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ASYMPTOTICS IN FINITE MONOIDAL CATEGORIES

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ABSTRACT. We give explicit formulas for the asymptotic growth rate of the number of summands in tensor powers in certain monoidal categories with finitely many indecomposable objects, and related structures.

CONTENTS

1. INTRODUCTION

Let $R = (R, C)$ be a *finite based* $\mathbb{R}_{\geq 0}$*-algebra* with basis $C = \{1 = c_0, \dots, c_{r-1}\}$ (recalled in Section 3 together with some other notions used in this introduction). Recall that we thus have

$$c_i c_j = \sum_k m_{i,j}^k \cdot c_k \quad \text{with} \quad m_{i,j}^k \in \mathbb{R}_{\geq 0}. \tag{1A.1}$$

Iterating this gives us coefficients $m_{i,j,\dots,l}^k \in \mathbb{R}_{\geq 0}$. Similarly, for $c = a_0 \cdot c_0 + \dots + a_{r-1} \cdot c_{r-1}, d = d_0 \cdot c_0 + \dots + d_{r-1} \cdot c_{r-1} \in \mathbb{R}_{\geq 0}C$ we get, for example, $cd = \sum_k a_i d_j m_{i,j}^k \cdot c_k$ with $a_i d_j m_{i,j}^k \in \mathbb{R}_{\geq 0}$.

Fix $c \in \mathbb{R}_{\geq 0}C$. We write $m_n^*(c)$ for these coefficients as they appear in c^n where $* \in \{0, \dots, r-1\}$. Define

$$b_n^{R,c} := \sum_* m_n^*(c) = \text{total sum of the coefficients } m_n^*(c).$$

Moreover, we define the function

$$b^{R,c} \colon \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}, n \mapsto b^{R,c}(n) := b_n^{R,c}.$$

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We are interested in the *asymptotic behavior* of the function $b^{R,c}(n)$. The main question we address is:

Question 1. Find an explicit formula $a(n)$ such that

$$b^{R,c}(n) \sim a(n),$$

where we write $\sim$ for asymptotically equal.

We answer Question 1 as follows.

The (transposed) *action matrix* of $c = a_0 \cdot c_0 + \cdots + a_{r-1} \cdot c_{r-1} \in \mathbb{R}_{\geq 0}C$ is the matrix $(\sum_i a_i m_{i,j}^k)_{k,j}$. Abusing language, we will call the submatrix of it corresponding to the connected component of 1 also the action matrix and use this below.

Assume that the *Perron–Frobenius theorem* holds, that is the action matrix of $c \in \mathbb{R}_{\geq 0}C$ has a leading eigenvalue $\lambda_0 = \mathrm{PFdim}\, c$ of multiplicity one that we call the *Perron–Frobenius dimension* of c. Moreover, the action matrix has some period $h \in \mathbb{Z}_{\geq 0}$ such that $\lambda_k = \zeta^k \mathrm{PFdim}\, c$, where $\zeta = \exp(2\pi i/h)$ and $k \in \{1, \dots, h-1\}$, are precisely the other eigenvalues of absolute value $\mathrm{PFdim}\, c$. We will drop this assumption in Section 3.

Let us denote the right (the one with $Mv_i = \lambda_i \cdot v_i$) and left (the one with $w_i^T M = \lambda_i \cdot w_i^T$) eigenvectors by v_i and w_i, normalized such that $w_i^T v_i = 1$. Let $v_i w_i^T[1]$ denote taking the sum of the first column of the matrix $v_i w_i^T$, and let $\overline{\mathbb{Z}}$ denote the algebraic integers. Define

(1A.2)

$$a(n) = \big(v_0 w_0^T[1] \cdot 1 + v_1 w_1^T[1] \cdot \zeta^n + v_2 w_2^T[1] \cdot (\zeta^2)^n + \cdots + v_{h-1} w_{h-1}^T[1] \cdot (\zeta^{h-1})^n\big) \cdot (\mathrm{PFdim}\, c)^n \in \overline{\mathbb{Z}}.$$

Let λ^{sec} be any second largest eigenvalue of the action matrix of c. We will prove (see Section 3):

Theorem 1. *We have*

$$b^{R,c}(n) \sim a(n),$$

and the convergence is geometric with ratio $|\lambda^{sec}/\mathrm{PFdim}\, c|$. *In particular,*

$$\beta^{R,c} := \lim_{n\to\infty} \sqrt[n]{b_n^{R,c}} = \mathrm{PFdim}\, c.$$

The reason why Theorem 1 is interesting from the categorical point of view is the following. For us a *finite monoidal category* is a category such that:

(i) It is monoidal.

(ii) It is additive Krull–Schmidt.

(iii) It has finitely many (isomorphism classes of) indecomposable objects.

Example 1. Here are a few examples:

(a) Let G be a finite group and consider $\mathbf{Rep}(G) = \mathbf{Rep}(G, \mathbb{C})$ the category of finite dimensional complex representations of G. This is a prototypical example of a finite monoidal category.

(b) More generally, all fusion categories are finite monoidal.

(c) For a finite group G, and arbitrary field, we can consider finite dimensional projective $\mathbf{Proj}(G)$ or injective $\mathbf{Inj}(G)$ representations. These are finite monoidal categories. More generally, one can take any finite dimensional Hopf algebra instead of a finite group.

(d) If we assume that a Hopf algebra H is of finite type, then we can even consider $\mathbf{Rep}(H)$ (finite dimensional H-representations). An explicit and nonsemisimple example over $\mathbb{C}$ is the Taft algebra by [CVOZ14, Theorem 2.5].

(e) In any additive Krull–Schmidt monoidal category one can take $\{\mathtt{X}^{\otimes d}|d \in \mathbb{Z}_{\geq 0}\}^{\oplus,\subset\oplus}$, the additive idempotent completion of the full subcategory generated by an object $\mathtt{X}$, as long as this has finitely many indecomposable objects. Explicitly, for a finite group G one can take any two dimensional G-representation for $\mathtt{X}$, which follows from [Alp79]. There are many more examples, see e.g. [Cra13].

(f) Consider Soergel bimodules $\mathbf{SBim}(W)$ as in [Soe92]. These are finite monoidal categories if $W = (W, S)$ is of finite Coxeter type.

There are of course many more examples.

The following is very easy and omitted:

Lemma 1. *The additive Grothendieck ring of a finite monoidal category is a finite based $\mathbb{R}_{\geq 0}$-algebra with basis given by the classes of indecomposable objects.* □

Fix a finite monoidal category $\mathbf{C}$ and an object $\mathtt{X} \in \mathbf{C}$. Following [COT23], we define

$$b_n^{\mathbf{C},\mathtt{X}} := \#\text{indecomposable summands in } \mathtt{X}^{\otimes n} \text{ counted with multiplicities.}$$

Note that $a(n)$ has an analog in this context, denoted by the same symbol, obtained for the (transposed) action matrix for left tensoring. Similarly as before we also have λ^{sec}. We then get:

Theorem 2. *Under the same assumption as in Theorem 1, we have*

$$b^{\mathbf{C},\mathtt{X}}(n) \sim a(n),$$

and the convergence is geometric with ratio $|\lambda^{sec}/\mathrm{PFdim}\,\mathtt{X}|$. In particular,

$$\beta^{\mathbf{C},\mathtt{X}} := \lim_{n\to\infty} \sqrt[n]{b_n^{\mathbf{C},\mathtt{X}}} = \mathrm{PFdim}\,\mathtt{X}.$$

Proof. From Theorem 1 and Lemma 1. □

In the next section we will discuss examples of Theorem 2, and then we will prove Theorem 1. We also generalize these two theorems in Section 3 by getting rid of the assumption on the action matrix. Before that, let us finish the introduction with some (historical) remarks.

Remark 1.

(a) To study asymptotic properties of tensor powers is a rather new subject and most things are still quite mysterious. Let us mention a few facts that are known. An early reference we know is [Bia93], which studies questions similar to the one in this note but for Lie algebras, and this was carried on in several works such as [PR20]. As another example, the paper [BS20] studies the growth rate of the dimensions of the non-projective part of tensor powers of a representation of a finite group. More generally, the paper [CEO23b] studies, working in certain tensor categories, the growth rates of summands of categorical dimension prime to the underlying characteristic. The paper [COT23] studies the growth rate of all summands, while [KST22] studies the Schur–Weyl dual question.

(b) Theorem 1 and Theorem 2 generalize [CEO23a, Proposition 2.1]. And for us one of the main features of that proposition is its simplicity, having a simple statement and proof. As we will see, the same is true for Theorem 1 and Theorem 2 as well: Clearly, the statements themselves are (surprisingly) simple yet general. Moreover, the proof of Theorem 1, and therefore the proof of Theorem 2 as well, is rather straightforward as soon as the key ideas are in place.

(c) The second statements in Theorem 1 and Theorem 2 were already observed in [COT23] (in the setting of [COT23] the Perron–Frobenius dimension agrees with the usual dimension), but the (finer) asymptotic behavior appears to be new.

Finally, let us mention that similar questions have been studied much earlier, see for example [AE81] for a related notion involving length of projective resolutions, or [LS77] for counting and Young diagrams.

2. Examples

Let us call Theorem 1 and Theorem 2 our *main theorem(s)* or MT for short. To underpin the explicit nature of these theorems, we now list examples MT applies. We also add that all the below can be double checked using the code on [LTV23]. That page also contains a (potentially empty) Erratum.

Let us briefly explain why MT can be used in all the examples discussed below: For Section 2A this follows since our assumption on V implies that the action matrix is irreducible. For all other examples a direct calculation verifies that the action matrices satisfy the Perron–Frobenius theorem.

2A. **Finite groups.** Let G be a finite group. Given a finite dimensional complex G-representation V, denote by $Z_V(G) \subset G$ the subgroup consisting of elements of g that acts as a scalar on V and by $\omega_V(g) \in \mathbb{C}$ the corresponding scalar. If V is simple, then ω_V is known as the *central character* of V.

Suppose that V is a faithful G-representation. Since V is faithful we get that $Z_V(G)$ is a subgroup of $Z(G)$ and also that the action graph of tensoring with V is connected (in the oriented sense). Then MT implies:

$$a(n) = \left(\frac{1}{\#G} \sum_{g \in Z_V(G)} \Big(\sum_{L \in \mathrm{S}(G)} \omega_L(g) \dim_{\mathbb{C}} L \Big) \cdot \omega_V(g)^n \right) \cdot (\dim_{\mathbb{C}} V)^n, \tag{2A.1}$$

where $\mathrm{S}(G) = \{\text{simple } G\text{-representations}\}/\cong$. This follows directly from MT after recalling the connection from Perron–Frobenius theory to character theory as explained in e.g. [EGNO15, Chapter 3 and Example 4.5.5]. To elaborate a bit, the Perron–Frobenius dimension in this case is just the dimension, and the leading eigenvector corresponds to the regular G-representation.

Remark 2. If V is not faithful, then the action graph of tensoring with V needs not to be connected, but that is not an issue in MT. Thus, the assumption that V is faithful can be easily relaxed.

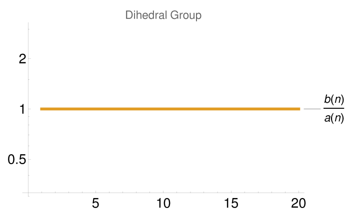
Remark 3. Alternatively one can prove (2A.1) using character theory, similarly to [CEO23a, Proposition 2.1]. (2A.1) still generalizes [CEO23a, Proposition 2.1].

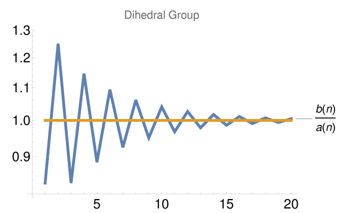
Let us give a few explicit examples.

Example 2 (Dihedral groups)**.** Let $m \in \mathbb{Z}_{\geq 3}$ and let G be the *dihedral group* of order $2m$. Let $m' = m/2$, if m is even, and $m' = (m-1)/2$, if m is odd. Choose V any faithful representation of dimension 2 of G. Then (2A.1) gives the formulas

$$a(n) = \begin{cases} \frac{m+1}{2m} \cdot 2^n & \text{if } m \text{ is odd,} \\ \frac{m+2}{2m} \cdot 2^n & \text{if } m \text{ is even and } m' \text{ is odd,} \\ \left(\frac{(m+2)}{2m} \cdot 1 + \frac{1}{m} \cdot (-1)^n\right) \cdot 2^n & \text{if } m \text{ is even and } m' \text{ is even.} \end{cases}$$

Two explicit examples are $m \in \{4, 5\}$ and V is the G-representation corresponding to rotation by $2\pi/m$. Then:

$m = 4$: 

, $m = 5$: 

.

Here and throughout, we display the graphs of $b(n)/a(n)$ in the usual way but log plotted (on the y-axis). Moreover, for $m = 4$ we have $b(n) = a(n)$ and we will omit plots in case that happens.

The next example can be seen as a $p > 2$ version of Example 2.

Example 3 (Extraspecial groups)**.** Let p be a prime and $m \in \mathbb{Z}_{\geq 1}$. Recall that a p-group of order p^{1+2m} is called *extraspecial* if its center $Z(G)$ is of order p and the quotient $G/Z(G)$ is a p-elementary abelian group. For each p and m, there exists two isomorphism classes of extraspecial groups of order p^{1+2m}, and they have the same character table. Thus, by (2A.1) we can take any of these two without difference. In the special case $p = 2$ and $m = 1$ we recover the dihedral group and the quaternion group of order 8.

Fix now an extraspecial group G of order p^{1+2m}. The simple G-representations are given as follows:

(i) There are p^{2m} nonisomorphic one dimensional representations that arise from the representations of $G/Z(G)$.

(ii) There are $p-1$ nonisomorphic simple representations of dimension p^m which are characterized by their central character.

Choose V any of the simple G-representation of dimension p^m. Then $Z_V(G) = Z(G)$ and (2A.1) gives

$$a(n) = \begin{cases} (p^m)^n & \text{if } p \mid n, \\ (p^m)^{n-1} & \text{otherwise.} \end{cases}$$

It turns out that this formula is not only asymptotic: we have $b(n) = a(n)$. This is due to the fact that the character of V vanishes outside of $Z(G)$.

Example 4 (Imprimitive complex reflection groups)**.** Let d and m be integers in $\mathbb{Z}_{\geq 1}$ and consider the *imprimitive complex reflection group* $G = G(d, 1, m)$. This group can be seen as the group of m-by-m monomial matrices with entries being dth roots of unity. For $d = 1$ we recover the symmetric group, covered by [CEO23a, Example 2.3], and if $d = 2$ we recover the Weyl group of type B_m.

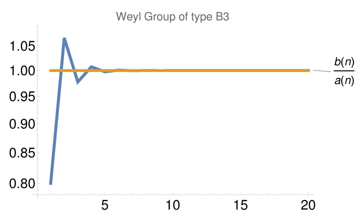
Choose V the standard representation given by the matrix description of G. Then (2A.1) gives a formula akin to [CEO23a, Example 2.3] which we decided not to write down as it's a bit tedious.

In any case, for the special cases $d \in \{1, 2\}$ and $m = 2$, or $d = 2$ and $m = 4$ we get

$$\begin{cases} d = 1, \\ m = 3 \end{cases} : a(n) = \frac{2}{3} \cdot 3^n, \quad \begin{cases} d = 2, \\ m = 3 \end{cases} : a(n) = \frac{5}{12} \cdot 3^n,$$
$$\begin{cases} d = 2, \\ m = 4 \end{cases} : a(n) = \left(\frac{19}{96} \cdot 1 + \frac{1}{32} \cdot (-1)^n \right) \cdot 4^n.$$

We get the plots

$$\begin{cases} d = 2, \\ m = 3 \end{cases} :$$



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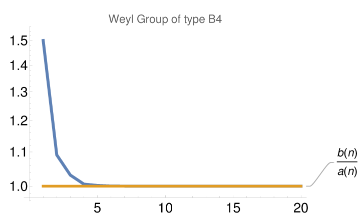
$$\begin{cases} d = 2, \\ m = 4 \end{cases} :$$



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Moreover, the formula $a(n) = \frac{2}{3} \cdot 3^n$ is exact for $d = 1$ and $m = 3$.

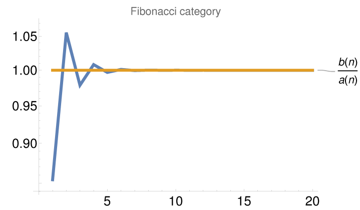
2B. **Fusion categories.**

This section discusses fusion categories over $\mathbb{C}$ different from $\mathbf{Rep}(G)$.

Example 5 (Fibonacci category)**.** Let $\mathscr{F}$ be the *Fibonacci category*, see for example [EGNO15, Exercise 8.18.7] where $\mathscr{F}$ is denoted by $\mathcal{YL}_+$ (or $\mathcal{YL}_-$, depending on conventions). All we need to know is that $\mathscr{F}$ is $\otimes$-generated by one object $\mathtt{X}$ with action matrix $M(\mathtt{X}) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

We want to estimate $b^{\mathscr{F},\mathtt{X}}(n)$. To this end, the eigenvalues of $M(\mathtt{X})$ are the two roots of $x^2 = x + 1$, in particular, $\mathrm{PFdim}\,\mathtt{X} = \phi$, the golden ratio. Its Perron–Frobenius eigenvectors are $v = w = \left(\frac{\sqrt{5}-1}{\sqrt{10-2\sqrt{5}}}, \sqrt{\frac{1}{10}(\sqrt{5}+5)}\right)^T$ and we therefore get

$$a(n) = \frac{1}{10}(\sqrt{5}+5)\cdot\phi^n = \frac{1}{\sqrt{5}}\phi^{n-1},$$



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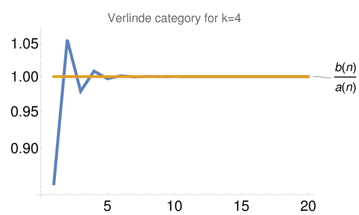
from MT. Note that the classical asymptotic for the Fibonacci numbers is $\frac{1}{\sqrt{5}}\phi^n$ and not $\frac{1}{\sqrt{5}}\phi^{n-1}$, but $b^{\mathscr{F},\mathtt{X}}(n)$ is equal to the $(n+1)$th Fibonacci number and hence the off-by-one-error in the exponent.

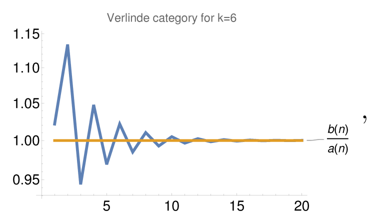
Example 6 (Verlinde category)**.** We now consider the *Verlinde category* $\mathbf{Ver}_k(\mathrm{SL}_2)$ for $k \in \mathbb{Z}_{\geq 2}$, see for example [EGNO15, Section 8.18.2] (denoted differently therein). This fusion category has k simple objects, and we take the generating object $\mathtt{X}$ of categorical dimension $2\cos\big(\pi/(k+1)\big)$. The case $k = 2$ compares to super vector spaces.

The action matrix for $\mathtt{X}$ has the type A Dynkin diagram as its associated graph, and the eigenvalues and eigenvectors of this graph are well-known, see for example [Smi70]. In particular, $\mathrm{PFdim}\,\mathtt{X} = 2\cos\big(\pi/(k+1)\big)$. Let $q = \exp\big(\pi i/(k+1)\big)$. Then MT gives us

$$a(n) = \begin{cases} \frac{[1]_q+\cdots+[k]_q}{[1]_q^2+\cdots+[k]_q^2}\cdot\big(2\cos(\pi/(k+1))\big)^n & \text{if } k \text{ is even,} \\ \Big(\frac{[1]_q+\cdots+[k]_q}{[1]_q^2+\cdots+[k]_q^2}\cdot 1 + \frac{[1]_q-[2]_q+\cdots-[k-1]_q+[k]_q}{[1]_q^2+\cdots+[k]_q^2}\cdot(-1)^n\Big) & \\ \qquad\qquad\cdot\big(2\cos(\pi/(k+1))\big)^n & \text{if } k \text{ is odd.} \end{cases}$$

Here $[a]_q$ denotes the ath quantum number evaluated at q. We get, for example:

$k = 4\colon$


, $k = 6\colon$


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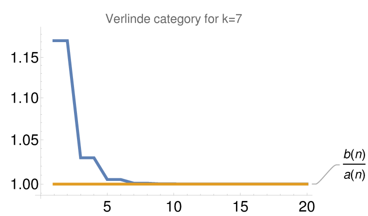
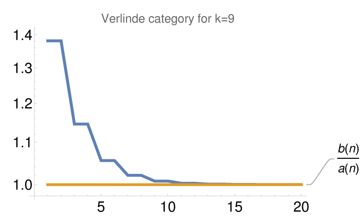
$k = 7\colon$


, $k = 9\colon$


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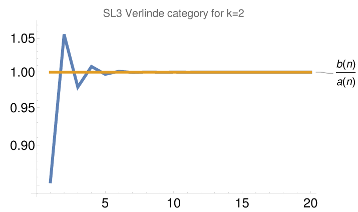
Moreover, for $k \in \{3, 5\}$ the formula $a(n)$ is spot on.

Example 7 (Higher rank Verlinde categories)**.** Verlinde categories can be defined for all simple Lie algebras as quotients of representations of quantum groups at a root of unity as explained in [AP95]. Let us focus in this example on $\mathbf{Ver}_k(\mathrm{SL}_3)$, the one for the special linear group of rank three (with k determined as e in [MMMT20, Section 2]).

For $\mathbf{Ver}_k(\mathrm{SL}_3)$ we take the generating object $\mathtt{X}$ corresponding to the vector representation of $\mathrm{SL}_3(\mathbb{C})$. Its action matrix is the oriented version of the graph displayed in [MMMT20, Fig. A1] with the orientation as in [MMMT20, (3-1)]. Using this, and omitting $k = 1$ since this is trivial, MT gives

$$k=2\colon a(n)=\frac{1}{10}(\sqrt{5}+5)\cdot\phi^k,\quad k=3\colon a(n)=1/2\cdot 2^n,$$
$$k=4\colon a(n)=\frac{1}{7}\Big(2+2\cos\big(\frac{3\pi}{7}\big)\Big)\cdot\Big(1+2\cos\big(\frac{2\pi}{7}\big)\Big)^n,$$

$k = 2\colon$ 

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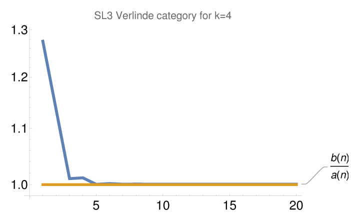
$k = 4\colon$ 

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For $k = 3$ the displayed formulas are exact. Moreover, one can find $a(n)$ explicitly in general using the formulas in [Zub98] or [MMMT20, Section 2].

2C. **Nonsemisimple examples.** We now discuss two nonsemisimple examples.

Example 8 ($\mathrm{SL}_2(\mathbb{F}_p)$ in defining characteristic)**.** Let our ground field be $\mathbb{F}_p$ for some prime $p > 2$. We consider the finite group $\mathrm{SL}_2(\mathbb{F}_p)$ and its representations over $\mathbb{F}_p$. Take $V = \mathbb{F}_p^2$ to be the vector representation of $\mathrm{SL}_2(\mathbb{F}_p)$. In this case the action matrices are exemplified by

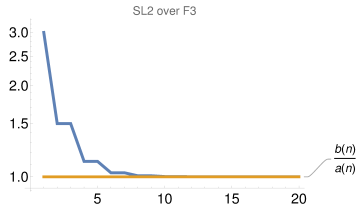
$$p=3\colon \begin{pmatrix} 0&1&0&0&0\\ 1&0&0&0&0\\ 0&1&0&3&0\\ 0&0&1&0&1\\ 0&0&0&1&0 \end{pmatrix},\quad p=5\colon \begin{pmatrix} 0&1&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&2&0&1&0\\ 0&0&0&0&1&0&1&0&0\\ 0&0&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&1&0 \end{pmatrix},\quad p=7\colon \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&2&0&0&0&1&0\\ 0&0&0&0&0&0&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&1&0 \end{pmatrix}.$$

These can be described as follows. The matrix is the one obtained as a $(2p-1)$-by-$(2p-1)$ cut-off of the matrix for the infinite group over $\overline{\mathbb{F}_p}$ that can be obtained from [STWZ23, Proposition 4.4], together with an extra entry 1 in position $(p, 2p-2)$.

Then MT gives

$$a(n)=\left(\frac{1}{2p-2}\cdot 1+\frac{1}{2p^2-2p}\cdot(-1)^n\right)\cdot 2^n.$$

Explicitly, for $p \in \{3,5\}$ we get

$p = 3$: 

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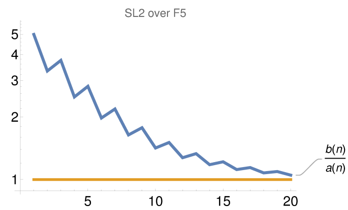
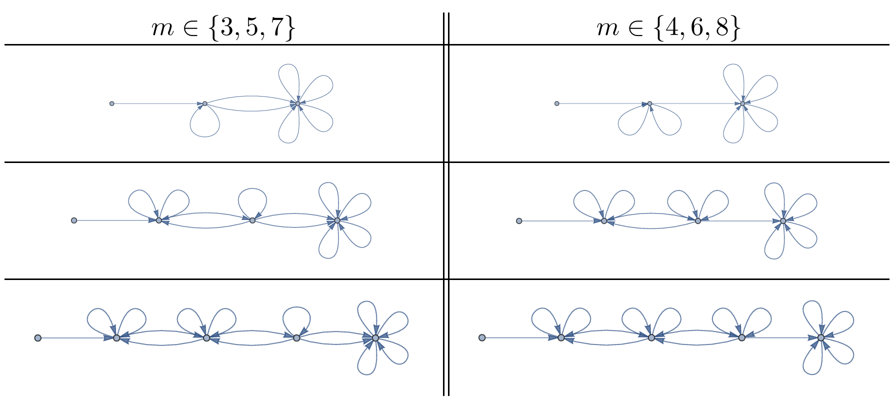
$p = 5$: 

.

The convergence is rather slow (but still geometric).

Example 9 (Dihedral Soergel bimodules)**.** We now look at the category of dihedral Soergel bimodules as studied in details in, for example, [Eli16], [MT19] or [Tub22]. In particular, [Tub22, Section 3C] lists all the formulas relevant for MT. To get a finite based $\mathbb{R}_{\geq 0}$-algebra we collapse the grading, meaning we specialize [Tub22, Section 3C] at $q = 1$.

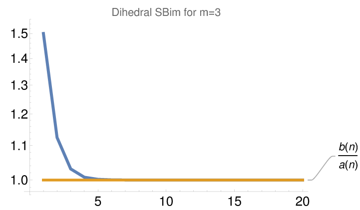
Fix $\langle s,t | s^2 = t^2 = (st)^m \rangle$ as the presentation for the dihedral group of order $2m$ where $m \in \mathbb{Z}_{\geq 3}$. Let us take $\mathtt{X}$ to be the Bott–Samelson generator for st. By the explicit formulas in [Tub22, Section 3C], the action graph of $\mathtt{X}$ is almost the same as the action graph of tensoring with $\mathbb{C}^3$ as a $\mathrm{SO}_3(\mathbb{C})$-representation. The first ones are (read in two columns):



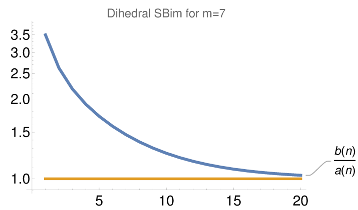
The pattern generalizes. It is then easy to show that the leading eigenvalues are always 4 and the absolute values of all other eigenvalues are strictly smaller.

Moreover, MT gives:

$$a(n) = \frac{1}{2m} \cdot 4^n,$$

$m = 3$: 

,

$m = 7$: 

.

The rate of convergence is rather slow for $m \gg 0$.

3. Generalizations and proofs

We will prove several versions of Theorem 1.

3A. **Perron–Frobenius theory.** We start with the main player, the *Perron–Frobenius theorem*. To this end, recall that one can associate an oriented and weighted graph, its *adjacency graph*, to an m-by-m matrix $M = (m_{ij})_{1 \leq i,j \leq m} \in \mathrm{Mat}_m(\mathbb{R}_{\geq 0})$ as follows:

(i) The vertices are $\{1, \dots, m\}$.

(ii) There is an edge with weight m_{ij} from i to j.

We call a nonzero matrix $M \in \mathrm{Mat}_m(\mathbb{R}_{\geq 0})$ *irreducible* if its associated graph is connected in the oriented sense (this is called *strongly connected*). Recall that in this note a right eigenvector satisfies $Mv = \lambda \cdot v$, and a left eigenvector satisfies $w^T M = \lambda \cdot w^T$.

Theorem 3 (Perron–Frobenius theorem part I)**.** *Let $M \in \mathrm{Mat}_m(\mathbb{R}_{\geq 0})$ be irreducible.*

(a) *M has a* Perron–Frobenius eigenvalue, *that is, $\lambda \in \mathbb{R}_{>0}$ such that $\lambda \geq |\mu|$ for all other eigenvalues μ. This eigenvalue appears with multiplicity one, and all other eigenvalues with $\lambda = |\mu|$ also appear with multiplicity one.*

(b) *There exists $h \in \mathbb{Z}_{\geq 1}$, the* period, *such that all eigenvalues μ with $\lambda = |\mu|$ are $\exp(k2\pi i/h)\lambda$ for $k \in \{0, \dots, h-1\}$. We call these* pseudo-dominant eigenvalues.

(c) *The eigenvectors, left and right, for the Perron–Frobenius eigenvalue can be normalized to have entries in $\mathbb{R}_{\geq 0}$.*

Proof. Well-known. See for example, Frobenius' paper 92 in Band 3 of [Fro68]. (This is the paper "Über Matrizen aus nicht negativen Elementen".) ☐

Fix a function $f\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$. We say that $f(n)$ converges *geometrically to* $a \in \mathbb{R}$ *with ratio* $\beta \in [0,1)$ if for all $\gamma \in (\beta, 1)$ we have that $\{(f(n)-a)/\gamma^n\}_{n \in \mathbb{Z}_{\geq 0}}$ is bounded. We, abusing language, will call the infimum of all ratios *the* ratio of convergence.

For two matrices of the same size let $\sim$ mean that they are asymptotically equal entrywise (note that below $v_i w_i^T$ are matrices). For such matrices we apply the definition of geometric convergence entrywise with the ratio being the maximum of the entrywise ratios. The following accompanies Theorem 3:

Theorem 4 (Perron–Frobenius theorem part II)**.** *Let $M \in \mathrm{Mat}_m(\mathbb{R}_{\geq 0})$ be irreducible, λ be its Perron–Frobenius eigenvalue and h be its period. Let $\zeta = \exp(2\pi i/h)$. For each $k \in \{0, \dots, h-1\}$, choose a left eigenvector v_k and a right eigenvector w_k with eigenvalue $\zeta^k \lambda$, normalized such that $w_k^T v_k = 1$.*

Then we have:

$$M^n \sim v_0 w_0^T \cdot \lambda^n + v_1 w_1^T \cdot (\zeta\lambda)^n + v_2 w_2^T \cdot (\zeta^2 \lambda)^n + \dots + v_{h-1} w_{h-1}^T \cdot (\zeta^{h-1}\lambda)^n.$$

Moreover, the convergence is geometric with ratio $|\lambda^{sec}/\lambda|$, where λ^{sec} is any second largest (in the sense of absolute value) eigenvalue.

Proof. This is known, but proofs are a bit tricky to find in the literature, so we give one. The proof also shows where the vectors v_i and w_i come from.

For any $\mu \in \mathbb{C}$, let V_μ be the generalized eigenspace of $V = \mathbb{C}^m$ associated to the eigenvalue μ. Then we have

$$V = \bigoplus_{k=0}^{h} V_{\zeta^k \lambda} \oplus \bigoplus_{\mu, |\mu| < \lambda} V_\mu.$$

By Theorem 3, the space $V_{\zeta^k\lambda}$ is the eigenspace associated to the eigenvalue $\zeta^k \lambda$ and $v_k w_k^T$ is the projection onto that subspace.

This implies that we have

$$M^n = v_0 w_0^T \cdot \lambda^n + v_1 w_1^T \cdot (\zeta\lambda)^n + v_2 w_2^T \cdot (\zeta^2 \lambda)^n + \dots + v_{h-1} w_{h-1}^T \cdot (\zeta^{h-1}\lambda)^n + R(n),$$

where $R(n)$ is the multiplication action of M^n onto the rest. Since the eigenvalues μ of M on the rest satisfies $|\mu| < \lambda$, we have $R(n)/\lambda^n \to_{n\to\infty} 0$ geometrically with ratio $|\lambda^{sec}/\lambda|$. ∎

For a general matrix $M \in \mathrm{Mat}_m(\mathbb{R}_{\geq 0})$ things change, but not too much:

Theorem 5 (Perron–Frobenius theorem part III)**.** *Let $M \in \mathrm{Mat}_m(\mathbb{R}_{\geq 0})$.*

(a) *M has a* Perron–Frobenius eigenvalue*, that is, $\lambda \in \mathbb{R}_{\geq 0}$ such that $\lambda \geq |\mu|$ for all other eigenvalues μ.*

(b) *Let s be the multiplicity of the Perron–Frobenius eigenvalue. There exists $(h_1, \dots, h_s) \in \mathbb{Z}_{\geq 1}^s$, the* periods*, such that all eigenvalues μ with $\lambda = |\mu|$ are $\exp(k2\pi i/h_\ell)\lambda$ for $k \in \{0, \dots, h_\ell - 1\}$, for some period. We call these* pseudo-dominant eigenvalues*.*

(c) *The eigenvectors, left and right, for the Perron–Frobenius eigenvalues can be normalized to have entries in $\mathbb{R}_{\geq 0}$.*

(d) *Let $h = \mathrm{lcm}(h_1, \dots, h_s)$, and let ν be the maximal dimension of the Jordan blocks of M containing λ. There exist matrices $S^i(n)$ with polynomial* *entries of degree* $\leq (\nu - 1)$ *for* $i \in \{0, \dots, h-1\}$ *such that*

$$\lim_{n\to\infty} |\,(M/\lambda)^{hn+i} - S^i(n)| \to 0 \quad \forall i \in \{0, \dots, h-1\},$$

and the convergence is geometric with ratio $|\lambda^{sec}/\lambda|^h$*. There are also explicit formulas for the matrices* $S^i(n)$*, see* [Rot81, Section 5].

Proof. This can be found in [Rot81]. See also [Hog07, Section I.10] (in the second version) for a useful list of properties of nonnegative matrices. □

3B. **Three versions of the *MT*.** We recall based algebras. These algebras originate in work of Lusztig on so-called *special* representations of Weyl groups [Lus79]. We follow [KM16, Section 2] with our definition.

Let $\mathbb{K} \subset \mathbb{C}$ be a unital subring. A $\mathbb{K}$-algebra R with a finite $\mathbb{K}$-basis $C = \{1 = c_0, \dots, c_{r-1}\}$ is called a *finite based* $\mathbb{R}_{\geq 0}$*-algebra* if all structure constants are in $\mathbb{R}_{\geq 0}$ with respect to the basis C. That is, (1A.1) holds.

The underlying ring $\mathbb{K}$ is allowed to be different from $\mathbb{R}$ or $\mathbb{C}$, but it needs to contain the structure constants of course. When the structure constants are in $\mathbb{Z}_{\geq 0} \subset \mathbb{R}_{\geq 0}$ a popular choice for the ground ring is $\mathbb{K} = \mathbb{Z}$.

Example 10. Examples include:

(a) The Grothendieck rings of all the examples in Example 1. In these examples one often takes $\mathbb{K} = \mathbb{Z}$, but other rings are allowed as well.

(b) Group or more general semigroup algebras for finite groups or semigroups.

(c) There are many interesting infinite examples coming from skein theory, see e.g. [Thu14].

Decategorifications are our main examples where $\mathbb{R}_{\geq 0}$ can be replaced by $\mathbb{Z}_{\geq 0}$.

A finite based $\mathbb{R}_{\geq 0}$-algebra is actually a pair (R, C), but we will write R for short. Next, fix such an R and $c \in \mathbb{R}_{\geq 0}C$. In this setting we can define the *(pre) action matrix* $M'(c)_{k,j} = \sum_i a_i m_{i,j}^k \in \mathbb{R}_{\geq 0}$. The *action matrix* $M(c)$ is then the adjacency matrix for the connected component, in the nonoriented sense, of the identity $1 \in C$ in the adjacency graph of $M'(c)$. Note that $M(c) \in \mathrm{Mat}_m(\mathbb{R}_{\geq 0})$ is a submatrix of $M'(c) \in \mathrm{Mat}_r(\mathbb{R}_{\geq 0})$ for some $1 \leq m \leq r$.

We give three versions of *MT*, stated in terms of finite based $\mathbb{R}_{\geq 0}$-algebras. The categorical version then follows immediately from Lemma 1.

Theorem 6 (Version 1)**.** *Fix a finite based* $\mathbb{R}_{\geq 0}$*-algebra* R*, and* c *a* $\mathbb{R}_{\geq 0}$*-linear combination of elements from* C*. Assume that the action matrix* $M(c)$ *is* ***irreducible****. Then Theorem 1 holds with* $a(n)$ *as in* (1A.2).

Proof. Consider the following matrix equation:

$$M(c)c(n-1) = c(n),$$

where $c(k) = \big(c_0(k), \dots, c_{r-1}(k)\big) \in \mathbb{R}^r_{\geq 0}$ are vectors such that their ith entry is the multiplicity of c_i in c^k, and $c(0) = (1, 0, \dots, 0)^T$ with the one is in the slot of $c_0 = 1$. This equation holds by the definition of the action matrix. Iterating this process, we get

$$M(c)^n c(0) = c(n).$$

Note that $M(c)^n c(0)$ is the same as taking the first column of $M(c)^n$. Hence,

$$b^{R,c}(n) = M(c)^n[1]$$

in the notation of the introduction. Thus, Theorem 4 implies the result. □

Remark 4. Theorem 6 is sufficient for many example. Explicitly, Theorem 6 works for all *transitive* finite based $\mathbb{R}_{\geq 0}$-algebras. Examples include all finite monoidal categories that are rigid by [EGNO15, Proposition 4.5.4].

We say that $M \in \mathrm{Mat}_m(\mathbb{R}_{\geq 0})$ has the *Perron–Frobenius property* if its Perron–Frobenius eigenvalue has multiplicity one.

Theorem 7 (Version 2)**.** *Fix a finite based $\mathbb{R}_{\geq 0}$-algebra R, and c an $\mathbb{R}_{\geq 0}$-linear combination of elements from C. Assume that the action matrix $M(c)$ has the* ***Perron–Frobenius property****. Then Theorem 1 holds with $a(n)$ as in* (1A.2)*.*

Proof. The iteration works as in the proof of Theorem 6, so let us focus on the growth rate. We will use Theorem 5 for $s = 1$. This implies that $\nu = 1$, by its definition. In particular, we only have $S^i(n)$ with entries of degree zero, so these are matrices that do not depend on n, so we can simply write S^i. We will argue that they are essentially the matrices $v_i w_i^T$.

Precisely, as follows from [Rot81, Section 5], we have

$$\begin{aligned} S^i &= v_0 w_0^T + v_1 w_1^T \cdot \zeta^i + v_2 w_2^T \cdot \zeta^{2i} + \cdots + v_{h-1} w_{h-1}^T \cdot \zeta^{(h-1)i} \\ &= v_0 w_0^T + v_1 w_1^T \cdot \zeta^{nh+i} + v_2 w_2^T \cdot \zeta^{2(nh+i)} + \cdots + v_{h-1} w_{h-1}^T \cdot \zeta^{(h-1)(nh+i)i}. \end{aligned}$$

Now we apply Theorem 5.(c). □

Remark 5. Theorem 7 is the version we used in Section 2.

Recall that the polynomials $S^i(n)$ are explicitly given in [Rot81, Section 5] and define:

$$a(n) = \frac{1}{h} \sum_{i=0}^{h-1} \sum_{j=0}^{h-1} S^j\big(\lfloor n/h \rfloor\big) \cdot \zeta^{i(n-j)}. \tag{3B.1}$$

Theorem 8 (Version 3)**.** *Fix a finite based $\mathbb{R}_{\geq 0}$-algebra R, and c an $\mathbb{R}_{\geq 0}$-linear combination of elements from C. Then Theorem 1 holds with $a(n)$ as in* (3B.1)*.*

Proof. Observing that $1 + \zeta^i + \cdots + \zeta^{(h-1)i} = 0$ if $i \not\equiv 0 \bmod h$, this follows as for the previous theorems. □

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