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A CHARACTERIZATION OF WHITNEY FORMS

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ABSTRACT. We give a characterization of Whitney forms on an n-simplex σ and prove that for every real valued simplicial k-cochain c on σ, the form Wc is the unique differential k-form φ on σ with affine coefficients that pulls back to a constant form of degree k on every k-face τ of σ, and satisfies $\int_\tau \varphi =< c, \tau >$.

1. Introduction

Whitney forms have been extraordinarily useful in several areas of mathematics: algebraic topology [8], [6]; global analysis and spectral geometry [4], [3]; numerical electromagnetism[1], [2]; vibrations of thin plates [7]. Their definition in Whitney's book [9, p. 140] appears somewhat mysterious. Attempts to gain a better insight into the definition have continued up to now. For example, the recent paper of Lohi and Kettunen [5] contains *three different equivalent definitions.* In this note we give a conceptual, easily stated characterization of Whitney forms.

We consider a triangulated differentiable manifold M of n dimensions with a triangulation $h : K \longrightarrow M$, cf. [9, p. 124]. We use Whitney's terminology exactly. Thus h is a homeomorphism of a simplicial complex K onto the manifold M with the additional property that for every closed n-simplex σ of K there exists a coordinate system χ_σ defined in an open neighborgood U_σ of the image $h(\sigma)$ so that the composition $\chi_\sigma \circ (h \,|\, \sigma)$ is an affine map of σ into $\mathbb{R}^n$. One often identifies K with M via h which usually does not lead to any confusion. We will do so here as well and regard a simplex σ as a subset of K, M or $\mathbb{R}^n$ without explicitly mentioning identifications given by h or χ_σ.

Now the Whitney form Wc corresponding to the cochain $c \in C^k(K)$ is an assignment of a smooth k-form ω_σ s to each closed n-simplex σ that satisfies certain compatibility conditions. Namely, if τ is a common face of two top dimensional faces σ_1 and σ_2, then the pull-backs to τ of ω_{σ_1} and ω_{σ_2} coincide. Thus to describe the Whitney form Wc it suffices to give a description of $Wc \,|\, \sigma = \omega_\sigma$ for every simplex σ of top dimension. Note that the homeomorphism h defines an affine structure on σ and the induced affine structures on common faces of two n-simplexes agree. Thus the concept of an affine function on a simplex is well-defined and so is a notion of a "constant" form of degree k on a k-simplex.

From now on we work on a *fixed* n-simplex σ. Our characterization of Wc is stated precisely in the Theorem below. It asserts that Wc restricted to σ is the

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unique k-form on σ with affine coefficients and constant pull-backs to k-faces whose integrals over k-faces τ are prescribed by the values $\langle c, \tau \rangle$ of c on τ.

2. Proof of the Theorem

A simplex $\tau = [p_0, p_1, \ldots, p_k]$ of k dimensions is a convex hull of $k + 1$ points in general position in $\mathbb{R}^n$. In particular, every simplex is closed. We will consider a fixed n-simplex σ together with all its k-faces τ with $0 \leq k \leq n$. Thus a point $q \in \sigma$ is a convex linear combination

$$q = m_0 p_0 + m_1 p_1 + \ldots + m_n p_n$$
$$m_i \geq 0 \quad \text{for} \quad i = 0, 1 \ldots, n$$
$$m_0 + m_1 \ldots + m_n = 1$$

and the barycentric coordinate functions $\nu_i(q)$ are defined by

$$\nu_i(q) = m_i.$$

We observe that, if $q = (x^1, x^2, \ldots, x^n)$ the barycentric coordinates are affine functions of $x^1, x^2, \ldots, x^n$ i.e. are of the form $a_1 x^1 + a_2 x^2 + \ldots + a_n x^n + b$. We regard all simplices as oriented with the orientation determined by the order of vertices with the usual convention that $-\tau$ is τ with the opposite orientation and that under a permutation of vertices the orientation changes by the sign of the permutation. A cochain c of degree k is then defined as a formal linear combination with real coefficients of duals τ^* of the k-faces τ of σ and we denote by $C^k(\sigma) = C^k$ the space of all such cochains. If $c = \sum_\tau a_\tau \tau^*$ we will write $a_\tau = \langle c, \tau \rangle$. Finally, we will denote by $\Lambda^k(\sigma) = \Lambda^k$ the space of all smooth exterior differential forms of degree k on the simplex σ. With this notation, one defines the Whitney mapping

$$W : C^k \longrightarrow \Lambda^k$$

for all $k = 0, 1, \ldots n$, cf. [9] or [3] for a detailed discussion. We will call forms in the image of W the Whitney forms. It follows immediately from the definition that the Whitney forms when expressed in terms of the coordinates of $\mathbb{R}^n$ have affine coefficients. We abuse the language and say that a form $\eta \in \Lambda^k(\tau)$ is constant if it is a constant multiple of the Euclidean volume element on τ. For clarity, we emphsize that by the integral of a k-form over a submanifold of k dimensions we always mean the integral of the pull-back of the form to the submanifold via the inclusion map. Thus, for example, in (3) below $\int_\tau \omega = \int_\tau \iota_\tau^* \omega$. After these preliminaries we state our theorem.

Theorem. *Let σ be a simplex of n dimensions and c a cochain of degree k on σ. Wc is the unique k-form ω on σ satisfying the following conditions.*

(1) *ω has affine coefficients.*
(2) *The pull-back $\iota_\tau^* \omega$ is constant for every k-dimensional face τ of σ, where $\iota_\tau : \tau \hookrightarrow \sigma$ denotes the inclusion map.*
(3) *$\int_\tau \omega = \langle c, \tau \rangle$ for every k-face τ of σ.*

Proof. We first observe that without any loss of generality we can assume that σ is the standard simplex in $\mathbb{R}^n$ i.e. is given by

$$\sigma = \left\{ (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n \mid x^i \geq 0 \quad \text{for} \quad i = 1, 2, \ldots n\,; \quad \sum_{i=0}^{n} x^i \leq 1 \right\}.$$

Thus $\sigma = [0, e_1, e_2 \ldots, e_n]$ where e_i is the point on the i-th coordinate axis with $x^i = 1$. The barycentric coordinate functions restricted to σ are then given by

$$(1) \qquad \nu_0 = 1 - (x^1 + x^2 + \ldots + x^n) \quad \text{and} \quad \nu_i = x^i \quad \text{for} \quad i = 1, 2, \ldots n.$$

We first do a quick dimension count that makes the theorem plausible. The dimension of the space of k-forms with affine coefficients on σ is $\binom{n}{k}(n+1)$. Requiring that $\iota_\tau^*\omega$ is *constant* on a k-simplex τ imposes k conditions and the number of k-faces of an n-simplex is $\binom{n+1}{k+1}$. Thus, the dimension of the space of k-forms satisfying (1) and (2) above ought to be

$$\binom{n}{k}(n+1) - \binom{n+1}{k+1}k = \binom{n+1}{k+1}.$$

This last integer is the number of k-faces of σ, i.e. the dimension of the space $C^k(\sigma)$ of k-cochains.

It is instructive to consider the simplest cases $k = 0$ and $k = n$ of the theorem. A 0-cochain is a sum $c = \sum a_i p_i^*$ and

$$\begin{aligned} Wc &= a_0\nu_0 + a_1\nu_1 + \ldots a_n\nu_n \\ &= a_0\left(1 - \sum_{i=1}^n x^i\right) + \sum_{i=1}^n a_i x^i \\ &= a_0 + \sum_{i=1}^n (a_i - a_0)x^i \end{aligned}$$

is the unique affine function f taking prescribed values $f(p_i) = \int_{p_i} f = \langle c, p_i\rangle$, where the integration of a form of degree 0 over a vertex is just the evaluation.

If $k = n$, σ is the only face of dimension n so every cochain is a multiple of σ^*. For $c = \sigma^*$, we have

$$\begin{aligned} Wc &= W\sigma^* \\ &= \left(n!\sum_{j=0}^n (-1)^j \nu_j d\nu_0 \wedge \ldots \wedge \widehat{d\nu_j} \wedge \ldots \wedge d\nu_n\right) \\ &= n! dx^1 \wedge \ldots \wedge dx^n \end{aligned}$$

where we used the explicit expressions of the barycentric coordinates (1) in terms of the coordinates $x^1, \ldots, x^n$ and the hat over a factor means that the factor is omitted. Since the volume of the standard n-simplex in $\mathbb{R}^n$ is equal to $1/n!$, $\int_\sigma W(\sigma^*) = \langle\sigma^*, \sigma\rangle = 1$, $W\sigma^*$ is the unique constant form with prescribed integral equal to one.

We now consider the case when $1 \le k \le n-1$. We will write Λ_e^k for the space of k-forms on σ with affine coefficients and with constant pull-backs to k-faces of σ. It is obvious from the definition of Wc and from (1) that Wc has affine coefficients on σ for every $c \in C^k(\sigma)$. Similarly, since $\iota_\tau^* W(c)$ is a form of maximal degree on τ, the calculation above, with k replacing n, shows that $\iota_\tau^* W(c)$ is constant on τ for every k-face τ of σ. It follows that $WC^k \subset \Lambda_e^k$. Now let $\varphi \in \Lambda_e^k$. We use the restriction of the de Rham map $R : \Lambda^k(\sigma) \longrightarrow C^k(\sigma)$,

$$\langle R\omega, \tau\rangle = \int_\tau \omega,$$

to Λ^k_e and consider the difference $\eta = \varphi - WR\varphi$. Clearly, $\eta \in \Lambda^k_e$. Moreover basic properties of the Whitney mapping (cf. [3, 9]) imply that $R\eta = R\varphi - RWR\varphi = R\varphi - R\varphi = 0$, i.e. η integrates to zero on every k-face of σ. Since the pull-back $\iota^*\eta$ is constant on every such face τ, $\iota^*_\tau\eta$ vanishes identically on every k-face τ. Thus to show that $\varphi = WR\varphi$ (which would prove our theorem) it suffices to show that every form $\eta \in \Lambda^k_e$, whose pull-backs to all k-faces vanish, is itself identically zero on σ. Let η be such a form. We express it in the standard coordinates of $\mathbb{R}^n$ as follows.

$$\eta = \sum_I (b_I + a_{I,1}x^1 + \ldots + a_{I,n}x^n)dx^I. \tag{2}$$

Here I is a multi-index $I = (i_1 < i_2 < \ldots < i_k)$, $1 \le i_j \le n$ for every j and $dx^I = dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$. We will abuse the notation at times and think of I as a set. Fix a multi-index J and consider the coordinate plane of the variables $x^{j_1}, x^{j_2}, \ldots, x^{j_k}$.

Let τ_J denote the k-face of σ contained in that plane. By assumption $\iota^*_{\tau_J}\eta$ is identically zero. The variables x_t for $t \notin J$ vanish in this plane so that

$$\iota^*_{\tau_J}\eta = \sum_{t\in J}(a_{J,t}x^t + b_J)dx^J \equiv 0. \tag{3}$$

Since J was arbitrary, $b_J = 0$ and $a_{J,t} = 0$ for all J and all $t \in J$. It follows that we can rewrite (2) on σ as follows.

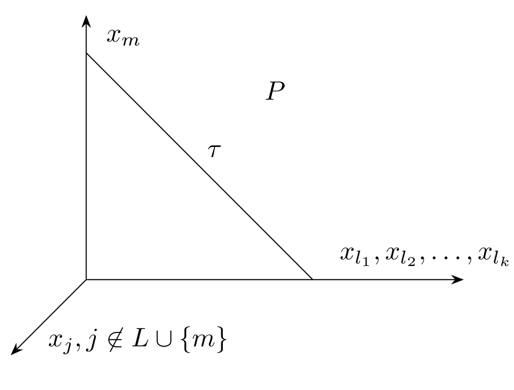
$$\eta = \sum_I \sum_{j\notin I} a_{I,j}x^j dx^I. \tag{4}$$

Again, fix the multi-index L, an integer $m \notin L$, $1 \le m \le n-1$, and the simplex $\tau = [e_m, e_{l_1}, \ldots, e_{l_k}]$. τ is a k-simplex in the $(k+1)$-plane P with coordinates $x^m, x^{l_1}, \ldots, x^{l_k}$ as in the figure below. Recall that on τ, $x^{l_1}, \ldots, x^{l_k}$ can be taken as local coordinates since

$$x^m = 1 - (x^{l_1} + \ldots + x^{l_k}). \tag{5}$$

Moreover,

$$dx^m = -(dx^{l_1} + \ldots + dx^{l_k}). \tag{6}$$



We express the pull-back $\iota^*_\tau\eta$ in terms these coordinates using (5) and (6). Observe that if $I \cup \{j\} \ne L \cup \{m\}$ one of the indices in $I \cup \{j\}$ is not in $L \cup \{m\}$. The corresponding variable is identically zero on the plane P so that the summand

$a_{I,j}x^j dx^I$ vanishes on P and is therefore equal to zero when pulled back to τ. Therefore

$$\iota_\tau^*\eta = \sum_{I\cup\{j\}=L\cup\{m\}} a_{I,j}x^j dx^I. \tag{7}$$

Now consider the summand with $I = L$ and $j = m$. The coefficient of dx^L in this term is

$$a_{L,m}x^m + a_{L,l_1}x^{l_1} + \ldots + a_{L,l_k}x^{l_k}$$

and we use (5) to eliminate x^m.

Thus, on τ, the coefficient in question can be written as

$$a_{L,m} - a_{L,m}\sum_{s=1}^{k} x^{l_s} + a_{L,l_1}x^{l_1} + \ldots + a_{L,l_k}x^{l_k}.$$

Remaining terms in the sum (7) have $j \neq m$. It follows that, for those terms, x^j is one of $x^{l_1}, \ldots, x^{l_k}$ and x^m enters only into the differential monomial dx^I from which it can be eliminated using (6). It follows that

$$\iota_\tau^*\eta = (\, a_{L,m} + \textit{linear terms}\,)\, dx^L.$$

Since $\iota_\tau^*\eta$ is assumed to be identically zero, $a_{L,m} = 0$. L and m were fixed but arbitrary so that $\eta \equiv 0$. ☐

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