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CRITICAL POINT COUNTS IN KNOT COBORDISMS: ABELIAN AND METACYCLIC INVARIANTS

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ABSTRACT. For a pair of knots K_1 and K_0, we consider the set of four-tuples of integers (g, c_0, c_1, c_2) for which there is a cobordism from K_1 to K_0 of genus g having c_i critical points of each index i. We describe basic properties that such sets must satisfy and then build homological obstructions to membership in the set. These obstructions are determined by knot invariants arising from cyclic and metacyclic covering spaces.

1. INTRODUCTION

Given a pair of knots K_1 and K_0 in S^3, let $\mathcal{G}(K_1, K_0)$ denote the set of all four-tuples (g, c_0, c_1, c_2) of nonnegative integers for which there is a smooth orientable cobordism from K_1 to K_0 of genus g having c_i critical points of each index i. Our goal is to identify ways in which classical knot theory can provide constraints on this set. The value of c_1 is determined by those of g, c_0 and c_2, so our investigation is reduced studying the sets $\mathcal{G}_g(K_1, K_0)$ consisting of nonnegative pairs (c_0, c_2) for which there is a genus g cobordism from K_1 to K_0 having c_0 and c_2 critical points of index 0 and 2, respectively.

A number of well-studied problems can be formulated in terms of $\mathcal{G}(K_1, U)$, where U is the unknot: related topics include the knot four-genus, the slice-ribbon conjecture, problems related to the ribbon-number of ribbon knots, and general unknotting operations. The set $\mathcal{G}_0(K_1, K_0)$ is related to knot concordances and in particular to the existence and properties of ribbon concordances. Papers that touch on aspects of these topics include [1, 2, 5, 8, 14, 16, 19–22, 26–28, 34, 36]. Through the use of cyclic branched covers, this study is related to the study of the handlebody structure of cobordisms between three-manifolds, as presented, for instance, in [2].

We have several goals. The first is simply to present this perspective on knot cobordism. Next, we describe how homological invariants of cyclic branched covers of knots provide constraints on the sets $\mathcal{G}(K_1, K_0)$; this work consists of extensions of known results concerning ribbon disks and concordances to the setting of cobordisms. Our use of equivariant homology groups lets us further refine our results. After this, we consider the use of metacyclic invariants; these arise from cyclic covers of cyclic branched covers. Finally, we list some problems that arise from this perspective.

Summary of results. In seeking invariants from $M_n(K)$, the n–fold cyclic branched cover of a knot K, or from a q–fold cyclic cover of $M_n(K)$, one faces a series of choices: the values of n and q; the coefficients $\mathbb{F}$ for the homology groups; and

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the choice of which q–fold cover to consider. There is also a decision as to whether to take into account the module structure of the homology, viewed as an $\mathbb{F}[\mathbb{Z}_n]$–module or $\mathbb{F}[\mathbb{Z}_q]$–module. As has been done in the past, we will follow a path that is sufficiently complicated to illustrate the techniques but is simple enough to avoid technicalities. For instance, we will work with knots for which the associated $\mathbb{F}[\mathbb{Z}_n]$–modules are of a simple form.

Our main result that is based on cyclic branched covers is the following.

Theorem 5.1. *Suppose that Σ is a cobordism from K_1 to K_0. Then for all n, for all primes p satisfying $p-1 \equiv 0 \mod n$, and for all $\zeta \in \mathbb{F}_p$ satisfying $\zeta^n = 1$, we have*

$$c_0(\Sigma) \geq \frac{\beta_1^\zeta(M_n(K_1), \mathbb{F}_p) - \beta_1^\zeta(M_n(K_0), \mathbb{F}_p)}{2} - g(\Sigma).$$

In this statement, $\beta_1^\zeta(M_n(K_1), \mathbb{F}_p)$ is the dimension of the ζ–eigenspace of the $\mathbb{Z}_n$–action on $H_1(M_n(K), \mathbb{F}_p)$, where $\zeta \in \mathbb{F}_p$ is an n–root of unity in the finite field with p elements. Averaging over the set of n–roots of unity yields the following simplier, but often weaker, result.

Corollary 5.4. *Under the conditions of Theorem* 5.1*,*

$$c_0(\Sigma) \geq \frac{\beta_1(M_n(K_1), \mathbb{F}_p) - \beta_1(M_n(K_0), \mathbb{F}_p)}{2(n-1)} - g(\Sigma).$$

A simple application of Corollary 5.4 concerns 3–stranded pretzel knots: $P_k = P(2k+1, -2k-1, 2k+1)$. These are ribbon knots. It follows from Corollary 5.4 that if $2i+1$ and $2j+1$ are distinct primes, then there is a genus g cobordism from αP_i to βP_j having with $c_0 \geq 0$ and $c_2 \geq 0$ critical points of index 0 and 2, respectively, if and only if $c_0 \geq \alpha - g$ and $c_2 \geq \beta - g$. This is proved using 2–fold branched covers.

We will also present an example that depends on the full strength of Theorem 5.1, using higher-fold covers and the eigenspace splitting. The example is built from the knot 10_{153}, which is a ribbon knot with ribbon number 1 (see [23]). We show that there exists a genus g surface in B^4 bounded by $\alpha 10_{153}$ having c_0 and c_2 index 0 and index 2 critical points, respectively, if and only if $c_0 \geq \alpha + 1 - g$.

Examples in which metacyclic covers yield stronger results will be built from knots $K(k, J)$ illustrated on the left in Figure 1. In the figure, the right band is knotted. For instance, in the case that the knot J is the figure eight knot, the portion of the knot within the box labelled "J" would be replaced with the knotted band illustrated on the right in Figure 1. The left band has $-k$ full twists and the right band has $k+1$ full twists. If J is a ribbon knot, then this knot is ribbon: the simple closed curve that goes over each band once in opposite directions has framing 0 and has knot type J. This family is of interest because the Seifert form of $K(k, J)$ is independent of the choice of J, and thus no homological invariants arising from cyclic branched covers can be used to distinguish a pair $K(k, J_1)$ and $K(k, J_2)$. However, the branched cyclic covers $M_n(K(k, J))$ themselves have cyclic covers, and the homology of these iterated covers does depend on J. In Section 8 we will explore these examples in detail, focusing on the case of $k = 1$ and J is a multiple of either $K(1, U) = 6_1$ or $K(2, U) = 10_3$. The obstructions we develop are determined from 3–fold cyclic covers of the 2–fold branched cover of S^3, but the proofs of the results require that we consider covers of order 3^b for some unknown value of b. This is a reflection of an underlying issue that first appeared in [6].

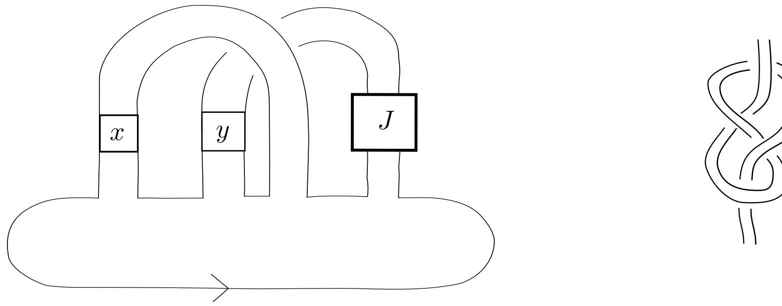

FIGURE 1. Basic knot $K(k, J)$. The two strands of the right hand band are knotted. In the case that J is the figure 8 knot, the box labeled "J" is replaced with the figure on the right. In the diagram, $x = -k$ and $y = k + 1$ denote full twists.

2. THE SET $\mathcal{G}_g(K_1, K_0)$

In this section, we present in detail the knot invariants of interest and describe some of their basic properties.

2.1. The definition of $\mathcal{G}_g(K_1, K_0)$. We view knots as smooth oriented diffeomorphism classes of pairs (S, K) where S is diffeomorphic to S^3 and K is diffeomorphic to S^1. We will be using the shorthand notation $K \subset S^3$ or simply K for such a pair; $-K$ denotes the pair $(-S, -K)$. A cobordism from a knot K_1 to a knot K_0 consists of a smooth oriented surface $\Sigma \subset S^3 \times [0, 1]$ for which $\partial(S^3 \times [0, 1], \Sigma) = -(S^3, K_0) \bigsqcup (S^3, K_1)$. (In particular, $\Sigma \cap (S^3 \times \{1\}) = K_1$.) We will assume that Σ is *connected.* We will also restrict our attention to *Morse cobordisms*, those for which the projection $\Sigma \to [0, 1]$ is a Morse function.

Viewing Σ as a twice punctured surface of genus g, we have that $\beta_1(\Sigma) = 2g+1$; alternatively, $g = (\beta_1(\Sigma) - 1)/2$. We will write $g(\Sigma)$ for the value of g.

We let $c_0(\Sigma), c_1(\Sigma)$, and $c_2(\Sigma)$ denote the number of local minima, saddle points, and local maxima of the projection of Σ to $[0, 1]$, respectively. The height function on Σ determines a handlebody structure on (Σ, K_0) having c_0, c_1, and c_2 handles of dimensions 0, 1, and 2, respectively. We will move between the Morse function and the handlebody decomposition without further comment.

If $g(\Sigma) = 0$, then Σ is called a *concordance.* If $c_2(\Sigma) = 0$, then Σ is called a *ribbon cobordism.*

An Euler characteristic argument shows that for a genus g cobordism with c_0, c_1, and c_2 critical points of each index, we have $c_1 = c_2 + c_0 + 2g$. Thus, to understand the counts of critical points of possible cobordisms, or equivalently the number of handles in the corresponding handlebody structure, we do not need to keep track of the value of c_1. (Many past papers focus on c_1, for instance in studying the ribbon number of ribbon knots, but notice that if there is a cobordism from K_1 to K_0 with c_1 saddle points, there is also a cobordism from K_0 to K_1 with c_1 saddle points; we can more readily highlight the asymmetry of the general problem by using c_0 and c_2.)

Definition 2.1. For knots K_1 and K_0, set

- $\mathcal{G}_g(K_1, K_0)$=$\{(c_0(\Sigma), c_2(\Sigma)) \big| \Sigma$ is a cobordism from K_1 to K_0 with $g(\Sigma)$=$g\}$ $\subset (\mathbb{Z}_{\geq 0})^2$.
- $\mathcal{G}(K_1, K_0) = \{(g, c_0, c_1, c_2) \bigm| (c_0, c_2) \in \mathcal{G}_g(K_1, K_0)$ and $c_1 = c_2 + c_0 + 2g\} \subset (\mathbb{Z}_{\geq 0})^4$.

2.2. **Elementary properties of $\mathcal{G}_g(K_1, K_0)$.** We begin with Proposition 2.2, a restatement of the definition of ribbon cobordism.

Proposition 2.2. *There exists a $c_0 \geq 0$ such that $(c_0, 0) \in \mathcal{G}_g(K_1, K_0)$ if and only if there exists a genus g ribbon cobordism from K_1 to K_0.*

A cobordism can be modified by adding a pair of critical points of indices 0 and 1, or of indices 1 and 2, without altering the genus. Thus we have the next result.

Proposition 2.3. *For a pair of knots K_1 and K_0, if $(c_0, c_2) \in \mathcal{G}_g(K_1, K_0)$, then $(c_0 + i, c_2 + j) \in \mathcal{G}_g(K_1, K_0)$ for all $i, j \geq 0$.*

It follows that each $\mathcal{G}_g(K_1, K_0)$ is a finite union of quadrants, $\bigcup_\alpha \mathcal{Q}(a_\alpha, b_\alpha)$, where
$$\mathcal{Q}(a, b) := \{(i, j) \bigm| i \geq a \text{ and } \ j \geq b \ \}.$$
Figure 2 illustrates the union of quadrants $\mathcal{Q}(2, 3) \bigcup \mathcal{Q}(5, 1)$.

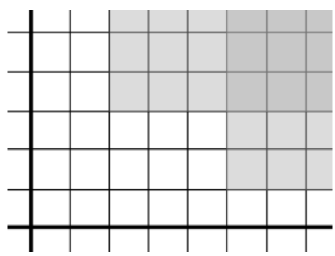

FIGURE 2. Graph of $\mathcal{Q}(2, 3) \bigcup \mathcal{Q}(5, 1)$

If for some pair of knots K_1 and K_0 and $g \geq 0$, the graphic in Figure 2 represents $\mathcal{G}_g(K_1, K_0)$, then the fact that there are no point on either axis implies that there does not exist a genus g ribbon cobordism from K_1 to K_0 or from K_0 to K_1.

Next, we observe the most basic ways in which points in $\mathcal{G}_g(K_1, K_0)$ determine points in $\mathcal{G}_{g+1}(K_1, K_0)$

Proposition 2.4. *For a pair of knots K_1 and K_0, suppose that $(c_0, c_2) \in \mathcal{G}_g(K_1, K_0)$.*

(1) *If $c_0 > 0$, then $(c_0 - 1, c_2) \in \mathcal{G}_{g+1}(K_1, K_0)$.*
(2) *If $c_2 > 0$, then $(c_0, c_2 - 1) \in \mathcal{G}_{g+1}(K_1, K_0)$.*

Proof. In terms of cross-sections of the cobordism, an index 0 critical point at height t corresponds to the addition of an unknotted, unlinked component to the cross-section of Σ as the height increases past t. The same addition can be realized by performing a trivial band move to the cross-section at height just below t. This corresponds to adding a critical point of index 1 in exchange for eliminating the index 0 critical point. It increases the genus by 1. A similar construction eliminates index 2 critical points. □

Example 2.5. Figure 3 illustrates how a point in $\mathcal{G}_g$ generates points in $\mathcal{G}_{g+1}$. In this example, the point $(4,2) \in \mathcal{G}_0$. Using Propostion 2.4 we see that $\{(3,2),(4,1)\} \subset \mathcal{G}_1$. This in turn implies that $\{(2,2),(3,1),(4,0)\} \subset \mathcal{G}_2$. It next follows that $\{(1,2),(2,1),(3,0)\} \subset \mathcal{G}_3$. As a consequence, we have $\{(0,2),(1,1),(2,0)\} \subset \mathcal{G}_4$ and then that $\{(0,1),(1,0)\} \subset \mathcal{G}_5$. Finally, $(0,0) \in \mathcal{G}_g$ for all $g \geq 6$.

In this example, if the first figure represents $\mathcal{G}_0$ for some pair of knots, we are not asserting the remaining diagrams illustrate the $\mathcal{G}_g$, but only that they represent subsets of the $\mathcal{G}_g$. Example 5.6 in Section 5 we will show that $\mathcal{G}(K_1, K_0)$ can be strictly larger than the set guaranteed by Proposition 2.4.

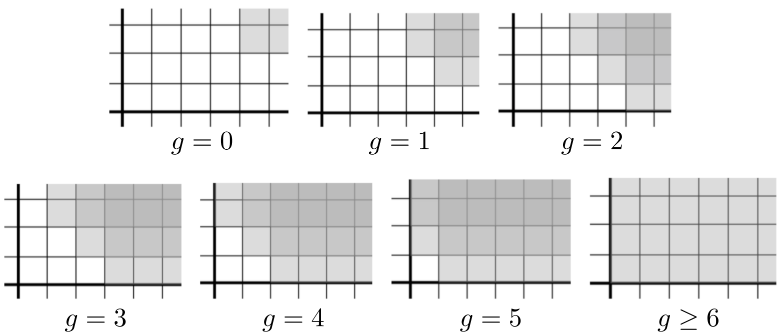

FIGURE 3. Possible sets $\mathcal{G}_g(K_1, K_2)$

2.3. **The set of $\mathcal{G}(K_1, K_2)$ and the associated sequence.** It is apparent that each $\mathcal{G}_g$ is determined by a unique finite set of points and that for large g, $\mathcal{G}_g$ consists of the entire quadrant. This is summarized in Theorem 2.6.

Theorem 2.6. *Each set $\mathcal{G}(K_1, K_0)$ is determined by a finite sequence*

$$\mathcal{S}(K_1, K_0) = \big((g_1, a_1, b_1), (g_2, a_2, b_2), (g_3, a_3, b_3), (g_4, a_4, b_4), \ldots, (g_k, 0, 0)\big)$$

of elements in $(\mathbb{Z}_{\geq 0})^3$ which is lexicographically ordered. There is a unique minimal length such sequence.

As an example, some of the terms of the lexigraphically ordered sequence corresponding to the regions in Figure 3 are

$$\big((0,4,2),(1,3,2),(1,4,1),(2,2,2),(2,3,1),(2,4,0),\ldots,(5,0,1),(5,1,0),(6,0,0)\big).$$

A general problem that seems to be beyond currently available techniques is to determine if there are any constraints on the sequences that can arise from a pair of knots other than those that are a consequence of Propositions 2.3 and 2.4. For instance, the ribbon conjecture can be stated as the following: if $(0, c_0, c_2) \in \mathcal{S}(K, U)$ for some c_0 and c_2, then $(0, c_0', 0) \in \mathcal{S}(K, U)$ for some c_0'. The generalized ribbon conjecture states that if $(g, c_0, c_1) \in \mathcal{S}(K, U)$ for some g, c_0 and c_2, then $(g, c_0', 0) \in \mathcal{S}(K, U)$ for some c_0'. See Section 10 for a further discussion.

2.4. **The case of K_0 is unknotted.** Understanding $\mathcal{G}_g(K, U)$ is equivalent to analyzing surfaces bounded by K in B^4. Given a knot $K \subset S^3$, we let $\Sigma \subset B^4$ with $\partial\Sigma = K$. We will assume the radial function is Morse on Σ; hence, we can define the count of critical points as before.

Definition 2.7. For knot a knot K, set

$$\mathcal{B}_g(K) = \{(c_0(\Sigma), c_2(\Sigma)) \mid \Sigma \subset B^4,\ \partial\Sigma = K,\ \text{and } g(\Sigma) = g\}.$$

The following is clear.

Proposition 2.8. *For any knot K, $(c_0, c_2) \in \mathcal{B}_g(K)$ if and only if $(c_0 - 1, c_2) \in \mathcal{C}_g(K, U)$.*

The sets $\mathcal{G}_g(K_1, K_0)$ and $\mathcal{B}_g(K_1 \# -K_0)$ are related, but note that in considering $\mathcal{G}_g(K_1 \# -K_0)$ we have lost the asymmetry of the general problem. Let $\mathrm{b}(K)$ denote the minimum number of index 0 critical points in a ribbon disk for $K \# -K$. This invariant is related to classical three-dimensional knot invariants. For instance, let $\mathrm{br}(K)$ denote the bridge index of K. A ribbon disk for $K \# -K$ with $c_0 = \mathrm{br}(K)$ and $c_1 = \mathrm{br}(K) - 1$ is easily constructed; thus $\mathrm{b}(K) \le \mathrm{br}(K)$. Results concerning the interplay between these invariants appear in [21, Section 1]. See also Problem (4) in Section 10.

Given a cobordism from K_1 to K_0, we can start with the ribbon surface for $K_0 \# -K_0$ to build a slicing surface for $K_1 \# -K_0$: use the cobordism to change $K_1 \# -K_0$ into $K_0 \# -K_0$ and then attach a slice disk. This leads to the next result.

Theorem 2.9. *If $(c_0, c_2) \in \mathcal{G}_g(K_1, K_0)$, then $(c_0 + \mathrm{b}(K_0), c_2) \in \mathcal{B}_g(K_1 \# -K_0)$*

In the reverse direction, given a surface bounded by $K_1 \# -K_0$, we can build a cobordism from K_1 to K_0: build a cobordism from K_1 to $K_1 \# -K_0 \# K_0$ and then cap it off with the surface bounded by $K_1 \# -K_0$. This yields the following.

Theorem 2.10. *If $(c_0, c_2) \in \mathcal{B}_g(K_1 \# -K_0)$, then $(c_0 - 1, c_2 + \mathrm{b}(K_0)) \in \mathcal{G}_g(K_1, K_0)$.*

3. Covering spaces and equivariant homology theory

In this section, we set up the notation for covering spaces and the general theory of the associated equivariant homology theory. We then consider a technical issue that arises from the following situation. A homomorphism $\rho\colon \pi_1(X) \to \mathbb{Z}_m$ determines a homomorphism $\rho\colon \pi_1(X) \to \mathbb{Z}_{km}$ for any k via inclusion; we will need to understand relationships between the equivariant homology groups of the associated m–fold and km–fold cyclic covers.

3.1. Cyclic covers of knots.

Let $K \subset S^3$ be a knot and let $\Sigma \subset S^3 \times [0,1]$ be a cobordism between knots.

Definition 3.1.

- $M_n(K)$ will denote n–fold cyclic cover of S^3 branched over K.
- $\widetilde{K}$ denotes the preimage of K in $M_n(K)$.
- $W_n(\Sigma)$ and $\widetilde{\Sigma}$ denote the n–fold cyclic cover of $S^3 \times [0,1]$ branched over Σ and the preimage of Σ.
- $M_\infty(K)$ and $W_\infty(\Sigma)$ will denote the infinite cyclic covers of $S^3 \setminus K$ and $\big(S^3 \times [0,1]\big) \setminus \Sigma$.

3.2. **Covering space theory.** For any group Γ, let K_Γ denote an Eilenberg-MacLane space for Γ and let E_Γ denote its universal cover. All spaces X considered here will be connected manifolds and covering spaces will be abelian, so we need not discuss details about the underlying point set topology and basepoint issues.

If X is connected and $\rho\colon \pi_1(X) \to \Gamma$ is a homomorphism, then it induces a map $X \to K_\Gamma$. The pullback of $E_\Gamma \to K_\Gamma$ to X is a covering space $\widetilde{X}_\rho$. Points in the preimage of a basepoint in $\widetilde{X}_\rho$ correspond to elements of Γ and components of $\widetilde{X}_\rho$ corresponds to cosets of $\rho(\pi_1(X)) \subset \Gamma$.

3.3. **Cyclic groups acting on vector spaces.** We collect here some basic algebraic results. Details can be found in texts such as [7]. Let $\mathbb{F}$ be a field of characteristic p, possibly 0, and let $\overline{\mathbb{F}}$ denote its algebraic closure. For any $m > 0$, the polynomial $t^m - 1 \in \overline{\mathbb{F}}[t]$ factors as $\prod_{0\le i<m}(t-\zeta_i)$, where the ζ_i are not necessarily distinct. For instance, $t^2 - 1 = (t-1)^2 \in \mathbb{F}_2[t]$.

Proposition 3.2. *If p does not divide m, then the roots ζ_i of $t^m - 1$ in $\overline{\mathbb{F}}$ are distinct.*

Proof. Suppose that $(t-\zeta_i)^2$ is a factor of $t^m - 1$. Then applying the product rule in computing the derivative of $t^m - 1$, we see that $(t-\zeta_i)$ is a factor of mt^{m-1}. However, since p does not divide m, the polynomial mt^{m-1} is nontrivial and does not have any nonzero roots. □

Proposition 3.3. *Suppose that V is a finite dimensional $\mathbb{F}$–vector space and T is a linear transformation of V satisfying $T^m - Id = 0$. If the characteristic of $\mathbb{F}$ does not divide m, then $V_s = \mathbb{F}_s \otimes V$ splits into eigenspaces under the induced action of T, where $\mathbb{F}_s \subset \overline{\mathbb{F}}$ is the splitting field for $t^m - 1$.*

Proof. View V_s as a module over $\mathbb{F}_s[t]$ by letting t act by the transformation T. Then since $\mathbb{F}_s[t]$ is a PID, V_s splits as a direct sum of modules $\mathbb{F}_s[t]/\langle f_j(t)\rangle$ where the $f_j(t)$ are powers of irreducible polynomials. We have $\left(\prod_i (t-\zeta_i)\right) V_s = (t^m-1)V_s = 0$. In particular, $(t^m-1)\mathbb{F}_s[t]/\langle f_j(t)\rangle = 0$. Hence, each $f_j(t)$ must be a factor of $t^m - 1$ in $\mathbb{F}_s[t]$; that is, it must be the form $f_j(t) = (t-\zeta_i)$ for some i. Each summand of the form $\mathbb{F}_s[t]/(t-\zeta_i)$ corresponds to a one-dimensional eigenspace for T having eigenvalue ζ_i. □

Continuing the notation of the previous results we have the following.

Proposition 3.4. *Let $\phi\colon \mathbb{F}_s \to \mathbb{F}_s$ be an automorphism of the extension $\mathbb{F}_s/\mathbb{F}$ for which $\phi(\zeta_i) = \zeta_j$. Then ϕ induces an isomorphism $\Phi\colon V_s \to V_s$ that carries the ζ_i eigenspace of T to the ζ_j eigenspace.*

Proof. The action of T on $V_s = \mathbb{F}_s \otimes V$ is given by $T(f\otimes v) = f \otimes T(v)$. The action of Φ is given by $\Phi(f\otimes v) = \phi(f)\otimes v$. These two actions commute. Notice that for any $\overline{v} \in V_s$ and $a \in \mathbb{F}_s$, we have $\Phi(a\overline{v}) = \phi(a)\Phi(\overline{v})$; this follows from the observation that $\Phi(a(f\otimes v)) = \Phi(af\otimes v)) = \phi(af)\otimes v = \phi(a)\phi(f)\otimes v = \phi(a)(\phi(f)\otimes v) = \phi(a)\Phi(f\otimes v)$.

Let $\overline{v} \in V_s$ be a ζ_i-eigenvector. Then $T(\Phi(\overline{v})) = \Phi(T(\overline{v})) = \Phi(\zeta_i\overline{v}) = \zeta_j\Phi(\overline{v})$, showing that $\Phi(\overline{v})$ is a ζ_j-eigenvector for T, as desired. □

Note. In the case that the characteristic p of $\mathbb{F}$ does not divide m, the roots of $t^m - 1$ form a cyclic group of order m under multiplication. To see this, note that the set of roots forms an abelian group of order m, and if not cyclic, it would contain a subgroup isomorphic to $\mathbb{Z}_q \oplus \mathbb{Z}_q$ for some prime q. This would imply that the polynomial $t^q - 1$ would have at least q^2 distinct roots, which is not possible. There is a Galois automorphism Φ carrying ζ_i to ζ_j if and only if ζ_j and ζ_j are of the same order.

3.4. **The equivariant homology of CW-complexes with cyclic group actions.** Let $\widetilde{X}$ be a connected finite CW-complex and let $T\colon \widetilde{X} \to \widetilde{X}$ be an order m homeomorphism that preserves the CW-structure.

We work with a field $\mathbb{F}$ of characteristic that does not divide m. Then the complex $C_*(\widetilde{X}, \mathbb{F}_s)$ splits into eigenspaces which we denote $C_*^{\zeta_i}(\widetilde{X}, T, \mathbb{F}_s)$ and the homology groups $H_k(\widetilde{X}, \mathbb{F}_s)$ split into eigenspaces denoted $H_k^{\zeta_i}(\widetilde{X}, T, \mathbb{F}_s)$. The dimension of $H_k^{\zeta_i}(\widetilde{X}, T, \mathbb{F}_s)$ will be denoted $\beta_k^{\zeta_i}(\widetilde{X}, T, \mathbb{F}_s)$. When possible, we will suppress the appearance of the T in the notation when it is understood and we will supress subscripts that indicate gradings when possible.

Theorem 3.5. *For all i, we have $H^{\zeta_i}(\widetilde{X}, \mathbb{F}_s) \cong H(C^{\zeta_i}(\widetilde{X}, \mathbb{F}_s))$.*

Proof. There are two splittings of $H(\widetilde{X}, \mathbb{F}_s)$:

$$\oplus_{i=0}^{m-1} H^{\zeta_i}(\widetilde{X}, \mathbb{F}_s) \cong H(\widetilde{X}, \mathbb{F}_s) \cong H(C(\widetilde{X}, \mathbb{F}_s)) \cong \oplus_{i=0}^{m-1} H(C^{\zeta_i}(\widetilde{X}, \mathbb{F}_s)).$$

Any cycle in $C^{\zeta_i}(\widetilde{X}, \mathbb{F}_s)$ represents a class in $H^{\zeta_i}(\widetilde{X}, \mathbb{F}_s)$ and thus under the isomorphism from the direct sum on the right side to the one on the left, $H(C^{\zeta_i}(\widetilde{X}, \mathbb{F}_s))$ maps to $H^{\zeta_i}(\widetilde{X}, \mathbb{F}_s)$. In general, if an isomorphism of vectors spaces $\oplus_i A_1 \to \oplus_i B_i$ maps A_i to B_i for all i, then it induces isomorphisms $A_i \to B_i$ for all i. $\square$

3.5. **Cyclic covers of CW-complexes.** In the case that $\widetilde{X}$ is an m–fold cyclic cover of a finite CW-complex X, we can consider the homology of $\widetilde{X}$ with the lifted CW-structure and with T a generating deck transformation.

For each cell $x \in X$ we choose a lift $\widetilde{x}$ in $\widetilde{X}$.

Theorem 3.6. *The complex $C^{\zeta_i}(\widetilde{X}, T, \mathbb{F}_s)$ has basis the set $\{\sum_{k=0}^{m-1} \zeta_i^{-k} T^k(\widetilde{x})\}_{x \in X}$.*

Proof. As an $\mathbb{F}_s[t]$–module, $C(\widetilde{X}, \mathbb{F}_s)$ has basis $\{\widetilde{x}\}$. To determine the eigenspace decomposition of the action, we can focus on the eigenspace of the restriction of the action to the m–dimensional summand generated by the orbit of a single $\widetilde{x}$.

For each x, the action of T on the orbit of $\widetilde{x}$ is algebraically the same as that of t on $\mathbb{F}_s[t]/\langle t^m - 1\rangle$. A simple calculation shows that $\sum_{k=0}^{m-1} \zeta_i^{-k} t^k(\widetilde{x})$ is a ζ_i–eigenvector for this action. Given that the dimension of $\mathbb{F}_s[t]/\langle t^m - 1\rangle$ over $\mathbb{F}_s$ is m and we have found m eigenvectors with distinct eigenvalues, each of these eigenvectors must form the basis for a one-dimensional eigenspace. $\square$

Corollary 3.7. *For all m–roots of unity ζ_i, $\dim C_k^{\zeta_i}(\widetilde{X}, T, \mathbb{F}_s) = \dim C_k(X)$.*

There is a special case of interest to us. Enumerate the ζ_i so that $\zeta_0 = 1$. The map $\tau\colon x \to \sum T^k(\widetilde{x})$ is easily seen to commute with the boundary map and so yields a chain map $\tau\colon C(X, \mathbb{F}_s) \to C^{\zeta_0}(\widetilde{X}, \mathbb{F}_s)$ called the transfer map. We have the following.

Corollary 3.8. *For all k, $\tau_*\colon H_k(X, \mathbb{F}_s) \to H_k^{\zeta_0}(\widetilde{X}, \mathbb{F}_s)$ is an isomorphism.*

Proof. By Theorem 3.6 the map τ is surjective. Let $\pi\colon C(\widetilde{X}) \to C(X)$ be the map induced by projection. The composition $\pi \circ \tau\colon C_k(X, \mathbb{F}_s) \to C_k(X, \mathbb{F}_s)$ is multiplication by m, and since the characteristic of $\mathbb{F}_s$ does not divide m, this map is injective. In particular, τ is injective. Thus, τ is a chain isomorphism and induces an isomorphism on homology. $\square$

3.6. **Relations between equivariant Betti numbers.** For a connected finite CW complex X, suppose that $\rho\colon \pi_1(X) \to \mathbb{Z}_m$ is a homomorphism. Let $\rho'\colon \pi_1(X) \to \mathbb{Z}_{km}$ be induced by the inclusion $\mathbb{Z}_m \subset \mathbb{Z}_{km}$.

Theorem 3.9. *With the conditions given above, the induced km–fold cover of X is the disjoint union of k copies of the m–fold cover of X:*

$$\widetilde{X}_{\rho'} \cong \widetilde{X}_\rho \sqcup \widetilde{X}_\rho \sqcup \cdots \sqcup \widetilde{X}_\rho.$$

The order km deck transformation shifts each summand to the next. The last summand is mapped to the first via the order m deck transformation of $\widetilde{X}_\rho$.

Proof. This result follows from standard covering space theory. $\square$

Theorem 3.10. *Suppose that $\rho\colon \pi_1(X) \to \mathbb{Z}_m$ is a homomorphism and $\rho'\colon \pi_1(X) \to \mathbb{Z}_{km}$ is the composition of ρ with the inclusion $\mathbb{Z}_m \subset \mathbb{Z}_{km}$. Let T_ρ be the order m deck transformation of $\widetilde{X}_\rho$ and let $T_{\rho'}$ be the order km deck transformation of $\widetilde{X}_{\rho'}$. Then the k power of $T_{\rho'}$ is a transformation of order m and*

$$\beta_i^\zeta(\widetilde{X}_{\rho'}, T_{\rho'}^k, \mathbb{F}) = k\beta_i^\zeta(\widetilde{X}_\rho, T_\rho, \mathbb{F}).$$

Proof. The action of the k power of $T_{\rho'}$ leaves invariant each factor $\widetilde{X}_\rho$ in the decomposition given by Theorem 3.9, $\widetilde{X}_\rho \sqcup \widetilde{X}_\rho \sqcup \cdots \sqcup \widetilde{X}_\rho$. It restricts to each factor to be the deck transformation of $\widetilde{X}_\rho$. $\square$

3.7. **Pairs of spaces.** Let (X, Y) be a CW–pair and let $\rho\colon H_1(X) \to \mathbb{Z}_m$. Then there is an associated covering space pair $\widetilde{(X, Y)}$ and we can consider the equivariant relative homology groups of this cover. All the results of this section carry over to this relative setting.

3.8. **Computing the equivariant homology for spaces associated to knots.** For any given knot, the computation of $\beta_i^\zeta(M_n(K), \mathbb{F}_p)$ is fairly straightforward, using little more that what is covered in, say, the text by Rolfsen [33]. The computation of the metacyclic invariants can be technically challenging; in particular, they are not determined by a Seifert matrix. For this reason, we will restrict our examples to those for which for which the computation is quickly accessible.

4. Handlebody structure

Theorem 4.1. *The pair $(W_n(\Sigma) \setminus \widetilde{\Sigma}, M_n(K_0) \setminus \widetilde{K})$ has a relative handlebody decomposition with:*

- *$nc_0(\Sigma)$ 1–handles.*
- *$nc_1(\Sigma)$ 2–handles.*
- *$nc_2(\Sigma)$ 3–handles.*

Proof. See, for instance, [13, Proposition 6.2.1] for a description of the handlebody structure on $(S^3 \times [0, 1]) \setminus \Sigma$. That structure lifts to the covering space. $\square$

Theorem 4.2. *The pair $(W_n(\Sigma), M_n(K_0))$ has a relative handlebody decomposition with:*

- $nc_0(\Sigma)$ 1–*handles.*
- $nc_1(\Sigma)$ 2–*handles.*
- $nc_2(\Sigma) + 2g(\Sigma)$ 3–*handles.*

Proof. We have that $(W_n(\Sigma), M_n(K_0))$ is built from $(W_n(\Sigma) \setminus \widetilde{\Sigma}, M_n(K_0) \setminus \widetilde{K}_0)$ via handle additions. For each i–handle in Σ there is an $(i+2)$–handle added.

The surface Σ can be built with one 0–cell and $\beta_1(\Sigma)$ 1–cells. The 0–cell and the first 1–cell comprise K_0. Hence, in building $(W_n(\Sigma), M_n(K_0))$ the added 2–handle and the first 3–handle complete the construction of a product neighborhood of $M_n(K_0)$. There remain $(\beta_1(\Sigma) - 1)$ 3–handles to add. Finally, $\beta_1(\Sigma) - 1 = 2g$. $\square$

5. Homological constraints arising from cyclic branched covers

5.1. **Homological constraints.** In this section, we will denote the order n deck transformation of $M_n(K)$ by T. That is, no confusion should result by using the symbol T without notating its dependence on K and n. We will work with finite fields of prime order, $\mathbb{F}_p$, that contain primitive n–roots of unity; that is, $p - 1 \equiv 0 \mod n$. Unless specified, we will not assume that a given n–root of unity ζ is primitive. The main result of this section is the next theorem. Notice that the coefficients are in $\mathbb{F}_p$ rather than the splitting field. This is made possible by the assumption that $p - 1$ is divisible by n; the order of the multiplicative group of nonzero elements in $\mathbb{F}_p$ is cyclic of order $p - 1$, so if n divides $p - 1$ then it contains n elements of order n, and thus $\mathbb{F}_p$ is the splitting field for $t^n - 1$. Notice also that if n divides $p - 1$, then n and p are relatively prime.

Theorem 5.1. *Suppose that Σ is a cobordism from K_1 to K_0. Then for all n, for all prime p satisfying $p - 1 \equiv 0 \mod n$, and for all $\zeta \in \mathbb{F}_p$ satisfying $\zeta^n = 1$, we have*

$$c_0(\Sigma) \geq \frac{\beta_1^\zeta(M_n(K_1), T, \mathbb{F}_p) - \beta_1^\zeta(M_n(K_0), T, \mathbb{F}_p)}{2} - g(\Sigma).$$

Before proving this, we isolate the case $\zeta = 1$ in a lemma and then prove another lemma that will simplify our exposition.

Lemma 5.2. *Let K be a knot and let $\{n, p\}$ be a relatively prime pair. Then the 1–eigenspace of the deck transformation acting on $H_1(M_n(K), \mathbb{F}_p)$ is trivial.*

Proof. The proof is a slight generalization of that for Corollary 3.8; we must now take into account the branch set. We can still define the transfer map on the chain level. For each cell x in the decomposition of (S^3, K), the map τ is defined by $\tau(x) = \sum_{j=0}^{n-1} T^j \widetilde{x}$, where $\widetilde{x}$ is a chosen lift of x. It is clear the image of τ is contained in the 1–eigenspace. To verify that it is surjective, one considers cells that are in, or not in, K, separately.

For x not in K, $\tau(x)$ is the generator of the 1–eigenspace of the action restricted to the orbit of $\widetilde{x}$. This follows from Theorem 3.6 in the case of $\zeta_i = 1$.

If x is in K, then the orbit of $\widetilde{x}$ is a single cell which is a 1–eigenvector. Since p does not divide n, $\tau(x) = nx$ is nontrivial.

Again denoting the chain map induced by the projection map from $M_n(K)$ to S^3 by π, we have that the composition $\pi \circ \tau$ is multiplication by n on the chain level. Thus, it is a chain isomorphism, implying that τ is injective and thus an isomorphism from the chain complex of (S^3, K) to the 1–eigenspace of the action of T on the associated cellular decomposition of $M_n(K)$. The result now follows from the fact that $H_1(S^3, \mathbb{F}_p) = 0$. $\square$

Lemma 5.3. *Let (W, M) be a CW–pair supporting an action T of $\mathbb{Z}_n$. Suppose that $\mathbb{F}$ is a field containing $n-1$ distinct elements $\zeta \neq 1$ satisfying $\zeta^n = 1$. Finally, assume that T preserves the components of M; that is, that T_* acts trivially on $H_0(M, \mathbb{F})$. Then with $\mathbb{F}$–coefficients,*

$$\beta_1(W) \leq \beta_1^\zeta(M) + \dim(C_1^\zeta(W, M))$$

and

$$\beta_1(W) \geq \beta_1^\zeta(M) + \dim(C_1^\zeta(W, M)) - \dim(C_2^\zeta(W, M)).$$

Proof. Removing cells of dimension 3 or higher does not affect any of the terms in the statement, so we can assume that W is a 2–complex. In the proof, to simplify the presentation we suppress the $\mathbb{F}$ in notation for chain complexes, homology groups, and Betti numbers.

The group $H_0^\zeta(M) = 0$. Thus, from the long exact sequence, we have

$$H_2^\zeta(W, M) \to H_1^\zeta(M) \to H_1^\zeta(W) \to H_1^\zeta(W, M) \to 0.$$

From this it follows that

$$\beta_1^\zeta(W) = \beta_1^\zeta(W, M) + \beta_1^\zeta(M) - \dim\big(\mathrm{Image}(H_2^\zeta(W, M) \to H_1^\zeta(M))\big). \tag{1}$$

Since $\beta_1^\zeta(W, M) \leq \dim(C_1^\zeta(W, M))$, the first inequality in the statement of the lemma is immediate.

We have $\dim\big(\mathrm{Image}(H_2^\zeta(W, M) \to H_1^\zeta(M))\big) \leq \beta_2^\zeta(W, M)$; substituting into Equation 1 yields

$$\beta_1^\zeta(W) \geq \beta_1^\zeta(M) + \beta_1^\zeta(W, M) - \beta_2^\zeta(W, M).$$

We have that $c_0^\zeta(W, M) = 0$, so a standard Euler characteristic argument implies that $\beta_1^\zeta(W, M) - \beta_2^\zeta(W, M) = \dim\big(C_1^\zeta(W, M)\big) - \dim\big(C_2^\zeta(W, M)\big)$. Hence,

$$\beta_1^\zeta(W) \geq \beta_1^\zeta(M) + \dim\big(C_1^\zeta(W, M)\big) - \dim\big(C_2^\zeta(W, M)\big),$$

as desired. ☐

Proof of Theorem 5.1. To simplify notation, we let $W = W_n(\Sigma) \setminus \widetilde{\Sigma}$, $\partial_0 W = M_n(K_0) \setminus \widetilde{K}_0$ and $\partial_1 W = M_n(K_1) \setminus \widetilde{K}_1$.

The 1–handles and 2–handles in the relative handlebody structure on $(W, \partial_0 W)$ are each freely permuted by the action of the generating deck transformation T. That is, for $i = 1$ and $i = 2$ we have that as an $\mathbb{F}_p[\mathbb{Z}_n]$–module the CW–chain complex $C_i(W, \partial_0 W, \mathbb{F}_p)$ is free; it is isomorphic to c_{i-1} copies of $\mathbb{F}_p[\mathbb{Z}_n]$. We identify $\mathbb{F}_p[\mathbb{Z}_n]$ with $\mathbb{F}_p[T]/\left<1 - T^n\right>$, where the action of the generator of $\mathbb{Z}_n$ is given by multiplication by T. Each of these splits into n eigenspaces; letting ξ be a primitive n–root of unity,

$$\mathbb{F}_p[T]/\left<1 - T^n\right> \cong \oplus_{i=0}^{n-1} \mathbb{F}_p[T]/\left<\xi^i - T\right>.$$

We have that $\zeta = \xi^i$ for some i, so the ζ–eigenspace of the relative CW–chain complex of $(W, \partial_0 W)$ has c_0 generators in dimension 1 and c_1 generators in dimension (2). That is, $\dim(C_i^\zeta(W, \partial_0 W, \mathbb{F}_p)) = c_{i-1}$. The first inequality of Lemma 5.3 gives

$$\beta_1^\zeta(W, T, \mathbb{F}_p) \leq \beta_1(\partial_0 W, T, \mathbb{F}_p) + c_0(\Sigma).$$

We have a similar construction of W starting with $\partial_1 W = M_n(K_1) \setminus \widetilde{K}_1$. In this case, we have $\dim(C_1^\zeta(W, \partial_1 W)) = c_2$ and $\dim(C_2^\zeta(W, \partial_1 W)) = c_1$. Using the second inequality in Lemma 5.3,

$$\beta_1^\zeta(W,T,\mathbb{F}_p) \geq \beta_1^\zeta(\partial_1 W,T,\mathbb{F}_p) + c_2(\Sigma) - c_1(\Sigma).$$

Combining these, we see that

$$\beta_1^\zeta(\partial_0 W,\mathbb{F}_p) + c_0(\Sigma) \geq \beta_1^\zeta(\partial_1 W,\mathbb{F}) + c_2(\Sigma) - c_1(\Sigma).$$

Recall that $c_1(\Sigma) = c_0(\Sigma)+c_2(\Sigma)+2g(\Sigma)$. The previous inequality can be rewritten as

$$\beta_1^\zeta(\partial_0 W,T,\mathbb{F}_p) + c_0(\Sigma) \geq \beta_1(\partial_1 W,T,\mathbb{F}_p) + c_2(\Sigma) - (c_0(\Sigma) + c_2(\Sigma) + 2g(\Sigma)).$$

This inequality simplifies to give

$$c_0(\Sigma) \geq \frac{\beta_1^\zeta(\partial_1 W,T,\mathbb{F}_p) - \beta_1^\zeta(\partial_0 W,T,\mathbb{F}_p)}{2} - g(\Sigma).$$

The proof is finished by noting that completing the covers to form the branched cyclic covers adds generators to the CW–complex that are all in the 1–eigenspace and thus do not change the computation. □

Early work [28] studying ribbon knots provided homological constraints on the minimum number of index 1 critical points in a ribbon disk based on the homology of the 2–fold branched covers. The next theorem is a fairly simple generalization of such results. Notice that we do not restrict to the ribbon situation, 2–fold covers, or the case of $g = 0$.

Corollary 5.4. *Under the conditions of Theorem* 5.1*,*

$$c_0(\Sigma) \geq \frac{\beta_1(M_n(K_1),\mathbb{F}_p) - \beta_1(M_n(K_0),\mathbb{F}_p)}{2(n-1)} - g(\Sigma).$$

Proof. The proof consists of summing over the $n-1$ eigenspaces. By Lemma 5.2 the 1–eigenspace is trivial. □

Example 5.5. Let P_k denote the pretzel knot $P(2k+1,-2k-1,2k+1)$. These are ribbon knots having Seifert form

$$\begin{pmatrix} 0 & k \\ k+1 & 0 \end{pmatrix}.$$

Each bounds a ribbon disk with one saddle point and two minimum. For a knot with Seifert form V, the homology of its 2–fold branched cover is presented by $V + V^{\mathsf{T}}$. Thus, we have $H_1(M_2(P_k)) \cong \mathbb{Z}_{2k+1} \oplus \mathbb{Z}_{2k+1}$. (Presentations of the homology of general cyclic branched covers are described in Theorem A.1.) In the examples that follow, we consider P_1 and P_2, and so have $H_1(M_2(P_1)) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$ and $H_1(M_2(P_2)) \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5$

We want to consider the sets $\mathcal{G}_g(nP_1, mP_2)$ and for convenience assume that $n \geq m$. This example presents the case of $g = 0$ and the next considers $g > 0$.

Our first observation is that aP_k bounds a ribbon disk with a saddle points and $a+1$ minima. From this it is easily seen that there is a concordance Σ from nP_1 to mP_2 with $c_0(\Sigma) = n$, $c_1(\Sigma) = n+m$, and $c_2(\Sigma) = m$. That is, $(n,m) \in \mathcal{G}_0(nP_1, mP_2)$.

Using $\mathbb{Z}_3$–coefficients in Corollary 5.4, we see that

$$c_0(\Sigma) \geq \frac{2n-0}{2} - 0 = n.$$

Similar, working with $\mathbb{Z}_5$–coefficients we have $c_2(\Sigma) \geq m$. Thus, $\mathcal{G}_0(nP_1, mP_2)$ is precisely the quadrant with vertex (n, m), that is $\mathcal{Q}(n, m)$.

Example 5.6. If $m > 0$ and $n > 0$, then by Proposition 2.4 we have $\mathcal{Q}(n-1, m) \cup \mathcal{Q}(n, m-1) \subset \mathcal{G}_1(nP_1, mP_2)$. Here we show that this is a proper containment, that in fact, $\mathcal{G}_1(nP_1, mP_2) = \mathcal{Q}(n-1, m-1)$.

The construction of a cobordism is simple. In the initial concordance that we built, the local maxima were at levels below the local minima. Because of this, the concordance can be modified by replacing disk neighborhoods of a maximum point and a minimum point by an annulus near an increasing path from the maximum to the minimum. The effect is to decrease both c_0 and c_2 by 1 in exchange for increasing the genus by 1; that is $\mathcal{G}_1(nP_1, mP_2) \subset \mathcal{Q}(n-1, m-1)$. Corollary 5.4 immediately implies that this inclusion must be an equality.

The process can be repeated to prove that for $g \leq m$ we have $\mathcal{G}_g(nP_1, mP_2) = \mathcal{Q}(n-g, m-g)$.

Finally, Proposition 2.4 implies that for $m \leq g \leq n$ we have $\mathcal{G}_g(nP_1, mP_2) = \mathcal{Q}(n-g, 0)$. For $g \geq n$ we have $\mathcal{G}_g(nP_1, mP_2) = \mathcal{Q}(0, 0)$.

Figure 4 illustrates the sets $\mathcal{G}_g(4P_1, 2P_2)$.

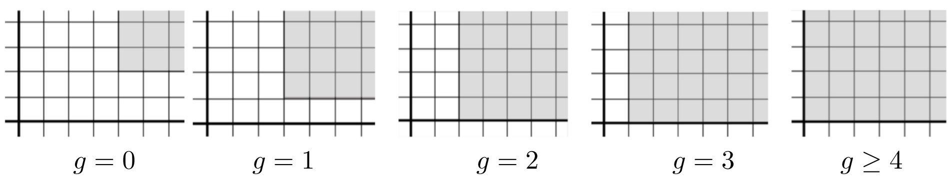

FIGURE 4. $\mathcal{G}_g(4P_1, 2P_2)$

Example 5.7. Let $K = 10_{153}$. We consider cobordisms to the unknot. For this knot $H_1(M_2(K)) = 0$ and $H_1(M_5(K)) \cong \mathbb{Z}_{11} \oplus \mathbb{Z}_{11}$. Clearly Theorem 5.1 and Corollary 5.4 provide no information in the case of 2–fold covers. Using 5–fold covers does. The four primative 5–roots of unity in $\mathbb{F}_{11}$ are $3, 4, 5$, and 9.

We first observe that there are precisely two nontrivial eigenspaces for the $\mathbb{Z}_5$ action on $\mathbb{Z}_{11} \oplus \mathbb{Z}_{11}$, each 1–dimensional, as can be seen as follows. Clearly there are at most two nontrivial eigenspaces. Poincaré duality implies that if there is a ζ–eigenvector, there is also a ζ^{-1}–eigenvector; we present a proof of this in the appendix as Lemma B.1.

Using either eigenvalue, Theorem 5.1 implies that for any cobordism from nK to the unknot, we have $c_0(\Sigma) \geq n - g$. Corollary 5.4 yields the weaker result that $c_0(\Sigma) \geq n/2 - g$. The improvement by a factor of two is expected, since two of the eigenspaces are trivial and two have dimension 1.

6. The infinite cyclic cover and the Alexander module

It has been known that the rank of the Alexander module of a knot has an upper bound that is determined by the genus of a surface bounded by the knot in B^4 and the critical point structure of that surface. We now generalize that observation, focusing on cobordisms.

Recall that $M_\infty(K_i)$ and $W_\infty(\Sigma)$ represent the infinite cyclic covers of the complements of the K_i and Σ. In general, suppose we have a finite CW–complex X

and a homomorphism $\rho\colon H_1(X) \to \mathbb{Z}$; then ρ induces an infinite cyclic cover $\widetilde{X}_\rho$. The group $H_1(\widetilde{X}_\rho, \mathbb{Q})$ is a finitely generated module over the PID $\mathbb{Q}[t, t^{-1}]$. We denote this module by $A(X, \rho, \mathbb{Q}[t, t^{-1}])$. There is splitting

$$A(X, \rho, \mathbb{Q}[t, t^{-1}]) \cong \bigoplus_{j=1}^{n} \mathbb{Q}[t, t^{-1}] / \left\langle f_j(t) \right\rangle,$$

where f_j divides f_{j+1} for $j < n$. This splitting is unique and the value of n is called the rank of the module.

Definition 6.1. Let X be a space (or pair of spaces) supporting a map $\rho\colon H_1(X) \to \mathbb{Z}$ with associated infinite cyclic cover $\widetilde{X}_\rho$. We denote by $\beta_i(\widetilde{X}_\rho, \rho, \mathbb{Q}[t, t^{-1}])$ the $\mathbb{Q}[t, t^{-1}]$–rank of $A(X, \rho, \mathbb{Q}[t, t^{-1}])$. When ρ is implicit, it is dropped from the notation.

For the complements of the K_i and of Σ there are canonical maps of the first homology to $\mathbb{Z}$, and thus we can suppress the ρ in our notation. The infinite cyclic cover $W_\infty(\Sigma)$ is built from the infinite cyclic cover $M_\infty(K_0)$ by adding the lifts of c_0 1–handles, followed by c_1 2–handles, and then 3–handles. There is a similar decomposition arising for $M_\infty(K_1)$. The proof of Theorem 5.1 carries over to this setting, yielding the following result.

Theorem 6.2. *Suppose that Σ is a cobordism from K_1 to K_0. Then*

$$c_0(\Sigma) \geq \frac{\beta_1(M_\infty(K_1), \mathbb{Q}[t, t^{-1}]) - \beta_1(M_\infty(K_0), \mathbb{Q}[t, t^{-1}])}{2} - g(\Sigma).$$

This result can be strengthened by focusing on the direct sum decomposition of the module $A(X, \rho, \mathbb{Q}[t, t^{-1}])$ that corresponds to irreducible elements in $\mathbb{Q}[t, t^{-1}]$. For any irreducible polynomial f we can set $\beta_i^f(\widetilde{X}, \rho, \mathbb{Q}[t, t^{-1}])$ to be the rank of the f–primary summand of $A(X, \rho, \mathbb{Q}[t, t^{-1}])$. The proof of the following result is much the same as that for the previous theorem. (As an alternative, one can switch to the ring $\mathbb{Q}[t, t^{-1}]_{(f)}$, which denotes the localization at f, that is, the ring formed from $\mathbb{Q}[t, t^{-1}]$ by adding a multiplicative inverse to all nontrivial elements g that are relatively prime to f. This is a PID with a unique prime, represented by f.)

Theorem 6.3. *Suppose that Σ is a genus g cobordism from K_1 to K_0. Then for any irreducible polynomial $f \in \mathbb{Q}[t, t^{-1}]$,*

$$c_0(\Sigma) \geq \frac{\beta_1^f(M_\infty(K_1), \mathbb{Q}[t, t^{-1}]) - \beta_1^f(M_\infty(K_0), \mathbb{Q}[t, t^{-1}])}{2} - g(\Sigma).$$

Corollary 6.4 is immediate.

Corollary 6.4. *If knots K and J have nontrivial Alexander polynomials with a pair of distinct irreducible factors, then for any cobordism Σ from from nK to mJ we have*

$$c_0(\Sigma) \geq n/2 - g$$

and

$$c_2(\Sigma) \geq m/2 - g.$$

For related results in the case of ribbon concordances, see [9].

7. Knots $K(1,\alpha 6_1)$, $K(1,\beta 10_3)$, and their associated metacycle covers

A *metacyclic* invariant of a knot K, or of a surface $\Sigma \subset S^3 \times [0,1]$, is one that is derived from a cyclic cover of a cyclic branched cover of K or Σ. The use of such invariants in knot theory already appears in early work, such as Reidemeiser's 1932 book [31, 32]. The role of such invariants in concordance first appeared in the work of Casson and Gordon [6]. That paper, which introduced what is now called Casson-Gordon theory, was restricted to 2–bridge knots $B((2k+1)^2, 2)$. We will build our examples using the 2–bridge knots $B((2k+1)^2, 2k)$. The reason for the different choice is that Casson and Gordon were interested in showing that particular knots are not slice; we want to start with knots that are slice and explore their slice disks and concordances between them.

Our examples are built from two knots from this family: $K(1,U) = B(9,2) = 6_1$ and $K(2,U) = B(25,4) = 10_3$, but further examples are easily constructed.

Figure 1 gives an illustration of a knot $K(k,J)$. For J unknotted, this is $B((2k+1)^2, 2k)$. We can think of $K(1,J)$ as being built from $B((2k+1)^2, 2k)$ by removing a neighborhood of a circle α linking the right band in the Seifert surface shown in Figure 1 (for which the right band unknotted) and replacing that neighborhood with the complement of the knot J in S^3. Viewed as knots in S^3, α and J have meridians and longitudes. The identification of the boundaries of their complements identifies the meridian of each with the longitude of the other. This creates a new knot in S^3, formed from $K(1,U)$ by tying the knot J in a band on the Seifert surface, as desired. We will focus on two specific examples: $K(1,\alpha 6_1)$ and $K(1,\beta 10_3)$, where α and β are nonnegative integers.

7.1. Ribbon disks for $K(k,J)$.

Theorem 7.1. *If J is ribbon and bounds a ribbon disk with n minima, then $K(k,J)$ is ribbon, bounding a ribbon disk with $2n$ minima.*

Proof. A simple closed curve γ that passes once over each of the bands of the genus one Seifert surface for $K(k,J)$ in opposite directions has framing zero and has the knot type of J. A ribbon disk for $K(k,J)$ is built by first removing an annular neighborhood of γ on the Seifert surface for $K(k,J)$. The boundary of this annulus is a pair of parallel curves on the surface, each of knot type J. Those two curves can be capped off in the four-ball with parallel copies of the ribbon disk in the 4–ball for J. The resulting surface is a disk; since the ribbon surface for J has n minima, using two copies of the ribbon disk yields a surface with $2n$ minima. $\square$

7.2. The 2–fold branched cover of $K(1,J)$. An algorithm of Akbulut-Kirby [4] provides a surgery diagram of the 2–fold branched cover of $K(1,J)$, as shown on the left in Figure 5; $M_2(K(1,J))$ is given as surgery on a two-component link, with one of the components unknotted and the other representing $J \# J^r$, where J^r denotes J with its string orientation reversed. Since all the knots J we consider are reversible, we have left out the superscript "r" and do not orient the circles labeled with J. Also, we can write $2J$ rather then $J \# J^r$ when needed. As described in, for instance, [33], that surgery diagram can be modified to appear as in the diagram on the right. This illustrates the 2–fold branched cover as formed from the lens space $L(9,2)$ by removing two parallel copies of a core circle and replacing each with a copy of the complement of J.

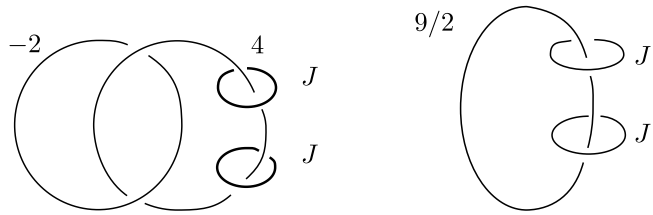

FIGURE 5. Branched cyclic cover of $K(1,J)$, where $J = J^r$

As an immediate consequence, we have the following result, which also follows from the fact that that the homology is presented by $V + V^{\mathsf{T}}$, where V is the Seifert matrix for $K(1,J)$ given by $V = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$.

Theorem 7.2. *For all J, $H_1(M_2(K(1,J))) \cong \mathbb{Z}_9$.*

7.3. **The metacyclic cover of $K(1,J)$.** The first homology of $M_2(K(1,J))$ is isomorphic to $\mathbb{Z}_9$. Thus it has a unique connected cyclic 3–fold cover $\widetilde{M_2^3}(K(1,J))$. We need to understand the construction of this space from the lens space $L(3,2)$. Here is a summary of the result we will use.

Lemma 7.3. *The space $\widetilde{M_2^3}(K(1,J))$ is built from $L(3,2)$ by removing tubular neighborhoods of two homologically essential parallel curves, $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$, and replacing each with copies of the 3–fold cyclic cover of $S^3 - J$. The attaching map for each identifies a lift of a longitude of J with a meridian of $\widetilde{\alpha}_1$ and identifies the preimage of a meridian of J with a parallel curve to $\widetilde{\alpha}_i$.*

Proof. Results such as this appear in the work of Gilmer [12] and Litherland [25] in greater generality, in which rather than the 3–fold cover an infinite cover is analyzed. Here is an outline of the proof.

First, we set up some notation. Let α_1 and α_2 be the curves that are labeled J in Figure 5. Viewed as curves in S^3, these have meridians and longitudes, which we will denote m_1, l_1, m_2, and l_2. The 3–fold cyclic of $L(9,2)$ is $L(3,2)$. The curves α_i, now viewed as in $L(9,2)$, each have connected preimages which we denote $\widetilde{\alpha}_i$. The longitude of α_i has a connected preimage that we denoted $\widetilde{l}_i$. The preimage of its meridian has three components; we choose one to call $\widetilde{m}_i$.

We denote a meridian and a longitude of J by μ and λ. We view these as curves on the torus boundary of $S^3 - N(J)$. The preimage of μ in the 3–fold cyclic cover, $\widetilde{S^3 - N(J)}$, is connected; we call it $\widetilde{\mu}$. We denote by $\widetilde{\lambda}$ one of the components of the preimage of λ.

The diagram below offers a schematic of the covering spaces. The union on the bottom row represents $M_2(K(1,J))$; on the top row we have $\widetilde{M_2^3}(K(1,J))$. The downward pointing arrows represent the covering spaces. These maps commute with the attaching maps described above that are used to form the union. This completes a summary of the proof of the lemma.

$$\begin{array}{ccc}
\big(L(3,2) - \{\widetilde{\alpha}_i\}\big) & \cup & \big(\widetilde{S^3 - J} \cup \widetilde{S^3 - J}\big) \\
\big\downarrow & & \big\downarrow \\
\big(L(9,2) - \{\widetilde{\alpha}_i\}\big) & \cup & \big(S^3 - J \cup S^3 - J\big)
\end{array}$$

$\square$

7.4. **The homology of the metacyclic cover of** $K(1,J)$**.** As described in Lemma 7.3, the 3–fold cyclic cover of $M_2(K(1,J))$ is built from the 3–fold cyclic cover of $L(9,2)$, which is the lens space $L(3,2)$, by removing a pair of parallel core circles and replacing them with copies of $M_3(J) \setminus \widetilde{J}$. This is illustrated in Figure 6. We will thus need the following.

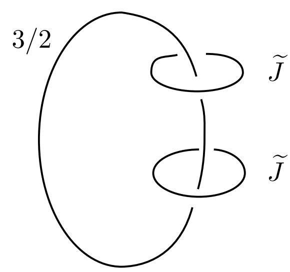

FIGURE 6. The 3–fold cyclic cover of the 2–fold branched cover of $K(1,J)$

Theorem 7.4. *There is an isomorphism* $H_1(\widetilde{M_2^3}(K(1,J))) \cong \mathbb{Z}_3 \oplus H_1(M_3(J))^2$.

Proof. For any knot J, let X_1 and X_2 be copies of the 3–fold cyclic cover of $S^3 \setminus J$. We have $H_1(X_i) \cong \mathbb{Z} \oplus H_1(M_3(J))$.

The torus boundary of X_i has natural boundary curves, m_i, and l_i, lifts of the meridian and longitude of J. The curve m_i represents an element of infinite order in $\mathbb{Z} \oplus H_1(M_3(J))$, and with the appropriate choice of basis represents $1 \oplus 0$. The curve l_i is null-homologous in X_i, bounding a lift of a Seifert surface.

In Figure 6 the curves m_i and l_i are attached to the longitude and meridian, respectively, of the curves labeled $\widetilde{J}$. (Notice that there is an interchange of meridian and longitude.)

One can now undertake a Mayer-Vietoris argument. The covering space is split into four components by the three evident tori in Figure 6: that is, the peripheral tori to the three curves illustrated. As just described, two are related to the 3–fold covers of J, one is a solid torus with core γ (corresponding to the 3/2–surgery), and one is the compliment of the three component link that is illustrated, having homology generated by three meridians, which we denote α_0, β_1 and β_2, corresponding to the 3/2–surgery curves and the two $\widetilde{J}_3$. We let $T = H_1(M_3(J))$. Via the Mayer-Vietoris sequence, we see the homology is a quotient of

$$\big(\mathbb{Z}(\alpha) \oplus \mathbb{Z}(\beta_1) \oplus \mathbb{Z}(\beta_2)\big) \oplus \big(\mathbb{Z}(m_1) \oplus T\big) \oplus \big(\mathbb{Z}(m_2) \oplus T\big) \oplus \mathbb{Z}(\gamma).$$

The identification along the three tori, each with rank two first homology, introduces six relations. Taking them in order, meridian first and initially along the surgery torus, yields the following, where we write l_1 despite it equaling 0, to make the gluing maps more evident:

- $\alpha = -2\gamma$
- $\beta_1 + \beta_2 = 3\gamma$
- $\beta_1 = l_1$
- $\alpha = m_1$
- $\beta_2 = l_2$
- $\alpha = m_2$.

None of these involve the summand $T \oplus T$ and so, in effect, they are relations defining a quotient of $\mathbb{Z}^6 \cong \langle \alpha, \beta_1, \beta_2, m_1, m_2, \gamma \rangle$. A simple exercise shows the quotient is isomorphic to $\mathbb{Z}_3$, as desired. □

To apply this result, we will use the following and its immediate corollary, showing that in our case $T \cong (\mathbb{Z}_7)^2$ or $T \cong (\mathbb{Z}_{19})^2$, depending on whether $J = 6_1$ or 10_3.

Lemma 7.5. *$H_1(M_3(6_1)) \cong \mathbb{Z}_7 \oplus \mathbb{Z}_7$ and $H_1(M_3(10_3)) \cong \mathbb{Z}_{19} \oplus \mathbb{Z}_{19}$.*

Proof. This is a standard knot theoretic computation; see, for instance [33]. More generally, in Theorem A.1 in the appendix it is shown that for q odd, $H_1(M_q(B((2k+1)^2, 2k))) \cong \mathbb{Z}_{(k+1)^q - k^q} \oplus \mathbb{Z}_{(k+1)^q - k^q}$. □

Corollary 7.6. *$H_1(\widetilde{M}^3_2(K(1, 6_1))) \cong \mathbb{Z}_3 \oplus (\mathbb{Z}_7)^4$ and $H_1(\widetilde{M}^3_2(K(1, 10_3))) \cong \mathbb{Z}_3 \oplus (\mathbb{Z}_{19})^4$.*

7.5. **The eigenvalue decomposition of $H_1(\widetilde{M}^3_2(K(1, J)))$.** For any field $\mathbb{F}$, there is an action of $\mathbb{Z}_3$ on $H_1(\widetilde{M}^3_2(K(1, J)), \mathbb{F})$. In the case that $\mathbb{F}$ contains a primitive 3–root of unity ζ, the homology $H_1(\widetilde{M}(K(1, J)), \mathbb{F})$ splits into eigenspaces, as described in Section 3. Note that $\mathbb{F}_7$ and $\mathbb{F}_{19}$ both contain such roots of unity. When no confusion can result, we will use the same symbol ζ to denote a primitive cube root of unity in $\mathbb{F}_7$ and in $\mathbb{F}_{19}$.

Theorem 7.7. *With the set-up described above:*

- $\beta_1^\zeta(\widetilde{M}^3_2(K(1, \alpha 6_1)), \mathbb{F}_7) = 2\alpha.$
- $\beta_1^\zeta(\widetilde{M}^3_2(K(1, \alpha 6_1)), \mathbb{F}_{19}) = 0.$
- $\beta_1^\zeta(\widetilde{M}^3_2(K(1, \beta 10_3)), \mathbb{F}_7) = 0.$
- $\beta_1^\zeta(\widetilde{M}^3_2(K(1, \beta 10_3)), \mathbb{F}_{19}) = 2\beta.$

Proof. Considering the $\mathbb{F}_7$–homology, we have $H_1(\widetilde{M}^3_2(K(1, \alpha 6_1)), \mathbb{F}_7) \cong (\mathbb{F}_7)^{2\alpha}$ arises entirely from the 2α copies of $M_3(J) \setminus \widetilde{J}$ that appear in the covering space. Thus the proof of the first statement comes down to analyzing the eigenspace splitting of the $\mathbb{Z}_3$–action on $M_3(J) \cong \mathbb{F}_7 \oplus \mathbb{F}_7$. We claim that the 1–eigenspace is trivial and the ζ–eigenspaces and ζ^{-1}–eigenspaces are both 1–dimensional. This can be shown with an explicit computation, or one can argue abstractly, as follows. A transfer argument, using the branched covering map $M_3(J) \to S^3$ shows that the 1–eigenspace is trivial. Poincaré duality implies that the ζ–eigenspace and ζ^{-1}–eigenspace are isomorphic (see Lemma B.1 for a proof).

Similar arguments give the remaining statements. □

7.6. **Metacyclic covers of** $nK(1, \alpha 6_1)$ **and** $mK(1, \beta 10_3)$**.** In this section we need to understand the construction of cyclic covers of connected sums of three-manifolds. The details are intricate, but the following example should clarify the approach. Consider the schematic diagram in Figure 7. In this diagram we illustrate a 5–fold cover, rather than the 3–fold cover we will use in our later examples. The figure at the bottom indicates the connected sum of manifolds N_1, N_2, and N_3. A connected sum is formed by removing a ball from each manifold and replacing it with a copy of $S^2 \times [0,1]$. The segments in the diagram represent copies of $S^2 \times [0,1]$. Suppose that we have a homomorphism from the first homology of the connected sum to $\mathbb{Z}_5$ that restricts to be surjective on the first two summands and is trivial on the third. Then the induced 5–fold covering space is formed from 5–fold covers of N_1 and N_2, along with copies of N_3 as illustrated in the top diagram. This space is diffeomorphic to $\widetilde{N_1} \# \widetilde{N_2} \# 5N_3 \# 4(S^1 \times S^2)$.

The homology of the cover is the direct sum of the homology groups of $\widetilde{N_1}$, $\widetilde{N_2}$, five copies of the first homology of N_3, and an additional $\mathbb{Z}^4$ summand. As a $\mathbb{Z}[\mathbb{Z}_5]$–module, the $\mathbb{Z}^4$ summand is isomorphic to $\mathbb{Z}[t]/\langle 1 + t + \cdots + t^4\rangle$. If we reduce to $\mathbb{F}_p$ coefficients where $p \neq 5$, then the $(\mathbb{F}_p)^4$ summand factors as the direct sum of four nontrivial eigenspaces after tensoring with the splitting field of $t^5 - 1$ over $\mathbb{F}_p$, one for each nontrivial 1/5–root of unity.

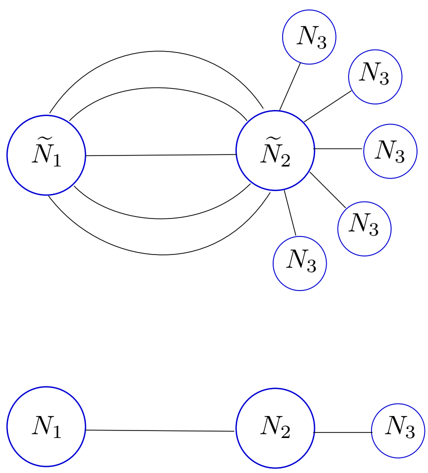

FIGURE 7. Schematic of a 5–fold cyclic cover of a connected sum

Moving on to the details of our specific example, let $\rho\colon H_1(M_2(nK(1,J))) \to \mathbb{Z}_3$ be nonzero on a of the natural $\mathbb{Z}_3$–summands and be 0 on $(n-a)$ of the summands. We need to understand the eigenspace decomposition of the homology of the associated cover.

Theorem 7.8.

(**A**) *Suppose that* $\rho\colon H_1(M_2(nK(1,\alpha 6_1))) \to \mathbb{Z}_3$ *is nonzero on* $a \geq 0$ *of the natural* $\mathbb{Z}_9$*–summands. Then*

- *If* $a \geq 1$*, then* $\beta_1^{\zeta}(\widetilde{M_2^3}(nK(1,\alpha 6_1)), \rho, \mathbb{F}_7) = 2a\alpha + a - 1$.
- *If* $a \geq 1$*,* $\beta_1^{\zeta}(\widetilde{M_2^3}(nK(1,\alpha 6_1)), \rho, \mathbb{F}_{19}) = a - 1$.
- *If* $a = 0$*, then* $\beta_1^{\zeta}(\widetilde{M_2^3}(nK(1,\alpha 6_1)), \rho, \mathbb{F}_7) = \beta_1^{\zeta}(\widetilde{M_2^3}(nK(1,\alpha 6_1)), \rho, \mathbb{F}_{19}) = 0$.

Similarly,

(**B**) *Suppose that* $\rho\colon H_1(M_2(nK(1,\beta 10_3)) \to \mathbb{Z}_3$ *is nonzero on* $a' \geq 0$ *of the natural* $\mathbb{Z}_9$*–summands. Then*

- *If* $a' \geq 1$*, then* $\beta_1^\zeta(\widetilde{M}_2^3(nK(1,\beta 10_3)),\rho,\mathbb{F}_{19}) = 2a'\beta + a' - 1$.
- *If* $a' \geq 1$, $\beta_1^\zeta(\widetilde{M}_2^3(nK(1,\beta 10_3)),\rho,\mathbb{F}_7) = a' - 1$.
- *If* $a' = 0$*, then* $\beta_1^\zeta(\widetilde{M}_2^3(nK(1,\beta 10_3)),\rho,\mathbb{F}_{19}) = \beta_1^\zeta(\widetilde{M}_2^3(nK(1,\beta 10_3)),\rho,\mathbb{F}_7) = 0$.

Proof. We consider the first statement in part (**A**). The 3–fold cover $\widetilde{M}_2^3(nK(1,\alpha 6_1))$ is a connected sum containing: a copies of the nontrivial cover $\widetilde{M}_2^3(K(1,\alpha 6_1))$; an additional $3(n-a)$ copies of the trivial cover, $M_2(nK(1,\alpha 6_1))$; and $(a-1)$ copies of $S^1 \times S^2 \mathbin{\#} S^1 \times S^2$. By Theorem 7.7, each of the a copies of $\widetilde{M}_2^3(K(1,\alpha 6_1))$ contributes 2α to the Betti number β_1^ξ, explaining the $2a\alpha$. The $(a-1)$ copies of $S^1 \times S^2$ each contribute 1 to the Betti number β_1^ξ. This completes the discussion of the first of the six cases of the theorem. The others are addressed in a similar way. ☐

8. Cobordisms between $nK(1,\alpha 6_1)$ and $mK(1,\beta 10_3))$

To simplify the discussion, we will assume that $n \geq m > 0$. Let Σ be a genus g cobordism from $nK(1,\alpha 6_1)$ and $mK(1,\beta 10_3))$. We continue to denote the 2–fold cover of $S^3 \times [0,1]$ branched over Σ by $W_2(\Sigma)$; this is a cobordism from $M_2(nK(1,\alpha 6_1))$ to $M_2(mK(1,\beta 10_3))$.

8.1. **Gilmer's results on surfaces in B^4 bounded by knots.** We begin with a summary of key results of Gilmer [11, Lemma 1 and Theorem 1]. Recall that for a rational homology three-sphere M, the linking form β is a nonsingular pairing $H_1(M) \times H_1(M) \to \mathbb{Q}/\mathbb{Z}$. Let M be a rational homology three-sphere bounding a four-manifold W. Then Gilmer's Lemma 1 states the following.

Lemma 8.1. *The linking form for* M *splits as a direct sum* $\beta_1 \oplus \beta_2$ *defined with respect to a splitting* $A_1 \oplus A_2$ *of* $H_1(M)$*. The splitting is such that* A_1 *has a presentation of rank* $\beta_2(W)$ *and* β_2 *vanishes on a subgroup* $\mathcal{M} \subset A_2$ *where* $\left|\mathcal{M}\right|^2 = \left|A_2\right|$*. Furthermore, the homomorphism* $H_1(M) \to \mathbb{Q}/\mathbb{Z}$ *defined by linking with any element of* $\mathcal{M}$ *extends to a homomorphism defined on* $H_1(W)$*.*

In general, for a finite abelian group G with nonsingular linking form lk, a subroup $\mathcal{M}$ upon which the linking form restricts to be 0 and for which $\left|\mathcal{M}\right|^2 = \left|G\right|$ is called a *metabolozier* for (G, lk).

The second result that we use is the next lemma, which is contained in the proof of Gilmer's Theorem 1.

Lemma 8.2. *If* W *is the* 2*–fold branched cover of* B^4 *branched over a surface of genus* g *bounded by a knot, then* $\beta_2(W) = 2g$*.*

In our setting we can apply these two results to attain the following.

Theorem 8.3. *Let* K_0 *and* K_1 *be knots such that* $H_1(M_2(K_0)) \cong H_1(M_2(K_1)) \cong \mathbb{Z}_9$*, so that* $H_1(nM_2(K_1)) \oplus H_1(-mM_2(K_0)) \cong (\mathbb{Z}_9)^n \oplus (\mathbb{Z}_9)^m$*. Furthermore, assume that there is a genus* g *cobordism* Σ *from* nK_0 *to* nK_1 *and that* $n + m \geq 2g$*. For some* $\epsilon \geq 0$*, the linking form on this group splits off a summand that is isomorphic to* $(\mathbb{Z}_9)^{n+m-2g+\epsilon}$ *which contains a metabolizer* $\mathcal{M} \subset H$*, all elements of which* *define homomorphisms that extend to the 2–fold branched cover of the cobordism. In particular, the order of $\mathcal{M}$ is at least 3^{n+m-2g}.*

8.2. **Extending homomorphisms from $H_1(M_2(nK(1,\alpha 6_1)))$ to $H_1(W_2(\Sigma))$.**

Theorem 8.4. *Suppose that a surface Σ is a genus g cobordism from $nK(1,\alpha 6_1)$ to $mK(1,\beta 10_3)$ and assume that $n > 2g$. Then there exists a surjective homomorphism $\rho\colon H_1(M_2(nK(1,6_1))) \to \mathbb{Z}_3$ that extends to a homomorphism $\rho'\colon H_1(W_2(\Sigma)) \to \mathbb{Z}_{3^b}$ for some b.*

Proof. Abbreviate $nK(1,\alpha 6_1)$ by nK_1 and $mK(1,\beta 10_3)$ by mK_2. We then have that $H_1(M_2(K_1)) \oplus H_1(-M_2(K_2)) \cong (\mathbb{Z}_9)^{n+m}$.

As described in Theorem 8.3, there exists a set $\mathcal{M}$ of homomorphisms mapping $H_1(nM_2(K_1)) \oplus H_1(-M_2(mK_0))$ to $\mathbb{Q}/\mathbb{Z}$ that extend to homomorphisms on $H_1(W_2(\Sigma))$. The order of $\mathcal{M}$ is 3^{n+m-2g} and the order of a metabolizer for $H_1(-M_2(mK_0))$ is 3^m. It follows that if $3^{n+m-2g} > 3^m$, then some element in $\mathcal{M}$ is not contained in $0 \oplus H_1(-M_2(mK_0))$ and thus must be nontrivial on $H_1(M_2(mK_1))$. This will occur as long as $n > 2g$. Call one such element $\rho \in \mathcal{M}$ and let ρ' denote an extension of ρ to $H_1(W_2(\Sigma))$.

The image of ρ' is a finite cyclic subgroup $G \subset \mathbb{Q}/\mathbb{Z}$. Projecting G to its 3–primary summand does not change its restriction to the boundary, so we can assume that ρ' takes values in $\mathbb{Z}_{3^b}$ for some b. If ρ is not of order 3, then it can be multipled by 3 so that it does have order 3. □

8.3. **The 3^b–fold cyclic cover of $W(\Sigma)$.** Let $\pi\colon \widetilde{W}^3_2(\Sigma) \to W_2(\Sigma)$ denote the 3^b–fold cyclic cover of $W_2(\Sigma)$ associated to the homomorphism ρ' defined above. We let $\partial_1(\widetilde{W}^3_2) = \pi^{-1}(M_2(nK(1,\alpha 6_1)))$ and $\partial_0(\widetilde{W}^3_2) = \pi^{-1}(M_2(nK(1,\alpha 10_3)))$.

We can now apply Theorem 3.10 and Theorem 7.8. Let ζ be a primitive 3–root of unity and consider the $\mathbb{Z}_3$–action on $\partial_1(\widetilde{W}^3_2)$, the 3^{b-1} power of order 3^b deck transformation, which we denote by $T = S^{3^{b-1}}$.

Theorem 8.5. *Assume that the restriction $\rho\colon M_2(nK(1,\alpha 6_1)) \to \mathbb{Z}_3$ is nonzero on $a \geq 1$ of the n summands. Also suppose that the restriction is nonzero on $a' \geq 0$ of the m summands of $H_1(M_2(mK(\beta 10_3))$.*

- $\beta^\zeta_1(\partial_1(\widetilde{W}^3_2), T, \mathbb{F}_7) = 3^{b-1}\beta^\zeta_1(\widetilde{M}^3_2(nK(1,\alpha 6_1)), \mathbb{F}_7) = 3^{b-1}(2a\alpha + a - 1)$.
- $\beta^\zeta_1(\partial_0(\widetilde{W}^3_2), T, \mathbb{F}_7) = 3^{b-1}(a'-1)$ *if* $a' \geq 1$ *and* $\beta^\zeta_1(\partial_0(\widetilde{W}^3_2), \mathbb{F}_7) = 0$ *if* $a' = 0$.

Applying a relative version of Corollary 3.7 along with Theorem 4.2 gives the next result.

Theorem 8.6. *Let $C^\zeta_i(\widetilde{W}^3_2, \partial_0(\widetilde{W}^3_2), \mathbb{F}_7)$ be the ζ–eigenspace of the CW–chain group under the $\mathbb{Z}_3$–action given as the 3^{b-1}–power of its deck transformation. Then*

- $\dim C^\zeta_1(\widetilde{W}^3_2, \partial_0(\widetilde{W}^3_2), T, \mathbb{F}_7) = 3^{b-1}(2c_0(\Sigma))$.
- $\dim C^\zeta_2(\widetilde{W}^3_2, \partial_0(\widetilde{W}^3_2), T, \mathbb{F}_7) = 3^{b-1}(2c_1(\Sigma))$.
- $\dim C^\zeta_3(\widetilde{W}^3_2, \partial_0(\widetilde{W}^3_2), T, \mathbb{F}_7) = 3^{b-1}(2c_2(\Sigma) + 2g(\Sigma))$.

Theorem 8.7. *Let Σ be a genus g cobordism from $nK(1,\alpha 6_1)$ to $mK(1,\beta 10_3)$. Assume that $n > 2g$. Then*

$$c_0(\Sigma) \geq \frac{2\alpha + 1 - m}{4} - g.$$

Proof. The proof is much like the one for Theorem 5.1. We work with the ζ–eigenspaces of the $\mathbb{Z}_3$–actions. Consider the fact that $\widetilde{W}_2^3$ is built from $\partial_0(\widetilde{W}_2^3)$. We have

$$\beta_1^\zeta(\widetilde{W}_2^3, T, \mathbb{F}_7) \leq 3^{b-1}(m-1) + 3^{b-1}(2c_0(\Sigma)).$$

The first summand comes from the homology of the boundary, using the fact that in Theorem 8.5 we have $a' - 1 \leq m$. Turning the bordism upside down and using the fact that $\widetilde{W}_2^3$ is built from $\partial_1(\widetilde{W}_2^3)$ by adding 1–handles and 2–handles that correspond to the index two and index one critical points of Σ, respectively, we find that

$$\beta_1^\zeta(\widetilde{W}_2^3, T, \mathbb{F}_7) \geq 3^{b-1}(2a\alpha + a - 1) + 3^{b-1}(2c_2(\Sigma)) - 3^{b-1}(2c_1(\Sigma)).$$

Together, these inequalities imply

$$(2a\alpha + a - 1) + 2c_2(\Sigma) - (2c_1(\Sigma)) \leq (m-1) + (2c_0(\Sigma)).$$

We have that $c_1(\Sigma) = c_0(\Sigma) + c_2(\Sigma) + 2g(\Sigma)$. Substituting yields

$$(2a\alpha + a - 1) + 2c_2(\Sigma) - 2(c_0(\Sigma) + c_2(\Sigma) + 2g(\Sigma)) \leq (m-1) + 2c_0(\Sigma).$$

This simplifies to give

$$c_0(\Sigma) \geq \frac{2a\alpha + a - m}{4} - g.$$

Finally, since $a \geq 1$, we have the desired result:

$$c_0(\Sigma) \geq \frac{2\alpha + 1 - m}{4} - g.$$

□

8.4. **Strengthening the bounds.** The difference between the lower bound provided by Theorem 8.7 and the best upper bound that we can prove with a realization result is quite large. For instance, we have the following realization result.

Theorem 8.8. *If $g \leq \min\{n(2\alpha+1), m(2\beta+1)\}$, then there is a genus g cobordism Σ from $nK(\alpha 6_1)$ to $mK(\beta 10_3)$ satisfying*

$$c_0(\Sigma) = n(2\alpha+1) - g \text{ and } c_0(\Sigma) = m(2\beta+1) - g.$$

Proof. The construction given in Example 5.6 can be easily modified to produce the result. What is essential is that the canonical ribbon disks can be pieced together to form a concordance in which the local maxima are beneath the local minima. □

A limitation in this theorem is the absence of n in the bound on c_0 given Theorem 8.7. We want to explore this briefly. We have an inclusion of $(\mathbb{Z}_3)^{m+n}$ into a group with nonsingular linking form:

$$(\mathbb{Z}_3)^n \oplus (\mathbb{Z}_3)^m \subset (\mathbb{Z}_9)^n \oplus (\mathbb{Z}_9)^m.$$

We have assumed that $n + m > 2g$ and identified a subgroup $\mathcal{M} \subset (\mathbb{Z}_9)^n \oplus (\mathbb{Z}_9)^m$ of order 3^{n+m-2g} upon which the linking form is identically 0. In the proof of Theorem 8.7, we used the fact that if $n > 2g$ then $\mathcal{M} \cap (\mathbb{Z}_3)^n \oplus 0$ is nontrivial. But in fact, if n is large in comparison to m and g, then the rank of the intersection $\mathcal{M} \cap (\mathbb{Z}_3)^n \oplus 0$ must be large as well; in particular, rather than use $a \geq 1$ in the argument, we could find metabolizing elements for which a is much larger. Similarly, we used the obvious fact that $a' \leq m$; with care, we could also show that it is possible to assume that a' is close to 0. We have opted not to undertake the careful analysis of self-annihilating subgroups of the standard linking form on $(\mathbb{Z}_9)^{n+m}$ that is required to establish these better bounds.

9. Non-reversible knots

To conclude our presentation of examples, we consider a subtle family built from knots K and K^r, where K^r denotes the reverse of K. Such knots are difficult to distinguish by any means. For instance, all abelian invariants are identical for the two knots. It is not known at the moment whether any invariants that are built from the Heegaard Floer knot complex $\mathrm{CFK}^\infty(K)$ defined in [29], such as its involutive counterpart, defined in [18], can distinguish them. The successful application of metacyclic invariants to distinguishing knots from their reverses began with the work of Hartley, [17].

Figure 8 illustrates a knot that we will denote $P = P(J_1, J_2)$. The starting knot is the pretzel knot $P(3, -3, 3)$, and knots J_1 and J_2 are placed in the two bands. Notice that we have indicated the orientation of P. We let P^* denote reverse of the knot; that is, the knot with the same diagram except with the arrow reversed (the use of P^* rather than the more standard notation P^r will simplify some notation later on). These knots have formed the basis of a variety of concordance result related to reversibility; see, for instance, [24]. In past papers that used these knots, the J_1 were chosen so that the knots could be shown not to be concordant. We will let J_1 and J_2 be slice knots, so that they the P and P^* are themselves slice and our results apply to consider concordances between them.

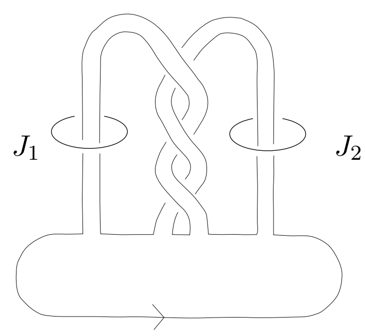

Figure 8. The knot $P(J_1, J_2)$

We will now briefly summarize the results of some calculations related to these knots, leaving the details to [24]. Our first observation concerns the choice of deck transformation for a cyclic branched cover a knot. Let $K \subset S^3$ be an oriented knot. The canonical map $\phi\colon \pi_1(S^3 \setminus K) \to \mathbb{Z}_n$ determines an index n subgroup of $\pi_1(S^3 \setminus K)$, and associated to this subgroup there is a cyclic branched cover, $M_n(K)$. This is an oriented manifold and is independent of the choice of string orientation of K. However, the map ϕ identifies the group of deck transformations of $M_n(K)$ with $\mathbb{Z}_n$. Reversal of the string orientation of K has the effect of changing the identification. To be more explicit, if m is an orientated meridian of K, viewed as a path, then a lifting of that path to $M_n(K)$ is a path α. The canonical generator T of the group of deck transformations has the property that $T(\alpha(0)) = \alpha(1)$. If the orientation of K is reversed, then the meridian is reversed, and for the new deck transformation T' we have $T'(\alpha(1)) = \alpha(0)$. That is, reversing the string orientation of K has the effect of inverting the canonical deck transformation.

If $x \in H_1(M_n(K), \mathbb{F}_p)$ is λ eigenvector for the deck transformation T, we have $T(x) = \lambda x$. If we replace K with its reverse and call the deck transformation T', then $T'(x) = T^{-1}(x) = \lambda^{-1}x$. That is, reversing K interchanges λ and λ^{-1} eigenspaces.

In our example of interest, the 3–fold cover has $H_1(M_3(P)) \cong \mathbb{Z}_7 \oplus \mathbb{Z}_7$. This group splits into a 2–eigenspace and a 4–eigenspace for the deck transformation using $\mathbb{F}_7$–coefficients. If z and w are linking circles to the two bands, with $\widetilde{z}$ and $\widetilde{w}$ being chosen lifts to $M_3(P)$, then the 2–eigenspace and 4–eigenspace are spanned by $\widetilde{z}$ and $\widetilde{w}$, respectively. For P^* we have $\widetilde{z}^*$ and $\widetilde{w}^*$ as eigenvectors, but because of the reversal, they are the 4–eigenvectors and 2–eigenvectors, respectively.

Let Σ be a cobordism from P to P^*. We let W_3' be the 3–fold cover of $S^3 \times [0,1]$ branched over Σ. By removing an arc from W_3' along the lift of Σ that joins its two boundary components, we build a four-manifold W_3 having boundary $M_3(P) \# -M_3(P^*)$. Lemma 8.1 asserts the existence of a subgroup $\mathcal{M} \subset H_1(M_3(P)) \oplus H_1(-M_3(P^*))$ with specified properites. Using the linking form to identify this subgroup with a group of homomorphisms to $\mathbb{Q}/\mathbb{Z}$, we have that all elements of $\mathcal{M}$ extend to define homomorphisms on $H_1(M_3(P)) \oplus H_1(-M_3(P^*))$ that extend to W_3. Within Gilmer's proof, one sees that since the deck transformation on ∂W_3 extends to W_3, the subgroup $\mathcal{M}$ must be invariant under the $\mathbb{Z}_3$–action and thus is spanned by eigenvectors. Here are the possibilities.

- $\mathcal{M}$ is a 2–eigenspace, spanned by $\{\widetilde{z}, \widetilde{w}^*\}$.
- $\mathcal{M}$ is a 4–eigenspace, spanned by $\{\widetilde{w}, \widetilde{z}^*\}$.
- $\mathcal{M}$ contains a nontrivial 2–eigenvector $a\widetilde{z}+b\widetilde{w}^*$ and a 4–eigenvector $c\widetilde{w}+b\widetilde{z}^*$.

We now wish to find obstructions based on the 7–fold cyclic covers of the spaces involved. There are three cases to consider. Here is a summary of what arises.

- **Case 1:** Considering the eigenvector $\widetilde{z}$, for the corresponding 7–fold cover of $M_3(P)$ the rank of the first homology will be determined, up to some constant, by the rank of the homology of $M_7(J_1)$. For the eigenvector $\widetilde{w}^*$, the corresponding cover of $M_3(P^*)$ will have first homology whose rank is determined, up to a constant, by the rank of the homology of $M_7(J_2)$. Computing the value of that constant arises is technical; details are presented in the appendix as Theorem C.1. What is essential here is that if the rank of $H_1(M_7(J_2), \mathbb{F}_p)$ is large (respectively 0), then the rank of the first homology of the 7–fold cover of $M_3(P)$ will be large (respectively, small).
- **Case 2:** This is similar, with the roles of J_1 and J_2 reversed.
- **Case 3:** The last case splits into subcases depending on whether the coefficients a, b, c, and d are zero or not. The most interesting case is when, say $a \neq 0 \neq b$. Then the homology of the corresponding 7–fold cover of $M_3(P)$ will involve the first homology of $M_7(J_1)$ and the homology of the corresponding 7–fold cover of $M_3(P^*)$ will also involve the first homology of $M_7(J_2)$.

From this it should be clear that by choosing J_1 and J_2 so that the rank of the first homology groups $H_1(M_7(J_1), \mathbb{F}_p)$ and $H_1(M_7(J_2), \mathbb{F}_{p'})$ are large for appropriate primes p and p', then regardless of which metabolizer arises, there will be obstructions to the values of $c_0(\Sigma)$ and $c_2(\Sigma)$ being small. This can be achieved by letting J_1 be a multiple of 6_1 and letting J_2 be a multiple of 10_3. Theorem A.1

shows $H_1(M_7(6_1)) \cong \mathbb{Z}_{127} \oplus \mathbb{Z}_{127}$ and $H_1(M_7(10_3)) \cong \mathbb{Z}_{2059} \oplus \mathbb{Z}_{2059}$. The number 2059 has prime factors 29 and 71. All of $\mathbb{F}_{127}$, $\mathbb{F}_{29}$ and $\mathbb{F}_{71}$ contain primitive 7–roots of unity.

To construct examples in Section 7, we used the fact that the 3–fold cover of $L(9,2)$ is $L(3,2)$. For carrying out an explicit computation here, we would need to know the homology of the 7–fold cover of $M_3(P)$ corresponding to each eigenspace of the $\mathbb{Z}_3$–action. Regardless of what there groups are, their ranks in comparison to the rank of $H_1(M_7(\alpha J_1))$ or $H_1(M_7(\beta J_2))$ will be small if α and β are large. This permits one to prove the following result.

Theorem 9.1. *For any nonnegative integers g, c_0 and c_2, there are positive integers α and β such that the knot $P = P(\alpha 6_1, \beta 10_3)$ has the following properties.*

- *$P(\alpha 6_1, b10_3)$ is a ribbon knot.*
- *Any genus g cobordism Σ from P to P^* has $c_0(\Sigma) \geq c_0$ and $c_2(\Sigma) \geq c_2$.*

10. Problems

(1) Is $\mathcal{G}_g(K_1, K_0)$ always a quadrant, of the form $\mathcal{Q}(a,b)$, for some a and b? In the case of $g = 0$, this would imply Gordon's Conjecture [16] recently proved by Agol [3]: If K_1 is ribbon concordant to K_0 and K_0 is ribbon concordant to K_1, then $K_1 = K_0$.

(2) A generalization of Gordon's conjecture is the following statement: if for some c_0 and c_2, $(c_0, 0) \in \mathcal{G}_g(K_1, K_0)$ and $(0, c_2) \in \mathcal{G}_g(K_1, K_0)$, then $\mathcal{G}_g(K_1, K_2) = \mathcal{Q}(0,0)$.

(3) If $(a+1, b+1) \in \mathcal{G}_g(K_1, K_0)$, then is $(a,b) \in \mathcal{G}_{g+1}(K_1, K_0)$?

(4) Recall that the bridge number of K is denoted $\mathrm{br}(K)$ and we defined $\mathrm{b}(K)$ to be the minimum number of index 0 critical points of a slice disk for $K \# -K$. It is elementary to show that $\mathrm{b}(K) \leq \mathrm{br}(K)$. It is also not difficult to construct ribbon knots K with large bridge index that bound disks in the four-ball with one saddle point. Using these knots we see that $\mathrm{br}(K) - \mathrm{b}(K)$ can be arbitrarily large.

For the torus knot $T_{2,3}$ we have $\mathrm{br}(T_{2,3}) = 2$ and it is elementary to see that $\mathrm{b}(T_{2,3}) = 2$. In fact, in [21] it is shown that for torus knots, $\mathrm{b}(K) = \mathrm{br}(K)$. Yet there are still basic examples that are unresolved: for $K = nT_{2,3}$ we have $\mathrm{br}(K) = n+1$; is it true that $\mathrm{b}(nT_{2,3}) = n+1$?

Appendix A. The knots $K(k, J)$

Here we summarize the computations required in Section 7 that determine the homology groups of covering spaces associated to $K(k,J)$. Recall that if J is unknotted, this is the two-bridge knot $B((2k+1)^2, 2k)$. It is the basic building block for the examples in Lemma 7.5.

A.1. **A Seifert surface for $K(k,J)$ and its Seifert form.** The knot $K(k,J)$ has a genus 1 Seifert surface F built by attaching two bands to a disk, one with framing $k+1$ and other with framing $-k$. One band has a knot J tied in it. This was illustrated in Figure 1. The Seifert matrix with respect to the natural basis $\{a, b\}$ of $H_1(F)$ is

$$A_k = \begin{pmatrix} k+1 & 1 \\ 0 & -k \end{pmatrix}.$$

The classes a and b are represented by simple closed curves on F representing the unknot and the knot J. If we change basis, letting $a' = a - b$ and $b' = b$ then the Seifert matrix becomes

$$B_k = \begin{pmatrix} 0 & k+1 \\ k & -k \end{pmatrix}.$$

These generators are still represented by simple closed curves, the first of which is unknotted and the second of which represents J.

A.2. **The homology of the cyclic branched covers of $K(k, J)$.** We next have the computation of the needed homology groups.

Theorem A.1. *Let $K(k, J)$ be as above. Then $H_1(M_2(K(k, J))) \cong \mathbb{Z}_{(2k+1)^2}$. For n odd, $H_1(M_n(K(k, J))) \cong \mathbb{Z}_d \oplus \mathbb{Z}_d$, where $d = (k+1)^n - k^n$.*

Proof. The homology group $H_1(M_2(K(k, J)))$ is presented by $A_k + A_k^{\mathsf{T}}$, where A^{T} denotes the transpose. This 2×2 matrix has one if its entries a 1, so it presents a cyclic group. The order of that group is the absolute value of the determinant of the matrix. As an alternative, the presence of J does not affect the Seifert matrix or the homology of the cover. If J is the unknot, then the 2–fold branched cover is the lens space $L((2k+1)^2, 2k)$.

The homology group $H_1(M_n(K(k, J)))$ can be computed using a formula of Seifert [35]; see [10] for a more recent treatment. In our notation, this result states that for a knot K with Seifert matrix B, $H_1(M_n(K))$ is presented by

$$\Gamma^n - (\Gamma - \mathrm{Id})^n,$$

where $\Gamma = (B^{\mathsf{T}} - B)^{-1} B^{\mathsf{T}}$.

In our case, one readily computes that

$$\Gamma = \begin{pmatrix} k+1 & -k \\ 0 & -k \end{pmatrix},$$

and thus we are interested in the group presented by

$$A_k = \begin{pmatrix} k+1 & -k \\ 0 & -k \end{pmatrix}^n - \begin{pmatrix} k & -k \\ 0 & -k-1 \end{pmatrix}^n.$$

For some b, this is of the form

$$A_k = \begin{pmatrix} (k+1)^n - k^n & b \\ 0 & (-k)^n - (-k-1)^n \end{pmatrix}.$$

Since n is odd, this can be rewritten as

$$A_k = \begin{pmatrix} (k+1)^n - k^n & b \\ 0 & (k+1)^n - k^n \end{pmatrix}.$$

With a bit more work we could show that $b = 0$, but instead we rely on a theorem of Plans [30] (or see [33, Chapter 8D]): the homology of an odd-fold cycle branched cover is a double. $\square$

A.3. **A number theoretic observation.** In our examples, we considered the cases of $H_1(M_3(K(1, U))) \cong \mathbb{Z}_7 \oplus \mathbb{Z}_7$ and $H_1(M_3(K(2, U))) \cong \mathbb{Z}_{19} \oplus \mathbb{Z}_{19}$. We observed that both $\mathbb{F}_7$ and $\mathbb{F}_{19}$ contain primitive 3–roots of unity, since $7 \equiv 1 \mod 3$ and $19 \equiv 1 \mod 3$. This is not a coincidence. Our examples were the cases of $p = 3$ and either $k = 1$ or $k = 2$ in Theorem A.2, which follows immediately from a standard application of the binomial theorem or from Fermat's Little Theorem.

Theorem A.2. *If p is prime, then for all k, $(k+1)^p - k^p \equiv 1 \mod p$.*

Appendix B. The eigenspace structure of $H_1(M_n(K), \mathbb{F}_p)$

In his survey paper on knot theory [15], Gordon used a duality argument to prove that the first homology of the infinite cyclic cover of a knot, viewed as a module over the ring $\mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t, t^{-1}]$, is isomorphic to its dual module, in which the action of t is replaced with the action of t^{-1}. A similar argument can be applied in the setting of n–fold cyclic branched covers. Here we give a simple proof of a consequence of such a result. Duality is still required to the extent that it implies that the linking form of a three-manifold is nonsingular.

Theorem B.1. *Assume that $H_1(M_n(K)) \cong \mathbb{F}_p^k$ for some k and prime p. Suppose that n divides $p-1$, so that $\mathbb{F}_p$ contains a primitive n–root of unity, ξ. Then $H_1(M_n(K))$ splits into a direct sum of ξ^i–eigenspaces, denoted E_i, under the action of the deck transformation T_*. In addition, E_0 is trivial and $E_i \cong E_{n-i}$ for all $i, 0 < i < n$.*

Proof. Since n divides $p-1$, we have that p does not divide n and Propostion 3.3 implies that $H_1(M_n(K))$ splits into eigenspaces. We focus on proving that $E_i \cong E_{n-i}$.

Let $\mathrm{lk}(x,y) \in \mathbb{F}_p$ denote the $\mathbb{F}_p$–valued linking form on $H_1(M_n(K))$. Recall that the linking form is symmetric, nonsingular and equivariant with respect to the action of a homeomorphism, in particular with respect to T_*.

Claim 1. The eigenspaces E_i and E_j are orthogonal with respect to the linking form unless $i = j = 0$ or $i = n - j$.

To see this, suppose that $x \in E_i$ and $y \in E_j$. Then

$$\xi^i \mathrm{lk}(x,y) = \mathrm{lk}(T_* x, y) = \mathrm{lk}(x, T_*^{-1} y) = \mathrm{lk}(x, \xi^{-j} y) = \xi^{-j} \mathrm{lk}(x,y).$$

It follows that $(\xi^i - \xi^{-j})\mathrm{lk}(x,y) = 0$. This can be rewritten as $(\xi^i - \xi^{n-j})\mathrm{lk}(x,y) = 0$. If $i \neq 0$, then $\xi^i - \xi^{n-j} \neq 0$ unless $i = n-j$. Thus, if $i \neq 0$ and $i \neq n-j$, then $\mathrm{lk}(x,y) = 0$.

Claim 2. E_0 is trivial. We can now write

$$H_1(M(K)) \cong E_0 \oplus E_{n/2} \bigoplus_{1 \le i < n/2} \big(E_i \oplus E_{n-i}\big).$$

(The summand $E_{n/2}$ exists if and only if n is even, in which case it represents the -1–eigenspace.)

If $x \in E_0$, then $x + T_* x + \cdots + T_*^{n-1} x = nx$ is in the image of the transfer map $\tau \colon H_1(S^3) \to H_1(M_n(K))$, and thus equals 0. We can write $p - 1 = nk$ for some k, and so $(p-1)x = 0$. But $p-1$ is relatively prime to p, and so we have $x = 0$, as desired.

Claim 3. $E_i \cong E_{n-i}$ for all $i, 0 < i < n$. This is automatic for $E_{n/2}$ in the case the n is even. We focus on a summand $E_i \oplus E_{n-i}$ for $1 \le i < n/2$.

Suppose that E_i is of dimension a and E_{n-i} is of dimension b. By choosing bases for these eigenspaces, the linking form can be represented by an $(a+b) \times (a+b)$ matrix with entries in $\mathbb{F}_p$. Both E_i and E_{n-i} are self-orthogonal, so there are blocks with all entries 0 of size $a \times a$ and $b \times b$. The nonsingularity implies that $a \le (a+b)/2$ and $b \le (a+b)/2$. This can occur only if $a = b$.

□

Appendix C. The 7–fold cover of the 3–fold cover of the nonreversible knot P in Section 9

In Section 9 we considered knots $P(J_1, J_2)$. We have that $H_1(M_3(P(J_1, J_2))) \cong \mathbb{Z}_7 \oplus \mathbb{Z}_7$. The homology is generated by classes represented by lifts $\widetilde{z}$ and $\widetilde{w}$ of meridians to the bands in the evident Seifert surface in Figure 8. A homomorphism $\phi\colon \mathbb{Z}_7 \oplus \mathbb{Z}_7 \to \mathbb{Z}_7$ determines a homomorphism $\phi\colon H_1(M_3(P(J_1, J_2))) \to \mathbb{Z}_7$ for all choices of J_1 and J_2. The corresponding 7–fold cyclic covers are denoted $\widetilde{M_3^7}(P(J_1, J_2), \phi)$.

We fix a prime p. In Theorem C.1, let $R_1 = \mathrm{rank}\big(H_1(M_7(J_1), \mathbb{F}_p)\big)$ and let $R_2 = \mathrm{rank}\big(H_1(M_7(J_1), \mathbb{F}_p)\big)$.

Theorem C.1. *There is a constant C that is independent of ϕ, p, J_1, and J_2 such that*

$$\big|\mathit{rank}(H_1(\widetilde{M_3^7}(P(J_1, J_2), \phi), \mathbb{F}_p\big) - \epsilon_1 R_1 - \epsilon_2 R_2\big| \leq C.$$

The value of ϵ_1 is 0 or 3, depending on whether ϕ is trivial or nontrivial on $\widetilde{z}$, and the value of ϵ_2 is 0 or 3, depending on whether ϕ is trivial or nontrivial on $\widetilde{w}$.

Proof. In Section 9 we saw that $\widetilde{z}$ and $\widetilde{w}$ are nonzero eigenvectors for the deck transformation. In particular, if the function ϕ is trivial on $\widetilde{z}$, it is trivial on the two translates of $\widetilde{z}$.

The preimage of $\widetilde{z}$ in $\widetilde{M_3^7}(P(U, U))$ consists of either seven curves, each trivially covering $\widetilde{z}$, or one curve that is a 7–fold covering of $\widetilde{z}$. Similarly for $\widetilde{w}$. We call these lifts $\widetilde{z}_i'$ and $\widetilde{w}_i'$, where the index set has either one or seven elements.

We then have that $\widetilde{M_3^7}(P(J_1, U))$ is built from $\widetilde{M_3^7}(P(U, U))$ by removing solid tori and replacing them with either copies of $S^3 - J_1$ (in the case that ϕ is trivial on $\widetilde{z}$) or with copies of $M_7(J_1) \setminus \widetilde{J}_1$ in the case that ϕ is nontrivial on $\widetilde{z}$. In the second case, there would be three such replacements, since $\widetilde{z}$ has two nontrivial translates. The construction of $\widetilde{M_3^7}(P(J_1, J_2))$ from $\widetilde{M_3^7}(P(J_1, U))$ is similar.

Let $\widetilde{S}$ denote the set of all curves $\widetilde{z}_i'$ and $\widetilde{w}_i'$ along with the lifts of the two nontrivial translates of w and z under the action of the deck transformation of $M_3(P(U, U))$. We now see that $\widetilde{M_3^7}(P(J_1, J_2))$ is built from copies of the following spaces by gluing along tori.

- $\widetilde{M_3^7}(P(U, U), \phi) \setminus \widetilde{S}$
- $S^3 \setminus J_1$
- $M_7(J_1) \setminus \widetilde{J}_1$
- $S^3 \setminus J_2$
- $M_7(J_2) \setminus \widetilde{J}_2$

The homology of $\widetilde{M_3^7}(P(J_1, J_2))$ can be computed from this decomposition. Since there is only a finite number of possible ϕ, there is a bound on the rank of $H_1(\widetilde{M_3^7}(P(U, U), \phi) \setminus \widetilde{S}, \mathbb{F}_p)$ that is independent of ϕ. There is also a bound on the number of components of the intersection (tori) of the various pieces, again that is independent of ϕ.

With this, we see that up to a constant that is independent of ϕ, the rank of the homology is determined by the rank of the homology of $M_7(J_1)$ and $M_7(J_2)$. Each appears nontrivially if and only if the constants ϵ_i in the statement of the theorem are nonzero, in which case the term appears three times in the Mayer-Vietoris sequence. $\square$

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