PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 11, Pages 36–46 (March 5, 2024)
https://doi.org/10.1090/bproc/207

SEPARATION OF HOMOGENEOUS CONNECTED LOCALLY COMPACT SPACES

VESKO VALOV

(Communicated by Julie Bergner)

ABSTRACT. We prove that any region Γ in a homogeneous n-dimensional and locally compact separable metric space X, where $n \geq 2$, cannot be irreducibly separated by a closed $(n-1)$-dimensional subset C with the following property: C is acyclic in dimension $n-1$ and there is a point $b \in C \cap \Gamma$ having a special local base $\mathcal{B}_C^b$ in C such that the boundary of each $U \in \mathcal{B}_C^b$ is acyclic in dimension $n-2$. In case X is strongly locally homogeneous, it suffices to have a point $b \in C \cap \Gamma$ with an ordinary base $\mathcal{B}_C^b$ satisfying the above condition. The acyclicity means triviality of the corresponding Čech cohomology groups. This implies all known results concerning the separation of regions in homogeneous connected locally compact spaces.

1. Introduction

By a *space* we mean a locally compact separable metric space, and *maps* are continuous mappings. We also consider reduced in dimension zero Čech cohomology groups $H^n(X;G)$ with coefficients from an Abelian group G. If G is the group of the integers $\mathbb{Z}$, we simply write $H^n(X)$. Recall that a space X is *separated by a set* $C \subset X$ if C is closed in X and $X \backslash C$ is the union of two disjoint open subsets G_1, G_2 of X. When C is the intersection of the closures $\overline{G}_1$ and $\overline{G}_2$, we say that C is an *irreducible separator*. A *partition* between two disjoint closed sets A, B in X is a closed set P such that $X \backslash P$ is the union of two open disjoint sets U, V in X such that $A \subset U$ and $B \subset V$. In such a case we say that P separates X between A and B, or A and B are separated in X by P. A *region in* X is an open connected subset of X. By $\dim X$ we denote the covering dimension of X, and $\dim_G X$ stands for the cohomological dimension of X with respect to a group G. The boundary of a given set $U \subset X$ in X is denoted by $\mathrm{bd}U$; if $U \subset C \subset X$, then $\mathrm{bd}_C U$ denotes the boundary of U in C. We say that a point $x \in X$ has a *special base* $\mathcal{B}_x$ if for any neighborhoods U, V of x in X with $\overline{U} \subset V$ there is $W \in \mathcal{B}_x$ such that $\mathrm{bd}W$ separates $\overline{V} \backslash U$ between $\mathrm{bd}\overline{V}$ and $\mathrm{bd}\overline{U}$.

One of the first results concerning the separation of homogenous metric spaces is the celebrated theorem of Krupski [11], [12] stating that every region in an n-dimensional homogeneous space cannot be separated by a subset of dimension $\leq n-2$. Kallipoliti and Papasoglu [9] established that any locally connected, simply connected, homogeneous metric continuum cannot be separated by arcs (according

Received by the editors March 27, 2023, and, in revised form, June 7, 2023, September 17, 2023, December 7, 2023, and December 20, 2023.

2020 *Mathematics Subject Classification.* Primary 55M10; Secondary 54F45.

Key words and phrases. Acyclic partitions, cohomological dimension, homogeneous spaces.

The author was partially supported by NSERC Grant 261914-19.

to Krupski's theorem, mentioned above, the Kallipoliti-Papasoglu result is interesting for spaces of dimension two). van Mill and the author [14] proved that the Kallipoliti-Papasoglu theorem remains true without simply connectedness, but requiring strong local homogeneity instead of homogeneity. Recall that a space X is *strongly locally homogeneous* if every point $x \in X$ has a base of open neighborhoods U such that for every $y, z \in U$ there is a homeomorphism h on X with $h(y) = z$ and h is the identity on $X \setminus U$. We say that such a homeomorphism h is *supported by* U. If for every $x, y \in X$ there is a homeomorphism h on X with $h(x) = y$, the spaces X is *homogeneous*.

In the present paper we establish Theorem 1.1 which captures all above-mentioned results:

Theorem 1.1. *Let Γ be a region in a homogeneous space X with $dimX = n \geq 2$. Then Γ cannot be irreducibly separated by any closed set $C \subset X$ with the following property:*

(i) $\dim C \leq n-1$ *and* $H^{n-1}(C) = 0$*;*
(ii) *There is a point $b \in C \cap \Gamma$ having a special base $\mathcal{B}_C^b$ in C with $H^{n-2}(\mathrm{bd}_C U) = 0$ for every $U \in \mathcal{B}_C^b$.*

If X is strongly locally homogeneous, condition (ii) *can be weakened to the following one:*

(iii) *There is $b \in C \cap \Gamma$ having an ordinary base $\mathcal{B}_C^b$ in C with $H^{n-2}(\mathrm{bd}_C U) = 0$, $U \in \mathcal{B}_C^b$.*

Remark 1.2. According to [18, Corollary 1.6], if X in Theorem 1.1 is a compactum with $H^n(X) \neq 0$, then X is not separated by any C satisfying condition (i).

Since $H^{k+1}(Y) = 0$ for any k-dimensional space Y, we have the following fact: If $\dim Y \leq n-2$, then $H^{n-1}(Y) = 0$ and every $x \in Y$ has a special base of neighborhoods U with $H^{n-2}(\mathrm{bd}U) = 0$ because every two closed subsets of Y can be separated by set A with $\dim A \leq n-3$. Moreover, any $(n-2)$-dimensional separator contains a closed subset which is an irreducible $(n-2)$-dimensional separator. Therefore, Theorem 1.1 implies directly that any region in a homogeneous n-dimensional space cannot be separated by a subset of dimension $\leq n-2$. Similar arguments show that if G is any countable Abelian group, then any homogeneous connected space of cohomological dimension $\dim_G X \leq n$ cannot be separated by a closed subset of dimension $\dim_G \leq n-2$ (this fact was established by different methods in [10]).

If a region Γ in a two-dimensional strongly locally homogeneous space is separated by an arc C, then there is a closed $C' \subset C$ irreducibly separating Γ, see [14]. Then $H^1(C') = 0$ and the point $b = \max\{x : x \in C'\}$ has a base $\mathcal{B}_{C'}$ such that $\mathrm{bd}_{C'}U$ is a point for all $U \in \mathcal{B}_{C'}$. Therefore, Theorem 1.1 also implies our result [14] with van Mill.

Theorem 1.1 is a particular case of Theorem 1.3 when $G = \mathbb{Z}$:

Theorem 1.3. *Let Γ be a region in a finite-dimensional homogeneous space X with* $\dim_G X = n \geq 2$*, where G is a countable Abelian group. Then Γ cannot be irreducibly separated by any closed set $C \subset X$ with the following property:*

(i) $\dim_G C \leq n-1$ *and* $H^{n-1}(C; G) = 0$*;*
(ii) *There is a point $b \in C \cap \Gamma$ having a special local base $\mathcal{B}_C^b$ in C with* $H^{n-2}(\mathrm{bd}_C U; G) = 0$ *for every $U \in \mathcal{B}_C^b$.*

If X is strongly locally homogeneous, the finite-dimensionality of X can be omitted and condition (ii) *can be weakened to the following one:*

(iii) *There is $b \in C \cap \Gamma$ having an ordinary base $\mathcal{B}_C^b$ in C with $H^{n-2}(\mathrm{bd}_C U) = 0$, $U \in \mathcal{B}_C^b$.*

Theorem 1.3 is established in Section 3. Section 2 contains some definitions and preliminary results.

2. Definitions and preliminary results

Recall that for any nontrivial Abelian group G the Čech cohomology group $H^n(X;G)$ is isomorphic to the group $[X, K(G,n)]$ of pointed homotopy classes of maps from X to $K(G,n)$, where $K(G,n)$ is a CW-complex of type (G,n), see [7]. It is also well known that the circle group $\mathbb{S}^1$ is a space of type $(\mathbb{Z},1)$. The cohomological dimension $\dim_G(X)$ is the largest number n such that there exists a closed subset $A \subset X$ with $H^n(X,A;G) \neq 0$. Equivalently, for a metric space X we have $\dim_G X \leq n$ if and only if for any closed pair $A \subset B$ in X the homomorphism $j^n_{B,A} : H^n(B;G) \to H^n(A;G)$, generated by the inclusion $A \hookrightarrow B$, is surjective, see [3]. This means that every map from A to $K(G,n)$ can be extended over B. For every G we have $\dim_G X \leq \dim_{\mathbb{Z}} X \leq \dim X$, and $\dim_{\mathbb{Z}} X = \dim X$ in case $\dim X < \infty$ [13] (on the other hand, there is an infinite-dimensional compactum X with $\dim_{\mathbb{Z}} X = 3$, see [2]).

Suppose (K,A) is a pair of compact sets in a space X with $\varnothing \neq A \subset K$. We say that K is an *k-cohomology membrane spanned on A for an element $\gamma \in H^k(A;G)$* if γ is not extendable over K, but it is extendable over every proper closed subset of K containing A. Here, $\gamma \in H^k(A;G)$ is *extendable over K* means that γ is contained in the image $j^k_{K,A}\big(H^k(K;G)\big)$. Concerning extendability, we are using the following simple fact:

Lemma 2.1. *Let A, B be closed sets in X with $X = A \cup B$. Then $\gamma \in H^k(A;G)$ is extendable over X if and only if $j^k_{A,\Gamma}(\gamma)$ is extendable over B, where $\Gamma = A \cap B$.*

Proof. This follows from the Mayer-Vietoris exact sequence

$$H^k(X;G) \xrightarrow{\varphi^k} H^k(A;G) \oplus H^k(B;G) \xrightarrow{\psi^k} H^k(\Gamma;G),$$

where $\varphi^k(\gamma) = (j^k_{X,A}(\gamma), j^k_{X,B}(\gamma))$ and $\psi^k(\gamma_1,\gamma_2) = j^k_{A,\Gamma}(\gamma_1) - j^k_{B,\Gamma}(\gamma_2)$. Indeed, suppose $\gamma_\Gamma = j^k_{A,\Gamma}(\gamma)$ is extendable over B. So, there is $\alpha \in H^k(B;G)$ with $j^k_{B,\Gamma}(\alpha) = \gamma_\Gamma$. Then, $\psi^k(\gamma,\alpha) = 0$, which implies the existence of $\beta \in H^k(X;G)$ such that $\varphi^k(\beta) = (\gamma,\alpha)$. This yields $j^k_{X,A}(\beta) = \gamma$. Hence, γ is extendable over X.

To prove the other implication, suppose $j^k_{X,A}(\beta) = \gamma$ for some $\beta \in H^k(X;G)$, and let $\alpha = j^k_{X,B}(\beta)$. Then $\psi^k(\gamma,\alpha) = 0$, which means that $j^k_{B,\Gamma}(\alpha) = j^k_{A,\Gamma}(\gamma)$. Therefore, $j^k_{A,\Gamma}(\gamma)$ is extendable over B. □

Lemma 2.2. *Let X be a homogeneous space with $\dim_G X = n > 1$. For every $x \in X$ there exists a compactum M containing x such that all sufficiently small neighborhoods W of x in X have the following property: For every open neighborhood V of x with $\overline{V} \subset W$ there exist a nontrivial $\gamma_V \in H^{n-1}(M \cap \mathrm{bd}\overline{V}; G)$ and an $(n-1)$-cohomology membrane $K_V \subset M \cap \overline{V}$ for γ_V spanned on $M \cap \mathrm{bd}\overline{V}$.*

Proof. Since X is a countable union of compact sets, there exists a compactum $Y \subset X$ with $\dim_G Y = n$ (otherwise, by the countable sum theorem for $\dim_G$,

$\dim_G X \leq n-1$). Since $\dim_G Y = n$ there exists a proper closed subset $F \subset Y$ and $\gamma \in H^{n-1}(F;G)$ such that γ is not extendable over Y. Using the continuity of Čech cohomology [17], we can apply Zorn's lemma to conclude there exists a minimal compact set $M \subset Y$ containing F such that γ is not extendable over M, but it is extendable over every proper closed subset of M containing F. Since X is homogeneous, we can assume that $x \in M \backslash F$. Now, let us show that any neighborhood W of x with $\overline{W} \subset X \backslash F$ is as required. Indeed, suppose V is an open neighborhood of x with $\overline{V} \subset W$. Then $M \backslash V$ is a proper closed subset of M containing F. Hence, there exists $\gamma' \in H^{n-1}(M \backslash V;G)$ extending γ such that γ' is not extendable over $M \cap \overline{V}$. Let $\gamma_V = j_{M \backslash V, M \cap \mathrm{bd}\overline{V}}(\gamma')$. Observe that $M = (M \backslash V) \cup (M \cap \overline{V})$ with $(M \backslash V) \cap ((M \cap \overline{V}) = M \cap \mathrm{bd}\overline{V}$. So, by Lemma 2.1, $M \cap \mathrm{bd}\overline{V}$ is nonempty, otherwise γ' would be extendable over M. By the same reason, γ_V is nontrivial and not extendable over $M \cap \overline{V}$. Therefore, there is a minimal closed set $K_V \subset M \cap \overline{V}$ containing $M \cap \mathrm{bd}\overline{V}$ such that γ_V is not extendable over K_V. Then K_V is an $(n-1)$-cohomology membrane for γ_V spanned on $M \cap \mathrm{bd}\overline{V}$. □

Proposition 2.3. *Let $A \subset P$ be a compact pair and γ be a nontrivial element of $H^{n-1}(A;G)$. Suppose there are closed subsets P_1, P_2 of P satisfying the following conditions:*

- *$P_1 \cup P_2 = P$ and $P_1 \cap P_2 = C \neq \varnothing$;*
- *γ is extendable over $P_i \cup A$ for each $i = 1, 2$, but γ is not extendable over P.*

Then $H^{n-1}(C, C \cap A; G) \neq 0$.

Proof. Consider the commutative diagram below whose rows are parts of Mayer-Vietoris exact sequences, while the columns are exact sequences for the corresponding couples:

$$\begin{array}{ccc}
H^{n-1}(P;G) & \xrightarrow{\varphi_P^{n-1}} & H^{n-1}(P_1;G) \oplus H^{n-1}(P_2;G) \\
\Big\downarrow j_{P,A}^{n-1} & & \Big\downarrow j_{P_1 \oplus P_2}^{n-1} \\
H^{n-1}(A;G) & \xrightarrow{\varphi_A^{n-1}} & H^{n-1}(A \cap P_1;G) \oplus H^{n-1}(A \cap P_2;G) \\
\Big\downarrow \partial_{P,A} & & \Big\downarrow \partial_{P_1 \oplus P_2} \\
H^n(P,A;G) & \xrightarrow{\varphi_{P,A}^{n}} & H^n(P_1, P_1 \cap A;G) \oplus H^n(P_2, P_2 \cap A;G)
\end{array}$$

Here, the maps $j_{P_1 \oplus P_2}^{n-1}$ and $\partial_{P_1 \oplus P_2}$ are defined by

$$j_{P_1, A \cap P_1}^{n-1} \oplus j_{P_2, A \cap P_2}^{n-1},$$
$$\partial_{P_1 \oplus P_2} = \partial_{P_1, P_1 \cap A} \oplus \partial_{P_2, P_2 \cap A}.$$

Recall also that $\varphi_P^{n-1} = (j_{P,P_1}^{n-1}, j_{P,P_2}^{n-1})$, the maps φ_A^{n-1}, $\varphi_{P,A}^{n}$ and φ_P^n are defined similarly.

Denote $\alpha_i = j_{A, A \cap P_i}^{n-1}(\gamma)$, $i = 1, 2$. Since γ is extendable over $A \cup P_i$, there exist $\gamma_i \in H^{n-1}(A \cup P_i)$ extending γ, i.e., $j_{A \cup P_i, A}^{n-1}(\gamma_i) = \gamma$. Let $\beta_i = j_{A \cup P_i, P_i}^{n-1}(\gamma_i)$. It follows from the Mayer-Vietoris exact sequence

$$H^{n-1}(A \cup P_i;G) \to H^{n-1}(A;G) \oplus H^{n-1}(P_i;G) \to H^{n-1}(A \cap P_i;G) \to \dots$$

that $j^{n-1}_{P_i,A\cap P_i}(\beta_i)=\alpha_i$ for every $i=1,2$. This implies $j^{n-1}_{P_1\oplus P_2}((\beta_1,\beta_2))=(\alpha_1,\alpha_2)$. Since the second column is a part of an exact sequence, the last equality yields $\partial_{P_1\oplus P_2}(\varphi^{n-1}_A(\gamma))=0$. Hence, $\varphi^n_{P,A}(\partial_{P,A}(\gamma))=0$. Note that $\widetilde{\gamma}=\partial_{P,A}(\gamma)\neq 0$ because the first column is exact and γ is not extendable over P.

Finally, since $\varphi^n_{P,A}(\widetilde{\gamma})=0$, Proposition 2.3 follows from the Mayer-Vietoris exact sequence

$$H^{n-1}(C,C\cap A;G)\xrightarrow{\triangle}H^n(P,A;G)\xrightarrow{\varphi^n_{P,A}}H^n(P_1,P_1\cap A;G)\oplus H^n(P_2,P_2\cap A;G).$$

□

Corollary 2.4. *Let X be a space with $\dim_G X=n$ and $C\subset X$ be a nonempty separator of X. If there is an open set U such that $C\subset U$ and $\overline{U}$ is an $(n-1)$-cohomology membrane for some $\gamma\in H^{n-1}(\mathrm{bd}\overline{U};G)$ spanned on $\mathrm{bd}\overline{U}$, then $H^{n-1}(C;G)\neq 0$.*

Proof. Let $A=\mathrm{bd}\overline{U}$ and P_1,P_2 be closed subsets of X with $P_1\cap P_2=C$ and $P_1\cup P_2=\overline{U}$. Since $H^{n-1}(C;G)=H^{n-1}(C,C\cap\mathrm{bd}\overline{U};G)$, the proof follows from Proposition 2.3. □

Corollary 2.4 implies the well-known fact [8] that $H^{n-1}(C;G)\neq 0$ for any compact separator C of $\mathbb{R}^n$. Indeed, take any ball $\mathbb{B}^n$ with $C\subset\mathrm{int}\,\mathbb{B}^n$.

By $X^*=X\cup\{\infty\}$ we denote the one-point compactification of a space X. By a *metric on X inherited from X^** we mean the restriction of dist to X, where dist is any metric on X^*.

The following version of Effros' theorem [6] is folklore. For the sake of completeness we include a proof.

Theorem 2.5. *Let X be a homogeneous locally compact space and ρ be a metric on X^*. Then for any $a\in X$ and $\varepsilon>0$ there exists $\delta>0$ such that for every $x\in X$ with $\rho(x,a)<\delta$ there exists a homeomorphism $h\colon X\to X$ with $h(a)=x$ and $\rho(h(y),y)<\varepsilon$ for all $y\in X$.*

Proof. Let $\mathcal{H}(X^*)$ be the space of all homeomorphism of X^*, endowed with the compact-open topology. Note that $\mathcal{H}(X^*)$ is a Polish group and its topology is generated by the metric $\hat{\rho}(f,g)=\sup\{\rho(f(x),g(x))\colon x\in X^*\}$. Therefore the set $\mathcal{H}_X$ consisting of all $h\in\mathcal{H}(X^*)$ with $h(\infty)=\infty$ is a closed subgroup of $\mathcal{H}(X^*)$, so $\mathcal{H}_X$ is also a Polish group. Recall that every homeomorphism h on X can be extended to a homeomorphism $\widetilde{h}\in\mathcal{H}_X$, so $\mathcal{H}_X\neq\varnothing$. Since the action $T^*\colon\mathcal{H}(X^*)\times X^*\to X^*$, $T^*(g,x)=g(x)$, is continuous, so is the action $T\colon\mathcal{H}_X\times X\to X$, $T(h,x)=h(x)$. Moreover, T is transitive because X is homogeneous. Hence, by [15, Theorem 1.1], T is micro-transitive, i.e., for every $x\in X$ and every neighbourhood U of the identity in $\mathcal{H}_X$ the set $Ux=\{h(x)\colon h\in U\}$ is a neighbourhood of x. This implies the statement of the theorem. □

3. Proof of Theorem 1.3

First, consider the case when X is homogeneous. Suppose $C\subset X$ is closed such that $C\cap\Gamma$ irreducibly separates Γ and satisfies conditions (i)–(ii). We are going to obtain a contradiction. To this end, fix a metric ρ on X inherited from the one-point compactification X^* of X. Then $\Gamma\backslash C=G_1\cup G_2$ with $C'=\overline{G}_1\cap\overline{G}_2\cap\Gamma\subset C$, where G_1,G_2 are disjoint open subsets of Γ. By Lemma 2.2, there exists a compactum M containing b such that all sufficiently small neighborhoods W of b satisfy the thesis of that lemma. We fix such a neighborhood W having a compact closure with $\overline{W} \subset \Gamma$.

Claim 1. Following the notations from Lemma 2.2, there exist sufficiently small neighborhoods V of b in X such that $\overline{V} \subset W$, $\dim_G \mathrm{bd}\overline{V} \leq n-1$ and $K_V \backslash \mathrm{bd}\overline{V}$ meets both sets G_1 and G_2.

Indeed, let $\varepsilon = \min\{\rho(\overline{W}, X\backslash\Gamma), \rho(b, X\backslash W)\}$ and $\delta > 0$ be a number from Theorem 2.5 corresponding to ε and the point b. Take any neighborhood V of b such that $\overline{V} \subset W$ and the diameter of V is less than δ. Since X is finite-dimensional, according to [4] and [5] we can suppose also that $\dim_G \mathrm{bd}\overline{V} \leq n-1$. Then there is a ϵ-small homeomorphism h on X so that $\overline{h(V)} \subset W$ and $h(b) \in K_V \backslash \mathrm{bd}\overline{V}$. Hence, considering the sets $h^{-1}(V)$ and $h^{-1}(K_V)$ instead of V and K_V, we can assume that $b \in K_V \backslash \mathrm{bd}\overline{V}$. Moreover, $\dim_G(K_V \backslash \mathrm{bd}\overline{V}) = n$, otherwise γ_V would be extendable over K_V.

Further, since $\dim_G C \leq n-1$ and $\dim_G(K_V \backslash \mathrm{bd}\overline{V}) = n$, $K_V \backslash \mathrm{bd}\overline{V}$ is not contained in C. So, $K_V \backslash \mathrm{bd}\overline{V}$ meets at least one G_i, $i = 1, 2$. If $K_V \backslash \mathrm{bd}\overline{V}$ intersects only G_1, then Theorem 2.5 allows us to push K_V towards G_2 by a small homeomorphism $h : X \to X$ such that $h(b) \in G_2$, $h(K_V \backslash \mathrm{bd}\overline{V}) \cap G_1 \neq \varnothing$ and $h(V)$ still contains b. This completes the proof of Claim 1.

Claim 2. Let V be a neighborhood of b satisfying Claim 1. If $H^{n-2}(C' \cap M \cap \mathrm{bd}\overline{V}; G) = 0$, we are done.

Indeed, let $A = M \cap \mathrm{bd}\overline{V}$, $P = K_V$ and $P_i = P \cap \overline{G}_i$, $i = 1, 2$. Clearly, then $P_1 \cup P_2 = P$ and $P_1 \cap P_2 = K_V \cap C'$. Hence, $A \cap P_1 \cap P_2 = C' \cap M \cap \mathrm{bd}\overline{V}$. Therefore, γ_V is a nontrivial element of $H^{n-1}(A; G)$ which is not extendable over P. Because $K_V \backslash \mathrm{bd}\overline{V}$ meets both sets G_1 and G_2, each $A \cup P_i$ is a proper subset of K_V containing A. So, γ_V is extendable over each $A \cup P_i$. Therefore, by Proposition 2.3, $H^{n-1}(C', C' \cap A) \neq 0$. On the other hand, we have the exact sequence

$$H^{n-2}(C' \cap A; G) \longrightarrow H^{n-1}(C', C' \cap A; G) \longrightarrow H^{n-1}(C'; G),$$

where $H^{n-2}(C' \cap A; G) = 0$. Since C' is a closed subset of C, $\dim_G C' \leq n-1$. The last inequality together with $H^{n-1}(C; G) = 0$ implies $H^{n-1}(C'; G) = 0$. Hence, $H^{n-1}(C', C' \cap A; G) = 0$, a contradiction. This completes the proof of Claim 2.

We use below the following notation: Suppose Π is partition in a space Z between two closed disjoint sets $P, Q \subset Z$. Then there are two open disjoint subset W_P, W_Q of Z containing P and Q, respectively, such that $Z \backslash \Pi = W_P \cup W_Q$. Then we denote $\Lambda_P = W_P \cup \Pi$ and $\Lambda_Q = W_Q \cup \Pi$.

Claim 3. Suppose that V is a neighborhood of b satisfying Claim 1. Then there is another neighborhood U of b with $\overline{U} \subset V$ such that:

(i) The element γ_V is extendable to an element $\gamma_{V,U} \in H^{n-1}(M(V,U); G)$, where $M(V,U) = \overline{V} \backslash U$;

(ii) The element $\gamma_U = j_{M(V,U),\mathrm{bd}\overline{U}}(\gamma_{V,U})$ is not extendable over the set $K_U = \mathrm{bd}\overline{U} \cup (K_V \cap \overline{U})$, but γ_U is extendable over each of the sets $K_{U,i} = \mathrm{bd}\overline{U} \cup (K_U \cap \overline{G}_i)$, $i = 1, 2$;

(iii) If Π separates $M(V,U)$ between $\mathrm{bd}\overline{U}$ and $\mathrm{bd}\overline{V}$, then there is $\gamma_\Pi \in H^{n-1}(\Pi; G)$ such that γ_Π is not extendable over $\Lambda_{\mathrm{bd}\overline{U}} \cup (K_V \cap \overline{U})$, but it is extendable over each set $\Lambda_{\mathrm{bd}\overline{U}} \cup (K_V \cap \overline{U} \cap \overline{G}_i)$, $i = 1, 2$.

Take points $x_i \in (K_V \backslash \mathrm{bd}\overline{V}) \cap G_i$, $i = 1,2$. Since K_V is an $(n-1)$-cohomology membrane for γ_V spanned on $M \cap \mathrm{bd}\overline{V}$ and each $K_{V,i} = (M \cap \mathrm{bd}\overline{V}) \cup (K_V \cap \overline{G}_i)$, $i = 1,2$, is a proper closed subset of K_V containing $M \cap \mathrm{bd}\overline{V}$, γ_V can be extended to $\gamma_i \in H^{n-1}(K_{V,i}; G)$, $i = 1,2$. Using that $\dim_G \mathrm{bd}\overline{V} \le n-1$, we can extend γ_V to an element $\gamma_V^* \in H^{n-1}(\mathrm{bd}\overline{V}; G)$. Then γ_V^* and γ_i provide elements $\gamma_i^* \in H^{n-1}(K_{V,i}^*; G)$, $i = 1,2$, where $K_{V,i}^* = \mathrm{bd}\overline{V} \cup (K_V \cap \overline{G}_i)$, such that $j_{K_{V,1}^*,\mathrm{bd}\overline{V}}(\gamma_1^*) = j_{K_{V,2}^*,\mathrm{bd}\overline{V}}(\gamma_2^*) = \gamma_V^*$.

Let K be a CW-complex of type $K(G, n-1)$ and the maps $f_V : \mathrm{bd}\overline{V} \to K$, $g_{V,i} : K_{V,i}^* \to K$ represent γ_V^* and γ_i^*, respectively, such that both restrictions $g_{V,1}|\mathrm{bd}\overline{V}$ and $g_{V,2}|\mathrm{bd}\overline{V}$ coincide with f_V. Since G is countable, K is also countable and homotopy equivalent to a metrizable simplicial complex. So, we can suppose that K is a metrizable simplicial complex, and let d be a metric on K. Because K is a neighborhood extensor for the class of metrizable spaces, there is an open cover ω of K such that any two ω-close maps from a given space Z into K are homotopic. Moreover, there is an open set Ω_i, $i = 1,2$, in X containing $K_{V,i}^*$ and a map $g_i : \overline{\Omega}_i \to K$ extending $g_{V,i}$ such that $\overline{\Omega}_i$ is compact. By the same reason, there is an open set $\Omega_0 \subset X$ with a compact closure and a map $g_0 : \overline{\Omega}_0 \to K$ extending f_V such that $\mathrm{bd}\overline{V} \subset \Omega_0$. We may assume that each $\overline{\Omega}_i$, $i = 0,1,2$, is contained in W. Since $\Theta = \bigcup_{i=0}^{i=2} g_i(\overline{\Omega}_i)$ is a compact subset of K, we can find $\eta > 0$ such that any two points $z_1, z_2 \in \Theta$ are contained in an element of ω provided $d(z_1, z_2) < \eta$. Then for every $i = 0,1,2$ there exist $\delta_i > 0$ such that $d(g_i(x), g_i(y)) < \eta/2$ for any $x, y \in \overline{\Omega}_i$ with $\rho(x,y) \le \delta_i$. Since the points x_1, x_2 and b belong to $V \backslash \mathrm{bd}\overline{V}$ and $\mathrm{bd}\overline{V} \subset \Omega_i$ for each $i = 0,1,2$, the number

$$\delta = \min\{\delta_i, \rho(\mathrm{bd}\overline{V}, X\backslash\Omega_i)/2, \rho(b, \mathrm{bd}\,\overline{V})/2, \rho(\{x_1\}\cup\{x_2\}, \mathrm{bd}\,\overline{V})/2 : i = 0,1,2\}$$

is positive, and let $U = \{x \in V : \rho(x, \mathrm{bd}\overline{V}) > \delta\}$. Clearly, U contains the points x_1, x_2, b. Moreover $\overline{U} \subset V$. Indeed, since $\overline{U} \subset \overline{V}$, if there is $x \in \overline{U} \backslash V$ then $x \in \mathrm{bd}\,\overline{V}$. So, $\rho(x, \mathrm{bd}\,\overline{V}) = 0$ which means that $x \notin \overline{U}$. Because $\delta \le \rho(\mathrm{bd}\overline{V}, X\backslash\Omega_i)/2$ for each $i = 0,1,2$, the set $\overline{V}\backslash U$ is contained in Ω_i. Hence, all maps g_i are well defined on $M(V,U) = \overline{V}\backslash U$. For every $x \in M(V,U)$ there exists $y \in \mathrm{bd}\,\overline{V}$ with $\rho(x,y) \le \delta$, and since $g_i(y) = f_V(y)$ for all i, we have $d(g_i(x), g_j(x)) < \eta$ for any $i,j \in \{0,1,2\}$ and $x \in M(V,U)$. This means that for all $x \in M(V,U)$ and $i,j \in \{0,1,2\}$ the points $g_i(x), g_j(x)$ belong to an element of ω. Therefore, for any closed set $B \subset M(V,U)$ the restrictions $g_{B,i} = g_i|B$, $i = 0,1,2$, are homotopic to each other and represent an element $\gamma_B \in H^{n-1}(B;G)$. In particular, all $g_{M(V,U),i}$ represent $\gamma_{V,U} \in H^{n-1}(M(V,U);G)$. Similarly, all maps $g_{\mathrm{bd}\overline{U},i}$ represent $\gamma_U \in H^{n-1}(\mathrm{bd}\overline{U};G)$. Moreover, since each $g_{M(V,U),i}$ extends $g_{B,i}$, we have $j_{M(V,U),B}(\gamma_{V,U}) = \gamma_B$ for all closed $B \subset M(V,U)$.

So, $j_{M(V,U),\mathrm{bd}\overline{V}}(\gamma_{V,U}) = \gamma_V^*$ and $j_{M(V,U),\mathrm{bd}\overline{U}}(\gamma_{V,U}) = \gamma_U$. This means that γ_V^* is extendable over $M(V,U)$. Hence, by Lemma 2.1, γ_V^* would be extendable over $M(V,U) \cup K_V$ provided γ_U is extendable over $K_U = \mathrm{bd}\overline{U} \cup (K_V \cap \overline{U})$. In such a case, γ_V would be extendable over K_V, a contradiction. Therefore, γ_U is not extendable over K_U.

Consider the sets $K_{U,i} = \mathrm{bd}\overline{U} \cup (K_U \cap \overline{G}_i)$, $i = 1,2$. Each $K_{U,i}$ is a proper closed subset of K_U because so is $K_V \cap \overline{U} \cap \overline{G}_i$ in $K_V \cap \overline{U}$. Observe also that $K_{V,i}^* \cup M(V,U) = M(V,U) \cup (K_V \cap \overline{U} \cap \overline{G}_i)$. On the other hand, Ω_i contains both $K_{V,i}^*$ and $M(V,U)$. So, Ω_i contains $K_{U,i}$, $i = 1,2$. Consequently, $g_i|K_{U,i}$ is well defined and extends $g_{\mathrm{bd}\overline{U},i}$. Since $g_{\mathrm{bd}\overline{U},i}$ represents γ_U, each $g_i|K_{U,i}$, $i=1,2$, represents an element $\mu_i \in H^{n-1}(K_{U,i};G)$ with $j_{K_{U,i},\mathrm{bd}\overline{U}}(\mu_i)=\gamma_U$. This means that γ_U is extendable over each of the sets $K_{U,i}$, $i=1,2$.

Finally, let $\Pi \subset M(V,U)$ be a closed set separating $M(V,U)$ between $\mathrm{bd}\overline{U}$ and $\mathrm{bd}\overline{V}$. Then $\Lambda_{\mathrm{bd}\overline{V}}$ contains both Π and $\mathrm{bd}\overline{V}$, $M(V,U)=\Lambda_{\mathrm{bd}\overline{V}}\cup\Lambda_{\mathrm{bd}\overline{U}}$ with $\Pi=\Lambda_{\mathrm{bd}\overline{V}}\cap\Lambda_{\mathrm{bd}\overline{U}}$. On the other hand, $j_{\Lambda_{\mathrm{bd}\overline{V}},\mathrm{bd}\overline{V}}(\gamma_{\Lambda_{\mathrm{bd}\overline{V}}})=\gamma_V^*$ means that γ_V^* is extendable over $\Lambda_{\mathrm{bd}\overline{V}}$. Since $j_{\Lambda_{\mathrm{bd}\overline{V}},\Pi}(\gamma_{\Lambda_{\mathrm{bd}\overline{V}}})=\gamma_\Pi$, according to Lemma 2.1, the assumption γ_Π is extendable over the set $\Lambda_{\mathrm{bd}\overline{U}}\cup(K_V\cap\overline{U})$ would imply that γ_V^* is extendable over $M(V,U)\cup K_V$, in particular γ_V would be extendable over K_V. So, γ_Π is not extendable over $\Lambda_{\mathrm{bd}\overline{U}}\cup(K_V\cap\overline{U})$. Let show that γ_Π is extendable over each set $\widetilde{\Lambda}_{\mathrm{bd}\overline{U},i}=\Lambda_{\mathrm{bd}\overline{U}}\cup(K_V\cap\overline{U}\cap\overline{G}_i)$, $i=1,2$. Observe that $\widetilde{\Lambda}_{\mathrm{bd}\overline{U},i}$ is contained in Ω_i, $i=1,2$. Consequently, each $h_i=g_i|\widetilde{\Lambda}_{\mathrm{bd}\overline{U},i}$ is well defined and extends $g_{\Pi,i}$. On the other hand, all $g_{\Pi,i}$, $i=0,1,2$, are homotopic to each other and represent γ_Π. Hence, each h_i, $i=1,2$, represents an element $\nu_i\in H^{n-1}(\widetilde{\Lambda}_{\mathrm{bd}\overline{U},i};G)$ with $j_{\widetilde{\Lambda}_{\mathrm{bd}\overline{U},i},\Pi}(\nu_i)=\gamma_\Pi$. Therefore, γ_Π is extendable over each $\widetilde{\Lambda}_{\mathrm{bd}\overline{U},i}$, $i=1,2$. This completes the proof of Claim 3.

Claim 4. Suppose U,V are neighborhoods of b satisfying the conditions from Claim 3. If $H^{n-2}(C'\cap K_V\cap\mathrm{bd}\overline{U};G)=0$, we are done.

Following the notations from Claim 3, denote $A=K_V\cap\mathrm{bd}\overline{U}$, $P=K_V\cap\overline{U}$ and $P_i=K_V\cap\overline{U}\cap\overline{G}_i$, $i=1,2$. Then $P_1\cup P_2=P$, $A\cap P_1\cap P_2=C'\cap K_V\cap\mathrm{bd}\overline{U}$. According to Claim 3, $\gamma_U\in H^{n-1}(\mathrm{bd}\overline{U};G)$ is not extendable over $K_U=\mathrm{bd}\overline{U}\cup(K_V\cap\overline{U})$. This means that the element $\mu_U=j_{\mathrm{bd}\overline{U},K_V\cap\mathrm{bd}\overline{U}}(\gamma_U)\in H^{n-1}(A;G)$ is not extendable over P. On the other hand, γ_U is extendable over each $K_{U,i}=\mathrm{bd}\overline{U}\cup(K_V\cap\overline{U}\cap\overline{G}_i)$, $i=1,2$. Consequently, μ_U is extendable over each $K_V\cap\overline{U}\cap\overline{G}_i$. To complete the proof of Claim 4, we can apply Proposition 2.3 as we did in Claim 2. This completes the proof of Claim 4.

Therefore, we can suppose everywhere below that there are two neighborhoods U,V of b satisfying the conditions of Claim 3 with $H^{n-2}(C'\cap M\cap\mathrm{bd}\overline{V};G)\neq 0$ and $H^{n-2}(C'\cap K_V\cap\mathrm{bd}\overline{U};G)\neq 0$. In particular, both $C\cap\mathrm{bd}\overline{V}$ and $C\cap\mathrm{bd}\overline{U}$ are nonempty.

Claim 5. Let V,U be neighborhoods of b satisfying the conditions from Claim 3 with $C\cap\mathrm{bd}\overline{V}\neq\varnothing\neq C\cap\mathrm{bd}\overline{U}$. Then there exists a partition Π in $M(V,U)=\overline{V}\backslash U$ between $\mathrm{bd}\overline{V}$ and $\mathrm{bd}\overline{U}$ such that $H^{n-2}(\Pi\cap C';G)=0$.

Consider the set $C\cap M(V,U)$ and its closed disjoint subsets $C\cap\mathrm{bd}\overline{V}$ and $C\cap\mathrm{bd}\overline{U}$. Since b has a special local base $\mathcal{B}_C^b$ in C, there is $W^*\in\mathcal{B}_C^b$ with $H^{n-2}(\mathrm{bd}_C W^*;G)=0$ such that $\mathrm{bd}_C W^*$ separates $C\cap M(V,U)$ between $C\cap\mathrm{bd}\overline{U}$ and $C\cap\mathrm{bd}\overline{V}$. By [16, Corollary 3.1.5], there exists a partition T in $M(V,V)$ between $\mathrm{bd}\overline{V}$ and $\mathrm{bd}\overline{U}$ such that $T\cap C\subset\mathrm{bd}_C W^*$. Hence, $\Pi=T\cup\mathrm{bd}_C W^*$ is a partition in $M(U,V)$ between $\mathrm{bd}\overline{V}$ and $\mathrm{bd}\overline{U}$ with $\Pi\cap C=\mathrm{bd}_C W^*$ and $H^{n-2}(\Pi\cap C;G)=0$. Finally, since $\Pi\subset\Gamma$, we have $\Pi\cap C=\Pi\cap C'$. This completes the proof of Claim 5.

Now, we can complete the proof of Theorem 1.3 when X is homogeneous. According to Claims 1–5, the proof is reduced to the assumption that there are two neighborhoods V,U of b and a partition Π of $M(V,U)$ between $\mathrm{bd}\overline{V}$ and $\mathrm{bd}\overline{U}$ such that $H^{n-2}(\Pi\cap C';G)=0$. Then, by Claim 3, there is $\gamma_\Pi\in H^{n-1}(\Pi;G)$ such that γ_Π is not extendable over $\Lambda_{\mathrm{bd}\overline{U}}\cup(K_V\cap\overline{U})$, but it is extendable over each set

$\Lambda_{\mathrm{bd}\overline{U}} \cup (K_V \cap \overline{U} \cap \overline{G}_i)$, $i = 1, 2$. In particular, γ_Π is extendable over each of the sets $\big(\Lambda_{\mathrm{bd}\overline{U}} \cup (K_V \cap \overline{U})\big) \cap \overline{G}_i$. We can apply Proposition 2.3 to obtain a contradiction. Indeed, denote $P = \Lambda_{\mathrm{bd}\overline{U}} \cup (K_V \cap \overline{U})$ and $P_i = P \cap \overline{G}_i$, $i = 1, 2$. Clearly $P_1 \cup P_2 = P$ and $P_1 \cap P_2 = \big(\Lambda_{\mathrm{bd}\overline{U}} \cup (K_V \cap \overline{U})\big) \cap C'$. Since $\Pi \subset \Lambda_{\mathrm{bd}\overline{U}}$ and $\Pi \cap \overline{U} = \varnothing$, $P_1 \cap P_2 \cap \Pi = \Pi \cap C'$. Finally, the exact sequence

$$H^{n-2}(C' \cap \Pi; G) \longrightarrow H^{n-1}(C', C' \cap \Pi; G) \longrightarrow H^{n-1}(C'; G)$$

shows that $H^{n-1}(C', C' \cap \Pi; G) = 0$ which contradicts Proposition 2.3. Therefore, the homogeneous case of Theorem 1.3 is established.

Consider now the case when X is strongly locally homogeneous.

Claim 6. The point b has a local base in X consisting of open sets V with $H^{n-2}(C' \cap \mathrm{bd}\overline{V}; G) = 0$.

Let W be an arbitrary neighborhood of b with $\overline{W} \subset \Gamma$. Since b has a base $\mathcal{B}_C^b$ in C consisting of sets U with $H^{n-2}(\mathrm{bd}_C U; G) = 0$, there is $U \in \mathcal{B}_C^b$ such that $\overline{U} \subset W$. Now, we use the following well-known fact [1]: If F is a closed subset of a metric space Z, then there is a correspondence $\mathrm{e} : \mathcal{T}(F) \to \mathcal{T}(Z)$ between the topologies of F and Z such that

$$\mathrm{e}(\Omega) \cap F = \Omega,\ \mathrm{e}(\Omega_1) \cap \mathrm{e}(\Omega_2) = \mathrm{e}(\Omega_1 \cap \Omega_2) \text{ and } \mathrm{e}(\varnothing) = \varnothing.$$

Such a correspondence is called a K_0*-function*. It is easily seen that $\overline{\mathrm{e}(\Omega)} \cap F = \overline{\Omega}$ for every open $\Omega \subset F$. Now, we consider a K_0-function $\mathrm{e} : \mathcal{T}(C' \cap \overline{W}) \to \mathcal{T}(\overline{W})$ and define $\mathrm{e}' : \mathcal{T}(C' \cap W) \to \mathcal{T}(W)$ by $\mathrm{e}'(\Omega) = \mathrm{e}(\Omega) \cap W$. Clearly, e' is also a K_0-function, and let $V = \mathrm{e}'(U)$. Then $b \in V$ and, according to the above-mentioned properties of K_0-functions, we have

$$\overline{V} \cap C' \subset \overline{\mathrm{e}(U)} \cap C' = \overline{\mathrm{e}(U)} \cap \overline{W} \cap C' = \overline{U}.$$

Since $U \subset \overline{V} \cap C'$, we obtain $\overline{V} \cap C' = \overline{U}$. Similarly, $V \cap C' = U$. Moreover, $U \cap \mathrm{bd}\,\overline{V} = \varnothing$ because $U \subset V$. So, $C' \cap \mathrm{bd}\,\overline{V} \subset \mathrm{bd}_{C'} U$. On the other hand, $V \cap C' = U$ implies that $\mathrm{bd}_{C'} U \subset \overline{V} \backslash V$. Therefore, $C' \cap \mathrm{bd}\,\overline{V} = \mathrm{bd}_{C'} U$. Clearly, $\mathrm{bd}_{C'} U = \mathrm{bd}_C U$. So, $H^{n-2}(C' \cap \mathrm{bd}\overline{V}; G) = 0$. This completes the proof of Claim 6.

Let W be as Lemma 2.2 and take another two neighborhoods V, U of b such that $\overline{U} \subset V \subset \overline{V} \subset W$, $H^{n-2}(C' \cap \mathrm{bd}\overline{V}; G) = 0$ and for every two points $x, y \in U$ there is a homeomorphism h on X with $h(x) = y$ and h is supported by U. According to the proof of Lemma 2.2, the element $\gamma \in H^{n-1}(F; G)$ is extendable to $\gamma' \in H^{n-1}(M \backslash U; G)$. Let $\gamma_V = j_{M\backslash U, M\cap \mathrm{bd}\overline{V}}(\gamma') \in H^{n-1}(M \cap \mathrm{bd}\overline{V}; G)$ and $\gamma_U = j_{M\backslash U, M\cap \mathrm{bd}\overline{U}}(\gamma') \in H^{n-1}(M \cap \mathrm{bd}\overline{U}; G)$. Then γ_V is not extendable over $M \cap \overline{V}$ (otherwise γ would be extendable over M). By the same reason, γ_U is not extendable over $M \cap \overline{U}$. Moreover, $\gamma_V = j_{M\cap(\overline{V}\backslash U), M\cap \mathrm{bd}\overline{V}}(\gamma'')$, where $\gamma'' = j_{M\backslash U, M\cap(\overline{V}\backslash U)}(\gamma')$. Hence, γ_V is extendable over $M \cap (\overline{V} \backslash U)$.

Let $A = (C' \cap \mathrm{bd}\overline{V}) \cup (M \cap \mathrm{bd}\overline{V})$. Since $\dim_G C' \cap \mathrm{bd}\overline{V} \le \dim_G C' \le n - 1$, there exists $\gamma_A \in H^{n-1}(A; G)$ extending γ_V. Observe that $A \cap C' = C' \cap \mathrm{bd}\overline{V}$ and $A \cap M = M \cap \mathrm{bd}\overline{V}$. Since γ_V is extendable over $M \cap (\overline{V} \backslash U)$, so is γ_A over $(M \cap (\overline{V} \backslash U)) \cup (C' \cap \mathrm{bd}\overline{V})$. On the other hand, γ_V being not extendable over $M \cap \overline{V}$ implies γ_A is not extendable over $(C' \cap \mathrm{bd}\overline{V}) \cup (M \cap \overline{V})$. Hence, there is an $(n-1)$-cohomology membrane $K_A \subset (M \cap \overline{V}) \cup (C' \cap \mathrm{bd}\overline{V})$ for γ_A spanned on A. Therefore, K_A meets $M \cap U$. We can suppose that $b \in K_A$. Indeed, if $b \notin K_A$

take a point $y \in K_A \cap U$ and a homeomorphism h on X supported by U with $h(y) = b$. Then $h(K_A)$ is an $(n-1)$-cohomological membrane for γ_A spanned on A and contains b.

We can suppose that $\dim_G \mathrm{bd}\overline{U} \le n-1$. Since $K_A \cap \overline{U} = (K_A \cap \mathrm{bd}\overline{U}) \cup (K_A \cap U)$ and $K_A \cap U$ is an F_σ-set, the assumption $\dim_G K_A \cap U \le n-1$ would imply that $\dim_G K_A \cap \overline{U} \le n-1$ (by the countable sum theorem for $\dim_G$). Then γ_U would be extendable over $K_A \cap \overline{U}$. Hence, because γ_A is extendable over $(M \cap (\overline{V} \setminus U)) \cup (C' \cap \mathrm{bd}\overline{V})$, we could extend γ_A over K_A. Therefore, $\dim_G K_A \cap U = n$. Consequently, $K_A \cap U$ meets at least one G_i, $i = 1, 2$. Suppose there is a point $x \in K_A \cap U \cap G_1$. Then $b \neq x$ because $b \notin G_1$. So, there exists a neighborhood U' of b such that $x \notin \overline{U}'$, $\overline{U}' \subset U$ and for every two points $x', y' \in U'$ there is a homeomorphism h' on X supported by U' with $h'(x') = y'$. Since $U' \cap G_2 \neq \varnothing$, we can push b by a homeomorphism φ on X supported by U' such that $\varphi(b) \in U' \cap G_2$. Then $\varphi(K_A)$ is an $(n-1)$-cohomology membrane for γ_A spanned on A meeting both G_i, $i = 1, 2$. Hence, γ_A is not extendable over $\varphi(K_A)$, but it is extendable over each $\varphi(K_A) \cap \overline{G}_i$. Finally, let $P = \varphi(K_A)$ and $P_i = \varphi(K_A) \cap \overline{G}_i$, $i = 1, 2$. Since $P_1 \cup P_2 = P$ and $A \cap P_1 \cap P_2 = A \cap C'$, we can apply Proposition 2.3 with $\gamma = \gamma_A$ to obtain that $H^{n-1}(C', A \cap C') \neq 0$. Finally, since $H^{n-2}(A \cap C'; G) = 0$ (recall that $A \cap C' = C' \cap \mathrm{bd}\overline{V}$), the exact sequence from the proof of Claim 2 implies $H^{n-1}(C', A \cap C') = 0$, a contradiction. This completes the proof of Theorem 1.3.

Acknowledgments

The author would like to express his gratitude to the referee for his/her valuable comments and suggestions. The author also thanks A. Dranishnikov and A. Karassev for their helpful discussions.

References

[1] Eric K. van Douwen, *Simultaneous linear extension of continuous functions*, General Topology and Appl. **5** (1975), no. 4, 297–319. MR380715

[2] A. N. Dranishnikov, *On a problem of P. S. Aleksandrov* (Russian), Mat. Sb. (N.S.) **135(177)** (1988), no. 4, 551–557, 560, DOI 10.1070/SM1989v063n02ABEH003290; English transl., Math. USSR-Sb. **63** (1989), no. 2, 539–545. MR942139

[3] Jerzy Dydak, *Cohomological dimension and metrizable spaces*, Trans. Amer. Math. Soc. **337** (1993), no. 1, 219–234, DOI 10.2307/2154319. MR1153013

[4] Jerzy Dydak and Akira Koyama, *Strong cohomological dimension*, Bull. Pol. Acad. Sci. Math. **56** (2008), no. 2, 183–189, DOI 10.4064/ba56-2-9. MR2431010

[5] Jerzy Dydak and Akira Koyama, *Cohomological dimension of locally connected compacta*, Topology Appl. **113** (2001), no. 1-3, 39–50, DOI 10.1016/S0166-8641(00)00018-3. Geometric topology: Dubrovnik 1998. MR1821845

[6] Edward G. Effros, *Transformation groups and C^*-algebras*, Ann. of Math. (2) **81** (1965), 38–55, DOI 10.2307/1970381. MR174987

[7] Peter J. Huber, *Homotopical cohomology and Čech cohomology*, Math. Ann. **144** (1961), 73–76, DOI 10.1007/BF01396544. MR133821

[8] Witold Hurewicz and Henry Wallman, *Dimension theory*, Princeton Mathematical Series, vol. 4, Princeton University Press, Princeton, NJ, 1941. MR6493

[9] Myrto Kallipoliti and Panos Papasoglu, *Simply connected homogeneous continua are not separated by arcs*, Topology Appl. **154** (2007), no. 17, 3039–3047, DOI 10.1016/j.topol.2007.06.016. MR2355509

[10] A. Karassev, P. Krupski, V. Todorov, and V. Valov, *Generalized Cantor manifolds and homogeneity*, Houston J. Math. **38** (2012), no. 2, 583–609. MR2954653

[11] Paweł Krupski, *Recent results on homogeneous curves and ANRs*, Topology Proc. **16** (1991), 109–118. MR1206457

[12] Paweł Krupski, *Homogeneity and Cantor manifolds*, Proc. Amer. Math. Soc. **109** (1990), no. 4, 1135–1142, DOI 10.2307/2048147. MR1009992

[13] V. I. Kuz'minov, *Homological dimension theory* (Russian), Uspehi Mat. Nauk **23** (1968), no. 5(143), 3–49. MR0240813

[14] Jan van Mill and V. Valov, *Homogeneous continua that are not separated by arcs*, Acta Math. Hungar. **157** (2019), no. 2, 364–370, DOI 10.1007/s10474-018-0881-0. MR3928779

[15] Jan van Mill, *A note on the Effros theorem*, Amer. Math. Monthly **111** (2004), no. 9, 801–806, DOI 10.2307/4145191. MR2104051

[16] Jan van Mill, *The infinite-dimensional topology of function spaces*, North-Holland Mathematical Library, vol. 64, North-Holland Publishing Co., Amsterdam, 2001. MR1851014

[17] Edwin H. Spanier, *Algebraic topology*, McGraw-Hill Book Co., New York-Toronto-London, 1966. MR210112

[18] V. Valov, *Homogeneous ANR-spaces and Alexandroff manifolds*, Topology Appl. **173** (2014), 227–233, DOI 10.1016/j.topol.2014.06.001. MR3227218

Department of Computer Science and Mathematics, Nipissing University, 100 College Drive, P.O. Box 5002, North Bay, Ontario P1B 8L7, Canada

Email address: `veskov@nipissingu.ca`