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EQUIVALENCE BETWEEN THE ENERGY DECAY OF FRACTIONAL DAMPED KLEIN–GORDON EQUATIONS AND GEOMETRIC CONDITIONS FOR DAMPING COEFFICIENTS

KOTARO INAMI AND SOICHIRO SUZUKI

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ABSTRACT. We consider damped s-fractional Klein–Gordon equations on $\mathbb{R}^d$, where s denotes the order of the fractional Laplacian. In the one-dimensional case $d = 1$, Green (2020) established that the exponential decay for $s \geq 2$ and the polynomial decay of order $s/(4-2s)$ hold if and only if the damping coefficient function satisfies the so-called geometric control condition. In this note, we show that the $o(1)$ energy decay is also equivalent to these conditions in the case $d = 1$. Furthermore, we extend this result to the higher-dimensional case: the logarithmic decay, the $o(1)$ decay, and the thickness of the damping coefficient are equivalent for $s \geq 2$. In addition, we also prove that the exponential decay holds for $0 < s < 2$ if and only if the damping coefficient function has a positive lower bound, so in particular, we cannot expect the exponential decay under the geometric control condition.

1. INTRODUCTION

We consider the following fractional damped Klein–Gordon equations on $\mathbb{R}^d$:

$$u_{tt}(t,x) + \gamma(x)u_t(t,x) + (-\Delta+1)^{s/2}u(t,x) = 0, \quad (t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^d, \tag{1.1}$$

where $s > 0$, and $0 \leq \gamma \in L^\infty(\mathbb{R}^d)$. Here we note that γu_t represents the damping force and the operator $(-\Delta+1)^{s/2}$ is defined by the Fourier transform on $L^2(\mathbb{R}^d)$;

$$(-\Delta+1)^{s/2}u \coloneqq \mathcal{F}^{-1}(|\xi|^2+1)^{s/2}\mathcal{F}u, \quad \xi \in \mathbb{R}^d.$$

We recast equation (1.1) as an abstract first-order equation for $U = (u, u_t)$:

$$U_t = \mathcal{A}_\gamma U, \quad \mathcal{A}_\gamma = \begin{pmatrix} 0 & I \\ -(-\Delta+1)^{s/2} & -\gamma(x) \end{pmatrix}, \tag{1.2}$$

then $\mathcal{A}_\gamma$ generates a C_0-semigroup $(e^{tA_\gamma})_{t\geq 0}$ on $H^{s/2}(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ (see [4]). Here the Sobolev space $H^r(\mathbb{R}^d)$ is defined by

$$H^r(\mathbb{R}^d) \coloneqq \left\{ u \in L^2(\mathbb{R}^d) : \|u\|_{H^r}^2 = \int_{\mathbb{R}^d} (|\xi|^2+1)^r |\mathcal{F}u(\xi)|^2 \, d\xi < \infty \right\}.$$

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In this paper, we discuss the decay rate of the energy

$$E(t) := \|e^{t\mathcal{A}_\gamma}(u(0), u_t(0))\|_{H^{s/2}\times L^2} = \left(\int_{\mathbb{R}^d} (|(-\Delta+1)^{s/4}u(t,x)|^2 + |u_t(t,x)|^2)\,dx\right)^{1/2}.$$

By standard calculus, we have $E(t) = E(0)$ if $\gamma \equiv 0$ and the exponential energy decay if $\gamma \equiv C > 0$. In recent works, the intermediate case, that is, the case that $\gamma = 0$ on a large set is studied:

Definition 1.1. We say that $\Omega \subset \mathbb{R}^d$ satisfies the *Geometric Control Condition* (GCC) if there exist $L > 0$ and $0 < c \leq 1$ such that for any line segments $l \in \mathbb{R}^d$ of length L, the inequality

$$\mathcal{H}^1(\Omega \cap l) \geq cL$$

holds, where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure.

Burq and Joly [2] proved that if γ is uniformly continuous and $\{\gamma \geq \varepsilon\}$ satisfies (GCC) for some $\varepsilon > 0$, then we have the exponential energy decay in the non-fractional case $s = 2$. After that, Malhi and Stanislavova [6] pointed out that (GCC) is also necessary for the exponential decay in the one-dimensional case $d = 1$:

Theorem 1.2 ([6, Theorem 1])**.** *Let $d = 1$, let $s = 2$, and let $0 \leq \gamma \in L^\infty(\mathbb{R})$ be continuous. Then the following conditions are equivalent:*

(1.3) *There exists $\varepsilon > 0$ such that the upper level set $\{\gamma \geq \varepsilon\}$ satisfies* (GCC).

(1.4) *There exist $C, \omega > 0$ such that whenever $(u(0), u_t(0)) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$,*

$$E(t) \leq C \exp(-\omega t) E(0)$$

holds for any $t \geq 0$.[1]

(1.5) $\lim_{t\to+\infty} \|e^{tA_\gamma}\|_{H^2\times H^1 \to H^1 \times L^2} = 0.$

Note that for $0 \leq \gamma \in L^\infty(\mathbb{R})$, condition (1.3) is also equivalent to that there exists $R > 0$ such that

$$\inf_{a\in\mathbb{R}} \int_{a-R}^{a+R} \gamma(x)\,dx > 0.$$

In another paper [7], Malhi and Stanislavova introduced the fractional equation (1.1) and showed that if γ is periodic, continuous and not identically zero, then we have the exponential decay for any $s \geq 2$ and the polynomial decay of order $s/(4-2s)$ for any $0 < s < 2$ in the case $d = 1$.

Remark. Nonzero periodic functions satisfy (GCC) in the case $d = 1$, but it is not true in the higher-dimensional case $d \geq 2$. Wunsch [10] showed that continuous periodic damping gives the polynomial energy decay of order $1/2$ for the non-fractional equation in the case $d \geq 2$. In addition, recently another proof and an extension to fractional equations of Wunsch's result were obtained by Täufer [9] and Suzuki [8], respectively. Note that these results for periodic damping are

[1] To be precise, the exponential decay estimate given in [6, Theorem 1] is a little weaker: $E(t) \leq C\exp(-\omega t)\|(u(0), u_t(0))\|_{H^2\times H^1}$. However, this is because the Gearhart–Prüss theorem in their paper ([6, Theorem 2]) is stated incorrectly. Using the theorem correctly (see Theorem 3.1), one can obtain the exponential decay estimate $E(t) \leq C\exp(-\omega t)E(0)$ as in (1.4).

established by reducing to estimates on the torus $\mathbb{T}^d$. Indeed, there are numerous studies on bounded domains; see references in [2] and [3], for example.

Green [4] improved results of Malhi and Stanislavova as follows:

Theorem 1.3 ([4, Theorem 1])**.** *Let $d = 1$, let $s > 0$ and let $0 \le \gamma \in L^\infty(\mathbb{R})$. Then the following conditions are equivalent:*

(1.3) *There exists $\varepsilon > 0$ such that the upper level set $\{\gamma \ge \varepsilon\}$ satisfies* (GCC).

(1.6) *There exist $C, \omega > 0$ such that whenever $(u(0), u_t(0)) \in H^s(\mathbb{R}) \times H^{s/2}(\mathbb{R})$,*

$$E(t) \le \begin{cases} (1+t)^{-\frac{s}{4-2s}} \|(u(0), u_t(0))\|_{H^s \times H^{s/2}} & \text{if} \quad 0 < s < 2, \\ C \exp(-\omega t) E(0) & \text{if} \quad s \ge 2 \end{cases}$$

holds for any $t \ge 0$.

In comparison with the result of [7], which states that (1.6) holds if γ is periodic, continuous and not identically zero, Theorem 1.3 refines this result by giving a necessary and sufficient condition for (1.6). Furthermore, Theorem 1.3 also improves the (1.3) $\iff$ (1.4) part of Theorem 1.2 by extending it to fractional equations and removing the continuity of γ, but on the other hand, it lacks the (1.5) $\implies$ (1.3), (1.4) part. One of our goal is to recover this part for fractional equations:

Theorem 1.4. *Let $d = 1$, let $s > 0$, and let $0 \le \gamma \in L^\infty(\mathbb{R})$. Then the following conditions are equivalent:*

(1.3) *There exists $\varepsilon > 0$ such that the upper level set $\{\gamma \ge \varepsilon\}$ satisfies* (GCC).

(1.6) *There exist $C, \omega > 0$ such that whenever $(u(0), u_t(0)) \in H^s(\mathbb{R}) \times H^{s/2}(\mathbb{R})$,*

$$E(t) \le \begin{cases} C(1+t)^{\frac{-s}{4-2s}} \|(u(0), u_t(0))\|_{H^s \times H^{s/2}} & \text{if} \quad 0 < s < 2, \\ C \exp(-\omega t) E(0) & \text{if} \quad s \ge 2 \end{cases}$$

holds for any $t \ge 0$.

(1.7) $\lim_{t \to +\infty} \|e^{tA_\gamma}\|_{H^s \times H^{s/2} \to H^{s/2} \times L^2} = 0.$

We also give the following result, which says that we cannot expect the exponential decay for $0 < s < 2$ under (GCC).

Theorem 1.5. *Let $d \ge 1$, let $0 < s < 2$, and let $0 \le \gamma \in L^\infty(\mathbb{R}^d)$. Then there exist $C, \omega > 0$ such that whenever $(u(0), u_t(0)) \in H^{s/2}(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$,*

$$E(t) \le C \exp(-\omega t) E(0)$$

holds for any $t \ge 0$ if and only if $\operatorname{ess\,inf}_{\mathbb{R}^d} \gamma > 0$.

Note that the "if" part easily follows by reducing to the constant damping case, so we will prove the "only if" part. Furthermore, we extend Theorem 1.4 to the higher-dimensional case $d \ge 2$ using a notion of *thickness*, which is equivalent to (GCC) in the case $d = 1$:

Definition 1.6. We say that a set $\Omega \subset \mathbb{R}^d$ is *thick* if there exists $R > 0$ such that

$$\inf_{a \in \mathbb{R}^d} m_d(\Omega \cap (a + [-R, R]^d)) > 0$$

holds, where m_d denotes the d-dimensional Lebesgue measure.

Then we have the following result:

Theorem 1.7. *Let $d \geq 2$, let $s \geq 2$, and let $0 \leq \gamma \in L^\infty(\mathbb{R}^d)$. Then the following conditions are equivalent:*

(1.8) *There exists $\varepsilon > 0$ such that the upper level set $\{\gamma \geq \varepsilon\}$ is thick.*

(1.9) *There exists $C > 0$ such that whenever $(u(0), u_t(0)) \in H^s(\mathbb{R}^d) \times H^{s/2}(\mathbb{R}^d)$,*

$$E(t) \leq \frac{C}{\log(e+t)} \|(u(0), u_t(0))\|_{H^s \times H^{s/2}}$$

holds for any $t \geq 0$.

(1.10) $\lim_{t\to+\infty} \|e^{tA_\gamma}\|_{H^s\times H^{s/2}\to H^{s/2}\times L^2} = 0.$

The implication (1.8) $\Longrightarrow$ (1.9) is a generalization of the result given by Burq and Joly [2]. They established (1.9) under the so-called network control condition, which is stronger than (1.8). Also, similar to the case $d = 1$, condition (1.8) is equivalent to that there exists $R > 0$ such that

$$\inf_{a\in\mathbb{R}} \int_{a+[-R,R]^d} \gamma(x)\, dx > 0.$$

Finally, we explain the organization of this paper. In Sections 2, 3, and 4, we will give proofs of Theorems 1.4, 1.5, and 1.7, respectively. To prove these theorems, we use a kind of uncertainty principle and results of the C_0 semigroup theory.

2. Proof of Theorem 1.4

To prove this theorem, we use the following result by Batty, Borichev, and Tomilov [1]:

Theorem 2.1 ([1, Theorem 1.4]). *Let A be a generator of a bounded C_0-semigroup $(e^{tA})_{t\geq 0}$ on a Banach space X, and $\lambda \in \rho(A)$. Then the following are equivalent:*

(2.1) $\sigma(A) \cap i\mathbb{R} = \emptyset$,

(2.2) $\lim_{t\to\infty} \|e^{tA}(\lambda - A)^{-1}\|_{\mathcal{B}(X)} = 0.$

In the case $A = \mathcal{A}_\gamma$, for $\lambda \in \rho(\mathcal{A}_\gamma)$, the map $(\lambda - \mathcal{A}_\gamma)^{-1} : H^{s/2}(\mathbb{R}) \times L^2(\mathbb{R}) \to H^s(\mathbb{R}) \times H^{s/2}(\mathbb{R})$ is surjective. Thus, we have:

Lemma 2.2 ([6, Corollary 2]). *For the semigroup $e^{t\mathcal{A}_\gamma}$ of the Cauchy problem* (1.2)*, the following are equivalent:*

(2.3) $\sigma(\mathcal{A}_\gamma) \cap i\mathbb{R} = \emptyset$,

(2.4) $\lim_{t\to\infty} \|e^{t\mathcal{A}_\gamma}\|_{H^s\times H^{s/2}\to H^{s/2}\times L^2} = 0.$

Proof of Theorem 1.4. It is enough to show that (1.7) $\Longrightarrow$ (1.3), since (1.3) $\Longleftrightarrow$ (1.6) is already known by Green [4] (Theorem 1.3) and (1.6) $\Longrightarrow$ (1.7) is trivial. Suppose that (1.7) holds, that is, $\lim_{t\to+\infty} \|e^{tA_\gamma}\|_{H^s\times H^{s/2}\to H^{s/2}\times L^2} = 0$. By Lemma 2.2, we have $i\mathbb{R} \subset \rho(\mathcal{A}_\gamma)$. This implies that for each $\lambda \in \mathbb{R}$, there exists some $c_0 > 0$ such that

$$c_0\|U\|_{H^{s/2}\times L^2} \leq \|(\mathcal{A}_\gamma - i\lambda I)U\|_{H^{s/2}\times L^2}$$

holds for any $U \in H^s(\mathbb{R}) \times H^{s/2}(\mathbb{R})$. Letting $u \in L^2(\mathbb{R}^d)$ and $U = ((-\Delta + 1)^{-s/4}u, iu)$, we obtain

$$\begin{aligned} 2c_0\|u\|_{L^2}^2 &\leq \|((-\partial_{xx}+1)^{s/4} - \lambda)u\|_{L^2}^2 + \|((-\partial_{xx}+1)^{s/4} - \lambda + i\gamma)u\|_{L^2}^2 \\ &\leq 3\|((-\partial_{xx}+1)^{s/4} - \lambda)u\|_{L^2}^2 + 2\|\gamma u\|_{L^2}^2. \end{aligned}$$

Now we consider the case $\lambda = 1$. Let $u \in H^{s/2}(\mathbb{R})$ satisfy $\operatorname{supp}\widehat{u} \subset [-D, D]$ for some $D > 0$, which is chosen later. For such u, we have

$$\|((-\partial_{xx}+1)^{s/4}-1)u\|_{L^2}^2 = \int_{-D}^{D} \left[(|\xi|^2+1)^{s/4}-1\right]^2 |\widehat{u}(\xi)|^2 d\xi$$
$$\leq \left[(D^2+1)^{s/4}-1\right]^2 \|u\|_{L^2}^2.$$

Hence, taking $D > 0$ small enough, we get some $c > 0$ such that

$$c\|u\|_{L^2} \leq \|\gamma u\|_{L^2}$$

holds for any $u \in H^{s/2}(\mathbb{R})$ satisfying $\operatorname{supp}\widehat{u} \subset [-D, D]$. Fix $f \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$ such that $\operatorname{supp}\widehat{f} \subset [-D, D]$ and write $f_a(x) := f(x-a)$ for each $a \in \mathbb{R}$, so that $\widehat{f_a}(\xi) = e^{ia\xi}\widehat{f}(\xi)$. Then, for each $a \in \mathbb{R}$ and $R > 0$, we have

$$0 < c\|f\|_{L^2} = c\|f_a\|_{L^2} \leq \|\gamma f_a\|_{L^2} = \left(\int_{[a-R,a+R]} + \int_{[a-R,a+R]^c}\right) |\gamma(x) f_a(x)|^2\, dx.$$

The second integral goes to 0 as $R \to +\infty$ since γ is bounded and $|f_a|^2$ is integrable, and this convergence is uniform with respect to a. Furthermore, for the first integral, we have

$$\int_{a-R}^{a+R} |\gamma(x) f_a(x)|^2\, dx \leq \|\gamma\|_{L^\infty} \|f\|_{L^\infty}^2 \int_{a-R}^{a+R} \gamma(x)\, dx,$$

since γ and f are bounded and $\|f_a\|_{L^\infty} = \|f\|_{L^\infty}$. Thus, there exists $R > 0$ such that

$$\inf_{a\in\mathbb{R}} \int_{a-R}^{a+R} \gamma(x)\, dx > 0$$

holds, which is equivalent to (1.3). ☐

3. Proof of Theorem 1.5

This section is based on the proof of Theorem 2 in Green [4]. To prove this theorem, we use the classical semigroup result by Gearhart, Prüss, and Huang:

Theorem 3.1 (Gearhart–Prüss–Huang)**.** *Let X be a complex Hilbert space and let $(e^{tA})_{t\geq 0}$ be a bounded C_0-semigroup on X with infinitesimal generator A. Then there exist $C, \omega > 0$ such that*

$$\|e^{tA}\| \leq C\exp(-\omega t)$$

holds for any $t \geq 0$ if and only if $i\mathbb{R} \subset \rho(A)$ and $\sup_{\lambda\in\mathbb{R}} \|(i\lambda - A)^{-1}\|_{\mathcal{B}(X)} < \infty$.

Proof of Theorem 1.5. We will prove the contraposition of the "only if" part of Theorem 1.5, that is, if the energy decays exponentially and $\operatorname{ess\,inf}_{x\in\mathbb{R}^d} \gamma(x) = 0$ holds, then $s \geq 2$. By the Gearhart–Prüss–Huang theorem and the exponential decay, there exists $c_0 > 0$ such that

$$c_0\|U\|_{H^{s/2}\times L^2}^2 \leq \|(\mathcal{A}_\gamma - i\lambda I)U\|_{H^{s/2}\times L^2}^2$$

holds for any $U \in H^{s/2}(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and any $\lambda \in \mathbb{R}$. Letting $u \in L^2(\mathbb{R}^d)$ and $U = ((-\Delta+1)^{-s/4}u, iu)$, we obtain

$$2c_0\|u\|_{L^2}^2 \leq \|((-\Delta+1)^{s/4} - \lambda)u\|_{L^2}^2 + \|((-\Delta+1)^{s/4} - \lambda + i\gamma)u\|_{L^2}^2$$
$$\leq 3\|(-\Delta+1)^{s/4} - \lambda\|_{L^2}^2 + 2\|\gamma u\|_{L^2}^2.$$

Now let $u \in L^2(\mathbb{R}^d)$ satisfy

$$\operatorname{supp} \widehat{u} \subset \{\xi \in \mathbb{R}^d : |(|\xi|^2+1)^{s/4} - \lambda| \le K\} =: A_\lambda(K)$$

for some K, which is chosen later. For such u, we have

$$\begin{aligned}\|((-\Delta+1)^{s/4} - \lambda)u\|_{L^2}^2 &= \int_{A_\lambda(K)} [(|\xi|^2+1)^{s/4} - \lambda]^2 |\widehat{u}(\xi)|^2 \, d\xi \\ &\le K^2 \|u\|_{L^2}^2.\end{aligned}$$

Hence, taking $K>0$ small enough, we get some $c>0$ such that

$$c\|u\|_{L^2}^2 \le \|\gamma u\|_{L^2}^2 \tag{3.1}$$

holds for any $u \in L^2(\mathbb{R}^d)$ satisfying $\operatorname{supp} \widehat{u} \subset A_\lambda(K)$ with some $\lambda \in \mathbb{R}$.

We prove $s \ge 2$ by contradiction. Assume that $s<2$. In this case, the thickness of the annulus $A_\lambda(K)$ is unbounded with respect to λ:

$$\lim_{\lambda\to\infty} \left| \sqrt{(\lambda+K)^{4/s} - 1} - \sqrt{(\lambda-K)^{4/s} - 1} \right| = \lim_{\lambda\to\infty} \frac{\lambda^{4/s-1}}{\lambda^{2/s}} = \infty.$$

Thus, inequality (3.1) holds for any $u \in L^2(\mathbb{R}^d)$ such that $\operatorname{supp} \widehat{u}$ is compact. To see this, notice that there exist $a \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$ satisfying $a + \operatorname{supp} \widehat{u} \subset A_\lambda(K)$ for such u. Therefore, letting $u_a(x) := e^{ia\cdot x} u(x)$, we have

$$c\|u\|_{L^2}^2 = c\|u_a\|_{L^2}^2 \le \|\gamma u_a\|_{L^2}^2 = \|\gamma u\|_{L^2}^2$$

since $\operatorname{supp} \widehat{u_a} = a + \operatorname{supp} \widehat{u} \subset A_\lambda(K)$.

Now note that $E_\varepsilon := \{x \in \mathbb{R}^d : \gamma(x) < \varepsilon\}$ has a positive measure for any $\varepsilon>0$, since $\operatorname{ess\,inf}_{x\in\mathbb{R}^d} \gamma(x) = 0$. For each $\varepsilon>0$, we take a subset $F_\varepsilon \subset E_\varepsilon$ such that $0 < m_d(F_\varepsilon) < \infty$. Take $R, \varepsilon > 0$ arbitrarily and set

$$f_\varepsilon := \chi_{F_\varepsilon} / \sqrt{m_d(f_\varepsilon)}, \quad g_{R,\varepsilon} := \mathcal{F}^{-1} \chi_{B(0,R)} \mathcal{F} f_\varepsilon,$$

where χ_Ω denotes the indicator function of $\Omega \subset \mathbb{R}^d$. By the definition, we have $\operatorname{supp} \widehat{g_{R,\varepsilon}} \subset B(0,R)$ and $g_{R,\varepsilon} \to f_\varepsilon$ as $R \to \infty$ in $L^2(\mathbb{R}^d)$. Therefore, applying inequality (3.1) to $g_{R,\varepsilon}$, we get

$$\begin{aligned}c\|g_{R,\varepsilon}\|_{L^2} &\le \|\gamma g_{R,\varepsilon}\|_{L^2} \\ &\le \|\gamma f_\varepsilon\|_{L^2} + \|\gamma(g_{R,\varepsilon} - f_\varepsilon)\|_{L^2} \\ &= \left(\frac{1}{m_d(F_\varepsilon)} \int_{F_\varepsilon} |\gamma(x)|^2 dx \right)^{1/2} + \|\gamma(g_{R,\varepsilon} - f_\varepsilon)\|_{L^2} \\ &\le \varepsilon + \|\gamma(g_{R,\varepsilon} - f_\varepsilon)\|_{L^2}.\end{aligned}$$

Taking the limit as $R \to +\infty$, we obtain

$$0 < c = c\|f_\varepsilon\|_{L^2} \le \varepsilon.$$

This is a contradiction since $\varepsilon > 0$ is arbitrary. □

4. Proof of Theorem 1.7

The proof of (1.10) $\implies$ (1.8) is similar to that of (1.7) $\implies$ (1.3) in Section 2, and the implication (1.9) $\implies$ (1.10) is trivial. Therefore, we will show that (1.8) $\implies$ (1.9). We use a kind of the uncertainty principle to obtain a certain resolvent estimate for the fractional Laplacian:

Theorem 4.1 ([5, Theorem 3]). *Let $\Omega \subset \mathbb{R}^d$ be thick. Then there exists a constant $C > 0$ such that for each $R > 0$, the inequality*

$$\|f\|_{L^2(\mathbb{R}^d)} \leq C \exp(CR)\|f\|_{L^2(\Omega)}$$

holds for any $f \in L^2(\mathbb{R}^d)$ satisfying $\operatorname{supp} \widehat{f} \subset B(0, R)$.

In order to obtain the logarithmic energy decay (1.9), we use the following result.

Theorem 4.2 ([2, Theorem 5.1]). *Let A be a maximal dissipative operator (and hence generate the C^0-semigroup of contractions $(e^{tA})_{t\geq 0}$) in a Hilbert space X. Assume that $i\mathbb{R} \subset \rho(A)$ and there exists $C > 0$ such that*

$$\|(A - i\lambda I)^{-1}\|_{\mathcal{B}(X)} \leq Ce^{C|\lambda|}$$

holds for any $\lambda \in \mathbb{R}$. Then, for each $k > 0$, there exists $C_k > 0$ such that

$$\|e^{tA}(I - A)^{-k}\|_{\mathcal{B}(X)} \leq \frac{C_k}{(\log(e + t))^k}$$

holds for any $t \geq 0$.

4.1. **Resolvent estimate.** The proof of these propositions are based on [4].

Proposition 4.3. *Let $s \geq 1$ and $\Omega \subset \mathbb{R}^d$ be thick. Then there exist $C, c > 0$ such that for all $f \in L^2(\mathbb{R}^d)$ and all $\lambda \geq 0$,*

$$c \exp(-C\lambda)\|f\|^2_{L^2(\mathbb{R}^d)} \leq \|((-\Delta + 1)^{s/2} - \lambda)f\|^2_{L^2(\mathbb{R}^d)} + \|f\|^2_{L^2(\Omega)}.$$

Proof of Proposition 4.3. Let $A_\lambda := \{\xi \in \mathbb{R}^d : |(|\xi|^2 + 1)^{1/2} - \lambda^{1/s}| \leq 1\}$. Since $A_\lambda \subset B(0, \lambda + 2)$ and Ω is thick, Theorem 4.1 implies that there exists $C > 0$ such that

$$\|f\|_{L^2(\mathbb{R}^d)} \leq C \exp(C\lambda)\|f\|_{L^2(\Omega)} \tag{4.1}$$

holds for any $\lambda \geq 0$ and any $f \in L^2(\mathbb{R}^d)$ satisfying $\operatorname{supp} \widehat{f} \subset A_\lambda$. Next, we set a projection $P_\lambda := \mathcal{F}^{-1}\chi_{A_\lambda}\mathcal{F}$, where χ_{A_λ} denotes the indicator function of A_λ. Then, since $P_\lambda f$ satisfies inequality (4.1) for each $f \in L^2(\mathbb{R}^d)$, we obtain

$$\begin{aligned}
\|f\|^2_{L^2(\mathbb{R}^d)} &= \|P_\lambda f\|^2_{L^2(\mathbb{R}^d)} + \|(I - P_\lambda)f\|^2_{L^2(\mathbb{R}^d)} \\
&\leq C \exp(C\lambda)\|P_\lambda f\|^2_{L^2(\Omega)} + \|(I - P_\lambda)f\|^2_{L^2(\mathbb{R}^d)} \\
&= C \exp(C\lambda)\|f - (I - P_\lambda)f\|^2_{L^2(\Omega)} + \|(I - P_\lambda)f\|^2_{L^2(\mathbb{R}^d)} \\
&\leq 2C \exp(C\lambda)\|f\|^2_{L^2(\Omega)} + 2C \exp(C\lambda)\|(I - P_\lambda)f\|^2_{L^2(\Omega)} \\
&\quad + \|(I - P_\lambda)f\|^2_{L^2(\mathbb{R}^d)} \\
&\leq 2C \exp(C\lambda)\|f\|^2_{L^2(\Omega)} + (2C \exp(C\lambda) + 1)\|(I - P_\lambda)f\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Also, by Lemma 1 in [4], we have

$$c\|(I - P_\lambda)f\|^2_{L^2(\mathbb{R}^d)} \leq \|((-\Delta + 1)^{s/2} - \lambda)f\|^2_{L^2(\mathbb{R}^d)}$$

for some $c > 0$ independent of λ. Therefore, we conclude that

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq C \exp(C\lambda) \left[\|((-\Delta + 1)^{s/2} - \lambda)f\|^2_{L^2(\mathbb{R}^d)} + \|f\|^2_{L^2(\Omega)}\right].$$

□

Proposition 4.4. *Let $s \geq 2$ and assume that $\Omega \subset \mathbb{R}^d$ is thick. Then there exist $C, c > 0$ such that for all $U = (u_1, u_2) \in H^s(\mathbb{R}^d) \times H^{s/2}(\mathbb{R}^d)$ and all $\lambda \in \mathbb{R}$,*

$$c \exp(-C|\lambda|)\|U\|^2_{H^{s/2}(\mathbb{R}^d)\times L^2(\mathbb{R}^d)} \leq \|(\mathcal{A}_0 - i\lambda I)U\|^2_{H^{s/2}(\mathbb{R}^d)\times L^2(\mathbb{R}^d)} + \|u_2\|^2_{L^2(\Omega)}.$$

Proof of Proposition 4.4. For $U = (u_1, u_2) \in H^s(\mathbb{R}^d) \times H^{s/2}(\mathbb{R}^d)$, we set

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} (-\Delta+1)^{s/4} & -i \\ (-\Delta+1)^{s/4} & i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

By the parallelogram law, we obtain

$$\|w_1\|^2_{L^2(\mathbb{R}^d)} + \|w_2\|^2_{L^2(\mathbb{R}^d)} = 2\|U\|^2_{H^{s/2}(\mathbb{R}^d)\times L^2(\mathbb{R}^d)}.$$

Moreover, we have

$$\begin{aligned}
&\|(\mathcal{A}_0 - i\lambda I)U\|^2_{H^{s/2}\times L^2} \\
&\quad = \|(-\Delta+1)^{s/2}(-i\lambda u_1 + u_2)\|^2_{L^2} + \|-(-\Delta+1)^{s/2}u_1 - i\lambda u_2\|^2_{L^2} \\
&\quad = \|-\lambda\frac{w_1+w_2}{2} + (-\Delta+1)^{s/2}\frac{w_1-w_2}{2}\|^2_{L^2} \\
&\qquad + \|-(-\Delta+1)^{s/2}\frac{w_1+w_2}{2} + \lambda\frac{w_1-w_2}{2}\|^2_{L^2} \\
&\quad = \|\lambda w_1 - (-\Delta+1)^{s/2}w_1\|^2_{L^2} + \|\lambda w_2 + (-\Delta+1)^{s/2}w_2\|^2_{L^2}.
\end{aligned}$$

For $\lambda \geq 0$, applying Proposition 4.3 to w_1 with $s/2$, we have

$$\begin{aligned}
&2c\exp(-C\lambda)\|U\|^2_{H^{s/2}\times L^2} \\
&\quad = c\exp(-C\lambda)(\|w_1\|^2_{L^2} + \|w_2\|^2_{L^2}) \\
&\quad \leq \|((-\Delta+1)^{s/4} - \lambda)w_1\|^2_{L^2} + \|w_1\|^2_{L^2(\Omega)} + c\exp(-C\lambda)\|w_2\|^2_{L^2} \\
&\quad \leq \|((-\Delta+1)^{s/4} - \lambda)w_1\|^2_{L^2} + 2\|w_1 - w_2\|^2_{L^2(\Omega)} + c\|w_2\|^2_{L^2} \\
&\quad \leq \|((-\Delta+1)^{s/4} - \lambda)w_1\|^2_{L^2} + c\|((-\Delta+1)^{s/4} + \lambda)w_2\|^2_{L^2} + 8\|u_2\|^2_{L^2(\Omega)} \\
&\quad \leq c\|(\mathcal{A}_0 - i\lambda I)U\|^2_{H^{s/2}\times L^2} + 8\|u_2\|^2_{L^2(\Omega)}.
\end{aligned}$$

For $\lambda < 0$, we get the same inequality replacing the role of w_1 with w_2. □

4.2. **Energy decay.** Finally we prove (1.8) $\implies$ (1.9). By assumption (1.8), $\Omega = \{\gamma \geq \varepsilon\}$ is thick for some $\varepsilon > 0$. Therefore, by Proposition 4.4, we have

$$\begin{aligned}
c\exp(-C|\lambda|)\|U\|^2_{H^{s/2}\times L^2} &\leq \|(\mathcal{A}_0 - i\lambda I)U\|^2_{H^{s/2}\times L^2} + \|u_2\|^2_{L^2(\Omega)} \\
&\leq 2\|(\mathcal{A}_\gamma - i\lambda I)U\|^2_{H^{s/2}\times L^2} + (2+\varepsilon^{-2})\|\gamma u_2\|^2_{L^2(\Omega)}.
\end{aligned}$$

Since $\mathcal{A}_0$ is skew-adjoint, we obtain

$$\mathrm{Re}\langle(\mathcal{A}_\gamma - i\lambda I)U, U\rangle = \mathrm{Re}\langle(\mathcal{A}_0 - i\lambda I)U, U\rangle - \langle\gamma u_2, u_2\rangle = -\|\sqrt{\gamma}u_2\|^2_{L^2}.$$

By the Cauchy–Schwarz inequality, we have

$$D\|\gamma u_2\|^2_{L^2} \leq \|\gamma\|_{L^\infty}\|\sqrt{\gamma}u_2\|^2_{L^2} \leq \frac{D^2\|\gamma\|^2_{L^\infty}\|(\mathcal{A}_\gamma - i\lambda)U\|^2_{H^{s/2}\times L^2}}{\delta} + \delta\|U\|^2_{H^{s/2}\times L^2}$$

for any $D, \delta > 0$. Taking $D = 2 + \varepsilon^{-2}$ and $\delta = c\exp(-C|\lambda|)/2$, we obtain

$$\begin{aligned} &c\exp(-C|\lambda|)\|U\|^2_{H^{s/2}\times L^2} \\ &\le 2\|(\mathcal{A}_\gamma - i\lambda I)U\|^2_{H^{s/2}\times L^2} + (2+\varepsilon^{-2})\|\gamma u_2\|^2_{L^2(\Omega)} \\ &\le 2\|(\mathcal{A}_\gamma - i\lambda I)U\|^2_{H^{s/2}\times L^2} + \frac{(2+\varepsilon^{-2})^2\|\gamma\|^2_{L^\infty}}{c\exp(-C|\lambda|)}\|(\mathcal{A}_\gamma - i\lambda I)U\|^2_{H^{s/2}\times L^2} \\ &\quad + \frac{1}{2}c\exp(-C|\lambda|)\|U\|^2_{H^{s/2}\times L^2}. \end{aligned}$$

By this inequality, we have

$$c\exp(-C|\lambda|)\|U\|^2_{H^{s/2}\times L^2} \le \|(\mathcal{A}_\gamma - i\lambda I)U\|^2_{H^{s/2}\times L^2},$$

here the constants c, C may differ from the previous ones. Applying Theorem 4.2 with $k = 1$, we conclude that (1.9) holds.

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Graduate School of Mathematics, Nagoya University, Furocho, Chikusa-ku, Nagoya, Aichi 464-8602, Japan

Email address: m21010t@math.nagoya-u.ac.jp

Department of Mathematics, Chuo University, 1-13-27, Kasuga, Bunkyo-ku, Tokyo 112-8551, Japan

Email address: soichiro.suzuki.m18020a@gmail.com