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EXISTENCE OF THE SOLUTIONS TO THE BROCARD–RAMANUJAN PROBLEM FOR NORM FORMS

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ABSTRACT. The Brocard–Ramanujan problem, which is an unsolved problem in number theory, is to find integer solutions (x, ℓ) of $x^2 - 1 = \ell!$. Many analogs of this problem are currently being considered. As one example, it is known that there are at most only finitely many algebraic integer solutions (x, ℓ), up to a unit factor, to the equations $N_K(x) = \ell!$, where N_K are the norms of number fields $K/\mathbf{Q}$. In this paper, we construct infinitely many number fields K such that $N_K(x) = \ell!$ has at least 22 solutions for positive integers ℓ.

1. INTRODUCTION

Brocard and Ramanujan independently considered the problem of determining all integer solutions (x, ℓ) of $x^2 - 1 = \ell!$ and conjectured that the only solutions are $(x, \ell) = (5, 4), (11, 5)$, and $(71, 7)$ [3,4,15]. As a generalization, it has been proposed that there are only finitely many solutions of the polynomial-factorial Diophantine equation

$$P(x) = \ell!, \tag{1.1}$$

where $P(x)$ is a polynomial of degree 2 or more with integer coefficients. The generalized Brocard–Ramanujan problem excludes the case $\deg P = 1$. In that case, we can observe that if $a_1 | a_0$, then equation $a_1 x + a_0 = \ell!$ has infinitely many solutions (x, ℓ), and otherwise has only finitely many solutions.

The Oesterlé–Masser conjecture, also known as the abc-conjecture, implies that polynomial-factorial equations (1.1) have only finitely many solutions. To explain the statement of the Oesterlé–Masser conjecture, we define the algebraic radical. For any nonzero integer n, the algebraic radical $\mathrm{rad}(n)$ is defined by

$$\mathrm{rad}(n) = \prod_{p|n} p,$$

where p runs through the prime factors of n. The Oesterlé-Masser conjecture states that for any $\varepsilon > 0$, there exists a positive constant $\beta(\varepsilon)$ such that

$$\max\{|a|, |b|, |c|\} < \beta(\varepsilon)\mathrm{rad}(abc)^{1+\varepsilon}$$

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for any triple (a, b, c) of non-zero coprime integers with $a + b = c$ [11, 12]. In the following, we summarize results from applying this conjecture to polynomial-factorial Diophantine equations.

First, Overholt showed that the weak form of Szpiro's conjecture implies that $x^2 - 1 = \ell!$ has only finitely many solutions [13]. We note that the Oesterlé-Masser conjecture implies the weak form of Szpiro's conjecture. More generally, Dąbrowski showed that if the weak form of Szpiro's conjecture is true, then for any integer A, the equation $x^2 - A = \ell!$ has only finitely many integer solutions (x, ℓ) [6]. He also showed that when A is not the square of an integer, this result becomes unconditional. As a generalization of these results, Luca showed that for any polynomial $P(x) \in \mathbf{Z}[x]$ with degree ≥ 2, the Oesterlé–Masser conjecture implies that the equation $P(x) = \ell!$ has only finitely many solutions (x, ℓ) [10].

Assuming an effective version of the Oesterlé–Masser conjecture, Browkin found that all the solutions to $x^2 - 1 = \ell!$ are the three conjectured ones given above. More precisely, if

$$\max\{|a|, |b|, |c|\} < \mathrm{rad}(abc)^{1.8}$$

for any triple (a, b, c) of nonzero coprime integers with $a + b = c$, then the complete list of solutions to $x^2 - 1 = \ell!$ are the conjectured ones, $(x, \ell) = (5, 4), (11, 5)$, and $(71, 7)$ [5].

There are many unconditional results. It is known that for $m \geq 3$, the equation $x^m + y^m = \ell!$ has no solution with $\gcd(x, y) = 1$ except for $(x, y, \ell) = (1, 1, 2)$ and the equation $x^m - y^m = \ell!$ has no solution with $\gcd(x, y) = 1$ except when $m = 4$ [7]. In 1973, Pollack and Shapiro showed that $x^4 - 1 = \ell!$ also has no solution [14]. We note that there are infinitely many solutions (x, y, ℓ) of $x^2 - y^2 = \ell!$. Indeed, for any $a \geq 4$, $(x, y, \ell) = (\frac{a!}{4} + 1, \frac{a!}{4} - 1, a)$ is a solution of $x^2 - y^2 = \ell!$.

Berend and Osgood showed that for any polynomial $P(x) \in \mathbf{Z}$ of degree 2 or more with integer coefficients, the equation $P(x) = \ell!$ has only a density 0 set of solutions ℓ [2], that is,

$$\lim_{n\to\infty} \frac{|\{\ell \leq n \mid \text{there exists } x \in \mathbf{Z} \text{ such that } P(x) = \ell!\}|}{n} = 0.$$

In 2006, Berend and Harmse considered several related problems. They showed that for any polynomial $P(x) \in \mathbf{Z}$ which is an irreducible polynomial or satisfies certain technical conditions, there exist only finitely many solutions of $P(x) = \ell!$ [1].

In a previous paper [17], we considered the number of solutions to the equation

$$\sum_{i=0}^{n} a_i x^i y^{n-i} = \ell!,$$

where $a_i \in \mathbf{Z}$. We dealt more generally with equations involving norm N_K of $K/\mathbf{Q}$ and the factorial operation. Let $\mathcal{O}_K$ be the ring of integers in number field K. Let $G(K)$ be the set of all complex embeddings σ from K to $\mathbf{C}$, that is, σ sending α_j to their conjugates and $\sigma(a) = a$ for all $a \in \mathbf{Q}$. Then the norm N_K is defined by

$$N_K(x) = \prod_{\sigma\in G(K)} \sigma(x).$$

We proved Theorem 1.1.

Theorem 1.1 ([17])**.** *The following hold.*

(1) *Let $F(x, y)$ be a homogeneous polynomial with* $\deg F \geq 2$. *If F is an irreducible polynomial or satisfies certain technical conditions, then there exist at most only finitely many ℓ such that $\ell!$ is represented by $F(x, y)$.*
(2) *There exist at most only finitely many ℓ such that $\ell!$ is represented by $N_K(x)$ with $x \in \mathcal{O}_K$.*

In this article, we study whether the set of all integer solutions to $N_K(x) = \ell!$ is empty. Let $S(K)$ be the set of all integers $\ell \geq 2$ such that there exists an element $x \in \mathcal{O}_K$ with $N_K(x) = \ell!$. For any number field K, we have $N_K(1) = 1!$. Therefore, we remove this from $S(K)$.

An outline for the rest of this article is as follows. In Section 2, we make some observations and prove that for a fixed integer $n \geq 2$, there are infinitely many number fields K with $[K : \mathbf{Q}] = n$ such that $S(K) \neq \emptyset$. In Section 3, we focus on radical extensions and compose solutions, and show that there are infinitely many number fields K with $\#S(K) \geq 21$. More precisely, we prove Theorem 1.2.

Theorem 1.2. *[Theorem 3.1] Let $K = \mathbf{Q}(\sqrt[n]{2})$. For any positive odd integers n relatively prime to 30, it holds that*

$$\{\ell \mid 2 \leq \ell \leq 22\} \subset S(K).$$

After we prove Theorem 1.2, we observe that there may exist many solutions to $N_K(x) = \ell!$ for radical fields $K = \mathbf{Q}(\sqrt[n]{a})$. In the appendix, we note that one can improve the upper bound of solutions to our equations by using the effective version of the Chebotarev density theorem shown by [18]. This improves a Bertrand-type estimate for prime ideals used to prove Theorem 1.1 in [17]. By following the proof of Theorem 1.1 in [17], we give an improvement for the upper bound of solutions to $N_K(x) = \ell!$.

2. Infiniteness

In this section, we deal with specific number fields and prove that for fixed integer $n \geq 2$, there are infinitely many number fields K with $[K : \mathbf{Q}] = n$ and $S(K) \neq \emptyset$.

Theorem 2.1. *For fixed integer n, there exist infinitely many number fields K such that $[K : \mathbf{Q}] = n$ and $S(K) \neq \emptyset$.*

Proof. For positive integer $\ell \geq 2$, we consider the prime factorization $\ell! = m^n k$, where k is the n-th power-free part of $\ell!$. By the Bertrand-Chebyshev theorem, for $x \geq 2$, there exists a prime p in the interval $(x, 2x)$. Therefore, $\ell!$ is not a powerful number and $k > 1$. Let $K = \mathbf{Q}(\sqrt[n]{-k})$. Then we have $m\sqrt[n]{-k} \in \mathcal{O}_K$ and $N_K((-1)^n m\sqrt[n]{-k}) = \ell!$. Therefore, $S(K) \neq \emptyset$. One can verify that there are infinitely many k appearing as the n-th power-free part of some $\ell!$. This ensures that there exist infinitely many number fields K such that $[K : \mathbf{Q}] = n$ and $S(K) \neq \emptyset$. □

The above theorem shows that there are infinitely many number fields K with $S(K) \neq \emptyset$. Subsequently, we consider the number of solutions $\#S(K)$. From the norm being multiplicative, the greater the class number is, the less the number of solutions is. The following examples describe $S(K)$ for imaginary quadratic fields K with class numbers 1 and 3.

Example 2.2 (Imaginary quadratic fields with class number 1).

d	$S(\mathbf{Q}(\sqrt{-d}))$	d	$S(\mathbf{Q}(\sqrt{-d}))$
1	$\{2,6\}$	7	$\{10,11\}$
2	$\{3,4\}$	11	$\{6\}$
3	$\{10\}$	19	$\{6,7,10,11\}$

If $d = 43, 67$, or 163, then $S(\mathbf{Q}(\sqrt{-d})) = \emptyset$.

Example 2.3 (Imaginary quadratic fields with class number 3).

d	$S(\mathbf{Q}(\sqrt{-d}))$	d	$S(\mathbf{Q}(\sqrt{-d}))$
23	$\{3,4\}$	139	$\{7,10,11\}$
31	$\{6,7,8,9,10\}$	283	$\{11\}$
59	$\{6,7,10\}$	307	$\{11\}$
83	$\{10,11,12\}$		

If $d = 107, 211, 331, 379, 499, 547, 643, 883$, or 907, then $S(\mathbf{Q}(\sqrt{-d})) = \emptyset$.

Theorem 2.4. *There exist infinitely many real quadratic fields $K = \mathbf{Q}(\sqrt{d})$ such that $2 \in S(K)$.*

Proof. Using $k^2 - 2$, we construct infinitely many real quadratic fields $K = \mathbf{Q}(\sqrt{d})$ such that $2 \in S(K)$. We decompose $k^2 - 2 = m^2 n$, with n being the square-free part of $k^2 - 2$. As $k^2 - 2 \equiv 2, 3 \mod 4$, we note that $m^2 \equiv 1 \mod 4$ and $n \equiv 2, 3 \mod 4$. Also, the norm of $\mathbf{Q}(\sqrt{n})$ is $N_K(x + y\sqrt{n}) = x^2 - ny^2$. We confirm that $N_K(k + m\sqrt{n}) = 2 = 2!$. Therefore, it suffices to show that there are infinitely many n appearing as the square-free part of $k^2 - 2$ for some k.

The quadratic reciprocity law implies that if a power of an odd prime p is a divisor of $k^2 - 2$, then $p \equiv 1, 7 \mod 8$. Now, we denote by g a primitive root for p^2. Let r be an integer such that $g^{2r} \equiv 2 \mod p^2$. Then we have that $g^{2r+(p-1)} \equiv 2 \mod p$ and $g^{2r+(p-1)} \not\equiv 2 \mod p^2$. Thus, when $k = g^{r+\frac{p-1}{2}}$, the prime p divides $k^2 - 2$ exactly once and $p|n$. This shows the desired conclusion. ☐

Theorem 2.4 shows that there are infinitely many quadratic fields K such that the equation $N_K(x) = 2!$ has an algebraic integer solution. Applying a similar argument for odd n and $k^n - \ell!$ in the proof of Theorem 2.4, we can prove that there are infinitely many fields K with $[K : \mathbf{Q}] = n$ such that the equations of the form $N_K(x) = \ell!$ have an algebraic integer solution. We note that if $\ell!$ is large, $k^n - \ell!$ may be an n-powerful number for small k. For example, $17^2 - 5! = 13^2$ and $31^2 - 5! = 29^2$.

3. Norm for radical extension fields

The finite extension L/K is said to be a radical extension if L is obtained by adjoining a root of a polynomial $x^n - a \in K[x]$. In this section, we study $S(K)$ for radical extension fields K. First, we focus on the fields $\mathbf{Q}(\sqrt[n]{2})$.

Theorem 3.1. *Let $K = \mathbf{Q}(\sqrt[n]{2})$. For any positive odd integers n relatively prime to 30, it holds that*

$$\{\ell \mid 2 \leq \ell \leq 22\} \subset S(K).$$

Proof. From the norm being multiplicative, it suffices to show that for any prime $p \leq 19$, there exists an $x \in \mathcal{O}_K$ with $N_K(x) = p$. We can first find that $N_K(\sqrt[n]{2}) = 2$. Since $N_K(x + \sqrt[n]{2^k}) = x^k + 2^k$, we have $N_K(1 + \sqrt[n]{2}) = 3, N_K(1 + \sqrt[n]{4}) = 5, N_K(-1 + \sqrt[n]{8}) = 7$, and $N_K(1 + \sqrt[n]{16}) = 17$. Combining $N_K(1 + \sqrt[n]{2}^5) = 33$ with $N_K(1 + \sqrt[n]{2}) = 3$, we obtain $N_K(1 - \sqrt[n]{2} + \sqrt[n]{4} - \sqrt[n]{8} + \sqrt[n]{16}) = 11$. Similarly, it holds that $N_K(1 - \sqrt[n]{4} + \sqrt[n]{16}) = 13$ by $N_K(1 + \sqrt[n]{4}^3) = 65$ and $N_K(1 + \sqrt[n]{4}) = 5$. From $N_K(1 + \sqrt[n]{8}^3) = 513$ and $N_K(1 + \sqrt[n]{8}) = 9$, we finally confirm that $N_K(1 - \sqrt[n]{8} + \sqrt[n]{64}) = 57 = 3 \cdot 19$. Therefore, we verify that for any integer $2 \leq \ell \leq 22$, the factorial $\ell!$ is represented as N_K. We conclude the assertion. □

Theorem 3.1 implies that there are infinitely many number fields K with $\#S(K) \geq 21$. For $K = \mathbf{Q}(\sqrt[7]{2})$, we confirm that $N_K(1 + \sqrt[7]{4} + \sqrt[7]{32}) = 23$. From the norm being multiplicative and Theorem 3.1, we obtain Theorem 3.2.

Theorem 3.2. *Let* $K = \mathbf{Q}(\sqrt[7]{2})$. *Then*

$$\{\ell \mid 2 \leq \ell \leq 28\} \subset S(K)$$

and $29 \notin S(K)$.

We will consider Theorem 3.1 and Theorem 3.2 in detail, but before that we revisit a basic result of Dedekind about the prime ideal factorization.

Theorem 3.3 (Dedekind)**.** *Let* $K = \mathbf{Q}(\alpha)$ *be a number field with* $\alpha \in \mathcal{O}_K$. *We denote by* $f(x) \in \mathbf{Z}[x]$ *the minimal polynomial of* α. *For any prime* $p \nmid [\mathcal{O}_K : \mathbf{Z}[\alpha]]$, *we decompose* f *into irreducible factors*

$$f(x) = \prod_{i=1}^{r} f_i(x)^{e_i} \mod p,$$

where the f_i *are distinct monic irreducible polynomials in* $\mathbf{F}_p[x]$. *Then the prime ideal factorization of* $p\mathcal{O}_K$ *is*

$$p\mathcal{O}_K = \prod_{i=1}^{r} \mathfrak{p}_i^{e_i}. \tag{3.1}$$

In particular, if $\mathcal{O}_K = \mathbf{Z}[\alpha]$, *the above factorization* (3.1) *holds for all* p.

Let $K = \mathbf{Q}(\sqrt[n]{a})$ with $[K : \mathbf{Q}] = n$ and $D(a, n) = [\mathcal{O}_K : \mathbf{Z}[\sqrt[n]{a}]]$. For any prime p with $\gcd(p-1, n) = 1$, there exists $b \in \mathbf{Z}$ with

$$b^n - a \equiv 0 \mod p.$$

We can check this by applying $\#(\mathbf{Z}/p\mathbf{Z})^\times = p-1$. From Theorem 3.3, for any prime $p \nmid D(a, n)$, there exists a prime ideal $\mathfrak{p}$ lying above p with ideal norm $\mathfrak{N}\mathfrak{p} = p$. Since the absolute value of discriminant of $\mathbf{Z}[\sqrt[n]{a}]$ is $a^{n-1}n^n$, if $p | D(a, n)$, then $p | an$. Thus, we conclude Theorem 3.4.

Theorem 3.4. *Let* a, n *be integers. If the class number of* $K = \mathbf{Q}(\sqrt[n]{a})$ *is* 1, *then*

$$\{\ell \mid 2 \leq \ell \leq P - 1\} \subset S(K),$$

where $P = P(n, a)$ *is the minimum of* $S_1(n, a) \cup S_2(n, a) \cup S_3(n, a)$, *in which*

$$S_1(n, a) = \{p : prime \mid \gcd(p-1, n) > 1\};$$

$$S_2(n, a) = \{p : prime \mid p^2 | a\};$$

$$S_3(n, a) = \{p : prime \mid p | n, a^{p-1} \equiv 1 \mod p^2\}.$$

Proof. As we remarked above, for any prime $p \leq P-1$ coprime to an, there exists a prime ideal $\mathfrak{p}$ lying above p with $\mathfrak{N}\mathfrak{p} = p$. By the assumption, $\mathfrak{p}$ is a principal ideal. Therefore, there exists an element $x \in \mathcal{O}_K$ such that $\mathfrak{p} = x\mathcal{O}_K$ and $N_K(x) = p$.

Next, we fix a prime $p \leq P-1$ with $p|a$. Since p divides a only once, the polynomial $x^n - a$ is an Eisenstein polynomial at p and p ramifies totally in $K = \mathbf{Q}(\sqrt[n]{a})$ [8, Theorem 24]. Thus, there exists a prime ideal $\mathfrak{p} = x\mathcal{O}_K$ lying above p with $N_K(x) = p$.

Finally, dealing only with primes $p|n$ such that $\gcd(p,a) = 1$ and $a^{p-1} \not\equiv 1 \mod p^2$, we fix such a prime $p \leq P-1$. We assume $p^r|n$ and $p^{r+1} \nmid n$. Combining these with the fact that the inertia degree is multiplicative, it suffices to show that p ramifies totally in $K = \mathbf{Q}(\sqrt[p^r]{a})$. Since $a^{p^r} \equiv a^p \mod p^2$ and $a^p \not\equiv a \mod p^2$, the polynomial $(x+a)^{p^r} - a = x^n + \cdots + a^{p^r} - a$ is an Eisenstein polynomial at p. As above, there exists a prime ideal $\mathfrak{p} = x\mathcal{O}_K$ lying above p with $N_K(x) = p$.

Hence, as the norm N_K is multiplicative, for $\ell \leq P-1$, there exists an $x \in \mathcal{O}_K$ such that $N_K(x) = \ell!$. □

A prime p satisfying $a^{p-1} \equiv 1 \mod p^2$ is called a Wieferich prime to base a. It has been conjectured that infinitely many Wieferich primes to base a exist for each positive integer a, but the only known Wieferich primes to base 2 are 1093 and 3511.

Example 3.5. Let $K = \mathbf{Q}(\sqrt[p]{2})$. Then the following hold.

(1) If $p = 17$, then $\{\ell \mid 2 \leq \ell \leq 102\} \subset S(K)$.
(2) If $p = 19$, then $\{\ell \mid 2 \leq \ell \leq 190\} \subset S(K)$.

We note that when $p = 17, 19$, the class number of $\mathbf{Q}(\sqrt[p]{2})$ is 1. Therefore, we can apply Theorem 3.4 to deduce the above results.

Appendix A. Solution-free region

In this appendix, we improve the upper bound of solutions to $N_K(x) = \ell!$ given in [17].

Theorem A.1 (cf. [17, Theorem 5.2])**.** *Let n and D be the degree and discriminant of K^{gal}. There exists an effectively computable constant $c > 0$ such that there is no solution to $N_K(x) = \ell!$ in*

$$\ell > \exp(cn(\log|D|)^2).$$

This bound is due to a Bertrand–Chebyshev type estimate for prime ideals corresponding to fixed conjugacy class of $\mathrm{gal}(K^{\mathrm{gal}}/\mathbf{Q})$. We used effective versions of the Chebotarev density theorem given by Lagarias and Odlyzko and obtained Bertrand-Chebyshev type estimates by a similar argument to [9,16]. Recently, Thorner and Zaman improved the Lagarias-Odlyzko result in [18], and we note that one can obtain a better upper bound of solutions to $N_K(x) = \ell!$ by using their result. We first prepare some notation to explain their result. Let L/K be a Galois extension of number fields with Galois group G. Then, for each conjugacy class C of G, we define $\pi_C(x)$ by

$$\pi_C(x) = \#\{\mathfrak{p} \subset \mathcal{O}_K \mid \mathfrak{p} \text{ is unramified in } L, [(\mathfrak{p}, L/K)] = C, \mathfrak{N}\mathfrak{p} \leq x\},$$

where $[(\mathfrak{p}, L/K)]$ is the conjugacy class of the Frobenius map corresponding to $\mathfrak{p}$.

Theorem A.2 ([18, Theorem 1.1]). *Let L/K be a Galois extension of number fields with Galois group G and $[L:\mathbf{Q}] = n_L$ and let D_L be the absolute value of the discriminant of L. Then there exist effectively computable positive constants c_1, c_2, and c_3 such that if $x > (D_L n_L^{n_L})^{c_2}$, then*

$$|\pi_C(x) - M(x)| \le M(x)E(x),$$

where

$$M(x) = \frac{|C|}{|G|}\left(\mathrm{Li}(x) - (-1)^{\varepsilon}\,\mathrm{Li}(x^{\beta})\right),$$

and

$$E(x) = c_1\left(\exp\left(-\frac{c_3 \log x}{\log(D_L n_L^{n_L})}\right) + \exp\left(-\sqrt{\frac{c_3 \log x}{n_L}}\right)\right).$$

We note that $\mathrm{Li}(x^{\beta})$ is only defined if there exists an exceptional real zero β of $\zeta_L(s)$. Also $\varepsilon = 0$ or 1 depending on L, K, and C.

One can show that $M(x)$ is a positive increasing function for all $x > 2$ and $E(x)$ is a positive decreasing function for all $x > 2$. In particular, we have Corollary A.3.

Corollary A.3. *With the same notation as in Theorem A.2, there exists a constant $c_4 > 0$ such that for $x > (D_L n_L^{n_L})^{c_4 \log\log(D_L n_L^{n_L})}$ we have*

$$|\pi_C(x) - M(x)| \le M(x)E(x) \text{ and } E(x) \ll \frac{1}{(\log x)^2}.$$

By using Corollary A.3, we obtain the following Bertrand-Chebyshev type estimate for a fixed conjugacy class.

Theorem A.4. *Let $L/\mathbf{Q}$ be a Galois extension with $[L:\mathbf{Q}] = k$ and D be the absolute value of the discriminant of L. For any $A > 1$, there exists an effectively computable constant $c(A) > 0$ depending only on A such that for $x > (Dk^k)^{c(A)\log\log(Dk^k)}$, there is a prime corresponding to a conjugacy class C of $\mathrm{gal}(L/\mathbf{Q})$ with $p \in (x, Ax]$.*

Proof. From Corollary A.3, for $x > (D_L n_L^{n_L})^{c_4 \log\log(D_L n_L^{n_L})}$, we get

$$\begin{aligned}\pi_C(Ax) - \pi_C(x) &\ge (M(Ax) - M(Ax)E(Ax)) - (M(x) + M(x)E(x))\\ &\ge (M(Ax) - M(x)) - 2M(Ax)E(x), \qquad \text{(A.1)}\end{aligned}$$

since $M(x)$ is a positive increasing function and $E(x)$ is a positive decreasing function for $x \ge 2$. For simplicity, we assume $\varepsilon = 0$. Then this implies that

$$M(Ax) - M(x) = \frac{|C|}{|G|}\left(\int_x^{Ax} \frac{dt}{\log t} - \int_{x^{\beta}}^{(Ax)^{\beta}} \frac{dt}{\log t}\right) = \frac{|C|}{|G|}\int_x^{Ax} \frac{1 - t^{\beta-1}}{\log t}\,dt. \qquad \text{(A.2)}$$

On the other hand, it follows that

$$\begin{aligned}M(Ax) &\le \frac{|C|}{|G|}\left(\int_2^{Ax} \frac{dt}{\log t} - \int_{2^{\beta}}^{(Ax)^{\beta}} \frac{dt}{\log t} + \int_{2^{\beta}}^{2} \frac{dt}{\log t}\right)\\ &\le \frac{|C|}{|G|}\left(\int_2^{Ax} \frac{1 - t^{\beta-1}}{\log t}\,dt + \frac{2 - 2^{\beta}}{\beta \log 2}\right). \qquad \text{(A.3)}\end{aligned}$$

Combining (A.2) and (A.3) with inequality (A.1), we find that

(A.4)

$$\frac{|G|}{|C|}(\pi_C(Ax)-\pi_C(x)) \geq \int_x^{Ax}\frac{1-t^{\beta-1}}{\log t}dt - 2E(x)\left(\int_2^{Ax}\frac{1-t^{\beta-1}}{\log t}dt + \frac{2-2^{\beta}}{\beta\log 2}\right).$$

It suffices to show that the right-hand side is positive for a sufficiently large x. Changing the variable by $t=e^u$, we have

$$\begin{aligned}\int_x^{Ax}\frac{1-t^{\beta-1}}{\log t}dt &= \int_{\log x}^{\log Ax}\frac{e^u-e^{\beta u}}{u}du\\ &\geq \int_{\log x}^{\log Ax}(1-\beta)\frac{e^u-1}{u}du\\ &\geq (1-\beta)\frac{(A-1)x-\log A}{\log Ax}\end{aligned}$$

and

$$\int_2^{Ax}\frac{1-t^{\beta-1}}{\log t}dt = \int_{\log 2}^{\log Ax}\frac{e^u-e^{\beta u}}{u}du \leq \int_{\log 2}^{\log Ax}(1-\beta)e^u du \leq (1-\beta)Ax.$$

As $\frac{2-2^{\beta}}{\beta\log 2} \leq 4(1-\beta)$ for $\frac{1}{2}\leq\beta\leq 1$, it follows by inequality (A.4) and Corollary A.3 that

$$\begin{aligned}\frac{|G|}{|C|}(\pi_C(Ax)-\pi_C(x)) &\geq (1-\beta)\left(\frac{(A-1)x-\log A}{\log Ax} - 2E(x)(Ax+4)\right)\\ &= (1-\beta)\left(\frac{(A-1)x}{\log Ax} + O_A\left(\frac{x}{(\log x)^2}\right)\right) > 0.\end{aligned}$$

This completes the proof of the theorem in the case $\varepsilon = 0$.

In the case $\varepsilon=1$, as a similar argument as shown above and a simple calculation lead to

$$\begin{aligned}\frac{|G|}{|C|}(\pi_C(Ax)-\pi_C(x)) &\geq \int_x^{Ax}\frac{1+t^{\beta-1}}{\log t}dt - 2E(x)\left(\int_x^{Ax}\frac{1+t^{\beta-1}}{\log t}dt - \frac{2-2^{\beta}}{\log 2}\right)\\ &\geq \int_x^{Ax}\frac{1-t^{\beta-1}}{\log t}dt - 2E(x)\left(\int_x^{Ax}\frac{1-t^{\beta-1}}{\log t}dt + \frac{2-2^{\beta}}{\log 2}\right)\end{aligned}$$

for sufficiently large x. Therefore, by applying the same argument with the case $\varepsilon=0$, we can confirm that $\pi_C(Ax)-\pi_C(x)>0$ for sufficiently large x independent of D and k.

This proves Theorem A.5. □

Following the proof of the finiteness of the solutions in [17, Theorem 4.1 and Theorem 5.2], we obtain the following theorem.

Theorem A.5. *Let n and D be the degree and discriminant of K^{gal}. There exists an effectively computable constant $c>0$ such that no solution to $N_K(x)=\ell!$ exists in*

$$\ell > (Dn^n)^{c\log\log(Dn^n)}.$$

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