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LIDSTONE INTERPOLATION III. SEVERAL VARIABLES

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Dedicated to Damien Roy for his 65-th birthday

Abstract. A polynomial in a single variable is uniquely determined by its derivatives of even order at 0 and 1. More precisely, such an univariate polynomial can be written and a finite sum of $f^{(2n)}(0)\Lambda_n(1-z)$ and $f^{(2n)}(1)\Lambda_n(z)$, $(n \geq 0)$, where the $\Lambda_n(z)$ are the Lidstone polynomials defined by the conditions

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k}\Lambda_n(0) = 0 \text{ and } \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k}\Lambda_n(1) = \delta_{k,n}, \quad k \geq 0,\ n \geq 0.$$

We generalize this theory to n variables, replacing the two points 0, 1 in $\mathbb{C}$ with $n+1$ points $\underline{e}_0, \underline{e}_1, \ldots, \underline{e}_n$ in $\mathbb{C}^n$, where $\underline{e}_0$ is the origin of $\mathbb{C}^n$ and $\underline{e}_1, \ldots, \underline{e}_n$ the canonical basis of $\mathbb{C}^n$. By selecting a suitable subset of even order derivatives at these $n+1$ points, we show that any polynomial in n variables has a unique expansion. We obtain generating series for these sequences of polynomials and we deduce an expansion for entire functions in $\mathbb{C}^n$ of exponential type $< \pi$. We extend to several variables results due to Lidstone [Proceedings Edinburgh Math. Soc., II. Ser. (2)2, 16-19 (1930)], Poritsky [Trans. Amer. Math. Soc. 34 (1932), pp. 274–331], Whittaker [Proc. London Math. Soc. (2) 36 (1934), pp. 451–469], Schoenberg [Bull. Amer. Math. Soc. 42 (1936), pp. 284–288], Buck [Proc. Amer. Math. Soc. 6 (1955), pp. 793–796]. We also show that our results are, to a certain extent, best possible.

1. The main results

Let n be a positive integer. For $\underline{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $\underline{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ and $\underline{t} = (t_1, \ldots, t_n) \in \mathbb{N}^n$, we set

$$|\underline{z}| = \max\{|z_1|, \ldots, |z_n|\}, \quad \|\underline{z}\| = |z_1| + \cdots + |z_n|, \quad \underline{\zeta}\underline{z} = \zeta_1 z_1 + \cdots + \zeta_n z_n,$$

$$\underline{t}! = t_1! \cdots t_n! \ \text{ and } \ \underline{z}^{\underline{t}} = \prod_{i=1}^{n} z_i^{t_i}.$$

We also use the notation

$$D^{\underline{t}} = \left(\frac{\partial}{\partial z_1}\right)^{t_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{t_n}.$$

We denote by $\mathbb{C}[\underline{z}]$ the vector space of polynomials in n variables. For $D \geq 0$, let $\mathbb{C}[\underline{z}]_{\leq D}$ denote the subspace of polynomials of total degree $\leq D$.

Let $(\underline{e}_1, \ldots, \underline{e}_n)$ denote the canonical basis of $\mathbb{C}^n$:

$$\underline{e}_i = (\delta_{ij})_{1 \leq j \leq n} \quad (i = 1, \ldots, n),$$

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where δ_{ij} is the Kronecker symbol. Further, set $\underline{e}_0 = (0, \ldots, 0)$. We denote by $2\mathbb{N}$ the set of even nonnegative integers.

Definition 1.1 will play a main role in this paper.

Definition 1.1. We denote by $\mathcal{T}$ the set of $(\underline{t}, i) \in \mathbb{N}^n \times \{0, 1, \ldots, n\}$ such that $\|\underline{t}\|$ is even and, for $i \geq 1$, $t_1, \ldots, t_i$ are even.

The initial step is the following result, which is easy to prove by induction on the number n of variables – see [6, Proposition 2].

Proposition 1.2. *If a polynomial $f \in \mathbb{C}[\underline{z}]$ satisfies*

$$(D^{\underline{t}} f)(\underline{e}_i) = 0$$

for all $(\underline{t}, i) \in \mathcal{T}$, then $f = 0$.

From Proposition 1.2 we will deduce the next result, which is our basis for interpolation. We use the Kronecker symbol

$$\delta_{\underline{\tau},\underline{t}} = \begin{cases} 1 & \text{if } \underline{\tau} = \underline{t}, \\ 0 & \text{otherwise} \end{cases}$$

and similarly for δ_{ij}.

Theorem 1.3. *For each $(\underline{t}, i) \in \mathcal{T}$, there is a unique polynomial $\Lambda_{\underline{t},i} \in \mathbb{C}[\underline{z}]$ satisfying, for all $(\underline{\tau}, j) \in \mathcal{T}$,*

$$(D^{\underline{\tau}} \Lambda_{\underline{t},i})(\underline{e}_j) = \delta_{\underline{\tau},\underline{t}} \delta_{ij}.$$

The total degree of $\Lambda_{\underline{t},i}$ is $\leq \|\underline{t}\| + 1$.

Here is the expansion formula for polynomials:

Corollary 1.4. *Any polynomial $f \in \mathbb{C}[\underline{z}]$ can be expanded in a unique way as a finite sum*

$$f(\underline{z}) = \sum_{(\underline{t},i) \in \mathcal{T}} (D^{\underline{t}} f)(\underline{e}_i) \Lambda_{\underline{t},i}(\underline{z}).$$

We now consider extensions of the unicity result in Proposition 1.2 and of the expansion formula in Corollary 1.4 from polynomials to entire functions of exponential type $< \pi$.

For $r > 0$ and for f an analytic function in a domain containing $\{\underline{z} \in \mathbb{C}^n \mid \|\underline{z}\| \leq r\}$, set

$$\|f\|_r = \sup_{\|\underline{z}\| = r} |f(\underline{z})|.$$

The order of an entire function f is

$$\varrho(f) = \limsup_{r \to \infty} \frac{\log \log \|f\|_r}{\log r}$$

and its exponential type

$$\tau(f) = \limsup_{r \to \infty} \frac{\log \|f\|_r}{r} \cdot$$

These definitions extend to several variables the definitions of [8, § 7] for two variables. For instance for $\underline{\zeta} \in \mathbb{C}^n$ the exponential type of the entire function $\underline{z} \mapsto \mathrm{e}^{\underline{\zeta} \underline{z}}$ is $|\underline{\zeta}|$. If we were using the norm $|\underline{z}|$ in place of $\|z\|$ in the definition of $\|f\|_r$, the order would be the same, but not the exponential type: for $\mathrm{e}^{\underline{\zeta} \underline{z}}$ it would be $\|\underline{\zeta}\|$.

Given an entire function f in $\mathbb{C}^n$ and a real number $\tau \geq 0$, we will say that f *has exponential type* $\leq \tau$ *in each of the variables* if, for any $i = 1, \dots, n$ and any $(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n) \in \mathbb{C}^{n-1}$, the function $z_i \mapsto f(z_1, \dots, z_n)$ has exponential type $\leq \tau$:

$$\limsup_{r\to\infty} \frac{1}{r} \log \sup_{|z_i|\leq r} |f(z_1, \dots, z_n)| \leq \tau.$$

If an entire function f in $\mathbb{C}^n$ has exponential type $\tau(f) \leq \tau$, then f has exponential type $\leq \tau$ in each of the variables. The converse is not true [8, § 7].

The next result is [6, Proposition 2], where it is proved by induction on the number of variables:

Proposition 1.5. *If an entire function f in $\mathbb{C}^n$ of exponential type $< \pi$ in each of the variables satisfies $(D^{\underline{t}} f)(\underline{e}_i) = 0$ for all $(\underline{t}, i) \in \mathcal{T}$, then $f = 0$.*

Proposition 1.5 is optimal for two different reasons. Firstly, it does not hold with a subset $\mathcal{T}'$ of $\mathcal{T}$ when $\mathcal{T} \setminus \mathcal{T}'$ is infinite, as shown by Proposition 1.6.

Proposition 1.6. *Let $\mathcal{T}'$ be a subset of $\mathcal{T}$ such that $\mathcal{T} \setminus \mathcal{T}'$ is infinite. Then there is an uncountable set of entire functions f of order 0, with Taylor expansion at the origin having rational coefficients, such that $(D^{\underline{t}} f)(\underline{e}_i) = 0$ for all $(\underline{t}, i) \in \mathcal{T}'$.*

The assumption on $\mathcal{T} \setminus \mathcal{T}'$ is necessary: for a subset $\mathcal{T}'$ of $\mathcal{T}$ such that $\mathcal{T} \setminus \mathcal{T}'$ is finite, the only entire functions of exponential type $< \pi$ in each of the variables such that $(D^{\underline{t}} f)(\underline{e}_i) = 0$ for all $(\underline{t}, i) \in \mathcal{T}'$ are the polynomials – this follows from Proposition 1.5. The proof of Proposition 1.6 is given in § 3.

Before giving the second reason for which Proposition 1.5 is optimal, let us state the following generalization of Corollary 1.4 to entire functions of exponential type $< \pi$.

Following [2, p. 27], we will say that a series of functions $\sum_\alpha a_\alpha(z)$ converges *normally* in an open subset Ω of $\mathbb{C}^n$ if $\sum_\alpha \sup_K |a_\alpha(z)|$ converges for every compact set $K \subset \Omega$. For instance [2, Theorem 2.2.6] an analytic function in a polydisc $\{\underline{z} \in \mathbb{C}^n \mid |z_j| < r_j,\ j = 1, \dots, n\}$ is the sum of its Taylor expansion at the origin with normal convergence in this polydisc.

Theorem 1.7. *Any entire function f in $\mathbb{C}^n$ of exponential type $< \pi$ can be written in a unique way as the sum of a series*

$$f(\underline{z}) = \sum_{(\underline{t},i)\in\mathcal{T}} (D^{\underline{t}} f)(\underline{e}_i) \Lambda_{\underline{t},i}(\underline{z}) \tag{1.1}$$

which is normally convergent in $\mathbb{C}^n$.

The case $n = 2$ of Theorem 1.7 is Theorem 7.1 of [8].

The proof of Theorem 1.7 (§ 6) is an application of the Laplace transform, once we prove the special case for the functions $f_{\underline{\zeta}}(\underline{z}) = \mathrm{e}^{\underline{\zeta}\underline{z}}$ for $\underline{\zeta} \in \mathbb{C}^n$ with $|\underline{\zeta}| < \pi$, which is the following:

Theorem 1.8. *For $\underline{z} \in \mathbb{C}^n$ and $\underline{\zeta} \in \mathbb{C}^n$ with $|\underline{\zeta}| < \pi$, we have*

$$\mathrm{e}^{\underline{\zeta}\underline{z}} = \sum_{\substack{\underline{t}\in\mathbb{N}^n \\ (\underline{t},0)\in\mathcal{T}}} \Lambda_{\underline{t},0}(\underline{z}) \underline{\zeta}^{\underline{t}} + \sum_{i=1}^{n} \mathrm{e}^{\zeta_i} \sum_{\substack{\underline{t}\in\mathbb{N}^n \\ (\underline{t},i)\in\mathcal{T}}} \Lambda_{\underline{t},i}(\underline{z}) \underline{\zeta}^{\underline{t}},$$

where the series in the right hand side are normally convergent in the open set $\{(\underline{\zeta}, \underline{z}) \in \mathbb{C}^n \times \mathbb{C}^n \mid |\underline{\zeta}| < \pi\}$.

For the proof of Theorem 1.8, we will produce in (§ 4) explicit formulae for the multivariate polynomials $\Lambda_{\underline{t},i}(\underline{z})$ $((\underline{t},i) \in \mathcal{T})$ in $\mathbb{C}[\underline{z}]$, in terms of the classical Lidstone univariate polynomials $\Lambda_{t,1}(z)$ and $\Lambda_{t,0}(z) = \Lambda_{t,1}(1-z)$ $(t \in 2\mathbb{N})$ in $\mathbb{C}[z]$ which we recall in § 2. We will deduce in § 5 explicit generating series for the 2^n families of polynomials $\Lambda_{\underline{t},i}(\underline{z})$, $(\underline{t},i) \in \mathcal{T}_{\underline{\alpha}}$, where $\mathcal{T}_{\underline{\alpha}}$ $(\underline{\alpha} \in (\mathbb{Z}/2\mathbb{Z})^n)$ is a partition of $\mathcal{T}$; the parity of the components $(t_1, \dots, t_n)$ of $\underline{t}$ for $(\underline{t},i) \in \mathcal{T}_{\underline{\alpha}}$ depend only on $\underline{\alpha}$.

The second reason for which Proposition 1.5 is optimal is that the upper bound π for the exponential type is best possible: in § 7 we give the following characterisation (Corollary 1.10) of the entire functions f in $\mathbb{C}^n$ of finite exponential type which satisfy $(D^{\underline{\tau}} f)(\underline{e}_j) = 0$ for all $(\underline{t},i) \in \mathcal{T}$.

Theorem 1.9. *Let K be a nonnegative integer. Let f be an entire function in $\mathbb{C}^n$ of finite exponential type $\leq \tau$ with $\tau < (K+1)\pi$. Then for $\underline{z} \in \mathbb{C}^n$ we have*

$$f(\underline{z}) = \sum_{(\underline{t},i)\in\mathcal{T}} (D^{\underline{t}} f)(\underline{e}_i) g_{\underline{t},i}(\underline{z}) + \sum_{k=1}^{K} \sum_{i=1}^{n} h_{k,i}(z_1, \dots, z_{i-1}, z_{i+1} \dots, z_n) \sin(k\pi z_i),$$

where the functions $g_{\underline{t},i}(\underline{z})$ are entire functions in $\mathbb{C}^n$, the series is normally convergent in $\mathbb{C}^n$ and $h_{k,i}$, $(k = 1, 2, \dots, K,\ i = 1, \dots, n)$ are entire functions of $n-1$ variables of exponential type $\leq \tau$.

Corollary 1.10. *Let f be an entire function in $\mathbb{C}^n$ of exponential type $\leq \tau$ with $\tau < (K+1)\pi$. Assume $(D^{\underline{t}} f)(\underline{e}_i) = 0$ for all $(\underline{t},i) \in \mathcal{T}$. Then there are entire functions of $n-1$ variables $h_{k,i}$ $(k = 1, 2, \dots, K,\ i = 1, \dots, n)$ of exponential type $\leq \tau$, such that*

$$f(\underline{z}) = \sum_{k=1}^{K} \sum_{i=1}^{n} h_{k,i}(z_1, \dots, z_{i-1}, z_{i+1} \dots, z_n) \sin(k\pi z_i).$$

2. Univariate Lidstone polynomials – a survey

We collect here some classical results on univariate Lidstone polynomials. Full proofs and references are given in [7].

Theorem 2.1 (G. J. Lidstone (1930) [3]). *There are two sequences of polynomials in a single variable, $\big(\Lambda_{t,0}(z)\big)_{t\in 2\mathbb{N}}$, $\big(\Lambda_{t,1}(z)\big)_{t\in 2\mathbb{N}}$, such that any polynomial $f \in \mathbb{C}[z]$ can be written as a finite sum*

$$f(z) = \sum_{t\in 2\mathbb{N}} f^{(t)}(0)\Lambda_{t,0}(z) + \sum_{t\in 2\mathbb{N}} f^{(t)}(1)\Lambda_{t,1}(z). \tag{2.1}$$

For $i = 0, 1$, the degree of $\Lambda_{t,i}$ is $t+1$. These polynomials are characterized by the following property: for t and τ in $2\mathbb{N}$, we have

$$\Lambda_{t,0}^{(\tau)}(0) = \delta_{t,\tau} \quad \text{and} \quad \Lambda_{t,0}^{(\tau)}(1) = 0$$

and

$$\Lambda_{t,1}^{(\tau)}(0) = 0 \quad \text{and} \quad \Lambda_{t,1}^{(\tau)}(1) = \delta_{t,\tau}.$$

It follows that $\Lambda_{t,0}(z) = \Lambda_{t,1}(1-z)$. In the literature, the polynomials $\Lambda_{2k,1}$ are denoted by Λ_k (they are the classical Lidstone polynomials), but for our generalization to several variables it is more convenient to use the present notation.

From Theorem 2.1 one deduces, for $t \in 2\mathbb{N}$, the recurrence formulae

$$\Lambda_{t,1}(z) = \frac{1}{(t+1)!}z^{t+1} - \sum_{\substack{\tau \in 2\mathbb{N} \\ 0 \le \tau \le t-2}} \frac{1}{(t-\tau+1)!}\Lambda_{\tau,1}(z) \tag{2.2}$$

and

$$\frac{z^t}{t!} = \Lambda_{t,0}(z) + \sum_{\substack{\tau \in 2\mathbb{N} \\ 0 \le \tau \le t}} \frac{1}{(t-\tau)!}\Lambda_{\tau,1}(z). \tag{2.3}$$

Lidstone's Theorem 2.1 for polynomials has been extended by H. Poritsky (1932) and J. M. Whittaker (1934) [4,9] to entire functions of sufficiently small exponential type.

Theorem 2.2. *The expansion* (2.1) *holds for any entire function f of exponential type $< \pi$, with normally convergent series over $\mathbb{C}$ in the right hand side of* (2.1).

Corollary 2.3. *Let f be an entire function of exponential type $< \pi$ satisfying $f^{(t)}(0) = f^{(t)}(1) = 0$ for all sufficiently large $t \in 2\mathbb{N}$. Then f is a polynomial.*

Another consequence of Theorem 2.2 is the following explicit expression for the generating series of the Lidstone polynomials:

$$\sum_{t \in 2\mathbb{N}} \Lambda_{t,1}(z)\zeta^t = \frac{\sinh(\zeta z)}{\sinh(\zeta)}. \tag{2.4}$$

From the estimates in [7, (15)] for the classical Lidstone polynomials, it follows that the series in the left hand side of (2.4) is normally convergent in $\{(z,\zeta) \in \mathbb{C}^2 \mid |\zeta| < \pi\}$.

Expansions similar to (2.1) hold for functions of finite exponential type, as shown by R. C. Buck in 1955 [1].

Theorem 2.4. *Let K be a positive integer. Let f be an entire function of finite exponential type $\tau(f) < (K+1)\pi$ and let $F(\zeta)$ be the Laplace transform of f. Then for $z \in \mathbb{C}$ we have*

$$f(z) = \sum_{t \in 2\mathbb{N}} f^{(t)}(0)g_t(1-z) + \sum_{t \in 2\mathbb{N}} f^{(t)}(1)g_t(z) + \sum_{k=1}^{K} C_k \sin(k\pi z),$$

where the series are normally convergent in $\mathbb{C}$, the functions g_t are entire and, for $\max\{K\pi, \tau(f)\} < r < (K+1)\pi$,

$$C_k = -k\mathrm{i}\int_{|\zeta|=r} \frac{1+(-1)^{k+1}e^{\zeta}}{\zeta^2 + k^2\pi^2} F(\zeta)\mathrm{d}\zeta \quad (1 \le k \le K).$$

Notice the assumption $\max\{\tau(f), K\pi\} < r < (K+1)\pi$ which replaces the erroneous condition $\tau(f) < r < (K+1)\pi$ of [7, Proposition 3], as pointed out in [8, § 8].

The following consequence of Theorem 2.4 was already proved by I. J. Schoenberg in 1936 [5]:

Corollary 2.5. *Let f be an entire function of finite exponential type $\tau(f)$ satisfying $f^{(t)}(0) = f^{(t)}(1) = 0$ for all $t \in 2\mathbb{N}$. Then there are complex numbers such that*

$$f(z) = \sum_{k=1}^{K} C_k \sin(k\pi z)$$

with $K \le \tau(f)/\pi$ and $C_1, \dots, C_K$ are the constants from Theorem 2.4.

3. Existence and unicity of the polynomials

Recall (Definition 1.1) that $\mathcal{T}$ denotes the set of $(\underline{t}, i)$ with $\underline{t} \in \mathbb{N}^n$, $\|\underline{t}\|$ even, $i \in \{0, 1, \dots, n\}$, which satisfy the additional condition, for $i \ge 1$, that $t_1, \dots, t_i$ are even.

Let $\Psi : \mathbb{N}^n \to \mathcal{T}$ be the map which sends $\underline{k} \in \mathbb{N}^n$ to $(\underline{k}, 0)$ for $\|\underline{k}\|$ even and to $(\underline{k} - \underline{e}_i, i)$ for $\|k\|$ odd, where $i \in \{1, \dots, n\}$ is the index such that $k_1, \dots, k_{i-1}$ are even and k_i is odd. We also define $\iota(\underline{k})$ as the integer i such that $\Psi(\underline{k}) = (\underline{k} - \underline{e}_i, i)$. This map Ψ is bijective, the inverse bijection is $(\underline{t}, i) \mapsto \underline{t} + \underline{e}_i$.

Proof of Proposition 1.2. For $\underline{t}$ and $\underline{k}$ in $\mathbb{N}^n$, we have

$$D^{\underline{t}} \underline{z}^{\underline{k}} = \begin{cases} \frac{\underline{k}!}{(\underline{k}-\underline{t})!} \underline{z}^{\underline{k}-\underline{t}} & \text{if } k_i \ge t_i \text{ for all } i = 1, \dots, n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Let $f(\underline{z}) = \sum_{k \in \mathbb{N}^n} a_{\underline{k}} \underline{z}^{\underline{k}} \in \mathbb{C}[\underline{z}]$. From (3.1) we deduce that for $\underline{t} \in \mathbb{N}^n$, we have

$$D^{\underline{t}} f(\underline{e}_0) = \underline{t}! a_{\underline{t}}$$

and that for $1 \le i \le n$, we have

$$D^{\underline{t}} f(\underline{e}_i) = \sum_{\ell \ge 0} \frac{(\underline{t} + \ell \underline{e}_i)!}{\ell!} a_{\underline{t} + \ell \underline{e}_i}.$$

Assume $(D^{\underline{t}} f)(\underline{e}_i) = 0$ for all $(\underline{t}, i) \in \mathcal{T}$. Let $K \ge 0$ be an even integer such that f has degree $\le K + 1$. We prove $f = 0$ by induction on K.

For $K = 0$, we have

$$f(\underline{z}) - f(\underline{e}_0) = \sum_{i=1}^{n} z_i (f(\underline{e}_i) - f(\underline{e}_0)). \tag{3.2}$$

Since $(\underline{e}_0, i) \in \mathcal{T}$ for $0 \le i \le n$, the assumption on f implies $f(\underline{e}_i) = 0$ for $0 \le i \le n$, hence $f = 0$.

Assume $K \ge 2$; recall that K is even. Let $(\underline{t}, i) \in \mathcal{T}$ satisfy $\|\underline{t}\| = K$. From $D^{\underline{t}} f(\underline{e}_0) = 0$ we deduce $a_{\underline{t}} = 0$. Since f has degree $\le K + 1$, for $1 \le i \le n$ we have

$$D^{\underline{t}} f(\underline{e}_i) = \underline{t}! a_{\underline{t}} + (\underline{t} + \underline{e}_i)! a_{\underline{t} + \underline{e}_i},$$

hence the condition $D^{\underline{t}} f(\underline{e}_i) = 0$ implies $a_{\underline{t} + \underline{e}_i} = 0$. Since Ψ is bijective, we deduce $a_{\underline{k}} = 0$ for all $\underline{k} \in \mathbb{N}^n$ with $\|\underline{k}\| \in \{K, K + 1\}$ and therefore f has degree $\le K - 1$. This completes the proof of the inductive argument. □

Let T be an even integer ≥ 0. Let $\mathcal{T}_T$ be the set of pairs $(\underline{t}, i) \in \mathcal{T}$ with $\|\underline{t}\| \le T$. By restriction, Ψ induces a bijective map from $\{\underline{k} \in \mathbb{N}^n \mid \|\underline{k}\| \le T + 1\}$ onto $\mathcal{T}_T$. Therefore Proposition 1.2 can be stated as:

Proposition 3.1. *For each even $T \ge 0$, the linear map which, to a polynomial $f \in \mathbb{C}[\underline{z}]_{\le T+1}$, associates the tuple*

$$\bigl(D^{\underline{t}} f(\underline{e}_i)\bigr)_{(\underline{t}, i) \in \mathcal{T}_T} \in \mathbb{C}^{\mathcal{T}_T}$$

is an isomorphism from $\mathbb{C}[\underline{z}]_{\le T+1}$ to $\mathbb{C}^{\mathcal{T}_T}$.

Proof of Theorem 1.3. The unicity follows from Proposition 1.2. We prove the existence together with the upper bound for the degree. Let T be an even integer ≥ 0. The inverse image under the isomorphism given by Proposition 3.1 of the canonical basis of $\mathbb{C}^{\mathcal{T}_T}$ is a family of polynomials $\Lambda_{\underline{t},i}$ in $\mathbb{C}[\underline{z}]_{\leq T+1}$ which satisfies, for each $(\underline{\tau}, j) \in \mathcal{T}$ with $\|\underline{\tau}\| \leq T+1$,

$$(D^{\underline{\tau}}\Lambda_{\underline{t},i})(\underline{e}_j) = \delta_{\underline{\tau},\underline{t}}\delta_{ij}.$$

Since $\Lambda_{\underline{t},i}$ has total degree $\leq T+1$, it also satisfies $(D^{\underline{\tau}}\Lambda_{\underline{t},i}) = 0$ for all $\underline{\tau}$ with $\|\underline{\tau}\| > T+1$.

In a more explicit way, we deduce from Proposition 3.1 that the square matrix

$$\Big((D^{\underline{\tau}}\underline{z}^{\underline{k}})(\underline{e}_j)\Big)_{\substack{(\underline{\tau},j)\in\mathcal{T}_T\\ \|\underline{k}\|\leq T+1}}$$

is regular. Denote by

$$\Big(a_{\underline{t},i}^{(\underline{k})}\Big)_{\substack{\|\underline{k}\|\leq T+1\\ (\underline{t},i)\in\mathcal{T}_T}}$$

the inverse matrix. The polynomials $\Lambda_{\underline{t},i}$ are nothing else than

$$\Lambda_{\underline{t},i}(\underline{z}) = \sum_{\|\underline{k}\|\leq T+1} a_{\underline{t},i}^{(\underline{k})}\underline{z}^{\underline{k}}.$$

□

Proof of Corollary 1.4. Under the assumptions of Corollary 1.4, it follows from Theorem 1.3 that the polynomial

$$g(\underline{z}) = f(\underline{z}) - \sum_{(\underline{t},i)\in\mathcal{T}} (D^{\underline{t}}f)(\underline{e}_i)\Lambda_{\underline{t},i}(\underline{z})$$

satisfies

$$(D^{\underline{t}}g)(\underline{e}_i) = 0 \text{ for all } (\underline{t}, i) \in \mathcal{T},$$

hence $g = 0$ as shown by Proposition 1.2. □

In the case $n = 1$, we recover the Lidstone univariate polynomials classically denoted $\Lambda_m(z)$: for $m \geq 0$ and $z \in \mathbb{C}$,

$$\Lambda_{2m,0}(z) = \Lambda_m(1-z), \quad \Lambda_{2m,1}(z) = \Lambda_m(z).$$

They also occur in the relations, for $m \geq 0$, $i = 1, \ldots, n$ and $\underline{z} \in \mathbb{C}^n$,

$$\Lambda_{2m\underline{e}_i,i}(\underline{z}) = \Lambda_m(z_i).$$

For $\underline{t} = \underline{e}_0$ we have (compare with (3.2))

$$\Lambda_{\underline{e}_0,0}(\underline{z}) = 1 - z_1 - \cdots - z_n, \quad \Lambda_{\underline{e}_0,i}(\underline{z}) = z_i \text{ for } i = 1, \ldots, n.$$

Proof of Proposition 1.6. Let $(P_n)_{n\geq 0}$ be an infinite sequence of polynomials among the $\Lambda_{\underline{t},i}$, $(\underline{t}, i) \in \mathcal{T} \setminus \mathcal{T}'$, where the sequence d_n of the degree of P_n is increasing. Let $(c_n)_{n\geq 0}$ be a sequence of positive rational numbers satisfying

$$|P_n|_r \leq c_n r^{d_n}$$

for all $r \geq 1$ and $n \geq 0$. For $n \geq 0$, set

$$u_n = \frac{1}{c_n(d_n!)^2}.$$

The series
$$\sum_{n\ge 0} u_n P_n(\underline{z})$$
is normally convergent in $\mathbb{C}^n$ and its sum $f(\underline{z})$ is an entire function of order 0. From the normal hence uniform convergence we deduce, for all $(\underline{t}, i) \in \mathcal{T}$,
$$(D^{\underline{t}} f)(\underline{e}_i) = \begin{cases} u_n & \text{if } \Lambda_{\underline{t},i} = P_n,\\ 0 & \text{otherwise.}\end{cases}$$
As a consequence we have $(D^{\underline{t}} f)(\underline{e}_i) = 0$ for all $(\underline{t}, i) \in \mathcal{T}'$.

Two distinct sequences $(c_n)_{n\ge 0}$ give rise to two distinct functions f, and consequently we obtain an uncountable set of such functions. □

4. Explicit formulae

Here are explicit formulae for the multivariate polynomials $\Lambda_{\underline{t},i}(\underline{z})$ ($(\underline{t}, i) \in \mathcal{T}$) in $\mathbb{C}[\underline{z}]$, in terms of the classical Lidstone univariate polynomials $\Lambda_{t,1}(z)$ and $\Lambda_{t,0}(z) = \Lambda_{t,1}(1 - z)$ ($t \in 2\mathbb{N}$) in $\mathbb{C}[z]$ introduced in § 2.

Theorem 4.1. *Let $(\underline{t}, i) \in \mathcal{T}$. If $i \in \{1, \dots, n\}$, we have*
$$\Lambda_{\underline{t},i}(\underline{z}) = \Lambda_{t_i,1}(z_i) \prod_{\substack{1\le j\le n\\ j\ne i}} \frac{z_j^{t_j}}{t_j!}\cdot \tag{4.1}$$
If $i = 0$, let us denote by $\nu \in \{0, 1, \dots, n\}$ the least integer ≥ 0 such that $t_{\nu+1}$ is odd, with $\nu = 0$ if t_1 is odd while $\nu = n$ if $t_1, \dots, t_n$ are all even. Then
$$\Lambda_{\underline{t},0}(\underline{z}) = \sum_{j=1}^{\nu} \Lambda_{t_j,0}(z_j) \prod_{\substack{1\le \ell\le n\\ \ell\ne j}} \frac{z_\ell^{t_\ell}}{t_\ell!} - (\nu - 1)\frac{\underline{z}^{\underline{t}}}{\underline{t}!}\cdot \tag{4.2}$$

We will prove Theorem 4.1 by using the following corollary of Theorem 1.3, where $\chi_{\mathcal{T}}$ denotes the characteristic function of $\mathcal{T}$: for $(\underline{t}, i) \in \mathbb{N}^n \times \{0, 1, \dots, n\}$, we write
$$\chi_{\mathcal{T}}(\underline{t}, i)\Lambda_{\underline{t},i}(\underline{z}) = \begin{cases} \Lambda_{\underline{t},i}(\underline{z}) & \text{if } (\underline{t}, i) \in \mathcal{T},\\ 0 & \text{if } (\underline{t}, i) \notin \mathcal{T}.\end{cases}$$

Lemma 4.2. *For $\underline{k} \in \mathbb{N}^n$, we have*
$$\frac{1}{\underline{k}!}\underline{z}^{\underline{k}} = \sum_{i=1}^{n}\sum_{\ell=0}^{k_i} \frac{1}{\ell!}\chi_{\mathcal{T}}(\underline{k} - \ell\underline{e}_i, i)\Lambda_{\underline{k}-\ell\underline{e}_i,i}(\underline{z}) + \begin{cases} \Lambda_{\underline{k},0}(\underline{z}) & \textit{if } \|\underline{k}\| \textit{ is even,}\\ 0 & \textit{if } \|\underline{k}\| \textit{ is odd.}\end{cases} \tag{4.3}$$

Proof of Lemma 4.2. We use Corollary 1.4 in the following equivalent form: for $\underline{k} \in \mathbb{N}^n$,
$$\underline{z}^{\underline{k}} = \sum_{i=0}^{n}\sum_{\underline{t}\in\mathbb{N}^n} (D^{\underline{t}}\underline{z}^{\underline{k}})(\underline{e}_i)\chi_{\mathcal{T}}(\underline{t}, i)\Lambda_{\underline{t},i}(\underline{z}).$$
From the relations (3.1) we deduce
$$(D^{\underline{t}}\underline{z}^{\underline{k}})(\underline{e}_0) = \delta_{\underline{t}\underline{k}}\underline{k}!,$$
hence
$$\sum_{\underline{t}\in\mathbb{N}^n} (D^{\underline{t}}\underline{z}^{\underline{k}})(\underline{e}_0)\chi_{\mathcal{T}}(\underline{t}, 0)\Lambda_{\underline{t},0}(\underline{z}) = \begin{cases} \underline{k}!\Lambda_{\underline{k},0}(\underline{z}) & \text{if } \|\underline{k}\| \text{ is even,}\\ 0 & \text{if } \|\underline{k}\| \text{ is odd.}\end{cases}$$

Let $i \in \{1, \dots, n\}$. The relations (3.1) yield

$$(D^{\underline{t}} \underline{z}^{\underline{k}})(\underline{e}_i) = \begin{cases} \frac{\underline{k}!}{(k_i - t_i)!} & \text{if } k_j = t_j \text{ for all } j \neq i \text{ and } t_i \leq k_i, \\ 0 & \text{otherwise,} \end{cases}$$

hence

$$\sum_{\underline{t} \in \mathbb{N}^n} (D^{\underline{t}} \underline{z}^{\underline{k}})(\underline{e}_i) \chi_{\mathcal{T}}(\underline{t}, i) \Lambda_{\underline{t},i}(\underline{z}) = \sum_{\ell=0}^{k_i} \frac{\underline{k}!}{\ell!} \chi_{\mathcal{T}}(\underline{k} - \ell \underline{e}_i, i) \Lambda_{\underline{k}-\ell \underline{e}_i,i}(z).$$

Lemma 4.2 follows. □

We deduce from Lemma 4.2 the following recurrence formulae, which extend to several variables the inductive formulae (2.2) and (2.3) for the classical Lidstone polynomials. Corollary 4.3 enables one to compute firstly $\Lambda_{\underline{t},i}$ for $i = 1, \dots, n$ by induction on $\|\underline{t}\|$ and next $\Lambda_{\underline{t},0}$. The computation of $\Lambda_{\underline{t},i}$ relies on formula (4.3) applied to $\underline{k} = \underline{t} + \underline{e}_i$, that is with $\underline{k}$ such that $\Psi(\underline{k}) = (\underline{t}, i)$, following the notation of § 3.

Corollary 4.3. *For $(\underline{t}, i) \in \mathcal{T}$ with $1 \leq i \leq n$, we have*

$$\Lambda_{\underline{t},i}(\underline{z}) = \frac{\underline{z}^{\underline{t}}}{\underline{t}!} \frac{z_i}{t_i + 1} - \sum_{\substack{m \in 2\mathbb{N} \\ 2 \leq m \leq t_i}} \frac{1}{(m+1)!} \Lambda_{\underline{t}-m\underline{e}_i,i}(\underline{z}). \tag{4.4}$$

For $i = 0$ and $\|\underline{t}\|$ even, we have

$$\Lambda_{\underline{t},0}(\underline{z}) = \frac{\underline{z}^{\underline{t}}}{\underline{t}!} - \sum_{j=1}^{\nu} \sum_{\substack{\ell \in 2\mathbb{N} \\ 0 \leq \ell \leq t_j}} \frac{1}{\ell!} \Lambda_{\underline{t}-\ell \underline{e}_j,j}(\underline{z}), \tag{4.5}$$

where ν is defined in the statement of Theorem 4.1.

In case $n = 1$, (4.4) is nothing else than the recurrence formulae (2.2), while (4.5) corresponds to (2.3) for the univariate Lidstone polynomials. In case $n = 2$, Corollary 4.3 reduces to [8, Lemma 3.1] for the bivariate polynomials.

Proof. Let $i \in \{1, \dots, n\}$ and let $(\underline{t}, i) \in \mathcal{T}$; hence $t_1, \dots, t_i$ are even and $\|\underline{t}\|$ is even. We use equation (4.3) with $\underline{k} = \underline{t} + \underline{e}_i$ (hence $k_i = t_i + 1$ and $\|\underline{k}\| = \|\underline{t}\| + 1$ are odd while $k_1, \dots, k_{i-1}$ are even):

$$\frac{\underline{z}^{\underline{k}}}{\underline{k}!} = \frac{\underline{z}^{\underline{t}}}{\underline{t}!} \frac{z_i}{t_i + 1} = \sum_{j=1}^{n} \sum_{\ell=0}^{k_j} \frac{1}{\ell!} \chi_{\mathcal{T}}(\underline{k} - \ell \underline{e}_j, j) \Lambda_{\underline{k}-\ell \underline{e}_j,j}(\underline{z}).$$

Let $j \in \{1, 2, \dots, n\}$ and $\ell \in \{0, \dots, k_j\}$ be such that $(\underline{k} - \ell \underline{e}_j, j) \in \mathcal{T}$; then $k_1, \dots, k_{j-1}$, $k_j - \ell$ and $\|\underline{k} - \ell \underline{e}_j\| = \|\underline{k}\| - \ell$ are even, hence ℓ is odd and therefore k_j also. Since $k_1, \dots, k_{j-1}$ are even and k_i is odd we have $i \geq j$. Since $k_1, \dots, k_{i-1}$ are even and k_j is odd we have $j \geq i$. Hence $j = i$. For $j = i$, we have $(\underline{k} - \ell \underline{e}_i, i) \in \mathcal{T}$ for all ℓ odd in the range $0 \leq \ell \leq k_j$. Equation (4.4) follows.

Let $\underline{t} \in \mathbb{N}^n$ with $\|\underline{t}\|$ even. Let $\nu \in \{0, 1, \dots, n\}$ be the index such that $t_1, \dots, t_\nu$ are even and $t_{\nu+1}$ is odd. We use (4.3) with $\underline{k} = \underline{t}$ (hence $\|\underline{k}\|$ is even):

$$\Lambda_{\underline{t},0}(\underline{z}) = \frac{1}{\underline{t}!} \underline{z}^{t} - \sum_{j=1}^{n} \sum_{\ell=0}^{t_j} \frac{1}{\ell!} \chi_{\mathcal{T}}(\underline{t} - \ell \underline{e}_j, j) \Lambda_{\underline{t}-\ell \underline{e}_j,j}(\underline{z}).$$

Let $j \in \{1, 2, \dots, n\}$ and $\ell \in \{0, \dots, t_j\}$ be such that $(\underline{t} - \ell \underline{e}_j, j) \in \mathcal{T}$. Now $\|\underline{t} - \ell \underline{e}_j\| = \|\underline{t}\| - \ell$ is even, hence $t_1, \dots, t_{j-1}$ and ℓ are even. Also $t_j - \ell$ is even, hence t_j is even. Since $t_1, \dots, t_j$ are even and $t_{\nu+1}$ is odd, we have $j \le \nu$. Finally for ℓ even in the range $1 \le j \le \nu$, we have $(\underline{t} - \ell \underline{e}_j, j) \in \mathcal{T}$. We deduce (4.5). □

Proof of Theorem 4.1. Let i satisfy $1 \le i \le n$. We prove (4.1) by induction on $\|\underline{t}\|$, starting with $\|\underline{t}\| = 0$, that is $\underline{t} = (0, \dots, 0) = \underline{e}_0$, for which $\Lambda_{\underline{e}_0, i}(\underline{z}) = z_i = \Lambda_{0,1}(z_i)$. Assume now $(\underline{t}, i) \in \mathcal{T}$ has $\|\underline{t}\| \ge 2$. From the induction hypothesis we deduce, for m even in the range $2 \le m \le t_i$,

$$\Lambda_{\underline{t} - m\underline{e}_i, i}(\underline{z}) = \Lambda_{t_i - m, 1}(z_i) \prod_{\substack{1 \le j \le n \\ j \ne i}} \frac{z_j^{t_j}}{t_j!}.$$

Using (4.4), we obtain

$$\Lambda_{\underline{t}, i}(\underline{z}) = \left(\frac{z_i^{t_i+1}}{(t_i+1)!} - \sum_{\substack{m \in 2\mathbb{N} \\ 2 \le m \le t_i}} \frac{1}{(m+1)!} \Lambda_{t_i - m, 1}(z_i) \right) \prod_{\substack{1 \le j \le n \\ j \ne i}} \frac{z_j^{t_j}}{t_j!}.$$

Now (4.1) follows from the recurrence formula (2.2).

We prove (4.2), again by induction, starting with $\|\underline{t}\| = 0$ for which we have $\nu = n$:

$$\Lambda_{\underline{e}_0, 0}(\underline{z}) = 1 - z_1 - \cdots - z_n = (1 - z_1) + \cdots + (1 - z_n) - (n-1) = \sum_{i=1}^n \Lambda_{0,0}(z_i) - (n-1).$$

Let $\underline{t} \in \mathbb{N}^n$ with $\|\underline{t}\|$ even ≥ 2. In (4.5), using (4.1), we substitute, for $1 \le j \le \nu$,

$$\Lambda_{\underline{t} - \ell\underline{e}_j, j}(\underline{z}) = \Lambda_{t_j - \ell, 1}(z_j) \prod_{\substack{1 \le i \le n \\ i \ne j}} \frac{z_i^{t_i}}{t_i!}.$$

Using (2.3) we obtain

$$\begin{aligned}
\Lambda_{\underline{t}, 0}(\underline{z}) &= \frac{\underline{z}^{\underline{t}}}{\underline{t}!} - \sum_{j=1}^{\nu} \sum_{\substack{\ell \in 2\mathbb{N} \\ 0 \le \ell \le t_j}} \frac{1}{\ell!} \Lambda_{t_j - \ell, 1}(z_j) \prod_{\substack{1 \le i \le n \\ i \ne j}} \frac{z_i^{t_i}}{t_i!} \\
&= \sum_{j=1}^{\nu} \left(\frac{z_j^{t_j}}{t_j!} - \sum_{\substack{\ell \in 2\mathbb{N} \\ 0 \le \ell \le t_j}} \frac{1}{\ell!} \Lambda_{t_j - \ell, 1}(z_j) \right) \prod_{\substack{1 \le i \le n \\ i \ne j}} \frac{z_i^{t_i}}{t_i!} - (\nu - 1) \frac{\underline{z}^{\underline{t}}}{\underline{t}!} \\
&= \sum_{j=1}^{\nu} \Lambda_{t_j, 0}(z_j) \prod_{\substack{1 \le i \le n \\ i \ne j}} \frac{z_i^{t_i}}{t_i!} - (\nu - 1) \frac{\underline{z}^{\underline{t}}}{\underline{t}!},
\end{aligned}$$

which completes the proof by induction of (4.2).

An alternative proof of Theorem 4.1 is by checking that the polynomials in the right hand sides of the formulae (4.1) and (4.2) satisfy the properties of Theorem 1.3 which give a characterisation of the polynomials $\Lambda_{\underline{t}, i}$. □

From Theorem 4.1 and [7, Equation (15)], we deduce, for $(\underline{t}, i) \in \mathcal{T}$ and $\underline{z} \in \mathbb{C}^n$,

$$
\begin{cases}
|\Lambda_{\underline{t},i}(\underline{z})| \le 2\pi^{-t_i} \mathrm{e}^{3\pi|z_i|/2} \displaystyle\prod_{\substack{1\le j\le n \\ j\ne i}} \frac{|z_j|^{t_j}}{t_j!}, \qquad i = 1, \dots, n, \\
|\Lambda_{\underline{t},0}(\underline{z})| \le (n-1)\dfrac{|\underline{z}^{\underline{t}}|}{\underline{t}!} + 2\mathrm{e}^{3\pi/2} \displaystyle\sum_{j=1}^{n} \pi^{-t_j} \mathrm{e}^{3\pi|z_j|/2} \prod_{\substack{1\le \ell\le n \\ \ell\ne j}} \frac{|z_\ell|^{t_\ell}}{t_\ell!}.
\end{cases}
\tag{4.6}
$$

5. Generating series

Let us write the conclusion of Theorem 1.8 as

$$
\mathrm{e}^{\underline{\zeta}\underline{z}} = \sum_{(\underline{t},i)\in\mathcal{T}} \Lambda_{\underline{t},i}(\underline{z}) \mathrm{e}^{\zeta_i} \underline{\zeta}^{\underline{t}}, \tag{5.1}
$$

where we set $\zeta_0 = 0$.

For $i \in \{0, 1, \dots, n\}$, define

$$
\mathcal{A}_i = \{\underline{\alpha} \in (\mathbb{Z}/2\mathbb{Z})^n \;\mid\; \|\underline{\alpha}\| = 0,\ \alpha_1 = \cdots = \alpha_i = 0\}
$$

and let

$$
\mathcal{A} = \{(\underline{\alpha}, i) \;\mid\; \underline{\alpha} \in \mathcal{A}_i,\ 0 \le i \le n\} \subset (\mathbb{Z}/2\mathbb{Z})^n \times \{0, 1, \dots, n\}.
$$

The number of elements in $\mathcal{A}_i$ is

$$
\begin{cases}
2^{n-1-i} \text{ for } 0 \le i \le n-1, \\
1 \text{ for } i = n,
\end{cases}
$$

hence the number of elements in $\mathcal{A}$ is 2^n. We now split $\mathcal{T}$ into a disjoint union of 2^n subsets $\mathcal{T}_{\underline{\alpha},i}$: for $(\underline{\alpha}, i) \in \mathcal{A}$, define

$$
\mathcal{T}_{\underline{\alpha},i} = \{(\underline{t}, i) \in \mathcal{T},\ t_j \bmod 2 = \alpha_j \text{ for } 1 \le j \le n\}.
$$

We now produce explicit formulae for the 2^n generating series indexed by $(\underline{\alpha}, i) \in \mathcal{A}$:

$$
M_{\underline{\alpha},i}(\underline{\zeta}, \underline{z}) = \sum_{\substack{\underline{t}\in\mathbb{N}^n \\ (\underline{t},i)\in\mathcal{T}_{\underline{\alpha},i}}} \Lambda_{\underline{t},i}(\underline{z}) \underline{\zeta}^{\underline{t}}.
$$

Using the notation $\zeta_0 = 0$ from above, we have

$$
\sum_{(\underline{t},i)\in\mathcal{T}} \Lambda_{\underline{t},i}(\underline{z}) \mathrm{e}^{\zeta_i} \underline{\zeta}^{\underline{t}} = \sum_{(\underline{\alpha},i)\in\mathcal{A}} \mathrm{e}^{\zeta_i} M_{\underline{\alpha},i}(\underline{\zeta}, \underline{z}). \tag{5.2}
$$

Definition 5.1. For $\underline{\alpha} \in \mathcal{A}_0$, we define $\nu(\underline{\alpha})$ as the largest index $\nu \in \{0, 1, \dots, n\}$ such that $\alpha_1 = \cdots = \alpha_\nu = 0$.

If $\alpha_1 = \cdots = \alpha_n = 0$ then $\nu(\underline{\alpha}) = n$. Otherwise $\nu(\underline{\alpha}) \le n-1$ and $\alpha_{\nu+1} = 1$. Also $\nu(\underline{\alpha}) = 0$ if and only if $\alpha_1 = 1$.

Theorem 4.1 can be stated as follows:

Theorem 5.2. *Let $(\underline{\alpha}, i) \in \mathcal{A}$ and let $(\underline{\zeta}, \underline{z}) \in \mathbb{C}^n \times \mathbb{C}^n$ with $|\underline{\zeta}| < \pi$.*

If $i \ne 0$, we have

$$
M_{\underline{\alpha},i}(\underline{\zeta}, \underline{z}) = \frac{\sinh(\zeta_i z_i)}{\sinh(\zeta_i)} \prod_{\substack{1\le j\le n, j\ne i \\ \alpha_j = 0}} \cosh(\zeta_j z_j) \prod_{\substack{1\le j\le n \\ \alpha_j = 1}} \sinh(\zeta_j z_j).
$$

If $i=0$, we have, with $\nu=\nu(\underline{\alpha})$,

$$M_{\underline{\alpha},0}(\underline{\zeta},\underline{z}) = \sum_{j=1}^{\nu} \frac{\sinh(\zeta_j(1-z_j))}{\sinh(\zeta_j)} \prod_{\substack{1\le \ell\le n, \ell\ne j\\ \alpha_\ell=0}} \cosh(\zeta_\ell z_\ell) \prod_{\substack{1\le \ell\le n\\ \alpha_\ell=1}} \sinh(\zeta_\ell z_\ell)$$
$$-(\nu-1) \prod_{\substack{1\le \ell\le n\\ \alpha_\ell=0}} \cosh(\zeta_\ell z_\ell) \prod_{\substack{1\le \ell\le n\\ \alpha_\ell=1}} \sinh(\zeta_\ell z_\ell),$$

and also

$$M_{\underline{\alpha},0}(\underline{\zeta},\underline{z}) = \prod_{\substack{1\le j\le n\\ \alpha_j=0}} \cosh(\zeta_j z_j) \prod_{\substack{1\le j\le n\\ \alpha_j=1}} \sinh(\zeta_j z_j)$$
$$-\sum_{j=1}^{\nu} \frac{\cosh(\zeta_j)}{\sinh(\zeta_j)} \sinh(\zeta_j z_j) \prod_{\substack{1\le \ell\le n, \ell\ne j\\ \alpha_\ell=0}} \cosh(\zeta_\ell z_\ell) \prod_{\substack{1\le \ell\le n\\ \alpha_\ell=1}} \sinh(\zeta_\ell z_\ell).$$

Proof. The formula for $i\ne 0$ and the first formula for $i=0$ follow from Theorem 4.1, the generating series (2.4) and the Taylor expansions of $\sinh(\zeta z)$ and $\cosh(\zeta z)$.

The last formula for $i=0$ follows from the previous one by using the identity

$$\sinh(\zeta(1-z)) = \sinh(\zeta)\cosh(\zeta z) - \sinh(\zeta z)\cosh(\zeta). \tag{5.3}$$

□

6. Entire functions of exponential type $<\pi$

This section is devoted to the proof of Theorem 1.8 first and then of Theorem 1.7. We start with the claims on the normal convergence of the series in these statements.

Proof of the normal convergence. If f is an entire function in $\mathbb{C}^n$ of exponential type $\tau(f)$, then

$$\limsup_{\|\underline{k}\|\to\infty} |D^{\underline{k}} f(\underline{z}_0)|^{\frac{1}{\|\underline{k}\|}} = \tau(f). \tag{6.1}$$

The proof given in [8, Lemma 7.1] for $n=2$ extends to all n.

Let $R>0$. According to (4.6), there exists $c=c(R)>0$ such that, for $(\underline{t},i)\in\mathcal{T}$, we have

$$\sup_{\|\underline{z}\|\le R} |\Lambda_{\underline{t},i}(\underline{z})| \le \frac{c}{\pi^{\|\underline{t}\|}}\cdot$$

From (6.1) we deduce that for κ in the range $\tau(f)<\kappa<\pi$, for sufficiently large $\|\underline{t}\|$, we have

$$|D^{\underline{t}} f(\underline{e}_i)| \le \kappa^{\|\underline{t}\|}. \tag{6.2}$$

Hence the series

$$\sum_{(\underline{t},i)\in\mathcal{T}} |(D^{\underline{t}} f)(\underline{e}_i)| \sup_{\|\underline{z}\|\le R} |\Lambda_{\underline{t},i}(\underline{z})| \quad \text{and} \quad \sum_{(\underline{t},i)\in\mathcal{T}} \sup_{\substack{\|\underline{\zeta}\|\le\kappa\\ \|\underline{z}\|\le R}} |\Lambda_{\underline{t},i}(\underline{z})\underline{\zeta}^{\underline{t}}|$$

converge. □

Proof of Theorem 1.8. Our goal is to prove the formula (5.1). Thanks to (5.2), it will be sufficient to check that $\mathrm{e}^{\underline{\zeta}\underline{z}}$ is given by the right hand side of (5.2).

We use the formula

$$\prod_{i=1}^{n}(a_i+b_i)=\sum_{J\subset\{1,2,\dots,n\}}\prod_{j\in J}a_j\prod_{j\notin J}b_j$$

with $a_i=\cosh(\zeta_i z_i)$ and $b_i=\sinh(\zeta_i z_i)$. Hence

$$\begin{aligned}\mathrm{e}^{\zeta_1z_1+\cdots+\zeta_nz_n}&=\prod_{i=1}^{n}(\cosh(\zeta_iz_i)+\sinh(\zeta_iz_i))\\&=\sum_{J\subset\{1,2,\dots,n\}}\prod_{j\in J}\cosh(\zeta_jz_j)\prod_{j\notin J}\sinh(\zeta_jz_j).\end{aligned}$$

For $(\underline{\alpha},i)\in\mathcal{A}$ and $1\le j\le n$, we define $A_{\underline{\alpha},i}(\underline{\zeta},\underline{z})$ and $B_{(\underline{\alpha},i),j}(\underline{\zeta},\underline{z})$ as follows. When $i\ge 1$, we set

$$A_{\underline{\alpha},i}(\underline{\zeta},\underline{z})=\sinh(\zeta_iz_i)\prod_{\substack{1\le j\le n,j\ne i\\ \alpha_j=0}}\cosh(\zeta_jz_j)\prod_{\substack{1\le j\le n\\ \alpha_j=1}}\sinh(\zeta_jz_j)$$

and

$$B_{(\underline{\alpha},i),j}(\underline{\zeta},\underline{z})=\delta_{i,j}A_{\underline{\alpha},i}(\underline{\zeta},\underline{z}).$$

When $i=0$, we set

$$A_{\underline{\alpha},0}(\underline{\zeta},\underline{z})=\prod_{\substack{1\le j\le n\\ \alpha_j=0}}\cosh(\zeta_jz_j)\prod_{\substack{1\le j\le n\\ \alpha_j=1}}\sinh(\zeta_jz_j)$$

and

$$B_{(\underline{\alpha},0),j}(\underline{\zeta},\underline{z})=\begin{cases}-\sinh(\zeta_jz_j)\displaystyle\prod_{\substack{1\le \ell\le n,\ell\ne j\\ \alpha_\ell=0}}\cosh(\zeta_\ell z_\ell)\prod_{\substack{1\le \ell\le n\\ \alpha_\ell=1}}\sinh(\zeta_\ell z_\ell) & \text{for } 1\le j\le\nu,\\ 0 & \text{for } \nu<j\le n,\end{cases}$$

where $\nu=\nu(\underline{\alpha})$ has been introduced in Definition 5.1. Recall $\zeta_0=0$ and notice that

$$\frac{\mathrm{e}^{\zeta}}{\sinh(\zeta)}=1+\coth(\zeta).$$

According to Theorem 5.2, for all $(\underline{\alpha},i)\in\mathcal{A}$ we have

$$\mathrm{e}^{\zeta_i}M_{\underline{\alpha},i}(\underline{\zeta},\underline{z})=A_{\underline{\alpha},i}(\underline{\zeta},\underline{z})+\sum_{j=1}^{n}\coth(\zeta_j)B_{(\underline{\alpha},i),j}(\underline{\zeta},\underline{z}).$$

The class modulo 2 of the number of sinh factors in the product $A_{\underline{\alpha},0}(\underline{\zeta},\underline{z})$ is $\|\underline{\alpha}\|$, hence the number of sinh factors is even. Among the 2^n products of the form

$$\prod_{j\in J}\cosh(\zeta_jz_j)\prod_{j\notin J}\sinh(\zeta_jz_j),$$

for J a subset of $\{1,2,\dots,n\}$, the $A_{\underline{\alpha},0}(\underline{\zeta},\underline{z})$ with $\underline{\alpha}\in\mathcal{A}_0$ are the 2^{n-1} such products where the number of sinh factors is even:

$$\begin{aligned}\sum_{\underline{\alpha}\in\mathcal{A}_0} A_{\underline{\alpha},0}(\underline{\zeta},\underline{z}) = \cosh(\zeta_1 z_1) &\sum_{\substack{I\subset\{2,\dots,n\}\\ |I|\text{ even}}} \prod_{j\in I}\sinh(\zeta_j z_j) \prod_{\substack{2\le j\le n\\ j\notin I}} \cosh(\zeta_j z_j)\\ &+\sinh(\zeta_1 z_1)\sum_{\substack{I\subset\{2,\dots,n\}\\ |I|\text{ odd}}} \prod_{j\in I}\sinh(\zeta_j z_j) \prod_{\substack{2\le j\le n\\ j\notin I}} \cosh(\zeta_j z_j).\end{aligned}$$

For $1\le i\le n$, the class modulo 2 of the number of sinh factors in

$$A_{\underline{\alpha},i}(\underline{\zeta},\underline{z}) = \left(\prod_{1\le j<i}\cosh(\zeta_j z_j)\right)\sinh(\zeta_i z_i)\prod_{\substack{i<j\le n\\ \alpha_j=1}}\sinh(\zeta_j z_j)\prod_{\substack{i<j\le n\\ \alpha_j=0}}\cosh(\zeta_j z_j)$$

is $1+\|\underline{\alpha}\|$, hence this number is odd; these $A_{\underline{\alpha},i}(\underline{\zeta},\underline{z})$ for $\underline{\alpha}\in\mathcal{A}_i$ are all the different products

$$\prod_{j\in J}\cosh(\zeta_j z_j)\prod_{j\notin J}\sinh(\zeta_j z_j),$$

which have an odd number of sinh factors and which are starting with

$$\cosh(\zeta_1 z_1)\cdots\cosh(\zeta_{i-1}z_{i-1})\sinh(\zeta_i z_i).$$

Hence the $A_{\underline{\alpha},i}(\underline{\zeta},\underline{z})$ for $(\underline{\alpha},i)\in\mathcal{A}$ with $1\le i\le n$ are the 2^{n-1} products

$$\prod_{j\in J}\cosh(\zeta_j z_j)\prod_{j\notin J}\sinh(\zeta_j z_j),$$

for J a subset of $\{1,2,\dots,n\}$, where the number of sinh factors is odd. Therefore

$$\sum_{(\underline{\alpha},i)\in\mathcal{A}} A_{\underline{\alpha},i}(\underline{\zeta},\underline{z}) = \sum_{J\subset\{1,2,\dots,n\}}\prod_{j\in J}\cosh(\zeta_j z_j)\prod_{j\notin J}\sinh(\zeta_j z_j) = \mathrm{e}^{\underline{\zeta}\underline{z}}.$$

We now fix $j\in\{1,2,\dots,n\}$. Let $(\underline{\alpha},i)\in\mathcal{A}$ such that $B_{(\underline{\alpha},i),j}\neq 0$. If $i=0$, then $\nu\ge 1$, $\alpha_{\nu+1}=1$, hence

$$B_{(\underline{\alpha},0),j}(\underline{\zeta},\underline{z}) = -\sinh(\zeta_j z_j)\prod_{\substack{1\le \ell\le n,\ell\neq j\\ \alpha_\ell=0}}\cosh(\zeta_\ell z_\ell)\prod_{\substack{1\le\ell\le n\\ \alpha_\ell=1}}\sinh(\zeta_\ell z_\ell).$$

If $i\ge 1$, the condition $B_{(\underline{\alpha},i),j}\neq 0$ implies $i=j$ and

$$B_{(\underline{\alpha},i),j}(\underline{\zeta},\underline{z}) = \sinh(\zeta_j z_j)\prod_{\substack{1\le \ell\le n,\ell\neq j\\ \alpha_\ell=0}}\cosh(\zeta_\ell z_\ell)\prod_{\substack{1\le\ell\le n\\ \alpha_\ell=1}}\sinh(\zeta_\ell z_\ell).$$

Hence for all $1\le j\le n$, we have

$$\sum_{(\underline{\alpha},i)\in\mathcal{A}} B_{(\underline{\alpha},i),j} = 0.$$

This completes the proof of Theorem 1.8. □

Proof of Theorem 1.7. Theorem 1.8 yields the formula (1.1) of Theorem 1.7 for the special case of the functions $\mathrm{e}^{\underline{\zeta}\underline{z}}$ with $\underline{\zeta}\in\mathbb{C}^n$, $|\underline{\zeta}|<\pi$. Once we know (1.1) for these functions $\mathrm{e}^{\underline{\zeta}\underline{z}}$, we deduce the general case by means of the Laplace transform in several variables as follows.

Let

$$f(\underline{z}) = \sum_{\underline{k}\in\mathbb{N}^n} \frac{a_{\underline{k}}}{\underline{k}!} \underline{z}^{\underline{k}}$$

be an entire function in $\mathbb{C}^n$ of exponential type $\tau(f)$. From (6.1) with $z_0 = 0$ it follows that the *Laplace transform* of f, viz. the function of n complex variables

$$F(\underline{\zeta}) = \sum_{\underline{k}\in\mathbb{N}^n} a_{\underline{k}} \zeta_1^{-k_1-1} \cdots \zeta_n^{-k_n-1}, \tag{6.3}$$

is analytic in the domain $\{\underline{\zeta} \in \mathbb{C}^n \mid |\zeta_i| > \tau(f),\ 1 \le i \le n\}$. Let $r > \tau(f)$. From Cauchy's residue theorem and from the normal convergence of the series (6.3) for F on $|\zeta_1| = \cdots = |\zeta_n| = r$ we deduce

$$f(\underline{z}) = \frac{1}{(2\pi\mathrm{i})^n} \int_{|\zeta_1|=r} \cdots \int_{|\zeta_n|=r} \mathrm{e}^{\underline{\zeta z}} F(\underline{\zeta}) \mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_n \tag{6.4}$$

and

$$D^{\underline{t}} f(\underline{z}) = \frac{1}{(2\pi\mathrm{i})^n} \int_{|\zeta_1|=r} \cdots \int_{|\zeta_n|=r} \underline{\zeta}^{\underline{t}} \mathrm{e}^{\underline{\zeta z}} F(\underline{\zeta}) \mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_n. \tag{6.5}$$

Assume $\tau(f) < \pi$. Let r satisfy $\tau(f) < r < \pi$. In (6.4) we replace $\mathrm{e}^{\underline{\zeta z}}$ by the formula of Theorem 1.8 :

$$f(\underline{z}) = \sum_{(\underline{t},i)\in\mathcal{T}} \Lambda_{\underline{t},i}(\underline{z}) \frac{1}{(2\pi\mathrm{i})^n} \int_{|\zeta_1|=r} \cdots \int_{|\zeta_n|=r} \underline{\zeta}^{\underline{t}} \mathrm{e}^{\zeta_i} F(\underline{\zeta}) \mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_n.$$

Using (6.5), we deduce

$$f(\underline{z}) = \sum_{(\underline{t},i)\in\mathcal{T}} \Lambda_{\underline{t},i}(\underline{z}) (D^{\underline{t}} f)(\underline{e}_i).$$

This completes the proof of Theorem 1.7. □

Remark. Denote by $\boldsymbol{\mu}_2$ the multiplicative group with two elements $\{-1, 1\}$. For $\underline{\gamma} \in \boldsymbol{\mu}_2^n$ and $\underline{\alpha} \in (\mathbb{Z}/2\mathbb{Z})^n$, write

$$\underline{\gamma}^{\underline{\alpha}} = \gamma_1^{\alpha_1} \cdots \gamma_n^{\alpha_n}.$$

Further, for $\underline{\gamma} \in \boldsymbol{\mu}_2^n$, $\underline{z}$ and $\underline{\zeta} \in \mathbb{C}^n$, set

$$\underline{\gamma\zeta} = (\gamma_1\zeta_1, \dots, \gamma_n\zeta_n) \in \mathbb{C}^n, \quad \underline{\gamma\zeta z} = \gamma_1\zeta_1 z_1 + \cdots + \gamma_n\zeta_n z_n \in \mathbb{C}$$

and for $i = 0, \dots, n$,

$$(\underline{\gamma\zeta})_i = \begin{cases} 0 & \text{for } i = 0, \\ \gamma_i\zeta_i & \text{for } 1 \le i \le n. \end{cases}$$

For $\underline{\gamma} \in \boldsymbol{\mu}_2^n$ we deduce from the definition of $M_{\underline{\alpha},i}$ in § 5

$$M_{\underline{\alpha},i}(\underline{\gamma\zeta}, \underline{z}) = \underline{\gamma}^{\underline{\alpha}} M_{\underline{\alpha},i}(\underline{\zeta}, \underline{z}),$$

and from Theorem 1.8 and (5.2)

$$\mathrm{e}^{\underline{\gamma\zeta z}} = \sum_{(\underline{\alpha},i)\in\mathcal{A}} \underline{\gamma}^{\underline{\alpha}} \mathrm{e}^{(\underline{\gamma\zeta})_i} M_{\underline{\alpha},i}(\underline{\zeta}, \underline{z}). \tag{6.6}$$

Let $\mathbf{E}(\underline{\zeta}, \underline{z})$ be the column vector with 2^n components $\mathrm{e}^{\underline{\gamma\zeta z}}$, $\underline{\gamma} \in \boldsymbol{\mu}_2^n$, and let $\mathbf{M}(\underline{\zeta}, \underline{z})$ be the column vector with 2^n components $M_{\underline{\alpha},i}(\underline{\zeta}, \underline{z})$, $(\underline{\alpha}, i) \in \mathcal{A}$. Then the equations

(6.6) can be written in matrix form $\mathbf{E}(\underline{\zeta},\underline{z}) = \mathbf{A}(\underline{\zeta})\mathbf{M}(\underline{\zeta},\underline{z})$, where $\mathbf{A}(\underline{\zeta})$ is the $2^n \times 2^n$ square matrix

$$\mathbf{A}(\underline{\zeta}) = \left(\underline{\gamma}^{\underline{\alpha}} \mathrm{e}^{(\underline{\gamma\zeta})_i}\right)_{\substack{\underline{\gamma}\in\boldsymbol{\mu}_2^n\\ (\underline{\alpha},i)\in\mathcal{A}}}.$$

Theorem 5.2 shows that for $0 < |\zeta_i| < \pi$ $(1 \le i \le n)$ the matrix $\mathbf{A}(\underline{\zeta})$ is regular and it gives implicitly a formula for its inverse. The case $n = 2$ is explained in [8, § 7].

7. Entire functions of finite exponential type

The proof of Theorem 1.9 is a generalization of the proof of Theorem 8.1 in [8].

Proof. Let $K \ge 1$. Set

$$A_K(\zeta, z) = 2\pi \sum_{k=1}^{K} \frac{(-1)^{k+1} k \sin(k\pi z)}{\zeta^2 + k^2\pi^2} \text{ and } B_K(\zeta, z) = -2\pi \sum_{k=1}^{K} \frac{k \sin(k\pi z)}{\zeta^2 + k^2\pi^2}.$$

We use the fact (see [7, § 8]) that the functions of two variables $G_K(\zeta, z)$ and $H_K(\zeta, z)$ defined by

$$\frac{\sinh(\zeta z)}{\sinh(\zeta)} = A_K(\zeta, z) + G_K(\zeta, z) \text{ and } \sinh(\zeta z)\coth(\zeta) = B_K(\zeta, z) + H_K(\zeta, z)$$

are analytic in the domain $\left\{(\zeta, z) \in \mathbb{C}^2 \ \mid\ |\zeta| < (K+1)\pi\right\}$. From Theorem 5.2 we deduce, for $(\underline{\alpha}, i) \in \mathcal{A}$, if $i \ge 1$,

$$M_{\underline{\alpha},i}(\underline{\zeta},\underline{z}) = \big(A_K(\zeta_i, z_i) + G_K(\zeta_i, z_i)\big) \prod_{\substack{1\le j\le n, j\ne i\\ \alpha_j=0}} \cosh(\zeta_j z_j) \prod_{\substack{1\le j\le n\\ \alpha_j=1}} \sinh(\zeta_j z_j)$$

and if $i = 0$

$$\begin{aligned} M_{\underline{\alpha},0}(\underline{\zeta},\underline{z}) = &\prod_{\substack{1\le j\le n\\ \alpha_j=0}} \cosh(\zeta_j z_j) \prod_{\substack{1\le j\le n\\ \alpha_j=1}} \sinh(\zeta_j z_j) \\ &- \sum_{j=1}^{\nu} \big(B_K(\zeta_j, z_j) + H_K(\zeta_j, z_j)\big) \prod_{\substack{1\le \ell\le n, \ell\ne j\\ \alpha_\ell=0}} \cosh(\zeta_\ell z_\ell) \prod_{\substack{1\le \ell\le n\\ \alpha_\ell=1}} \sinh(\zeta_\ell z_\ell). \end{aligned}$$

For $(\underline{t}, i) \in \mathcal{T}$ we define $g_{\underline{t},i}(\underline{z})$ by writing the Taylor expansions for $i \ge 1$,

$$\begin{aligned} G_K(\zeta_i, z_i) \sum_{\underline{\alpha}\in\mathcal{A}_i} &\left(\prod_{\substack{1\le j\le n, j\ne i\\ \alpha_j=0}} \cosh(\zeta_j z_j) \prod_{\substack{1\le j\le n\\ \alpha_j=1}} \sinh(\zeta_j z_j) \right) \\ &= \sum_{\substack{\underline{t}\in\mathbb{N}^n\\ (\underline{t},i)\in\mathcal{T}}} g_{\underline{t},i}(\underline{z}) \underline{\zeta}^{\underline{t}}, \end{aligned}$$

and for $i = 0$

$$\sum_{\underline{\alpha}\in\mathcal{A}_0}\left(\prod_{\substack{1\le j\le n\\ \alpha_j=0}}\cosh(\zeta_j z_j)\prod_{\substack{1\le j\le n\\ \alpha_j=1}}\sinh(\zeta_j z_j)\right.$$
$$\left. - \sum_{j=1}^{\nu} H_K(\zeta_j, z_j)\prod_{\substack{1\le \ell\le n,\ell\ne j\\ \alpha_\ell=0}}\cosh(\zeta_\ell z_\ell)\prod_{\substack{1\le \ell\le n\\ \alpha_\ell=1}}\sinh(\zeta_\ell z_\ell)\right)$$
$$= \sum_{\substack{\underline{t}\in\mathbb{N}^n\\ \|\underline{t}\|\in 2\mathbb{N}}} g_{\underline{t},0}(\underline{z})\underline{\zeta}^{\underline{t}}.$$

Finally we set

$$\chi(\underline{\zeta},\underline{z}) = \sum_{i=1}^{n} A_K(\zeta_i, z_i)\mathrm{e}^{\zeta_i}\sum_{\underline{\alpha}\in\mathcal{A}_i}\prod_{\substack{1\le j\le n, j\ne i\\ \alpha_j=0}}\cosh(\zeta_j z_j)\prod_{\substack{1\le j\le n\\ \alpha_j=1}}\sinh(\zeta_j z_j)$$
$$- \sum_{\underline{\alpha}\in\mathcal{A}_0}\sum_{j=1}^{\nu} B_K(\zeta_j, z_j)\prod_{\substack{1\le \ell\le n,\ell\ne j\\ \alpha_\ell=0}}\cosh(\zeta_\ell z_\ell)\prod_{\substack{1\le \ell\le n\\ \alpha_\ell=1}}\sinh(\zeta_\ell z_\ell),$$

so that Theorem 1.8 yields

$$\mathrm{e}^{\underline{\zeta}\underline{z}} = \sum_{(\underline{t},i)\in\mathcal{T}} \mathrm{e}^{\zeta_i} g_{\underline{t},i}(\underline{z})\underline{\zeta}^{\underline{t}} + \chi(\underline{\zeta},\underline{z}). \tag{7.1}$$

The series in the right hand side of (7.1) is normally convergent in $\{(\underline{\zeta},\underline{z})\in\mathbb{C}^n\times\mathbb{C}^n \mid |\underline{\zeta}| < (K+1)\pi\}$, since for $i = 0,\dots,n$, the series

$$\sum_{\substack{\underline{t}\in\mathbb{N}^n\\ (\underline{t},i)\in\mathcal{T}}} g_{\underline{t},i}(\underline{z})\underline{\zeta}^{t}$$

defines an analytic function in this domain [2, Theorem 2.2.6]. For the record, we point out that it follows from Cauchy's inequalities [2, Theorem 2.2.7] that there exists $c = c(R) > 0$ such that, for $(\underline{t}, i)\in\mathcal{T}$, we have

$$\sup_{\|\underline{z}\|\le R} |g_{\underline{t},i}(\underline{z})| \le \frac{c}{r^{\|\underline{t}\|}}. \tag{7.2}$$

For $1 \le i \le n$ and $\underline{z}\in\mathbb{C}^n$, we write $\underline{z}^{(i)}$ for $(z_1,\dots,z_{i-1},z_{i+1},\dots,z_n)\in\mathbb{C}^{n-1}$. For $1\le k\le K$ and $1\le i\le n$, define

$$\chi_{k,i}(\underline{\zeta},\underline{z}^{(i)}) = \frac{(-1)^{k+1}2\pi k\mathrm{e}^{\zeta_i}}{\zeta_i^2 + k^2\pi^2}\sum_{\underline{\alpha}\in\mathcal{A}_i}\prod_{\substack{1\le j\le n, j\ne i\\ \alpha_j=0}}\cosh(\zeta_j z_j)\prod_{\substack{1\le j\le n\\ \alpha_j=1}}\sinh(\zeta_j z_j)$$
$$+ \sum_{\substack{\underline{\alpha}\in\mathcal{A}_0\\ \nu(\underline{\alpha})\ge i}}\frac{2\pi k}{\zeta_i^2 + k^2\pi^2}\prod_{\substack{1\le j\le n, j\ne i\\ \alpha_j=0}}\cosh(\zeta_j z_j)\prod_{\substack{1\le j\le n\\ \alpha_j=1}}\sinh(\zeta_j z_j),$$

so that

$$\chi(\underline{\zeta}, \underline{z}) = \sum_{k=1}^{K} \sum_{i=1}^{n} \chi_{k,i}(\underline{\zeta}, \underline{z}^{(i)}) \sin(k\pi z_i). \tag{7.3}$$

For $k = 1, \dots, K$ and $i = 1, \dots, n$, the function $\chi_{k,i}(\underline{\zeta}, \underline{z}^{(i)})$ is analytic in the domain

$$\bigl\{(\underline{\zeta}, \underline{z}^{(i)}) \in \mathbb{C}^n \times \mathbb{C}^{n-1} \;\mid\; K\pi < |\zeta_j| < (K+1)\pi,\ (1 \le j \le n)\bigr\}.$$

Let f be an entire function in $\mathbb{C}^n$ of finite exponential type $\le \tau$ with $K\pi \le \tau < (K+1)\pi$. Denote by F the Laplace transform of f. In (6.4), we replace $\mathrm{e}^{\underline{\zeta}\underline{z}}$ with (7.1), where we substitute (7.3) for $\chi(\underline{\zeta}, \underline{z})$. We deduce the formula of Theorem 1.9, namely

$$f(\underline{z}) = \sum_{(\underline{t},i) \in \mathcal{T}} (D^{\underline{t}} f)(\underline{e}_i) g_{\underline{t},i}(\underline{z}) + \sum_{k=1}^{K} \sum_{i=1}^{n} h_{k,i}(\underline{z}^{(i)}) \sin(k\pi z_i),$$

with

$$h_{k,i}(\underline{z}^{(i)}) = \frac{1}{(2\pi \mathrm{i})^n} \int_{|\zeta_1|=r} \cdots \int_{|\zeta_n|=r} \chi_{k,i}(\underline{\zeta}, \underline{z}^{(i)}) F(\underline{\zeta}) \mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_n$$

for any r with $\tau < r < (K+1)\pi$. This function $h_{k,i}$ has exponential type $\le r$. Since this is true for all r in the range $\tau < r < (K+1)\pi$, this exponential type is $\le \tau$.

Let $R > 0$ and let κ, r satisfy $\tau(f) < \kappa < r < (K+1)\pi$. Using (6.2) and (7.2), we deduce that the series

$$\sum_{(\underline{t},i) \in \mathcal{T}} |(D^{\underline{t}} f)(\underline{e}_i)| \sup_{\|\underline{z}\| \le R} |g_{\underline{t},i}(\underline{z})|$$

converges. This proves the claim on the normal convergence of the series and completes the proof of Theorem 1.9. □

Finally, Corollary 1.10 follows immediately from Theorem 1.9.

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References

[1] R. C. Buck, *On n-point expansions of entire functions*, Proc. Amer. Math. Soc. **6** (1955), 793–796, DOI 10.2307/2032936. MR72947

[2] Lars Hörmander, *An introduction to complex analysis in several variables*, 3rd ed., North-Holland Mathematical Library, vol. 7, North-Holland Publishing Co., Amsterdam, 1990. MR1045639

[3] George James Lidstone. Notes on the extension of Aitken's theorem (for polynomial interpolation) to the Everett types. *Proceedings Edinburgh Math. Soc., II. Ser.* (2) 2, 16-19 (1930). JFM

[4] Hillel Poritsky, *On certain polynomial and other approximations to analytic functions*, Trans. Amer. Math. Soc. **34** (1932), no. 2, 274–331, DOI 10.2307/1989543. MR1501639

[5] I. J. Schoenberg, *On certain two-point expansions of integral functions of exponential type*, Bull. Amer. Math. Soc. **42** (1936), no. 4, 284–288, DOI 10.1090/S0002-9904-1936-06293-2. MR1563285

[6] Michel Waldschmidt, *On entire functions of several variables with derivatives of even order taking integer values*, J. Ramanujan Soc. Math. Math. Sci. **9** (2021), no. 1, 1–10. MR4400338

[7] Michel Waldschmidt, *Lidstone interpolation I. One variable*, Proceedings of the 87th Annual Conference of the Indian Mathematical Society, December 2021. *The Mathematics Student*, Volume **91** (Nos. 1-2), 2022, 79–94.

[8] Michel Waldschmidt, *Lidstone interpolation II. Two variables*, Accepted by the Asian-Eur. J. Math, (2023), 2350072, (22 pages), World Scientific Publishing Company, DOI: 10.1142/S1793557123500729.

[9] J. M. Whittaker, *On Lidstone's Series and Two-Point Expansions of Analytic Functions*, Proc. London Math. Soc. (2) **36** (1934), 451–469, DOI 10.1112/plms/s2-36.1.451. MR1575969

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