$\eta(\mathcal{G}_{\hat{S}}) = \eta(\mathcal{G}_{\hat{S}'}) \geq b_0(S') - 1,$$

since each tube increases the nullity by one. It follows that

$$\eta(\mathcal{G}_S) = \eta(\mathcal{G}_{\hat{S}}) - b_0(S) + 1 \geq b_0(S') - b_0(S) > 0,$$

contradicting the definite-ness of S.

Next, as in [6, Proposition 3.1], for any Seifert surface S' of L, we have

$$b_1(S') \geq |\sigma(L)| = b_1(S),$$

the last equality following from the fact that S is definite. This shows S has minimal b_1, and therefore maximal Euler characteristic. Finally, any Seifert surface S' with $\chi(S') = \chi(S)$ must have $b_1(S') = b_1(S) = |\sigma(L)|$, so must be definite as well. □

Theorem 3.2. *Suppose an oriented link $L \subset S^3$ satisfies the following conditions:*

- *The link L bounds a definite Seifert surface S.*
- *The branched double-cover $\Sigma(L)$ is an L-space.*

Then L does not admit a non-nugatory, cosmetic crossing change.

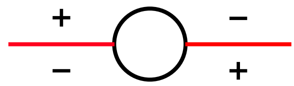


FIGURE 2. Two oriented lifts of $S - n(L)$, in a neighborhood of a meridian of $\partial n(\tilde{L})$

We note the second condition above implies L is non-split, since $\Sigma(L)$ is a rational homology sphere. Examples of non-alternating links which satisfy the hypotheses of Theorem 3.2 include the knots 9_{49}, 10_{134}, and 10_{142}. These knots are known to be quasi-alternating [3, 15], and hence have branched double-covers which are L-spaces. Further, each knot K satisfies $2g(K) = |\sigma(K)|$, g the genus of K, implying the existence of a definite Seifert surface. These examples were found with the help of KnotInfo [14].

Proof of Theorem 3.2. Let L be a link satisfying the hypotheses of the theorem, and let D be a cosmetic crossing disk for L. Let $K = \partial D$, and let $M = S^3 - n(K \cup L)$, where n indicates a regular neighborhood. Following [18], let M_{-1}, M_0, and M_∞ denote the result of filling M along $\partial n(K)$ by a solid torus with slope -1, 0, and ∞ respectively. Then $M_\infty = S^3 - n(L)$, and without loss of generality, M_{-1} is the result of performing the crossing change indicated by D. By assumption, $M_{-1} \cong M_\infty$.

Let $S \subset M$ be a Seifert surface for L which is taut in M. Shrinking D if necessary, we may assume that $S \cap D$ is a single arc γ, which is also a crossing arc for D. Scharlemann and Thompson prove that S is taut in at least two of M_{-1}, M_0, and M_∞ [18, Claim 1]. Thus S is taut in at least one of M_{-1} and M_∞, and since these manifolds are homeomorphic, S is taut in both. Let S denote the inclusion of S in M_∞, and let S' denote the inclusion of S in M_{-1}. It follows from Lemma 3.1 that both S and S' are definite.

We consider two cases.

Case 1 (The arc γ separates S). Let S'' be one of the components of $S - \gamma$, and let $\tilde{S}, \tilde{S}'', \tilde{L}, \tilde{\gamma} \subset \Sigma(L)$ denote the respective preimages of S, S'', L, and γ in the branched covering $\Sigma(L) \to S^3$. (Here we view S as a subset of S^3, rather than a subset of $S^3 - n(L)$.) Considering the classical construction of a branched cover from a Seifert surface [17], we see that $\tilde{S} - n(\tilde{L})$ consists of two lifted copies of $S - n(L)$; we orient these copies by lifting an orientation from $S - n(L)$. When restricted to a meridian circle of $\partial n(\tilde{L})$, the covering map $\Sigma(L) \to S^3$ has the form $z \mapsto z^2$. Thus, near such a meridian, the two components of $\tilde{S} - n(\tilde{L})$ are oriented as in Figure 2.

The surface $\tilde{S}$ is constructed by gluing the two lifted copies of $S - n(L)$ together along the annuli $\tilde{S} \cap n(\tilde{L})$. With Figure 2 in mind, by switching the orientation of one of the lifted copies, these annuli can be made to preserve orientation, and therefore $\tilde{S}$ is orientable. Since $\tilde{S}'' \subset \tilde{S}$, $\tilde{S}''$ is also orientable, and its boundary is exactly $\tilde{\gamma}$. The existence of $\tilde{S}''$ shows $\tilde{\gamma}$ is nullhomologous in $H_1(\Sigma(L))$, so Theorem 2.1 implies the crossing change is nugatory in this case.

Case 2 (The arc γ does not separate S). In this case, we choose a basis $a_1, \ldots, a_n$ for $H_1(S)$, represented by curves $\alpha_1, \ldots, \alpha_n \subset S$ respectively, such that α_1 intersects D one time, and $\alpha_i \cap D = \varnothing$ for $i \neq 1$. Let $G = (g_{ij})$ be the symmetric matrix representing the Gordon-Litherland form $\mathcal{G}_S$ in this basis. We also let $a_1, \ldots, a_n$ denote the same basis for $H_1(S')$, i.e. the basis induced by the inclusion $S \subset M \hookrightarrow M_{-1}$. Let $G' = (g'_{ij})$ be the corresponding matrix representing $\mathcal{G}_{S'}$.

We have $|\det(G)| = |\det(G')| = \det(L)$, and since $\mathcal{G}_S$ and $\mathcal{G}_{S'}$ are both definite of the same rank and sign, determined by $\sigma(L)$, $\det(G) = \det(G')$. Further, by inspecting how S changes in a neighborhood of D when (-1)-surgery is performed, we calculate that $g'_{11} + 2 = g_{11}$, and $g_{ij} = g'_{ij}$ for i and j not both equal to one. We consider computing the determinants of G and G' using a Laplace expansion along the top row—since the two quantities are equal, and the matrices differ at only one entry, we find

$$g_{11} \det(G_{11}) = g'_{11} \det(G'_{11}) = (g_{11} + 2) \det(G_{11}),$$

where G_{11} denotes the matrix formed by removing the first row and column of G. This matrix represents the restriction of $\mathcal{G}_S$ to the subspace of $H_1(S)$ spanned by $a_2, \ldots, a_n$; as the restriction of a definite form, this form is also definite, and hence $\det(G_{11}) \neq 0$. We conclude that

$$g_{11} = g_{11} + 2,$$

a contradiction which indicates this case cannot occur.

□

Proof of Corollary 1.3. Following the proof of Theorem 3.2, we obtain two taut Seifert surfaces for L, with the crossing arc γ embedded as a non-separating arc in each. Choosing the homology bases $a_1, \ldots, a_n$, as above, gives the desired Seifert matrices. □

Finally, we give a minor application of Corollary 1.3.

Corollary 3.3. *Suppose a knot $L \subset S^3$ admits a cosmetic, non-nugatory crossing change, and $\Sigma(L)$ is an L-space. Then, letting m denote the size of a minimal generating set for $H_1(\Sigma(L))$, we have $m < 2g(L)$.*

Proof. Let G and G' be the two matrices obtained in the proof of Theorem 3.2, representing two Gordon-Litherland forms of L with rank $2g(L)$. We use the fact that G and G' give presentations for the finite abelian group $H_1(\Sigma(L))$, and compute this group's invariant factors. For an invertible matrix A, let Γ_i^A denote the greatest common divisor of the determinants of the i-by-i minors of A, and let $\delta_i^A = \Gamma_i^A / \Gamma_{i-1}^A$. We recall, via the Smith normal form of A, that the invariant factors of the abelian group presented by A are given by the set of all δ_i^A not equal to 1.

Since G and G' have the same rank and present the same group, we have

$$\gcd_{ij}(g_{ij}) = \delta_1^G = \delta_1^{G'} = \gcd_{ij}(g'_{ij}).$$

Because $g_{11} = g'_{11} + 2$, δ_1^G divides 2. Additionally, since $\prod_i \delta_i^G = \det(L)$, and knots have odd determinant, we have $\delta_1^G = 1$. Thus $m < \mathrm{rk}(G) = 2g$, as desired. □

This result extends [1, Theorem 1.1(2)]. In general $m \leq 2g(L)$, but equality is occasionally attained. For example, the pretzel knot $K = P(9, 9, 9, 9, -27)$ is quasi-alternating by [3, Theorem 3.2(1)], hence has branched double-cover an L-space. The knot K has genus two and $H_1(\Sigma(K)) \cong \mathbb{Z}/9 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/99$, so Corollary

3.3 shows K does not admit cosmetic crossings. This example is easily generalized, for instance by considering the family of pretzel knots $P(m^2, m^2, m^2, m^2, -3m^2)$ with m odd, to produce many new examples of knots which do not admit cosmetic crossings. Choosing square numbers ensures the resulting pretzel knot is not included in the main theorem of [13].

Appendix A. Extending Theorem 2.1 to links

In what follows, let $L \subset S^3$ be a link, D a crossing disk, and γ the associated crossing arc. As above, let $\tilde{\gamma}$ denote the closed curve which is the preimage of γ in the branched cover $\Sigma(L)$.

The extension of Theorem 2.1 to links ultimately reduces to Proposition A.1.

Proposition A.1. *Suppose* $\det(L) \neq 0$*, and the crossing change associated with* D *is cosmetic. If* $\tilde{\gamma}$ *is a null-homologous unknot in* $\Sigma(L)$*, then* D *is nugatory.*

To complete the argument, the reader may consult the proof of [13, Thm. 2], using Proposition A.1 in place of [13, Prop. 12]. Our proof closely follows that of the latter proposition, and we set up some additional notation before sketching it. Let $B \subset S^3$ be a regular neighborhood of γ, chosen so that $B \cap D$ is a disk contained in $\text{int}(D)$, and so that $B \cap L$ consists of two arcs. Observe that the preimage $\tilde{B} \subset \Sigma(L)$ of B under the branched covering is a solid torus, and let $N = \Sigma(L) - \tilde{B}$. Since $\det(L) \neq 0$, $\Sigma(L)$ is a rational homology sphere, and a Mayer-Vietoris argument shows $H_2(N; \mathbb{Q}) \cong 0$ and $H_1(N; \mathbb{Q}) \cong \mathbb{Q}$. There is a unique slope λ_N of ∂N which generates the kernel of the inclusion-induced map $H_1(\partial N; \mathbb{Q}) \to H_1(N; \mathbb{Q})$. This slope λ_N is called the *rational longitude* of N; we refer the reader to [13, 20] for more details.

Proof. Let $\tilde{\Gamma} \subset \Sigma(L)$ be a disk with boundary $\tilde{\gamma}$; by definition, $\tilde{\Gamma} \cap \partial N$ is the rational longitude λ_N of N. Let τ denote the covering involution on $\Sigma(L)$. By the equivariant Dehn's Lemma, we may assume that either $\tau(\tilde{\Gamma}) \cap \tilde{\Gamma} = \varnothing$ or $\tau(\tilde{\Gamma}) = \tilde{\Gamma}$.

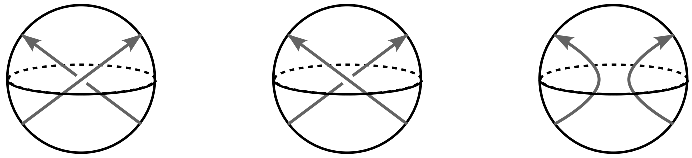
Suppose $\tau(\tilde{\Gamma}) \cap \tilde{\Gamma}$ is empty. This implies $\tilde{\Gamma}$ descends to a properly embedded disk Γ in $S^3 - B$. Since $\tilde{\Gamma}$ avoids the fixed-point set of τ, which is the preimage of L, the disk Γ is disjoint from L. To show D is nugatory, we will show that $\partial\Gamma$ is parallel to $D \cap \partial B$ in $\partial B - L$. If follows that ∂D bounds a disk disjoint from L, formed by gluing Γ to the annulus $D - B$. To show $\partial\Gamma$ and $D \cap \partial B$ are parallel in $\partial B - L$, it suffices to show that $D \cap \partial B$ lifts to λ_N in ∂N.

Let L_0 be the link formed by replacing the crossing ball B with the ball shown in Figure 3c, which we label B_0. Let Δ denote the Alexander polynomial, which satisfies the skein relation

$$\Delta_{L_+}(x) - \Delta_{L_-}(x) = -(x^{-1/2} - x^{-1/2})\Delta_{L_0}(x).$$

Since $L_+ = L_- = L$, we conclude $\Delta_{L_0} \equiv 0$. In particular, $\det(L_0) = \Delta_{L_0}(-1) = 0$, so $H_1(\Sigma(L_0))$ is infinite, and by Poincaré duality and the universal coefficient theorem, so is $H_2(\Sigma(L_0))$. Let $\tilde{B}_0$ be the preimage of B_0 in $\Sigma(L_0)$, which is equivalent to a Dehn filling of N along some slope γ_0. Using the fact that $H_2(N; \mathbb{Q}) \cong 0$, the Mayer-Vietoris theorem gives an exact sequence

$$0 \to H_2(\Sigma(L_0); \mathbb{Q}) \to H_1(\partial N; \mathbb{Q}) \to H_1(N; \mathbb{Q}) \oplus H_1(\tilde{B}_0; \mathbb{Q}).$$



(A) L_+ (B) L_- (C) L_0

FIGURE 3. Crossing balls

Let $a \in H_2(\Sigma(L_0);\mathbb{Q})$ be non-trivial, and let ∂a be its (non-trivial) image in $H_1(\partial N;\mathbb{Q})$. By exactness, ∂a is in the kernel of the second map, so ∂a is trivial in $H_1(\tilde{B}_0;\mathbb{Q})$ and $H_1(N;\mathbb{Q})$. Since ∂a is trivial in $H_1(\tilde{B}_0;\mathbb{Q})$, ∂a is a rational multiple of γ_0 (forgetting the orientation of the former). Since ∂a is trivial in $H_1(N;\mathbb{Q})$, ∂a is a rational multiple of λ_N. Thus $\gamma_0 = \lambda_N$.

We've shown the rational longitude of N corresponds to the slope γ_0 of the Dehn filling $\tilde{B}_0$. Since $D \cap B_0$ is a disk separating the two components of $L_0 \cap B_0$, $D \cap B_0$ lifts to a meridian disk of $\tilde{B}_0$, and $D \cap \partial B_0 = D \cap \partial B$ lifts to $\gamma_0 = \lambda_N$. This completes the proof in this case, and the case of $\tau(\tilde{\Gamma}) = \tilde{\Gamma}$ is handled just as in the proof of [13, Prop. 12]. □

Acknowledgments

The author thanks Jacob Caudell for introducing him to the cosmetic crossing conjecture, Josh Greene for helpful conversations, and an anonymous reviewer for insightful feedback and corrections.

References

[1] Cheryl Balm, Stefan Friedl, Efstratia Kalfagianni, and Mark Powell, *Cosmetic crossings and Seifert matrices*, Comm. Anal. Geom. **20** (2012), no. 2, 235–253, DOI 10.4310/CAG.2012.v20.n2.a1. MR2928712

[2] Cheryl Jaeger Balm and Efstratia Kalfagianni, *Knots without cosmetic crossings*, Topology Appl. **207** (2016), 33–42, DOI 10.1016/j.topol.2016.04.009. MR3501264

[3] Abhijit Champanerkar and Ilya Kofman, *Twisting quasi-alternating links*, Proc. Amer. Math. Soc. **137** (2009), no. 7, 2451–2458, DOI 10.1090/S0002-9939-09-09876-1. MR2495282

[4] Fyodor Gainullin, *Heegaard Floer homology and knots determined by their complements*, Algebr. Geom. Topol. **18** (2018), no. 1, 69–109, DOI 10.2140/agt.2018.18.69. MR3748239

[5] C. McA. Gordon and R. A. Litherland, *On the signature of a link*, Invent. Math. **47** (1978), no. 1, 53–69, DOI 10.1007/BF01609479. MR500905

[6] Joshua Evan Greene, *Alternating links and definite surfaces*, Duke Math. J. **166** (2017), no. 11, 2133–2151, DOI 10.1215/00127094-2017-0004. With an appendix by András Juhász and Marc Lackenby. MR3694566

[7] Joshua A. Howie, *A characterisation of alternating knot exteriors*, Geom. Topol. **21** (2017), no. 4, 2353–2371, DOI 10.2140/gt.2017.21.2353. MR3654110

[8] Tetsuya Ito, *Applications of the Casson-Walker invariant to the knot complement and the cosmetic crossing conjectures*, Geom. Dedicata **216** (2022), no. 6, Paper No. 63, 15, DOI 10.1007/s10711-022-00722-6. MR4475472

[9] Tetsuya Ito, *Cosmetic crossing conjecture for genus one knots with non-trivial Alexander polynomial*, Proc. Amer. Math. Soc. **150** (2022), no. 2, 871–876, DOI 10.1090/proc/15654. MR4356193

[10] Tetsuya Ito, *An obstruction of Gordian distance one and cosmetic crossings for genus one knots*, New York J. Math. **28** (2022), 175–181. MR4374147

[11] Efstratia Kalfagianni, *Cosmetic crossing changes of fibered knots*, J. Reine Angew. Math. **669** (2012), 151–164, DOI 10.1515/crelle.2011.148. MR2980586

[12] Rob Kirby, *Problems in low dimensional manifold theory*, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I., 1978, pp. 273–312. MR520548

[13] Tye Lidman and Allison H. Moore, *Cosmetic surgery in L-spaces and nugatory crossings*, Trans. Amer. Math. Soc. **369** (2017), no. 5, 3639–3654, DOI 10.1090/tran/6839. MR3605982

[14] Charles Livingston and Allison H. Moore, *Knotinfo: Table of knot invariants*, `knotinfo.math.indiana.edu`, June 2023.

[15] Ciprian Manolescu, *An unoriented skein exact triangle for knot Floer homology*, Math. Res. Lett. **14** (2007), no. 5, 839–852, DOI 10.4310/MRL.2007.v14.n5.a11. MR2350128

[16] Peter Ozsváth and Zoltán Szabó, *On the Heegaard Floer homology of branched double-covers*, Adv. Math. **194** (2005), no. 1, 1–33, DOI 10.1016/j.aim.2004.05.008. MR2141852

[17] Dale Rolfsen, *Knots and links*, Mathematics Lecture Series, No. 7, Publish or Perish, Inc., Berkeley, Calif., 1976. MR0515288

[18] Martin Scharlemann and Abigail Thompson, *Link genus and the Conway moves*, Comment. Math. Helv. **64** (1989), no. 4, 527–535, DOI 10.1007/BF02564693. MR1022995

[19] Ichiro Torisu, *On nugatory crossings for knots*, Topology Appl. **92** (1999), no. 2, 119–129, DOI 10.1016/S0166-8641(97)00238-1. MR1669827

[20] Liam Watson, *Surgery obstructions from Khovanov homology*, Selecta Math. (N.S.) **18** (2012), no. 2, 417–472, DOI 10.1007/s00029-011-0070-2. MR2927239

Department of Mathematics, Boston College, Chestnut Hill, Massachusetts, 02467
Email address: `boninger@bc.edu`