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STABLE PHASE RETRIEVAL AND PERTURBATIONS OF FRAMES

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Abstract. A frame $(x_j)_{j\in J}$ for a Hilbert space H is said to do phase retrieval if for all distinct vectors $x, y \in H$ the magnitudes of the frame coefficients $(|\langle x, x_j\rangle|)_{j\in J}$ and $(|\langle y, x_j\rangle|)_{j\in J}$ distinguish x from y (up to a unimodular scalar). A frame which does phase retrieval is said to do C-stable phase retrieval if the recovery of any vector $x \in H$ from the magnitude of the frame coefficients is C-Lipschitz. It is known that if a frame does stable phase retrieval then any sufficiently small perturbation of the frame vectors will do stable phase retrieval, though with a slightly worse stability constant. We provide new quantitative bounds on how the stability constant for phase retrieval is affected by a small perturbation of the frame vectors. These bounds are significant in that they are independent of the dimension of the Hilbert space and the number of vectors in the frame.

1. Introduction

Frames, like ortho-normal bases, give a continuous, linear, and stable reconstruction formula for vectors in a Hilbert space. The distinction between frames and bases is that frames allow for redundancy. That is, the coefficients used for reconstruction with a frame may be non-unique, and a frame for a finite dimensional Hilbert space will usually contain more vectors than the dimension. Frames have many applications in signal processing, physics, and engineering where one wishes to analyze or reconstruct a vector from a collection of linear measurements. However, in some situations such as X-ray crystallography and coherent diffraction imaging, one is only able to obtain the magnitude of each linear measurement. This loss of linearity makes the recovery of the vector much more difficult. As we have lost the phase of each measurement, the recovery of a vector from a collection of magnitudes of linear measurements is aptly named phase retrieval.

The importance of phase retrieval in applications has driven a significant amount of research on the mathematics of phase retrieval in frame theory, and we recommend [GKR] and [FaS] for surveys on the topic. A collection of vectors $(x_j)_{j\in J}$ in a Hilbert space H is called a frame of H if there are uniform constants $B \geq A > 0$

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called the *frame bounds*, such that

$$A\|x\|^2 \leq \sum_{j\in J} |\langle x, x_j\rangle|^2 \leq B\|x\|^2 \qquad \text{for all } x \in H. \tag{1.1}$$

A frame is called *tight* if the optimal frame bounds satisfy $A = B$, and a frame is called *Parseval* if $A = B = 1$. The *analysis operator* of a frame $(x_j)_{j\in J}$ of H is the map $\Theta : H \to \ell_2(J)$ given by $\Theta x = (\langle x, x_j\rangle)_{j\in J}$. That is, the analysis operator maps a vector to its sequence of frame coefficients.

Recall that the goal of phase retrieval is to recover a vector (up to a unimodular scalar) from the magnitude of its frame coefficients. This can be nicely expressed in terms of the analysis operator. We say that a frame $(x_j)_{j\in J}$ of a Hilbert space H with analysis operator $\Theta : H \to \ell_2(J)$ does *phase retrieval* if whenever $x, y \in H$ are such that $|\Theta x| = |\Theta y|$ we have that $x = \lambda y$ for some $|\lambda| = 1$. We may define an equivalence relation on H by $x \sim y$ if $x = \lambda y$ for some $|\lambda| = 1$. Then, $(x_j)_{j\in J}$ does phase retrieval is equivalent to $|\Theta| : H/\!\sim \to \ell_2(J)$ is one-to-one. Any application of phase retrieval will involve some error, and thus it is important that phase retrieval not only be possible but that it also be stable. We say that $(x_j)_{j\in J}$ does *C-stable phase retrieval* if the recovery of $[x]_\sim \in H/\!\sim$ from $|\Theta x| \in \ell_2(J)$ is C-Lipschitz. That is, a frame $(x_j)_{j\in J}$ of a Hilbert space H with analysis operator $\Theta : H \to \ell_2(J)$ does C-stable phase retrieval if

$$\min_{|\lambda|=1} \|x - \lambda y\|_H \leq C\big\||\Theta x| - |\Theta y|\big\|_{\ell_2(J)} = C\Big(\sum_{j\in J} \big||\langle x, x_j\rangle| - |\langle y, x_j\rangle|\big|^2\Big)^{1/2} \tag{1.2}$$
$$\text{for all } x, y \in H.$$

Let $(x_j)_{j\in J}$ be a frame of a Hilbert space H with optimal lower frame bound A and analysis operator $\Theta : H \to \ell_2(J)$. We have by (1.1) that the recovery of $x \in H$ from the frame coefficients Θx is $A^{-1/2}$-Lipschitz. Thus, if $(x_j)_{j\in J}$ does C-stable phase retrieval then $C \geq A^{-1/2}$.

In both theory and applications, one often doesn't work with the frame $(x_j)_{j\in J}$ itself, but instead with a frame $(y_j)_{j\in J}$ which is a small perturbation of $(x_j)_{j\in J}$. That is, one can think of moving each vector x_j a small amount to obtain the vector y_j. For example, the Fourier transform and Gabor transform are important tools in phase retrieval, but any implementation would require them to be discretized. A natural way to discretize a transform on a continuous domain is to choose a discrete ε-net in the domain and perturb each element in the domain to the center of an ε-ball. As another example, random constructions of frames are of fundamental importance in phase retrieval for finite dimensional Hilbert spaces [CL], [EM], [KL], and [KS]. Randomly sampling vectors on the unit sphere of a finite dimensional Hilbert space will produce a frame which is "close" to being tight with high probability [V], and the positive solution to the Paulsen problem gives that a frame of unit vectors which is close to being tight may be perturbed a small amount to be a tight frame of unit vectors [CC], [HM], and [KLLR]. This gives an example where one may improve a frame in some respects by perturbing it. Conversely, physical applications can involve setting up sensors which take linear measurements corresponding to some frame $(x_j)_{j\in J}$. Any such implementation will involve some error, and error of this form corresponds to the sensors giving measurements in terms of some perturbation $(y_j)_{j\in J}$ of $(x_j)_{j\in J}$ instead of $(x_j)_{j\in J}$ itself. Thus it is not only important to understand the frame properties of specific frames of interest, but also how those properties behave under perturbation.

Given a frame $(x_j)_{j\in J}$ for some Hilbert space H, Theorem 1.1 of Christensen [C] provides frame bounds for any perturbation $(y_j)_{j\in J}$ in terms of the frame bounds for $(x_j)_{j\in J}$ and how close $(y_j)_{j\in J}$ is to $(x_j)_{j\in J}$.

Theorem 1.1 ([C]). *Let $(x_j)_{j\in J}$ be a frame of a Hilbert space H with frame bounds $B \geq A > 0$. Let $A > \varepsilon > 0$ and $(y_j)_{j\in J} \subseteq H$ such that $\sum_{j\in J} \|x_j - y_j\|^2 < \varepsilon$. Then, $(y_j)_{j\in J}$ is a frame of H with upper frame bound $B\left(1+\sqrt{\frac{\varepsilon}{B}}\right)^2$ and lower frame bound $A\left(1-\sqrt{\frac{\varepsilon}{A}}\right)^2$.*

Christensen's perturbation theorem is of fundamental importance in frame theory, and it is natural to consider how the stability of phase retrieval is affected by perturbations of frame vectors. In [B], Balan proves that phase retrieval in finite dimensional Hilbert spaces is stable under small perturbations and provides a bound for $\varepsilon > 0$ in terms of properties of the frame $(x_j)_{j\in J}$. However, a different method of measuring stability is considered in [B]. We prove the following perturbation theorem for phase retrieval which is analogous to Christensen's perturbation theorem for frame bounds.

Theorem 1.2. *Let $(x_j)_{j\in J}$ be a frame of a finite dimensional Hilbert space H with frame bounds $B \geq A > 0$ which does C-stable phase retrieval. Let $\varepsilon > 0$ satisfy $\varepsilon < 2^{-4}C^{-4}B^{-1}$ and let $(y_j)_{j\in J} \subseteq H$ such that $\sum_{j\in J} \|x_j - y_j\|^2 < \varepsilon$. Then, $(y_j)_{j\in J}$ is a frame of H with upper frame bound $B\left(1+\sqrt{\frac{\varepsilon}{B}}\right)^2$ and lower frame bound $A\left(1-\sqrt{\frac{\varepsilon}{A}}\right)^2$ which does $C(1-4C^2\sqrt{\varepsilon B})^{-1/2}$-stable phase retrieval for H.*

Our proof of Theorem 1.2 relies on the recently proven theorem that if $x, y \in H$ and $(x_j)_{j\in J}$ is a frame of H with analysis operator Θ then there exists $x', y' \in \operatorname{span}\{x, y\}$ with $\langle x', y'\rangle = 0$ such that $\min_{|\lambda|=1} \|x - \lambda y\|^2 = \min_{|\lambda|=1} \|x' - \lambda y'\|^2 = \|x'\|^2 + \|y'\|^2$ and $\||\Theta x'| - |\Theta y'|\| \leq \||\Theta x| - |\Theta y|\|$ [AAFG]. Thus, when proving that $(x_j)_{j\in J}$ does C-stable phase retrieval, we only need to check that (1.2) holds for orthogonal vectors. In Section 3 we show that the stability condition in [B] provides a constant for the frame doing stable phase retrieval in $\ell_4(J)$. That is, we prove that if a frame $(x_j)_{j\in J} \subseteq H$ with analysis operator Θ satisfies the stability condition in [B] for a_0 then for all $x, y \in H$ we have that

$$
\begin{aligned}
(1.3) \qquad \min_{|\lambda|=1} \|x - \lambda y\|_H &\leq 2^{1/2} a_0^{-1/4} \big\||\Theta x| - |\Theta y|\big\|_{\ell_4(J)} \\
&= 2^{1/2} a_0^{-1/4} \Big(\sum_{j\in J} \big||\langle x, x_j\rangle| - |\langle y, x_j\rangle|\big|^4\Big)^{1/4}.
\end{aligned}
$$

As the $\ell_2(J)$-norm dominates the $\ell_4(J)$-norm, the conclusion of (1.3) is significantly stronger than (1.2). This is particularly apparent when considering the uniform stability of phase retrieval for frames of Hilbert spaces with arbitrary dimensions. There are random constructions of frames which provide constants $C > 0$ and $k \in \mathbb{N}$, so that for all $n \in \mathbb{N}$ there exists a Parseval frame $(x_j)_{j=1}^{kn}$ of ℓ_2^n which does C-stable phase retrieval [BW][CDFF][CL][EM][KL][KS]. However, ℓ_2^n is not uniformly isomorphic to a subspace of ℓ_4^{nk} [FLM], and thus for all $a_0 > 0$ and $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ so that every Parseval frame $(x_j)_{j=1}^{kn}$ of ℓ_2^n fails (1.3) for the value a_0. On the other hand, working in $\ell_4(J)$ or more generally $L_4(\mu)$ allows for the introduction of some powerful analytic methods [B][CPT].

Theorem 1.2 gives a solution to the problem of determining how perturbation affects the stability of phase retrieval of frames for finite dimensional Hilbert spaces. However, phase retrieval is often considered in more general settings, and there are many opportunities for considering the effect of perturbations on phase retrieval. In applications, one is usually interested in objects with some particular structure and it is not necessary that (1.2) be satisfied for all $x, y \in H$, but only for x and y in some subset of interest. Furthermore, it is often permissible to consider a weaker equivalence relation than $x \sim y$ if and only if $x = \lambda y$ for some $|\lambda| = 1$. For example, consider f and g to be audio recordings where the audio for f stops a full second before the audio for g starts. Then $f+g$, $f-g$, $-f+g$, and $-f-g$ would all sound the same and would all have the same absolute value. Thus, we may consider larger equivalence classes as we would be satisfied with obtaining any of those vectors when doing phase retrieval. This situation arises in many important contexts when studying phase retrieval and it is often possible to stably reconstruct the components of a signal which are supported on separated islands (even though it is not possible to determine the relative phase between different components) [ADGY], [CCSW], [CDDL], [FKM], [GR], and [GR2]. The problem of how stability is affected by perturbations in these kinds of circumstances remains an important open problem. It is known that if a frame does norm retrieval and not phase retrieval then norm retrieval is not stable under perturbations [HR], and it would be interesting to know how perturbations affect phase retrieval by projections [CGJT] and [EGK] and weak phase retrieval [BCGJT] as well. In [CDFF], [CPT], and [FOPT], stable phase retrieval is studied for infinite dimensional subspaces of Banach lattices, and the problem of how stability of phase retrieval is affected by perturbations is considered in Section 4 of [FOPT]. There are many opportunities for further research on how this may be quantified and how different Banach space geometry allows for more refined perturbation estimates. Phase retrieval has also been studied in the context of frames for Banach spaces [AG], and it would be interesting to determine how the perturbation ideas used in [BC] and [FOSZ] for frame pairs could be applied in this setting. We hope that this paper provides some inspiration to further study the relationship between perturbation and the stability of phase retrieval.

Note that in this paper we are considering only the stability of the recovery map $|\Theta x| \mapsto [x]_\sim$, and not how to implement it. There are many algorithms to implement phase retrieval in various contexts [ABFM], [CSV], [FKMS], [GKK], [NNB], [PBS], [PMVI], and [SBFIKSSZ]. The impact of measurement error has been studied for these algorithms, and it would be worthwhile to consider the effect of perturbation as well.

2. Perturbations of frames

Let $(x_j)_{j\in J}$ be a frame of a Hilbert space H with analysis operator $\Theta : H \to \ell_2(J)$ given by $\Theta x = (\langle x, x_j\rangle)_{j\in J}$. Let $C > 0$ be some constant. We say that $(x_j)_{j\in J}$

does *C-stable phase retrieval* if

$$\min_{|\lambda|=1}\|x-\lambda y\|_H \le C\big\||\Theta x|-|\Theta y|\big\|_{\ell_2(J)} = C\Big(\sum_{j\in J}\big||\langle x,x_j\rangle|-|\langle y,x_j\rangle|\big|^2\Big)^{1/2} \quad \text{for all } x,y\in H. \tag{2.1}$$

Proving that a frame does C-stable phase retrieval using the definition would require checking (2.1) for all pairs of vectors $x,y\in H$. However, Lemma 2.1 gives that we only need to check orthogonal pairs of vectors. This greatly simplifies calculations, as if x and y are orthogonal then $\min_{|\lambda|=1}\|x-\lambda y\|^2=\|x\|^2+\|y\|^2$.

Lemma 2.1 ([AAFG]). *Let $(x_j)_{j\in J}$ be a frame of a Hilbert space H. Then for all $x,y\in H$ there exists $x_o,y_o\in H$ with $\langle x_o,y_o\rangle=0$ so that*

$$\begin{gathered}\min_{|\lambda|=1}\|x-\lambda y\|_H=\|x_o-y_o\|_H\\ \text{and}\quad \big||\langle x_o,x_j\rangle|-|\langle y_o,x_j\rangle|\big| \le \big||\langle x,x_j\rangle|-|\langle y,x_j\rangle|\big| \text{ for all } j\in J.\end{gathered} \tag{2.2}$$

In particular, if $\Theta: H\to\ell_2(J)$ is the analysis operator of $(x_j)_{j\in J}$ then $(x_j)_{j\in J}$ does C-stable phase retrieval if and only if

$$\big(\|x_o\|_H^2+\|y_o\|_H^2\big)^{1/2}\le C\big\||\Theta x_o|-|\Theta y_o|\big\|_{\ell_2(J)} \quad \textit{for all } x_o,y_o\in H \textit{ with } \langle x_o,y_o\rangle=0. \tag{2.3}$$

We now restate and prove Theorem 1.2 from Section 1.

Theorem 2.2. *Let $(x_j)_{j\in J}$ be a frame of a finite dimensional Hilbert space H with frame bounds $B\ge A>0$ which does C-stable phase retrieval. Let $\varepsilon>0$ satisfy $\varepsilon<2^{-4}C^{-4}B^{-1}$ and let $(y_j)_{j\in J}$ be an indexed collection of vectors in H such that $\sum_{j\in J}\|x_j-y_j\|^2<\varepsilon$. Then, $(y_j)_{j\in J}$ is a frame of H with upper frame bound $B\big(1+\sqrt{\frac{\varepsilon}{B}}\big)^2$ and lower frame bound $A\big(1-\sqrt{\frac{\varepsilon}{B}}\big)^2$ which does $C(1-4\,C^2\sqrt{\varepsilon B})^{-1/2}$-stable phase retrieval for H.*

Proof. As $(x_j)_{j\in J}$ has lower frame bound A and does C-stable phase retrieval, we have that $C\ge A^{-1/2}$. Thus, $\varepsilon<2^{-4}C^{-4}B^{-1}\le 2^{-4}A^2B^{-1}\le 2^{-4}A$. Our bound on ε thus satisfies the hypothesis of Christensen's perturbation theorem (Theorem 1.1). Thus, $(y_j)_{j\in J}$ is a frame of H with upper frame bound $B(1+\sqrt{\frac{\varepsilon}{B}})^2$ and lower frame bound $A(1-\sqrt{\frac{\varepsilon}{B}})^2$. Let $\Theta_X: H\to\ell_2(J)$ be the analysis operator of $(x_j)_{j\in J}$ and let $\Theta_Y: H\to\ell_2(J)$ be the analysis operator of $(y_j)_{j\in J}$.

By Lemma 2.1, we only need to consider orthogonal vectors when proving that $(y_j)_{j\in J}$ does $C^2(1-4\,C^2\sqrt{\varepsilon B})^{-1/2}$-stable phase retrieval. Let $x,y\in H$ with $\langle x,y\rangle=0$. We have that

$$\sum_{j\in J}\Big||\langle x,y_j\rangle|-|\langle y,y_j\rangle|\Big|^2=\sum_{j\in J}|\langle x,y_j\rangle|^2-2\sum_{j\in J}|\langle x,y_j\rangle||\langle y,y_j\rangle|+\sum_{j\in J}|\langle y,y_j\rangle|^2. \tag{2.4}$$

We now compute bounds for each of the sums separately. We will do this by comparing each sum to the corresponding one with $(y_j)_{j\in J}$ replaced by $(x_j)_{j\in J}$.

$$\begin{aligned}
&\sum_{j\in J}|\langle x,y_j\rangle|^2-\sum_{j\in J}|\langle x,x_j\rangle|^2\\
&\quad=\sum_{j\in J}|\langle x,x_j-(x_j-y_j)\rangle|^2-\sum_{j\in J}|\langle x,x_j\rangle|^2\\
&\quad\geq\sum_{j\in J}\Big(|\langle x,x_j\rangle|-|\langle x,x_j-y_j\rangle|\Big)^2-\sum_{j\in J}|\langle x,x_j\rangle|^2\\
&\quad=-2\sum_{j\in J}|\langle x,x_j\rangle||\langle x,x_j-y_j\rangle|+\sum_{j\in J}|\langle x,x_j-y_j\rangle|^2\\
&\quad\geq-2\Big(\sum_{j\in J}|\langle x,x_j\rangle|^2\Big)^{1/2}\Big(\sum_{j\in J}|\langle x,x_j-y_j\rangle|^2\Big)^{1/2}\\
&\qquad+\sum_{j\in J}|\langle x,x_j-y_j\rangle|^2 \qquad\qquad \text{by Cauchy-Schwarz,}\\
&\quad\geq-2\Big(\sum_{j\in J}|\langle x,x_j\rangle|^2\Big)^{1/2}\Big(\sum_{j\in J}\|x\|^2\|x_j-y_j\|^2\Big)^{1/2}+\sum_{j\in J}|\langle x,x_j-y_j\rangle|^2\\
&\quad\geq-2B^{1/2}\|x\|\Big(\|x\|^2\sum_{j\in J}\|x_j-y_j\|^2\Big)^{1/2}\\
&\qquad+\sum_{j\in J}|\langle x,x_j-y_j\rangle|^2 \qquad\qquad \text{as } (x_j)_{j\in J} \text{ has upper frame bound } B\\
&\quad\geq-2\sqrt{\varepsilon B}\,\|x\|^2+\sum_{j\in J}|\langle x,x_j-y_j\rangle|^2 \qquad \text{as } \sum_{j\in J}\|x_j-y_j\|^2<\varepsilon.
\end{aligned}$$

Thus, we have that

$$\sum_{j\in J}|\langle x,y_j\rangle|^2\geq\sum_{j\in J}|\langle x,x_j\rangle|^2-2\sqrt{\varepsilon B}\|x\|^2+\sum_{j\in J}|\langle x,x_j-y_j\rangle|^2. \tag{2.5}$$

Likewise, we have that

$$\sum_{j\in J}|\langle y,y_j\rangle|^2\geq\sum_{j\in J}|\langle y,x_j\rangle|^2-2\sqrt{\varepsilon B}\|y\|^2+\sum_{j\in J}|\langle y,x_j-y_j\rangle|^2. \tag{2.6}$$

We now bound the remaining term.

$$\begin{aligned}
&\sum_{j\in J}|\langle x,y_j\rangle||\langle y,y_j\rangle|-\sum_{j\in J}|\langle x,x_j\rangle||\langle y,x_j\rangle|\\
&\quad=\sum_{j\in J}\Big|\langle x,x_j-(x_j-y_j)\rangle\langle y,x_j-(x_j-y_j)\rangle\Big|-\sum_{j\in J}|\langle x,x_j\rangle||\langle y,x_j\rangle|\\
&\quad\leq\sum_{j\in J}|\langle x,x_j-y_j\rangle\langle y,x_j\rangle|+|\langle x,x_j\rangle\langle y,x_j-y_j\rangle|+|\langle x,x_j-y_j\rangle\langle y,x_j-y_j\rangle|\\
&\quad\leq\Big(\sum_{j\in J}|\langle x,x_j-y_j\rangle|^2\Big)^{1/2}\Big(\sum_{j\in J}|\langle y,x_j\rangle|^2\Big)^{1/2}\\
&\qquad+\Big(\sum_{j\in J}|\langle x,x_j\rangle|^2\Big)^{1/2}\Big(\sum_{j\in J}|\langle y,x_j-y_j\rangle|^2\Big)^{1/2}+\sum_{j\in J}|\langle x,x_j-y_j\rangle\langle y,x_j-y_j\rangle|
\end{aligned}$$

$$
\begin{aligned}
&\leq \|x\| \Big(\sum_{j\in J} \|x_j - y_j\|^2 \Big)^{1/2} \Big(\sum_{j\in J} |\langle y, x_j\rangle|^2 \Big)^{1/2} \\
&\quad + \|y\| \Big(\sum_{j\in J} |\langle x, x_j\rangle|^2 \Big)^{1/2} \Big(\sum_{j\in J} \|x_j - y_j\|^2 \Big)^{1/2} + \sum_{j\in J} |\langle x, x_j - y_j\rangle \langle y, x_j - y_j\rangle| \\
&\leq \|x\| \varepsilon^{1/2} \Big(\sum_{j\in J} |\langle y, x_j\rangle|^2 \Big)^{1/2} + \|y\| \varepsilon^{1/2} \Big(\sum_{j\in J} |\langle x, x_j\rangle|^2 \Big)^{1/2} \\
&\quad + \sum_{j\in J} |\langle x, x_j - y_j\rangle \langle y, x_j - y_j\rangle| \\
&\leq \|x\| \sqrt{\varepsilon B} \|y\| + \|y\| \sqrt{\varepsilon B} \|x\| + \sum_{j\in J} |\langle x, x_j - y_j\rangle \langle y, x_j - y_j\rangle|.
\end{aligned}
$$

Thus, we have that

$$
\sum_{j\in J} |\langle x, y_j\rangle| |\langle y, y_j\rangle| \leq \sum_{j\in J} |\langle x, x_j\rangle| |\langle y, x_j\rangle| + 2\sqrt{\varepsilon B} \|x\| \|y\| + \sum_{j\in J} |\langle x, x_j - y_j\rangle \langle y, x_j - y_j\rangle|. \tag{2.7}
$$

By combining (2.5), (2.6), and (2.7) with (2.4) we have that

$$
\begin{aligned}
&\sum_{j\in J} \Big| |\langle x, y_j\rangle| - |\langle y, y_j\rangle| \Big|^2 \\
&= \sum_{j\in J} |\langle x, y_j\rangle|^2 - 2 \sum_{j\in J} |\langle x, y_j\rangle| |\langle y, y_j\rangle| + \sum_{j\in J} |\langle y, y_j\rangle|^2 \\
&\geq \sum_{j\in J} |\langle x, x_j\rangle|^2 - 2 \sum_{j\in J} |\langle x, x_j\rangle| |\langle y, x_j\rangle| + \sum_{j\in J} |\langle y, x_j\rangle|^2 - 2\sqrt{\varepsilon B} \|x\|^2 \\
&\quad - 4\sqrt{\varepsilon B} \|x\| \|y\| - 2\sqrt{\varepsilon B} \|y\|^2 + \sum_{j\in J} |\langle x, x_j - y_j\rangle|^2 \\
&\quad - 2 \sum_{j\in J} |\langle x, x_j - y_j\rangle \langle y, x_j - y_j\rangle| + \sum_{j\in J} |\langle y, x_j - y_j\rangle|^2 \\
&= \sum_{j\in J} \Big| |\langle x, x_j\rangle| - |\langle y, x_j\rangle| \Big|^2 - 2\sqrt{\varepsilon B} (\|x\| + \|y\|)^2 \\
&\quad + \sum_{j\in J} \Big| |\langle x, x_j - y_j\rangle| - |\langle y, x_j - y_j\rangle| \Big|^2 \\
&\geq C^{-2} \min_{|\lambda|=1} \|x - \lambda y\|^2 \\
&\quad - 2\sqrt{\varepsilon B} (\|x\| + \|y\|)^2 \qquad \text{as } (x_j)_{j\in J} \text{ does } C\text{-stable phase retrieval} \\
&\geq C^{-2} \min_{|\lambda|=1} \|x - \lambda y\|^2 - 4\sqrt{\varepsilon B} (\|x\|^2 + \|y\|^2) \qquad \text{by Jensen's Inequality} \\
&= C^{-2} \min_{|\lambda|=1} \|x - \lambda y\|^2 - 4\sqrt{\varepsilon B} \min_{|\lambda|=1} \|x - \lambda y\|^2 \qquad \text{as } x \text{ and } y \text{ are orthogonal} \\
&= C^{-2} (1 - 4\, C^2 \sqrt{\varepsilon B}) \min_{|\lambda|=1} \|x - \lambda y\|^2.
\end{aligned}
$$

We have that $\varepsilon < 2^{-4}C^{-4}B^{-1}$ and hence $C^{-2}(1-4\,C^2\sqrt{\varepsilon B})$ is positive. Thus, we have for every pair of orthogonal vectors $x, y \in H$ that

$$\min_{|\lambda|=1}\|x-\lambda y\| \le C(1-4C^2\sqrt{\varepsilon B})^{-1/2}\big\||\Theta_Y x|-|\Theta_Y y|\big\|.$$

Thus, the frame $(y_j)_{j\in J}$ does $C(1-4\,C^2\sqrt{\varepsilon B})^{-1/2}$-stable phase retrieval by Lemma 2.1. □

3. Stability comparisons

In [B], the value a_0 in Lemma 3.1 is used as a measurement for the stability of phase retrieval. Note that Lemma 3.1 is stated implicitly for complex Hilbert spaces, but it holds for real Hilbert spaces as well.

Lemma 3.1 ([B]). *Let $(x_j)_{j=1}^m$ be a frame for a finite dimensional Hilbert space H. Then $(x_j)_{j=1}^m$ does phase retrieval if and only if there is a constant a_0 so that for all $x, y \in H$ we have that*

$$\sum_{j=1}^m \big||\langle x, x_j\rangle|^2-|\langle y, x_j\rangle|^2\big|^2 \ge a_0\Big(\|x-y\|^2\|x+y\|^2-4(imag(\langle x, y\rangle))^2\Big). \tag{3.1}$$

Theorem 3.2 characterizes how the value of a_0 is affected by small perturbations.

Theorem 3.2 ([B]). *Let $X=(x_j)_{j=1}^m$ be a frame for a finite dimensional Hilbert space H with upper frame bound B which does phase retrieval. Let $a_0(X)$ be the constant given in Lemma 3.1. Let $\rho>0$ be given by*

$$\rho=\min\Big(\frac{1}{\sqrt{m}}, \frac{a_0(X)}{2\sqrt{2}(3B+2)^{3/2}}\Big). \tag{3.2}$$

Then if $Y=(y_j)_{j=1}^m \subseteq H$ satisfies that $\|x_j-y_j\|<\rho$ for all $1\le j\le m$ then $(y_j)_{j=1}^m$ is a frame of H which does phase retrieval and $\frac{1}{2}a_0(X)<a_0(Y)$.

Note that the value ρ stated in Theorem 3.2 depends on the number of frame vectors. However, the condition that $\rho \le m^{-1/2}$ is only used to guarantee that $\sum_{j=1}^m \|x_j-y_j\|^2 \le 1$. Thus, we can add this inequality directly to the hypothesis to obtain a perturbation theorem which provides a value for ρ which is independent of the number of frame vectors. This gives Corollary 3.3 which is more analogous to Theorem 1.1 and Theorem 1.2.

Corollary 3.3. *Let $X=(x_j)_{j=1}^m$ be a frame for a finite dimensional Hilbert space H with upper frame bound B which does phase retrieval. Let $a_0(X)$ be the constant given in Lemma 3.1. Let $\rho>0$ be given by*

$$\rho=\min\Big(1, \frac{(a_0(X))^2}{8(3B+2)^3}\Big). \tag{3.3}$$

If $Y=(y_j)_{j=1}^m \subseteq H$ satisfies $\sum_{j=1}^m \|x_j-y_j\|^2<\rho$ then $(y_j)_{j=1}^m$ is a frame of H which does phase retrieval and $\frac{1}{2}a_0(X)<a_0(Y)$.

Thus, both our measurement of the stability of phase retrieval given in (1.2) and the measurement given in (3.1) provide a similar perturbation theorem. The goal of this section is to compare these two measurements of stability. Note that if $(x_j)_{j=1}^m$ satisfies (3.1), then plugging in $y=0$ gives that $a_0\|x\|^4 \le \sum_{j=1}^m |\langle x, x_j\rangle|^4$. This provides a lower frame bound of $(x_j)_{j=1}^m$ being a p-frame for the value $p=4$. It is possible to consider p-frames for general Banach spaces, but we provide Definition 3.4 specifically in the context of finite dimensional Hilbert spaces.

Definition 3.4. Let H be a finite dimensional Hilbert space and let $1 \leq p < \infty$. A family $(f_j)_{j\in J} \subseteq H$ is called a p-frame of H with p-frame bounds $0 < A \leq B$ if

$$A\|x\| \leq \Big(\sum_{j\in J} |\langle x, f_j\rangle|^p\Big)^{1/p} \leq B\|x\| \qquad \text{for all } x \in H. \tag{3.4}$$

Essentially, a p-frame bounds the norm of a vector in terms of the p-norm of the frame coefficients [AST][CS]. Banach frames [AG][CCS][FG1][G] and associated spaces for Schauder frames [BF][BFL][CDOSZ][L] extend this further and apply more general Banach sequence space norms to the frame coefficients. Using p-norms and more general Banach lattice norms can be very useful when studying the stability of phase retrieval and are explicitly used in [AG][B][CPT][FG][FOPT]. Frames for Hilbert spaces are often constructed by sampling some continuous frame, such as the short-time Fourier transform or wavelet transform. It is known that every bounded continuous frame for a separable Hilbert space H may be sampled to obtain a discrete frame for H [FS]. For the case of finite dimensional Hilbert spaces, this may be quantified by their existing uniform constants $A, B, D > 0$ so that if H is n-dimensional and $(x_t)_{t\in\Omega}$ is a continuous Parseval frame of H over a probability space Ω with $\|x_t\| \leq \beta\sqrt{n}$ for all $t \in \Omega$ then there exist sampling points $(t_j)_{j=1}^m \subset \Omega$ with $m \leq D\beta^2 n$ so that $(m^{-1/2}x_{t_j})_{j=1}^m$ is a frame of H with lower frame bound A and upper frame bound B [LT]. Thus, it is possible to only consider the usual Hilbert space norm when sampling a continuous frame to construct a frame with certain desired frame bounds. In contrast to this, if one wishes to discretize a continuous frame to obtain a frame which does C-stable phase retrieval then it is necessary to control the bounds for discretizing both the L_2-norm and the L_1-norm on the range of the analysis operator [FG]. As another example, there exists a uniform constant $C > 0$ so that every finite dimensional Hilbert space has a Parseval frame which does C-stable phase retrieval. However, all known methods of constructing such frames involve sampling sub-Gaussian random vectors [BW][CDFF][CL][EM][KL][KS]. The property that a random vector $X = (x_t)_{t\in\Omega}$ in a Hilbert space H is sub-Gaussian is equivalent to the existence of a uniform constant $K > 0$ so that $\|(\langle x, x_t\rangle)_{t\in\Omega}\|_{L_p(\Omega)} \leq K\sqrt{p}\,\|x\|$ for all $x \in H$ and $p \geq 1$, and this property is extremely useful in proving concentration inequalities in high dimensional Hilbert spaces [V]. Essentially, one can think of the use of p-frame bounds in frame theory as corresponding to the use of higher-moment conditions in probability. We now generalize the definition of a frame doing C-stable phase retrieval to a p-frame doing C-stable phase retrieval in $\ell_p(J)$.

Definition 3.5. Let H be a finite dimensional Hilbert space and let $1 \leq p < \infty$. Let $(f_j)_{j\in J} \subseteq H$ be a p-frame of H with analysis operator $\Theta : H \to \ell_p(J)$ given by $\Theta x = (f_j(x))_{j\in J}$ for all $x \in H$. We say that $(f_j)_{j\in J}$ does C-stable phase retrieval in $\ell_p(J)$ if

$$\min_{|\lambda|=1} \|x - \lambda y\| \leq C\big\||\Theta x| - |\Theta y|\big\|_{\ell_p(J)} = C\Big(\sum_{j\in J} \big||f_j(x)| - |f_j(y)|\big|^p\Big)^{1/p} \tag{3.5}$$

$$\text{for all } x, y \in H.$$

In other words, if we define an equivalence relation $\sim$ on H by $x \sim y$ if and only if $x = \lambda y$ for some $|\lambda| = 1$ then a p-frame $(f_j)_{j\in J}$ of H with analysis operator $\Theta : H \to \ell_p(J)$ does C-stable phase retrieval in $\ell_p(J)$ means that the recovery of $[x]_\sim \in H/\sim$ from $|\Theta x| \in \ell_p(J)$ is a C-Lipschitz map from $|\Theta(H)|$ to $H/\sim$. Lemma 2.1 gives that $(f_j)_{j\in J}$ does C-stable phase retrieval in $\ell_p(J)$ if and only if (3.5) holds for orthogonal vectors in H. Note that if $p > 2$ then the $\ell_2(J)$-norm dominates the $\ell_p(J)$-norm and hence doing C-stable phase retrieval in $\ell_p(J)$ is a stronger condition than doing C-stable phase retrieval in $\ell_2(J)$. Proposition 3.6 relates the constant a_0 in (3.1) to the stability of phase retrieval in $\ell_4(J)$.

Proposition 3.6. *Let $(x_j)_{j\in J}$ be a frame of a Hilbert space H with Bessel bound B which satisfies* (3.1) *for the value a_0. Then $(x_j)_{j\in J}$ is a 4-frame of H with lower 4-frame bound $a_0^{1/4}$ and upper 4-frame bound $B^{1/2}$. Furthermore, $(x_j)_{j\in J}$ does $(2a_0^{-1}B)^{1/2}$-stable phase retrieval in $\ell_4(J)$.*

Proof. Let $x \in H$. By plugging $y = 0$ into (3.1), we have that

$$a_0^{1/4}\|x\| \le \Big(\sum_{j\in J} |\langle x, x_j\rangle|^4\Big)^{1/4} \le \Big(\sum_{j\in J} |\langle x, x_j\rangle|^2\Big)^{1/2} \le B^{1/2}\|x\|.$$

Thus, $(x_j)_{j\in J}$ is a 4-frame with lower 4-frame bound $a_0^{1/4}$ and upper 4-frame bound $B^{1/2}$. By Lemma 2.1, to prove that $(x_j)_{j\in J}$ does C-stable phase retrieval in $\ell_4(J)$ for $C = (2a_0^{-1}B)^{\frac{1}{2}}$, we only need to show that the inequality (3.5) holds for orthogonal vectors. We now fix $x, y \in H$ with $\langle x, y\rangle = 0$.

$$\begin{aligned}
a_0\|x+y\|^4 &\le \sum_{j\in J} \big||\langle x, x_j\rangle|^2 - |\langle y, x_j\rangle|^2\big|^2 \qquad \text{as } \langle x, y\rangle = 0\\
&= \sum_{j\in J} \big||\langle x, x_j\rangle| - |\langle y, x_j\rangle|\big|^2 \big||\langle x, x_j\rangle| + |\langle y, x_j\rangle|\big|^2\\
&\le \Big(\sum_{j\in J} \big||\langle x, x_j\rangle| - |\langle y, x_j\rangle|\big|^4\Big)^{1/2}\\
&\quad \times \Big(\sum_{j\in J} \big||\langle x, x_j\rangle| + |\langle y, x_j\rangle|\big|^4\Big)^{1/2} \qquad \text{by Cauchy-Schwarz,}\\
&\le \Big(\sum_{j\in J} \big||\langle x, x_j\rangle| - |\langle y, x_j\rangle|\big|^4\Big)^{1/2} \sum_{j\in J} \big||\langle x, x_j\rangle| + |\langle y, x_j\rangle|\big|^2\\
&\le \Big(\sum_{j\in J} \big||\langle x, x_j\rangle| - |\langle y, x_j\rangle|\big|^4\Big)^{1/2} \sum_{j\in J} 2\big(|\langle x, x_j\rangle|^2 + |\langle y, x_j\rangle|^2\big)\\
&\le \Big(\sum_{j\in J} \big||\langle x, x_j\rangle| - |\langle y, x_j\rangle|\big|^4\Big)^{1/2} 2B\big(\|x\|^2 + \|y\|^2\big)\\
&= 2B\|x+y\|^2\Big(\sum_{j\in J} \big||\langle x, x_j\rangle| - |\langle y, x_j\rangle|\big|^4\Big)^{1/2} \qquad \text{as } \langle x, y\rangle = 0.
\end{aligned}$$

Thus, we have that

$$\min_{|\lambda|=1} \|x - \lambda y\| = \|x+y\| \le (2a_0^{-1}B)^{1/2}\Big(\sum_{j=1}^{m} \big||\langle x, x_j\rangle| - |\langle y, x_j\rangle|\big|^4\Big)^{1/4}.$$

This proves that the frame $(x_j)_{j\in J}$ does $(2a_0^{-1}B)^{1/2}$-stable phase retrieval in $\ell_4(J)$. $\square$

It follows from classical results in Banach spaces that for all $2<p<\infty$ there is a universal constant $K>0$ so that if $(x_j)_{j=1}^m$ is a Parseval frame for an n-dimensional Hilbert space with lower p-frame bound A_p and upper p-frame bound B_p then $A_p n^{1/2} \leq K B_p m^{1/p}$ [FLM]. By Proposition 3.6, if a_0 satisfies (3.1) then $a_0 \leq K^4 m n^{-2}$. This gives a situation where the lower 4-frame bound is necessarily small and hence a_0 is small as well. However, the value a_0 can be small for other reasons independent of the 4-frame bounds. In Example 3.7 we show that it is possible to construct Parseval frames for even the 2-dimensional Hilbert space $\mathbb{C}^2$ which do uniformly stable phase retrieval, have uniform 4-frame bounds, but a_0 is arbitrarily small.

Example 3.7. Let $k\in\mathbb{N}$ and $p>2$. Consider the frame $(x_j)_{j\in J}$ of $\mathbb{C}^2$ defined by

$$(x_j)_{j\in J} := \big((k^{-1/2},0)\big)_{j=1}^k \sqcup \big((0,k^{-1/2})\big)_{j=1}^k \sqcup \big((1,1),(1,-1),(1,i),(1,-i)\big).$$

Then the following are all satisfied.

(1) $(x_j)_{j\in J}$ is a tight frame of $\mathbb{C}^2$ with frame bound 5.
(2) $(x_j)_{j\in J}$ has upper p-frame bound $5^{1/2}$ and lower p-frame bound 1.
(3) $(x_j)_{j\in J}$ does C_2-stable phase retrieval in $\ell_2(J)$ for some C_2 independent of k.
(4) If $C_p < (2k)^{1/2-1/p}$ then $(x_j)_{j\in J}$ does not do C_p-stable phase retrieval in $\ell_p(J)$.

In particular, for all $a_0>0$ if $k\in\mathbb{N}$ is chosen large enough then $(x_j)_{j\in J}$ does not satisfy (3.1) for that choice of a_0.

Proof. We first consider the case $k=1$ and denote

$$F := \{(1,0),(0,1),(1,1),(1,-1),(1,i),(1,-i)\}.$$

A direct calculation gives that F is a tight frame of $\mathbb{C}^2$ with frame bound 5. We now show that this frame does phase retrieval in $\mathbb{C}^2$. Let Θ be the analysis operator of F. Let $(a,b)\in\mathbb{C}^2$ with $(a,b)\neq(0,0)$. By scaling, we may assume without loss of generality that $a\in\mathbb{R}$. By Lemma 2.1 we need to show for all $c\geq 0$ that $|\Theta(a,b)| \neq |\Theta(c\overline{b},-ca)|$. For the sake of contradiction, we assume that $|\Theta(a,b)| = |\Theta(c\overline{b},-ca)|$.

As $|\langle (a,b),(1,0)\rangle| = |\langle (c\overline{b},-ca),(1,0)\rangle|$ we have that $|a|=|cb|$. Likewise, $|b| = |ca|$. Thus, $c=1$ and $|a|=|b|$.

As $|\langle (a,b),(1,1)\rangle| = |\langle (\overline{b},-a),(1,1)\rangle|$ we have that $Re(b)=0$. Likewise,

$$|\langle (a,b),(1,i)\rangle| = |\langle (\overline{b},-a),(1,i)\rangle|$$

implies that $Im(b)=0$. Hence $b=0$. This however contradicts that $|a|=|b|$.

We now have that the frame F does phase retrieval. Every frame which does phase retrieval for a finite dimensional Hilbert space does stable phase retrieval. Thus, there exists $C_2>0$ so that F does C_2-stable phase retrieval. Let $p>2$. Note that $\{(1,1),(1,-1),(1,i),(1,-i)\}$ is a tight frame with frame bound 4 and the norms on ℓ_p^4 and ℓ_2^4 are $4^{1/2-1/p}$ equivalent. Thus, $\{(1,1),(1,-1),(1,i),(1,-i)\}$ has lower p-frame bound $4^{1/2}\cdot 4^{1/p-1/2} = 4^{1/p} \geq 1$.

We now let $k\in\mathbb{N}$ and consider the frame

$$(x_j)_{j\in J} := \big((k^{-1/2},0)\big)_{j=1}^k \sqcup \big((0,k^{-1/2})\big)_{j=1}^k \sqcup \big((1,1),(1,-1),(1,i),(1,-i)\big).$$

That is, $(x_j)_{j\in J}$ can be thought of replacing the vectors $(1,0)$ and $(0,1)$ in F with k copies of $(k^{-1/2},0)$ and $(0,k^{-1/2})$ respectively. This will preserve all the frame properties which are measured in ℓ_2. In particular, $(x_j)_{j\in J}$ will be a tight frame with frame bound 5 and will do C_2-stable phase retrieval in $\ell_2(J)$. As $\{(1,1),(1,-1),(1,i),(1,-i)\}$ is a subset of $(x_j)_{j\in J}$, we have that 1 is a lower p-frame bound of $(x_j)_{j\in J}$. As $p>2$, we have that $5^{1/2}$ is an upper p-frame bound of $(x_j)_{j\in J}$. We now check the stability of phase retrieval of $(x_j)_{j\in J}$ in $\ell_p(J)$. We consider the orthogonal unit vectors $(1,0),(0,1)\in\mathbb{C}^2$.

$$\sum_{j\in J}\big||\langle(1,0),x_j\rangle|-|\langle(0,1),x_j\rangle|\big|^p=\sum_{j=1}^{k}(k^{-1/2})^p+\sum_{j=1}^{k}(k^{-1/2})^p+0+0+0+0=2k^{1-p/2}.$$

Note that for $x=(1,0)$ and $y=(0,1)$ we have that $\min_{|\lambda|=1}\|x-\lambda y\|=\sqrt{2}$ and hence

$$\min_{|\lambda|=1}\|x-\lambda y\|=2^{1/2-1/p}k^{1/2-1/p}\Big(\sum_{j\in J}\big||\langle x,x_j\rangle|-|\langle y,x_j\rangle|\big|^p\Big)^{1/p}.$$

As $p>2$, we have that the stability of $(x_j)_{j\in J}$ doing phase retrieval in $\ell_p(J)$ can be forced to be arbitrarily large. That is, if $C_p>0$ and $k\in\mathbb{N}$ is chosen large enough so that $C_p<2^{1/2-1/p}k^{1/2-1/p}$ then $(x_j)_{j\in J}$ does not do C_p-stable phase retrieval in $\ell_p(J)$. $\square$

We now show that the idea used in Example 3.7 will work for any n-dimensional Hilbert space with $n>1$.

Proposition 3.8. *There exists a uniform constant $C_2>0$ so that for all $p>2$, all $n\geq 2$, and all $C_p>0$ there exists a frame $(x_j)_{j\in J}$ of $\mathbb{C}^n$ so that*

1. *$(x_j)_{j\in J}$ is a tight frame of $\mathbb{C}^n$ with frame bound 2.*
2. *$(x_j)_{j\in J}$ has upper p-frame bound $2^{1/2}$ and lower p-frame bound $n^{1/p-1/2}$.*
3. *$(x_j)_{j\in J}$ does C_2-stable phase retrieval in $\ell_2(J)$.*
4. *$(x_j)_{j\in J}$ does not do C_p-stable phase retrieval in $\ell_p(J)$.*

Proof. There is a uniform constant $C_2>0$ so that for all $n\in\mathbb{N}$ there exists a Parseval frame $(z_j)_{j\in I}$ of $\mathbb{C}^n$ which does C_2-stable phase retrieval [KS]. Let $(e_j)_{j=1}^n$ be the unit vector basis for $\mathbb{C}^n$. We have that $(e_j)_{j=1}^n\sqcup(z_j)_{j\in I}$ is a tight frame of $\mathbb{C}^n$ with frame bound 2 and hence $(e_j)_{j=1}^n\sqcup(z_j)_{j\in I}$ has upper p-frame bound $2^{1/2}$ for all $p\geq 2$. As it contains the unit vector basis for $\mathbb{C}^n$, $(e_j)_{j=1}^n\sqcup(z_j)_{j\in I}$ has lower p-frame bound $n^{1/p-1/2}$.

We now let $k\in\mathbb{N}$ and consider the frame $(x_j)_{j\in J}$ which consists of $(e_j)_{j=1}^n$ and k copies of $(k^{-1/2}z_j)_{j\in I}$. We have that $(x_j)_{j\in J}$ will preserve all the frame properties of $(e_j)_{j=1}^n\sqcup(z_j)_{j\in I}$ which are measured in ℓ_2. In particular, $(x_j)_{j\in J}$ is a tight frame of $\mathbb{C}^n$ with frame bound 2 and does C_2-stable phase retrieval in $\ell_2(J)$. Furthermore, as $(x_j)_{j\in J}$ contains the unit vector basis of $\mathbb{C}^n$, we have that $(x_j)_{j\in J}$ has lower p-frame bound $n^{1/p-1/2}$.

We now consider the orthogonal vectors $x=2^{-1}(e_1+e_2)$ and $y=2^{-1}(e_1-e_2)$.

$$\begin{aligned}
&\sum_{j\in J}\big||\langle x,x_j\rangle|-|\langle y,x_j\rangle|\big|^p\\
&\quad=\sum_{j=1}^{n}\big||\langle x,e_j\rangle|-|\langle y,e_j\rangle|\big|^p+k\sum_{j\in I}\big||\langle x,k^{-1/2}z_j\rangle|-|\langle y,k^{-1/2}z_j\rangle|\big|^p\\
&\quad\leq 0+k^{1-p/2}\sum_{j\in I}|\langle x-y,z_j\rangle|^p\\
&\quad\leq k^{1-p/2}\Big(\sum_{j\in I}|\langle e_2,z_j\rangle|^2\Big)^{p/2}=k^{1-p/2}.
\end{aligned}$$

Thus, we have that

$$\min_{|\lambda|=1}\|x-\lambda y\|=1\geq k^{1/2-1/p}\Big(\sum_{j\in J}\big||\langle x,x_j\rangle|-|\langle y,x_j\rangle|\big|^p\Big)^{1/p}.$$

Hence, if C_p is any constant then we may choose $k\in\mathbb{N}$ large enough so that $(x_j)_{j\in J}$ does not do C_p-stable phase retrieval in $\ell_p(J)$. $\square$

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