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HOMOMORPHISM OBSTRUCTIONS FOR SATELLITE MAPS

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Abstract. A knot in a solid torus defines a map on the set of (smooth or topological) concordance classes of knots in S^3. This set admits a group structure, but a conjecture of Hedden suggests that satellite maps never induce interesting homomorphisms: we give new evidence for this conjecture in both categories. First, we use Casson-Gordon signatures to give the first obstruction to a slice pattern inducing a homomorphism on the topological concordance group, constructing examples with every winding number besides ± 1. We then provide subtle examples of satellite maps which map arbitrarily deep into the n-solvable filtration of Cochran, Orr, and Teichner [Ann. of Math. (2) 157 (2003), pp. 433–519], act like homomorphisms on arbitrary finite sets of knots, and yet which still do not induce homomorphisms. Finally, we verify Hedden's conjecture in the smooth category for all small crossing number satellite operators but one.

1. Introduction

A knot P in the parametrized solid torus $S^1 \times D^2$ defines a function on the set of knots in S^3 by the well-known satellite construction: given a knot K, let $i_K\colon S^1 \times D^2 \to \nu(K) \subseteq S^3$ be an identification of the standard solid torus with a 0-framed tubular neighborhood of K and define $P(K)$ to be $i_K(P)$ as illustrated in Figure 1. It is common to call P the *pattern* and K the *companion knot* of the satellite knot $P(K)$.

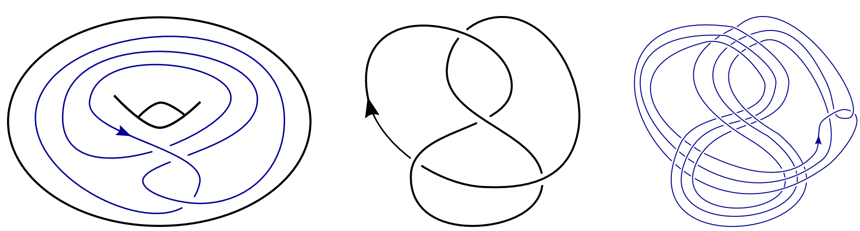

Figure 1. A pattern P (left) and companion knot K (center) combine to give the satellite knot $P(K)$ (right)

The map $K \mapsto P(K)$ descends to a well-defined function on the collection of (smooth or topological) concordance classes of knots. These satellite maps are essential tools in the modern study of knot concordance and in 3- and 4-manifold

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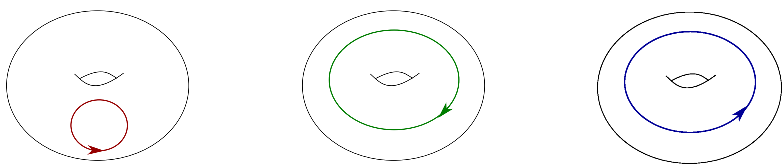
topology more generally. To sample just a few results, the satellite construction features prominently in the first evidence for a fractal structure on concordance [CHL11]; the first examples of non-smoothly concordant knots with homeomorphic 0-surgeries [Yas15]; and the first knots in homology spheres which do not bound PL discs in any contractible 4-manifold [Lev16]. As a result, satellite operations have become an object of study in their own right, with recent work in the area focusing on the existence of interesting bijective satellite maps [GM95, MP18], the behavior of the 4-genera of knots under satelliting [CH18, Pic19, Mil19, FMPC22], and on satellite maps with image of infinite rank [HPC21].

Nonetheless, a fundamental question remains almost entirely open. The collection of concordance classes of knots famously has the structure of an abelian group, with addition induced by connected sum and the inverse operation induced by taking the mirror-reverse of a knot, and it is natural to ask how a satellite operator interacts with this additional structure.

Question 1.1. When does a pattern induce a homomorphism of the concordance group?

The following three *standard* patterns evidently induce homomorphisms in both categories:



$K \mapsto U$, $\qquad K \mapsto K$, $\qquad K \mapsto K^r$.

Hedden conjectured that these are the only homomorphisms induced by the satellite operation.

Conjecture 1.2 ([BIR16, MPI16])**.** *Let P be a pattern which induces a homomorphism on the concordance group. Then P induces one of the three standard maps on concordance, i.e. the action of P is given by one of $[K] \mapsto [U]$, $[K] \mapsto [K]$, or $[K] \mapsto [K^r]$.*

We call a pattern P *slice* if $P(U)$ is a slice knot; this is an obvious prerequisite for a pattern to induce a homomorphism. Perhaps surprisingly, any slice pattern induces a homomorphism of Levine's algebraic concordance group [Lit84], and so 'looks like' a homomorphism from the perspective of classical invariants like the Tristram-Levine signatures and Alexander polynomial.

In this paper, we give new evidence for Conjecture 1.2 in both the smooth and topological categories. First, we give the first obstruction to a slice pattern P inducing a homomorphism on the topological concordance group.

Theorem A. *Let P be a pattern described by an unknot η in the complement of $P(U)$. Suppose that there exists some prime p dividing the winding number of P such that the lifts of η to the p-fold cyclic branched cover $\Sigma_p(P(U))$ generate the nontrivial group $H_1(\Sigma_p(P(U)))$. Then P does not induce a homomorphism on the topological concordance group.*

In Section 3 we give examples of patterns of every winding number besides ± 1 satisfying the conditions of Theorem A, as well as examples of patterns obstructed from acting as homomorphisms by Proposition 2.2, a stronger but harder to state version of Theorem A. We remark that the outstanding case of winding number ± 1 seems quite difficult: remarkably, there are no slice patterns of winding number 1 (respectively -1) that are known to not induce the identity (respectively reversal) map on the topological concordance group! Moreover, (non)-existence of such patterns is closely related to longstanding questions such as the topological Akbulut-Kirby and homotopy ribbon conjectures (see [GM95, MP18]).

The n-solvable filtration of [COT03] plays a central role in the current understanding of the topological knot concordance group; while we omit a precise definition, knots that are n-solvable for large $n \in \mathbb{N}$ are 'close' to being topologically slice. The Casson-Gordon style techniques of Theorem A cannot obstruct satellite maps with image deep in the n-solvable filtration from inducing homomorphisms. However, we apply results of [CHL11] to give many examples of patterns mapping arbitrarily deep in the filtration which do not induce homomorphisms.

Theorem B. *For any $n \in \mathbb{N}$, there exist infinitely many slice patterns P that have image contained within $\mathcal{F}_n$, the collection of n-solvable knots and yet do not induce homomorphisms on the topological concordance group.*

We also consider the extent to which non-standard patterns can act like homomorphisms on subsets of the concordance group, proving the following.

Theorem C. *Let $\{K_i\}_{i=1}^m$ be any finite collection of knots. Then there exists a slice pattern P that does not induce a homomorphism on the topological concordance group but which has the property that $P(K_i \# K_j)$ is smoothly concordant to $P(K_i) \# P(K_j)$ for all $1 \leq i, j \leq m$.*

We remark that in particular one can choose $\{K_i\}_{i=1}^m$ to be any finite 2-torsion subgroup of the concordance group, but it remains an interesting open question whether any non-standard pattern acts as a homomorphism when restricted to the subgroup $\{\#^n K\}_{n \in \mathbb{Z}}$ when K represents an infinite order element of the concordance group.

We conclude by switching to the smooth category, where we show that the Heegaard Floer knot invariant τ must be scaled by the (absolute value of the) algebraic winding number under the action of any pattern inducing a homomorphism on the smooth concordance group.

Theorem D. *If P is a pattern of winding number $w(P)$ that induces a homomorphism on the smooth concordance group, then for any knot K,*

$$\tau(P(K)) = |w(P)|\tau(K).$$

We then consider the 19 patterns which are presented by two-component links with at most 8 crossings, and almost completely verify Hedden's conjecture in that setting.

Theorem E. *Let P be a pattern presented by a link $P(U) \cup \eta$ with at most 8 crossings. Then P does not induce a homomorphism on the smooth concordance group, unless perhaps $P(U) \cup \eta = L8a9$, where it is unknown even if P acts by the identity.*

Note that previous work has used modern smooth technologies to show that the simplest non-standard patterns–the Whitehead pattern [Gom86], the Mazur pattern,[1] and the cable $C_{p,1}$ for $p > 1$ [Hed09]–do not induce homomorphisms on the smooth concordance group.

Remark 1.3. It remains almost entirely open whether a pattern is determined up to concordance in $(S^1 \times D^2) \times I$ by its action on the concordance group. The exception is the winding number 0 case in the topological category, where we know for example that the Whitehead double pattern is not concordant to the trivial pattern but does induce the zero map on topological concordance. One might therefore hope to strengthen Conjecture 1.2 to the statement that any pattern inducing a homomorphism must be concordant to a standard pattern, at least in the smooth category.

Organization of the paper

In Section 2, we state some useful results on the behavior of Casson-Gordon invariants under the satellite operation and derive Theorem A as a consequence of a stronger obstruction, Proposition 2.2. In Section 3 we provide examples of patterns satisfying the conditions of Theorem A and Proposition 2.2. In Section 4 we collect necessary background material on the n-solvable filtration and prove Theorems B and C. Finally, in Section 5 we prove Theorems D and E.

Conventions and notation

All manifolds are assumed to be compact and oriented. Moreover, each pattern P lives in a solid torus with a parametrization $S^1 \times D^2$ that inherits an orientation from those on S^1 and D^2 in the usual way. This care is necessary in order to specify the identification of $S^1 \times D^2$ with a 0-framed tubular neighborhood of K: we identify an oriented copy of $S^1 \times *$ with a 0-framed longitude of K that is oriented parallel to K.

We use $\mathcal{C}_s$ to denote the smooth concordance group, $\mathcal{C}_t$ the topological concordance group, and $\mathcal{C}$ when our statements hold in either category. Unless otherwise stated, all patterns are assumed to be slice in the appropriate category.

Given a pattern $P\colon S^1 \to S^1 \times D^2$, the class of $[P(S^1)]$ equals $m[S^1 \times \{*\}]$ in $H_1(S^1 \times D^2)$ for some $m \in \mathbb{Z}$. We call m the *algebraic winding number* of P and write $w(P) = m$. Given a pattern P with $w(P) = m$, the pattern P^{rev} with reversed orientation of P but the same identification and orientation of $S^1 \times D^2$ has $w(P^{rev}) = -m$ and the property that $P^{rev}(K)$ is isotopic to $P(K)^{rev}$ for all knots K. In particular, P induces a homomorphism of $\mathcal{C}$ if and only if P^{rev} does. For convenience, we therefore restrict to patterns of positive winding number.

2. A Casson-Gordon obstruction

Given a knot K and prime power $n \in \mathbb{N}$, the first homology group $H := H_1(\Sigma_n(K))$ of the n-fold cyclic branched cover comes with some additional structure. First, there is a nondegenerate symmetric form $\lambda\colon H \times H \to \mathbb{Q}/\mathbb{Z}$ called the *torsion linking form*. A *metabolizer* for (H, λ) is a subgroup $M \leq H$ such that $|M|^2 = |H|$ and $\lambda|_{M\times M} = 0$. There is a $\mathbb{Z}_n$ action on H induced by the action of covering transformations on $\Sigma_n(K)$, and a metabolizer is called *invariant* if this subgroup is set-wise preserved by the $\mathbb{Z}_n$-action. We remark that classical

[1] This follows immediately from [Lev16], though we expect it was known to the experts for some time before.

arguments (see [CG86]) imply that if K is slice then $(H_1(\Sigma_n(K)), \lambda)$ must have an invariant metabolizer.

Our first obstruction to a pattern inducing a homomorphism comes from Casson-Gordon signature invariants. We will not fully define these, noting only that to any knot K, prime powers p and q, and map $\chi\colon H_1(\Sigma_p(K)) \to \mathbb{Z}_q$, there is an associated *Casson-Gordon signature* $\sigma(K, \chi) \in \mathbb{Q}$ defined in terms of the Witt class of the twisted intersection form of some associated 4-manifold. Moreover, Casson-Gordon signatures obstruct topological sliceness as follows.

Theorem 2.1 ([CG86]). *Suppose K is a topologically slice knot. Then for every prime power p there exists an invariant metabolizer $M \leq H_1(\Sigma_p(K))$ such that if χ is a prime power order character with $\chi|_M = 0$ then $\sigma(K, \chi) = 0$.*

Theorem A is a consequence of the following more general obstruction.

Proposition 2.2. *Let P be a pattern described by an unknot η in the complement of $P(U)$. Let p be a prime dividing the winding number of P, let $H = H_1(\Sigma_p(P(U)))$, and denote the first homology classes represented by the p lifts of η to $\Sigma_p(P(U))$ by $z_1, \ldots, z_p \in H$.*

Suppose that for every invariant subgroup $M \leq H \oplus H \oplus -H$ that is a metabolizer for $\lambda \oplus \lambda \oplus -\lambda$ there exists a character $\chi = (\chi_1, \chi_2, \chi_3)\colon H \oplus H \oplus -H \to \mathbb{Z}_q$ with q a prime power and $\chi|_M = 0$ such that $\{\pm\chi_1(z_i)\}_{i=1}^p$, $\{\pm\chi_2(z_i)\}_{i=1}^p$, and $\{\pm\chi_3(z_i)\}_{i=1}^p$ are not identical when considered as sets with multiplicity. Then P does not induce a homomorphism on $\mathcal{C}_t$.

Proof of Theorem A, *assuming Proposition* 2.2. Let P and p be as in the statement of Theorem A and let $H := H_1(\Sigma_p(P(U)))$. Write $|H| = m^2$ for some $m > 1$, let q be a prime dividing m, and let $k \in \mathbb{N}$ be maximal such that q^k divides m.

For any subgroup S of $G := H \oplus H \oplus -H$, let S_q denote the q primary subgroup of S and define the q-primary annihilator of S to be

$$A_q(S) := \{\chi : H \oplus H \oplus -H \to \mathbb{Z}_{q^{6k}} \text{ such that } \chi|_S = 0\}.$$

Note that $|A_q(S)| = |(G/S)_q| = |G_q|/|S_q|$. Now let $M \leq H \oplus H \oplus -H$ be a metabolizer for the linking form and observe that $|M_q| = q^{3k} = |A_q(M)|$. Let $H^1 := H \oplus 0 \oplus 0$, and note that since $|H_q^{(1)}| = q^{2k}$, we have that $|A_q(H^1)| = q^{4k}$. Since $A_q(H^1)$ and $A_q(M)$ are both subgroups of $A_q(0)$, which has order $|G_q| = q^{6k}$, they must have non-zero intersection .

Let $\chi = (\chi_1, \chi_2, \chi_3)$ be a non-zero element of $A_q(H^1) \cap A_q(M)$. Since $\chi|_{H^1} = 0$, we have that $\{\pm\chi_1(z_i)\}_{i=1}^p = \{0\}_{i=1}^p$. However, since χ is non-zero and by assumption the lifts of η generate $H_1(\Sigma_p(P(U)))$, we must have either $\chi_2(z_i) \neq 0$ or $\chi_3(z_i) \neq 0$ for some $1 \leq i \leq p$. It follows that the sets $\{\pm\chi_1(z_i)\}_{i=1}^p$, $\{\pm\chi_2(z_i)\}_{i=1}^p$, and $\{\pm\chi_3(z_i)\}_{i=1}^p$ are not identical, and Proposition 2.2 applies to show that P does not induce a homomorphism. □

While Theorem A is a particularly simple condition to verify, Proposition 2.2 applies in a much broader range of settings, for instance in Example 3.4, when $P(U)$ is a composite knot with $H_1(\Sigma_p(P(U)))$ a non-cyclic module.

To prove Proposition 2.2, we will also need the following special case of Litherland's formula for the Casson-Gordon signatures of satellite knots.

Proposition 2.3 ([Lit84]). *Let P be a pattern described by an unknot η in the complement of $P(U)$. Let p be a prime power dividing the winding number of*

P. Then for any knot K, there is a canonical isomorphism $\alpha\colon H_1(\Sigma_p(P(K))) \to H_1(\Sigma_p(P(U)))$ that preserves linking forms, satisfies $\alpha(t_{P(K)} \cdot x) = t_{P(U)} \cdot \alpha(x)$ for all $x \in H_1(\Sigma_p(P(K)))$, and such that for any prime power order character $\chi\colon H_1(\Sigma_p(P(U))) \to \mathbb{Z}_q$ we have

$$\sigma(P(K), \chi \circ \alpha) = \sigma(P(U), \chi) + \sum_{i=1}^{p} \sigma_K(\omega_q^{\chi(\eta_i)}), \text{ where } \omega_q = e^{2\pi i/q}.$$

We remark that at first glance this result seems decidedly unhelpful in showing that P does not induce a homomorphism. Since P is a slice pattern, many of the $\sigma(P(U), \chi)$ terms must vanish, leaving us with the formula $\sigma(P(K), \alpha \circ \chi) = \sum_{i=1}^{p} \sigma_K(\omega_q^{\chi(\eta_i)})$. Since the Tristram-Levine signatures are additive with respect to connected sum of knots, we see that in many cases

$$\begin{aligned}\sigma(P(K_1 \# K_2), \alpha \circ \chi) &= \sum_{i=1}^{p} \sigma_{K_1}(\omega_q^{\chi(\eta_i)}) + \sum_{i=1}^{p} \sigma_{K_2}(\omega_q^{\chi(\eta_i)}) \\ &= \sigma(P(K_1), \alpha \circ \chi) + \sigma(P(K_2), \alpha \circ \chi)).\end{aligned}$$

Nonetheless, we are able to prove Proposition 2.2 as follows.

Proof of Proposition 2.2. Let P and p be as in the statement of the proposition, and define C to be the maximal value of $|\sigma(P(U), \chi)|$ ranging over all choices of nontrivial map $\chi : H_1(\Sigma_p(P(U))) \to \mathbb{Z}_r$ for a prime power r. Inductively pick even integers $m_1, \dots, m_{\lfloor q/2 \rfloor}$ such that $m_1 > 3C$ and $m_j > 3C + pm_{i-1}$ for $j > 1$ and even integers $n_1, \dots, n_{\lfloor q/2 \rfloor}$ such that $n_1 > 3C + pm_{\lfloor q/2 \rfloor}$ and $n_j > 3C + pm_{\lfloor q/2 \rfloor} + pn_{j-1}$ for $j > 1$. Let J and K be knots such that $\sigma_J(\omega_q^j) = m_j$ and $\sigma_K(\omega_q^j) = n_j$ for all $1 \leq j \leq \lfloor q/2 \rfloor$. This is possible by the proof of Theorem 1 of Cha-Livingston [CL04], see also [Mil19] for a similar argument.

Now let $L = P(K) \# P(J) \# - P(K \# J)$. In order to apply Theorem 2.1 to show that L is not slice, let M be a metabolizer of $H_1(\Sigma_p(L))$. By [Lit84], there is a canonical, covering transformation invariant, linking form preserving identification

$$\beta : H_1(\Sigma_p(L)) \xrightarrow{\cong} H_1(\Sigma_p(P(U))) \oplus H_1(\Sigma_p(P(U))) \oplus -H_1(\Sigma_p(P(U))).$$

such that for any $\chi = (\chi_1, \chi_2, \chi_3)$ we have

$$\sigma(L, \chi \circ \beta) = \sigma(P(J), \chi_1) + \sigma(P(K), \chi_2) - \sigma(P(K \# J), \chi_3).$$

Under this identification, $\beta(M)$ is an invariant metabolizer for $H \oplus H \oplus -H$, so by hypothesis we can let $\chi = (\chi_1, \chi_2, \chi_3)$ be a character to $\mathbb{Z}_q$ vanishing on $\beta(M)$ such that the sets $A_1 = \{\pm\chi_1(z_i)\}_{i=1}^p$, $A_2 = \{\pm\chi_2(z_i)\}_{i=1}^p$, and $A_3 = \{\pm\chi_3(z_i)\}_{i=1}^p$ are not identical. For $k = 1, 2, 3$ and $1 \leq j \leq \lfloor q/2 \rfloor$, define $\delta_k(j)$ to be the number of lifts of η sent by χ_k to $\pm j$. More formally,

$$\delta_k(j) := \#\{1 \leq i \leq p : \chi_k(z_i) = \pm j \in \mathbb{Z}_q\}.$$

By Proposition 2.3 and the above additivity of Casson-Gordon signatures with respect to connected sum, we therefore have that

$$\begin{aligned}\sigma(L,\chi\circ\beta) &= \sigma(P(J),\chi_1)+\sigma(P(K),\chi_2)-\sigma(P(K\#J),\chi_3)\\ &= C_\chi+\sum_{i=1}^{p}\sigma_J(\omega_q^{\chi_1(z_i)})+\sum_{i=1}^{p}\sigma_K(\omega_q^{\chi_2(z_i)})-\sum_{i=1}^{p}\sigma_{J\#K}(\omega_q^{\chi_3(z_i)}).\\ &= C_\chi+\sum_{j=1}^{\lfloor q/2\rfloor}(\delta_1(j)-\delta_3(j))m_j+\sum_{j=1}^{\lfloor q/2\rfloor}(\delta_2(j)-\delta_3(j))n_j,\end{aligned}$$

where $C_\chi=\sigma(P(U),\chi_1)+\sigma(P(U),\chi_2)-\sigma(P(U),\chi_3)$. Note that $|C_\chi|\leq 3C$.

As we will now explain, the choices of n_j and m_j appearing earlier in the proof ensure that the final non-vanishing term in the sum above is strictly larger in absolute value than the sum of the preceding terms, ensuring that $\sigma(L,\chi\circ\beta)\neq 0$. We split our argument into cases.

Suppose first that $A_2\neq A_3$. Let j_0 be the maximal j with $\delta_2(j)\neq\delta_3(j)$ and assume for convenience that $\delta_2(j_0)>\delta_3(j_0)$. (The argument for $\delta_3(j_0)>\delta_2(j_0)$ is exactly analogous). Then

$$\begin{aligned}\sigma(L,\chi\circ\beta) &= C_\chi+\sum_{j=1}^{\lfloor q/2\rfloor}(\delta_1(j)-\delta_3(j))m_j+\sum_{j=1}^{j_0-1}(\delta_2(j)-\delta_3(j))n_j+(\delta_2(j_0)-\delta_3(j_0))n_{j_0}\\ &\geq -3C-pm_{\lfloor q/2\rfloor}-pn_{j_0-1}+n_{j_0}>0,\ \text{as desired.}\end{aligned}$$

Now suppose that $A_2=A_3$ and hence that $A_1\neq A_3$. Let j_0 be the maximal j with $\delta_1(j)\neq\delta_3(j)$ and as before assume for convenience that $\delta_1(j_0)>\delta_3(j_0)$. In this case, we have that

$$\begin{aligned}\sigma(L,\chi\circ\beta) &= C_\chi+\sum_{j=1}^{j_0-1}(\delta_1(j)-\delta_3(j))m_j+(\delta_1(j_0)-\delta_3(j_0))m_{j_0}\\ &\geq -3C-pm_{j_0-1}+m_{j_0}>0,\ \text{as desired.}\end{aligned}$$

□

3. Examples of non-homomorphism satellite maps

An easy way to guarantee that the conditions of Theorem A are satisfied is to choose a slice knot $P(U)$ whose Alexander module is generated by the class of the winding number 0 curve η. One can then modify η to get a pattern of any winding number. More specifically, let P_n be the winding number n pattern of Figure 2, described by an unknot η in the complement of $P_n(U)$. Observe that $P_n(U)=6_1$ is slice for all $n\in\mathbb{N}\cup\{0\}$.

Proposition 3.1. *For any p dividing n, $H_1(\Sigma_p(P_n(U)))$ is a nontrivial group generated by the lifts of η to $\Sigma_p(P_n(U))$.*

Proof. Let p divide n. The left of Figure 3 depicts $P_n(U)$ in a nonstandard surgery description for S^3; this is simplified via isotopy to give the center picture. For p dividing n, we obtain a surgery diagram for $\Sigma_p(P_n(U))$ with $p+1$ surgery curves, as illustrated for $p=3$ in Figure 3.

Since the blue curve (called α) has writhe -4 and framing $+1$ in the center diagram and its p lifts $\alpha_1,\ldots,\alpha_p$ will have writhe 0, these lifts must be $+5$ framed.

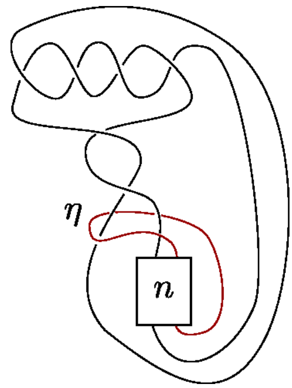


FIGURE 2. A winding number n pattern P_n in the solid torus $S^3 - \nu(\eta)$

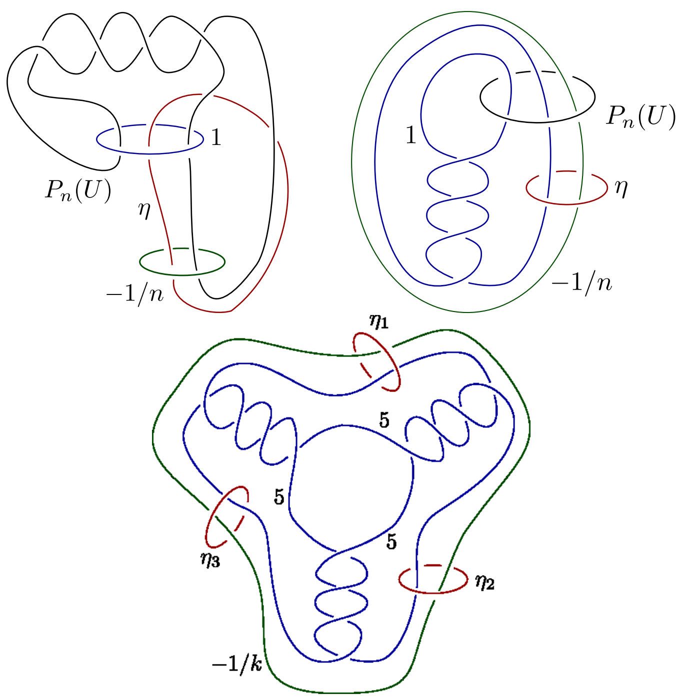


FIGURE 3. A surgery description for $P_n(U)$ (left) is isotoped to an alternate description (center) which lifts to give a surgery diagram for $\Sigma_p(P_n(U))$ for p dividing n (right, depicted for $p = 3$ and $n = 3k$).

Also for $i \neq j$ we have

$$\operatorname{lk}(\alpha_i, \alpha_j) = \begin{cases} -2 & \text{if } p > 2 \text{ and } j \equiv i \pm 1 \mod p \\ 0 & \text{if } p > 2 \text{ and } j \not\equiv i \pm 1 \mod p \\ -4 & \text{if } p = 2. \end{cases}$$

Note that the lifts $\eta_1, \ldots, \eta_p$ of the red curve η satisfy $\mathrm{lk}(\alpha_i, \eta_j) = \delta_{i,j}$. Finally, the single lift of the green curve β has framing $-1/k$, where $k = n/p$. Note that β does not link any of the α_i curves, so we can perform a Rolfsen twist along β to obtain a new surgery description without changing the framing of the α_i curves or the linking of the α_i curves with η_j curves (though while complicating the diagram significantly!) After this we see that $H_1(\Sigma_p(P_n(U)))$ is generated as a group by the meridians of $\alpha_1, \ldots, \alpha_p$, which are cyclically permuted by the covering transformation induced action. So $H_1(\Sigma_p(P_n(U)))$ is a cyclic $\mathbb{Z}[\mathbb{Z}_p]$-module. Also, observe that each η_i is homologous to the meridian of α_i and hence is a $\mathbb{Z}[\mathbb{Z}_p]$-generator for $H_1(\Sigma_p(P_n(U)))$. In particular, the collection $\eta_1, \ldots, \eta_p$ generates $H_1(\Sigma_p(P_n(U)))$ as a group.

It only remains to show that $|H_1(\Sigma_p(P_n(U)))| = (2^p-1)^2 > 1$. Observe that the group $H_1(\Sigma_p(P_n(U)))$ is presented by the linking matrix of the surgery description. For $p = 2$, this is $\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$, which is of order $9 = (2^2 - 1)^2$. For $p > 2$, this is a $p \times p$ matrix $A(p)$ with 5's on the main diagonal, -2's immediately above and below the main diagonal and in the upper right and lower left entries, and 0's everywhere else. More precisely,

$$A(p)_{i,j} = \begin{cases} 5 & \text{if } i = j \\ -2 & \text{if } i \equiv j \pm 1 \mod p \\ 0 & \text{else} \end{cases}.$$

We can check by hand that $\det(A(3)) = 49 = (2^3-1)^2$, $\det(A(4)) = 225 = (2^4-1)^2$, and $\det(A(5)) = 961 = (2^5-1)^2$. For $p \geq 1$ we define $B(p)$ to be the $p \times p$ matrix with

$$B(p)_{i,j} = \begin{cases} 5 & \text{if } i = j \\ -2 & \text{if } i = j \pm 1 \\ 0 & \text{else} \end{cases}.$$

For $p \geq 6$, it then follows from two cofactor expansions that

$$\det(A(p)) = 5b_{p-1} - 8b_{p-2} - 2^{p+1}, \text{ where } b_p := \det(B(p)). \tag{1}$$

Observe that $b_1 = 5$, $b_2 = 21$, and that for $p \geq 3$ we can perform two cofactor expansions to show that $b_p = 5b_{p-1} - 4b_{p-2}$. Some work with generating functions then shows that for $p \geq 1$ we have $b_p = \frac{1}{3}(2^{2p+2} - 1)$. Substituting this expression into Equation 1 gives that

$$\det(A(p)) = \frac{5}{3}(2^{2p} - 1) - \frac{8}{3}(2^{2p-2} - 1) - 2^{p+1} = (2^p - 1)^2. \qquad \square$$

Corollary 3.2. *The map induced by P_n on $\mathcal{C}$ is not a homomorphism for $|n| \neq 1$.*

Proof. This follows immediately from Theorem A in light of Proposition 3.1. $\square$

Note that P_1 is geometric winding number 1, and hence acts by connected sum with $P_1(U) = 6_1 \sim U$. However, P_{-1} is not geometric winding number ± 1. While P_{-1} is not topologically concordant to a core of $S^1 \times D^2$ in $(S^1 \times D^2) \times I$ and hence does not obviously act trivially, as noted in the introduction there are no known ways to show that a slice pattern of winding number ± 1 does not induce a standard map (i.e. identity or reversal) on $\mathcal{C}_t$.

Problem 3.3. Determine whether P_{-1} acts by the identity on $\mathcal{C}_t$ and, if not, determine whether it acts by a homomorphism.

Example 3.4. Let P be the winding number 2 pattern given in Figure 4, and note that $P(U) = 3_1 \# -3_1$. It is straightforward to verify, for example by building a

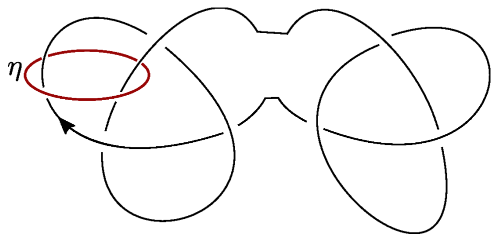


FIGURE 4. A winding number 2 pattern P with $P(U) = 3_1 \# -3_1$

surgery diagram for $\Sigma_2(P(U))$ as in Proposition 3.1, that $H = H_1(\Sigma_2(P(U))) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$, with generators x and y such that the linking form λ is given by the matrix $\frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Moreover, the curve η lifts to η_1 and η_2 in $\Sigma_2(P(U))$, where $[\eta_1] = x$ and $[\eta_2] = -x$ in H. In particular, the lifts of η to $\Sigma_2(P(U))$ certainly do not generate H, and so we cannot apply Theorem A. Let x_1, y_1, x_2, y_2, x_3, and y_3 be the natural generators for $\mathcal{H} = H \oplus H \oplus -H$, where $\Lambda = \lambda \oplus \lambda \oplus -\lambda$, the linking form on $\mathcal{H}$, is given with respect to our basis by

$$\frac{1}{3}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \oplus \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

A straightforward if tedious analysis of the order 27 subgroups of $\mathcal{H}$ gives us the following list of 48 metabolizers for Λ, where $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) \in (\pm 1)^3$.

$$M_1^\epsilon = \langle x_1 + \epsilon_1 y_1, x_2 + \epsilon_2 y_2, x_3 + \epsilon_3 y_3 \rangle,\ M_2^\epsilon = \langle x_1 + \epsilon_1 y_1, x_2 + \epsilon_2 x_3, y_2 + \epsilon_3 y_3 \rangle,$$
$$M_3^\epsilon = \langle x_1 + \epsilon_1 y_2, x_2 + \epsilon_2 y_1, x_3 + \epsilon_3 y_3 \rangle,\ M_4^\epsilon = \langle x_1 + \epsilon_1 y_2, x_2 + \epsilon_2 x_3, y_1 + \epsilon_3 y_3 \rangle,$$
$$M_5^\epsilon = \langle x_1 + \epsilon_1 x_3, x_2 + \epsilon_2 y_2, y_1 + \epsilon_3 y_3 \rangle,\ M_6^\epsilon = \langle x_1 + \epsilon_1 x_3, x_2 + \epsilon_2 y_1, y_2 + \epsilon_3 y_3 \rangle.$$

We now construct characters to $\mathbb{Z}_3$ vanishing on each M_j^ϵ satisfying the conditions of Proposition 2.2.

(1) If $M = M_1^\epsilon$ or $M = M_2^\epsilon$, let $\chi = (\chi_1, 0, 0)$, where $\chi_1(x_1) = -\epsilon_1$ and $\chi_1(y_1) = 1$. Then our collections $\{\chi_j(x_j), \chi_j(-x_j)\}$ for $j = 1, 2, 3$ are $\{-\epsilon_1, \epsilon_1\}$, $\{0, 0\}$ and $\{0, 0\}$.
(2) If $M = M_3^\epsilon$ or $M = M_4^\epsilon$, let $\chi = (\chi_1, \chi_2, 0)$ where χ_1 sends x_1 to $-\epsilon_1$ and y_1 to 0 and χ_2 sends x_2 to 0 and y_2 to 1. Then our collections $\{\chi_j(x_j), \chi_j(-x_j)\}$ for $j = 1, 2, 3$ are $\{-\epsilon_1, \epsilon_1\}$, $\{0, 0\}$ and $\{0, 0\}$.
(3) If $M = M_5^\epsilon$ or $M = M_6^\epsilon$, let $\chi = (\chi_1, \chi_2, 0)$, where χ_1 sends x_1 to 0 and y_1 to 1 and χ_2 sends x_2 to $-\epsilon_2$ and y_2 to 0. Then our collections $\{\chi_j(x_j), \chi_j(-x_j)\}$ for $j = 1, 2, 3$ are $\{0, 0\}$, $\{-\epsilon_2, \epsilon_2\}$, and $\{0, 0\}$.

Note that in this example it is important that we only need to consider metabolizers rather than arbitrary subgroups of the appropriate order, since certainly any character χ that vanishes on the order 27 subgroup $\langle x_1, x_2, x_3 \rangle$ will have $\{\chi_i(x_j), \chi_i(-x_j)\} = \{0, 0\}$ for all $j = 1, 2, 3$.

Remark 3.5. Proposition 2.2 can never be applied to a pattern P when any of the following conditions hold:

(1) The curve η is in the second derived subgroup of $S^3 \setminus P(U)$, and hence lifts to a null-homologous curve in every cyclic branched cover of $P(U)$.
(2) The knot $P(U)$ has $\Delta_{P(U)}(t) = 1$.
(3) The winding number of P is ± 1.

As discussed before Problem 3.3, the goal of obstructing a pattern from being a homomorphism in Case (3) is quite ambitious. However, in the next section we address Case (1) by giving examples of patterns for each $n \in \mathbb{N}$ which are described by curves lying in the nth derived subgroup of $S^3 \setminus P(U)$ and yet which do not induce homomorphisms on $\mathcal{C}_t$.

4. Other topological obstructions

We consider patterns R_J, which are described in Figure 5.

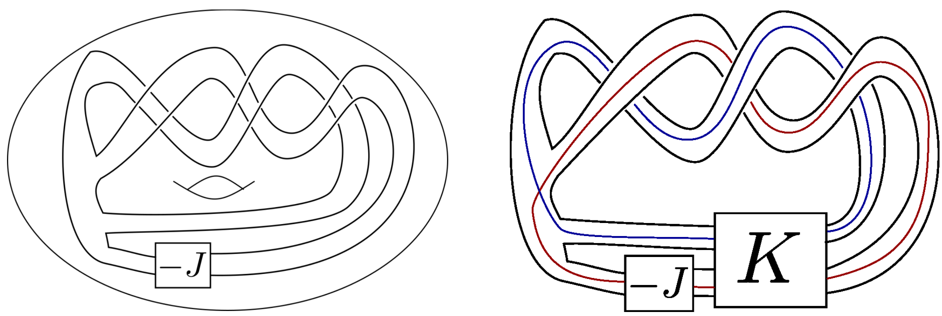


Figure 5. The pattern R_J (left) and the knot $R_J(K)$ (right), with two curves drawn on its genus 1 Seifert surface

Observe that $R_J(U)$ and $R_J(J)$ can both be seen to be smoothly slice as follows. The knot $R_J(K)$ always has a genus 1 Seifert surface with two 0-framed curves, shown in blue and red on the right of Figure 5. When $K = U$, the blue curve is an unknot and in particular is smoothly slice, so surgery of the pushed-in Seifert surface gives a smooth slice disc in B^4 for $R_J(U)$. Similarly, when $K = J$, the red curve has knot type $-J\#J$, which again is smoothly slice and so which can be surgered along to give a slice disc for $R_J(J)$. We now prove that there are many choices of knots $\{J_i\}$ such that arbitrarily many compositions of R_{J_i} maps still do not induce homomorphisms, and as a result give examples of non-homomorphism patterns which map deep into the n-solvable filtration of [COT03].

Remark 4.1. In fact, the pattern R_J never induces a homomorphism. We leave the details of this argument to the interested reader, noting that the classes of $H_1(\Sigma_2(R_J(U)))$ represented by the lifts of η are independent of J. It is not hard to verify as in Proposition 3.1 that $H_1(\Sigma_2(R_J(U)))$ is isomorphic to $H := \mathbb{Z}_3\langle x\rangle \oplus \mathbb{Z}_3\langle y\rangle$, and has linking form λ given with respect to these generators by the matrix $\frac{1}{3}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. Moreover, the infection curve η lifts to curves representing $x+y$ and $-x-y$ in $H_1(\Sigma_2(R_J(U)))$. An analysis of the metabolizers of $(H,\lambda) \oplus (H,\lambda) \oplus (H,-\lambda)$ as in Example 3.4 now shows that the conditions of Proposition 2.2 hold.

For appropriate choices of $\{J_i\}_{i\in\mathbb{Z}}$ (e.g. with increasingly large values of $\sigma_{J_i}(e^{2\pi i/3})$), it is not hard to prove that we get infinitely many winding number 0 patterns, distinct in their action on $\mathcal{C}_t$, which are obstructed from acting as homomorphisms by Proposition 2.2.

Theorem B follows immediately from Proposition 4.2.

Proposition 4.2. *For any $n \in \mathbb{N}$ and any choices of knots $K_1, \ldots, K_n$ there exists a winding number 0 pattern P such that*

(1) *$P(K)$ is smoothly slice for each $K = U, K_1, \ldots, K_n$.*
(2) *The image of P is contained in $\mathcal{F}_{n+1}$, the $(n+1)$th level of the solvable filtration.*
(3) *There exists $C > 0$ such that if $|\rho_0(K)| = \left|\int_{S^1} \sigma_\omega(K) d\omega\right| > C$ then $P(K) \notin \mathcal{F}_{n+2}$, and in particular is not topologically slice.*
(4) *P does not induce a homomorphism $\mathcal{C}_t \to \mathcal{C}_t$ (or even a homomorphism $\mathcal{C}_t \to C_t/\mathcal{F}_{n+2}$.)*

In fact, the solvable filtration is indexed by half integers, and we remark to the experts that Proposition 4.2 holds if $(n+2)$ is everywhere replaced by $(n+1.5)$. We have chosen to prove this slightly weaker statement for ease of exposition.

We will construct the patterns of Proposition 4.2 by using compositition of patterns.

Definition 4.3. Given patterns $P\colon S^1 \to S^1 \times D^2$ and $Q\colon S^1 \to S^1 \times D^2$, we define the *composite pattern* $P \circ Q$ as follows. Let $i_Q\colon S^1 \times D^2 \to S^1 \times D^2$ be an embedding of a standard tubular neighborhood of $Q(S^1)$. Then $P \circ Q$ is the pattern

$$P \circ Q\colon S^1 \xrightarrow{P} S^1 \times D^2 \xrightarrow{i_Q} S^1 \times D^2.$$

We remark that $(P \circ Q)(K)$ is always isotopic to $P(Q(K))$.

The following special case of a result of Cochran-Harvey-Leidy [CHL11] implies that any composition of R_J maps is highly nontrivial on concordance, even modulo terms of the n-solvable filtration of [COT03]. It uses the Blanchfield pairing Bl on the Alexander module of a knot, which takes values in $\mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$. We will not work with the Blanchfield pairing in detail, and therefore omit its definition: it suffices for our purposes to know that if the class of η generates the Alexander module then $\mathrm{Bl}(\eta, \eta)$ must be non-zero.

Theorem 4.4 ([CHL11])**.** *For each $i = 1, \ldots, n$, let R_i be a slice knot and η_i be an unknotted curve in the complement of R_i such that $\mathrm{lk}(R_i, \eta_i) = 0$ and $\mathrm{Bl}(\eta_i, \eta_i) \neq 0$. Let P_i be the pattern obtained by considering R_i in the solid torus $S^3 \smallsetminus \nu(\eta_i)$, and let $P = P_n \circ \cdots \circ P_1$.*

For any $k \in \mathbb{N}$, if $K \in \mathcal{F}_k$ then $P(K) \in \mathcal{F}_{n+k}$. Also, there exists $C > 0$ such that if K is a knot with $|\rho_0(K)| > C$ then $P(K) \notin \mathcal{F}_{n+1}$, and hence is not slice.

It is well-known that the pattern given by (R, η) in Figure 5 satisfies the Blanchfield pairing condition as well as evidently having $\mathrm{lk}(R, \eta) = 0$ [CHL11].

Corollary 4.5. *Let P be a composition of n patterns P_i described by unknotted curves η_i in the complement of $P_i(U)$ such that $[\eta_i] \in \pi_1(S^3 \smallsetminus \nu(P_i(U)))^{(1)}$ and $\mathrm{Bl}(\eta_i, \eta_i) \neq 0$ for all $i = 1, \ldots n$. Suppose that there is some knot K with $\rho_0(K) \neq 0$ such that $P(K)$ is slice. Then P does not induce a homomorphism on $\mathcal{C}$, even modulo $\mathcal{F}_{n+1}$.*

Proof. Let C be as in the statement of Theorem 4.4. Since $\rho_0(nK) = n\rho_0(K)$ and $|\rho_0(K)| > 0$, by taking n sufficiently large we have that $|\rho_0(nK)| > C$ and hence, by Theorem 4.4, that $P(nK) \neq 0 \in \mathcal{C}/\mathcal{F}_{n+1}$ and in particular is not topologically slice. Therefore P is not a homomorphism, since $P(K)$ is smoothly slice. □

This corollary implies that any composition of R_{J_i} patterns is not a homomorphism, as long as the 'innermost pattern' is based on a knot J_1 with $\rho_0(J_1) \neq 0$. Proposition 4.2 now follows quickly.

Proof of Proposition 4.2. Let $P_0 = R_{T_{2,3}}$ and $Q_0 = P_0$. Inductively for $i = 1, \dots, n$, let $P_i = R_{Q_{i-1}(K_i)}$ and let $Q_i = P_i \circ Q_{i-1}$. Let $P = Q_n = P_n \circ \dots \circ P_1 \circ P_0$. Observe that since $P_i(U) \sim U$ for all $i = 0, \dots, n$, we have that $P(U) \sim U$. Also, for each K_i we have that

$$\begin{aligned} P(K_i) &= (P_n \circ \dots \circ P_{i+1})(P_i(Q_{i-1}(K_i))) \\ &= (P_n \circ \dots \circ P_{i+1})(R_{Q_{i-1}(K_i)}(Q_{i-1}(K_i))) \sim (P_n \circ \dots \circ P_{i+1})(U) \sim U. \end{aligned}$$

For any knot K, the knot $P_0(K)$ is genus 1 and algebraically slice, hence by [DMOP19] is 1-solvable. Therefore, Theorem 4.4 implies that $P(K) = (P_n \circ \dots \circ P_1)(P_0(K))$ is $(n+1)$-solvable, and we have established condition (2). Theorem 4.4 also implies (3), and since $P(T_{2,3}) \sim U$ and $\rho_0(T_{2,3}) \neq 0$, Corollary 4.5 implies that (4) holds as well. □

We can also use iterated satellite constructions to give the first examples of non-standard patterns which behave like homomorphisms on arbitrary finite sets of knots.

Proof of Theorem C. Let $J_1, \dots, J_n$ be any finite list of knots. Apply Proposition 4.2 to the collection $\{T_{2,3}\} \cup \{J_i\}_{i=1}^n \cup \{J_i \# J_j\}_{i,j=1}^n$ to obtain a pattern P such that for any i and j

$$P(J_i \# J_j) \sim U \sim U \# U \sim P(J_i) \# P(J_j). \tag{2}$$

Corollary 4.5 implies that P does not induce a homomorphism of $\mathcal{C}_t$. □

We remark that the map induced by P is provably not even a homomorphism on the subgroup generated by the $\{J_i\}_{i=1}^m$, so long as one of the J_i has $\rho_0(J_i) \neq 0$. Forcing homomorphism-like behavior by sending many knots to the trivial class is not particularly exciting, but the fact that these examples are the first of their type should indicate the wide-open nature of Conjecture 1.2 and prompt Question 4.6.

Question 4.6. Does there exist a non-standard pattern P which acts by a homomorphism when restricted to some infinite subgroup of $\mathcal{C}$?

We remark that in Example 5.11 we exhibit a non-standard pattern which preserves amphichirality, thereby preserving all known examples of 2-torsion in $\mathcal{C}$. However, it does not seem likely that such a pattern P will have $P(K_1 \# K_2) \sim P(K_1) \# P(K_2)$ for K_1 and K_2 non-concordant amphichiral knots, and so the above question remains open.

We close by observing that all the obstructions discussed so far rely only on the homotopy class of η in the complement of $P(U)$. It is an interesting open question whether the map $P\colon \mathcal{C}_t \to \mathcal{C}_t$ is determined by this homotopy class (see Problem 3.5 of [AIM19]). If this were so, it would imply that our failure to give examples of patterns with $P(U)$ unknotted which do not induce homomorphisms on $\mathcal{C}_t$ is unsurprising, at least when $|w(P)| \leq 1$: when $P(U) = U$ the linking number of P with η determines the homotopy class.

On the other hand, we find it surprising that one cannot obstruct the $(p, 1)$ cable maps from inducing homomorphisms of $\mathcal{C}_t$, and state that as a worthwhile problem.

Problem 4.7. Determine whether the cable $C_{p,1}$ induces a homomorphism of $\mathcal{C}_t$ for $p > 1$.

5. Small patterns acting on $\mathcal{C}_s$

We conclude by considering patterns of small crossing number. Since the first nontrivial slice knot has 6 crossings (either the Stevedore's knot 6_1 or the square knot $T_{2,3}\# - T_{2,3}$), it is perhaps unsurprising that all of the patterns we consider have $P(U) = U$. As a consequence, none of our topological obstructions apply and so in this section we work only in the smooth category, in particular letting $\sim$ denote the equivalence relation of smooth concordance.

We rely heavily on the τ-invariant of Heegaard-Floer homology [OS03b], which vanishes on smoothly slice knots. We begin by reviewing the formula for the τ-invariant of $P(K)$ for certain prototypical patterns P of each winding number and arbitrary knots K.

Theorem 5.1 ([Hom14])**.** *Let $p > 1$. Then*

$$\tau(C_{p,1}(K)) = \begin{cases} p\tau(K), & \epsilon(K) \in \{0, 1\} \\ p\tau(K) + (p-1), & \epsilon(K) = -1. \end{cases}$$

Note that this gives an easy proof that $C_{p,1}$ does not induce a homomorphism on $\mathcal{C}_s$ as follows: Let K be any knot with $\epsilon(K) = +1$ (e.g. $K = T_{2,3}$). Then $\epsilon(-K) = -1$ and $C_{p,1}(K)\#C_{p,1}(-K)$ is not slice, since

$$\begin{aligned}\tau(C_{p,1}(K)\#C_{p,1}(-K)) = \tau(C_{p,1}(K)) + \tau(C_{p,1}(-K)) &= p\tau(K) + p\tau(-K) + p - 1 \\ &= p - 1 \neq 0.\end{aligned}$$

We also have the following similar results, which give analogous simple proofs that the Mazur and Whitehead patterns do not induce homomorphisms on $\mathcal{C}_s$.

Theorem 5.2 ([Lev16])**.** *Let* M *denote the positive Mazur pattern. Then*

$$\tau(\mathrm{M}(K)) = \begin{cases} \tau(K), & \tau(K) \leq 0 \text{ and } \epsilon(K) \in \{0, 1\} \\ \tau(K) + 1, & \text{else} \end{cases}.$$

Theorem 5.3 ([Hed07])**.** *Let* Wh *denote the positive Whitehead double pattern. Then*

$$\tau(\mathrm{Wh}(K)) = \begin{cases} 0, & \tau(K) \leq 0 \\ 1, & \tau(K) > 0 \end{cases}.$$

Somewhat surprisingly, even though these 'prototypical' patterns do not induce homomorphisms, we can still use these formulae for τ to prove Theorem D as follows.

Proof of Theorem D. Let P be a pattern of winding number $w(P)$ and suppose that P induces a homomorphism on $\mathcal{C}_s$. Let $p = |w(P)|$, and let $Q = C_{p,1}$ (if $p \geq 1$) and $Q = \mathrm{Wh}$ (if $p = 0$). We give the argument for $Q = C_{p,1}$, but an exactly analogous one works for Wh.

Since P and Q have the same winding number, by [CH18] there exists a constant $c > 0$ such that for all knots J,

$$|\tau(Q(J)\# - P(J))| \leq g_4(Q(J)\# - P(J)) \leq c.$$

Suppose now that $\epsilon(K) = -1$. It follows from the basic properties of ϵ that $\epsilon(nK) = -1$ for any $n > 0$. We therefore have the following, using in the first equality that P is a homomorphism:

$$\begin{aligned} c \geq |\tau(Q(nK)\# - P(nK))| &= |p\tau(nK) + (p-1) - \tau(nP(K))| \\ &= |n[p\tau(K) - \tau(P(K))] + p - 1|. \end{aligned}$$

Letting $n \to \infty$, we see that we must have $\tau(P(K)) = p\tau(K)$. The argument for K with $\epsilon(K) = +1$ or $\epsilon(K) = 0$ is analogous. □

We remark that it is perhaps an interesting problem to show the same result for Rasmussen's s-invariant, which shares many but not all formal properties with τ. Inspection of the proof shows that it would suffice to show that for each p there exists a constant $C(p)$ such that $|s(C_{p,1}(K)) - ps(K)| \leq C(p)$ for all knots K. The work of Van Cott [VC10] gives bounds for $s(C_{p,q}(K))$ as $q \to \infty$, which seem ill-suited to the case of interest. Of course, if one believes Conjecture 1.2, then the s-invariant analogue of Theorem D would be trivially true, independently of the behavior of s under cabling.

It will be useful for us to have a much weaker notion of preserving group structure.

Definition 5.4. An *pseudo-homomorphism* of a group G is a map $\phi\colon G \to G$ such that $\phi(e_G) = e_G$ and $\phi(g^{-1}) = \phi(g)^{-1}$ for all $g \in G$.

We can rephrase this in our context in a somewhat surprising way. For any pattern P and knot K, we have that $-P(K)$ is isotopic to $(-P)(-K)$. So P induces a pseudo-homomorphism on $\mathcal{C}$ if and only if $P(U) \sim U$ and $(-P)(K) = -(P(-K)) \sim P(K)$ for all K.

Corollary 5.5. *Let P be a winding number p pattern. Suppose that*

($p > 1$) *P can be changed to $C_{p,1}$ with any number of crossing changes $(+)$ to $(-)$ and strictly fewer than $\frac{p-1}{2}$ crossing changes $(-)$ to $(+)$.*

($p = 1$) *P can be changed to* M *with any number of crossing changes, all $(+)$ to $(-)$.*

($p = 0$) *P can be changed to* Wh *with any number of crossing changes, all $(+)$ to $(-)$.*

Then P does not induce a pseudo-homomorphism on $\mathcal{C}_s$.

Proof. Any of the above conditions implies that $\tau(P(T_{2,3})\#P(-T_{2,3})) > 0$, since if K_+ and K_- differ by changing a single crossing from $(+)$ to $(-)$ then

$$\tau(K_-) \leq \tau(K_+) \leq \tau(K_-) + 1.$$ □

We remark that Theorem D along with the crossing change inequality for τ implies that if P can be changed to $C_{p,1}$ with any number of crossing changes $(+)$ to $(-)$ and strictly fewer than $(p-1)$ crossing changes $(-)$ to $(+)$, then P is not a homomorphism. We will see in Example 5.11 that this weaker assumption does not obstruct P from inducing a pseudo-homomorphism.

The reference tables of LinkInfo [CL19] give 30 prime 2-component links which have diagrams with no more than 8 crossings, considered independently of orientation and without considering mirror images. By picking an unknotted component η of such a link, we obtain a pattern P in the solid torus $S^3 \smallsetminus \nu(\eta)$. There are 19 choices which define a slice pattern, coming from 18 different links. The link L8a1 is asymmetric, as detected by the multivariable Alexander polynomial, and hence defines two patterns which we call L8a1a and L8a1b.

Two of these patterns are standard, as depicted in Figure 6. (Note that in this

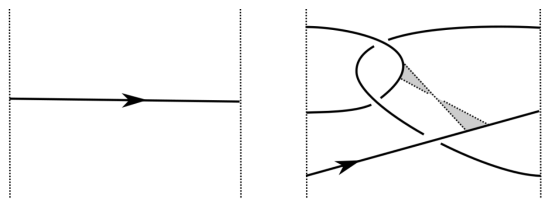


FIGURE 6. Small patterns concordant to a core: L2a1 (left) and L7a5 (right)

section, for efficiency's sake we depict patterns as living in $D^2 \times I$. An untwisted identification of $D^2 \times \{0\}$ with $D^2 \times \{1\}$ gives the pattern in the solid torus.)

Corollary 5.5 immediately implies that 12 of the remaining 17 do not induce pseudo-homomorphisms: the necessary crossing changes are illustrated in Figure 7.

This leaves us with 5 patterns to consider individually. We now give specific arguments to show that L8a1b, L8a8, and L8a10, depicted in Figure 8, do not induce pseudo-homomorphisms.

Example 5.6 (L8a1b does not induce a pseudo-homomorphism)**.** The crossing change inequality for τ generalizes to give the following result. (Note that one obtains a $(+)$ to $(-)$ crossing change by doing a $+1$ twist along a small linking number 0, geometric linking number 2 curve.)

Proposition 5.7 ([OS03a])**.** *Let K be a knot in S^3 and η be an unknot in the complement of K such that $\mathrm{lk}(K,\eta)=0$. Let K^+ be the knot obtained from K by doing a $+1$-twist along η. Then $\tau(K^+) \leq \tau(K) \leq \tau(K^+)+1$.*

Now, observe there is a +1-twist along a linking number 0 unknot that takes the pattern L8a1b to the positive Whitehead pattern, as illustrated in Figure 9. It follows that for any K, $\tau(\text{L8a1b(K)}) \geq \tau(\mathrm{Wh}(K))$, and so the arguments of the proof of Corollary 5.5 apply to show that L8a1b does not induce a pseudo-homomorphism

Example 5.8 (L8a8 does not induce a pseudo-homomorphism)**.** Since a single (+) to (-) crossing change takes $P=$ L8a8 to a core, we immediately have that $\tau(P(K)) \geq \tau(K)$ for all knots K. We will now show that this is not always equality, and therefore that for some knot J

$$\tau(P(J)\#P(-J)) = \tau(P(J)) + \tau(P(-J)) \geq \tau(P(J)) + \tau(-J) > \tau(J) + \tau(-J) = 0.$$

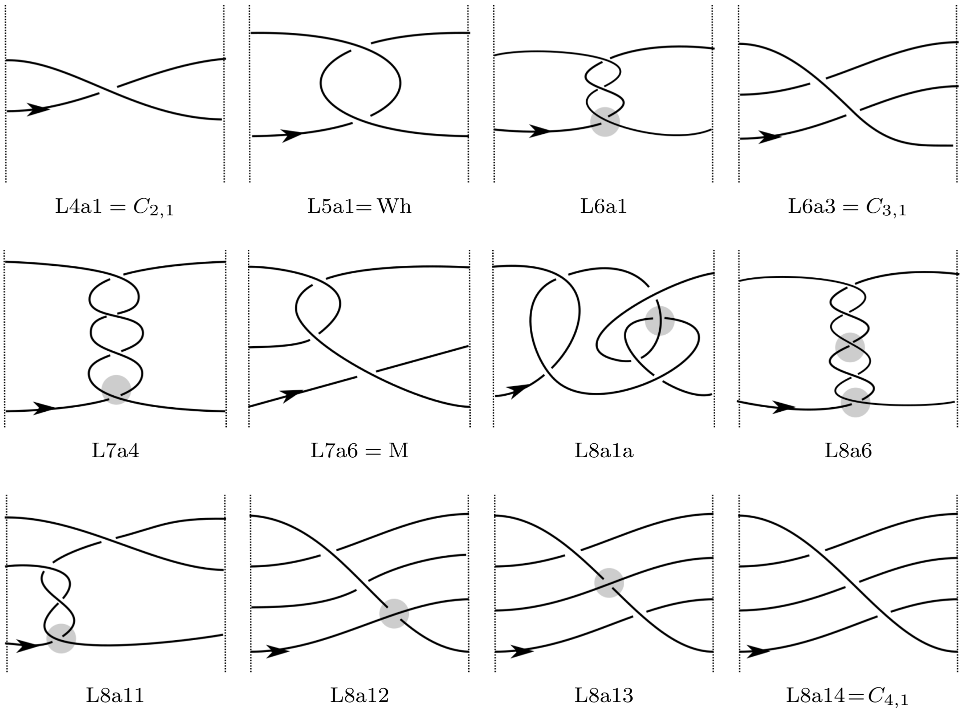


FIGURE 7. Small patterns satisfying the hypotheses of Corollary 5.5

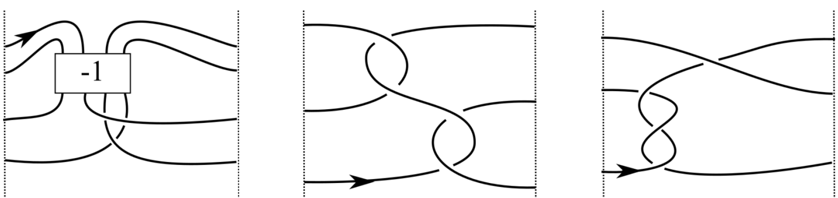


FIGURE 8. More small patterns: L8a1b (left), L8a8 (center), and L8a10 (right)

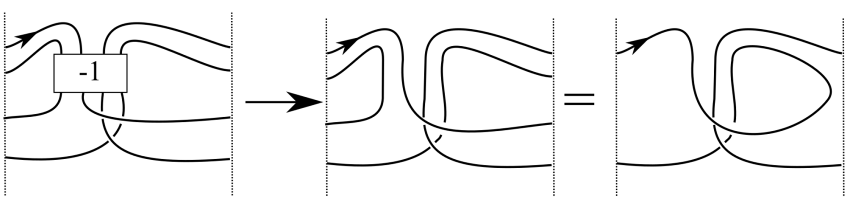


FIGURE 9. Twisting L8a1b (left) to Wh (right)

Observe that L8a8 has a Legendrian diagram (on the left of Figure 10) with Thurston-Bennequin number and rotation number equal to

$$tb(\mathcal{P}) = \text{writhe}(\mathcal{P}) - \#(\text{right cusps}) = 4 - 2 = 2.$$

$$rot(\mathcal{P}) = \frac{\#(\text{down cusps}) - \#(\text{up cusps})}{2} = \frac{2-2}{2} = 0.$$

There is also a Legendrian diagram for $J = T_{2,3}$ with

$$tb(\mathcal{J}) = \text{writhe}(\mathcal{J}) - \#(\text{right cusps}) = 3 - 3 = 0.$$
$$rot(\mathcal{J}) = \frac{\#(\text{down cusps}) - \#(\text{up cusps})}{2} = \frac{4-2}{2} = 1.$$

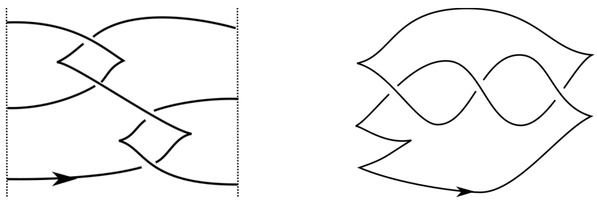


FIGURE 10. Legendrian diagrams of L8a8 (left) and a right-handed trefoil J (right)

As shown by Ng-Traynor [NT04], since $tb(\mathcal{J}) = 0$, we obtain a Legendrian diagram $\mathcal{P}(\mathcal{J})$ for $P(J)$ with

$$tb(\mathcal{P}(\mathcal{J})) = w(P)^2 tb(\mathcal{J}) + tb(\mathcal{P}) = (1)^2 \cdot 0 + 2 = 2$$
$$rot(\mathcal{P}(\mathcal{J})) = w(P) rot(\mathcal{J}) + rot(\mathcal{P}) = 1 \cdot 1 + 0 = 1.$$

We now apply the following result of Plamenevskaya.

Theorem 5.9 ([Pla04])**.** *Let $\mathcal{K}$ be a Legendrian representative of K. Then*

$$tb(\mathcal{K}) + |rot(\mathcal{K}| \leq 2\tau(K) - 1.$$

So for $J = T_{2,3}$ we have

$$\tau(P(J)) \geq (1/2)(tb(\mathcal{P}(\mathcal{J})) + rot(\mathcal{P}(\mathcal{J})) + 1) = (1/2)(2 + 1 + 1) = 2 > 1 = \tau(J).$$

Example 5.10 (L8a10 does not induce a pseudo-homomorphism)**.** Let $K = T_{2,3}$. We will use the alternate definition of pseudo-homomorphism and show that L8a10(K) and $(-$L8a10$)(K)$ are not concordant. Since a single $(-)$ to $(+)$ crossing change takes L8a10 to L6a2, we have that

$$\tau(\text{L8a10}(T_{2,3})) \leq \tau(\text{L6a2}(T_{2,3})) \leq 3\tau(T_{2,3}) - 1 = 2,$$

where the rightmost inequality comes from Equation 3 in Example 5.11.

A single $(+)$ to $(-)$ crossing change takes $-$L8a10 to the pattern R, as depicted in Figure 11. It follows that $\tau(-\text{L8a10}(K)) \geq \tau(R(K)$. We now argue as in Exam-

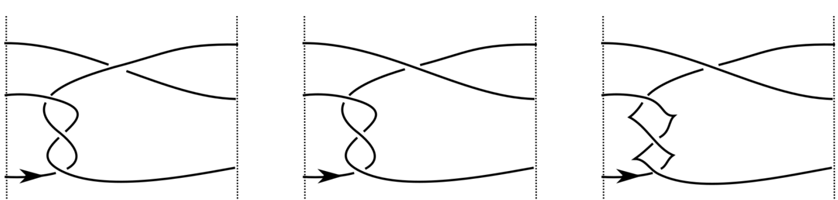


FIGURE 11. The patterns $-$L8a10 (left) and R (center), and a Legendrian realization $\mathcal{R}$ of R with $tb(\mathcal{R}) = 2$ and $rot(\mathcal{R}) = 0$ (right)

ple 5.8, using the Legendrian realization $\mathcal{R}$ of R on the right of Figure 11 to say that there is a Legendrian diagram $\mathcal{R}(\mathcal{K})$ for $R(K)$ with

$$tb(\mathcal{R}(\mathcal{K})) = 3^2 \cdot 0 + 2 = 2 \text{ and } rot(\mathcal{R}(\mathcal{K})) = 3 \cdot 1 + 0 = 3.$$

It therefore follows by Theorem 5.9 that

$$\tau(-\text{L8a10}(K)) \geq \tau(R(K)) \geq \frac{2+3+1}{2} = 3.$$

So L8a10(K) and ($-$L8a10)(K) are not concordant and L8a10 does not induce a pseudo-homomorphism.

The two remaining patterns, illustrated in Figure 12, are L6a2 and L8a9, both of which can be easily seen to induce pseudo-homomorphisms, since each is slice and amphichiral.

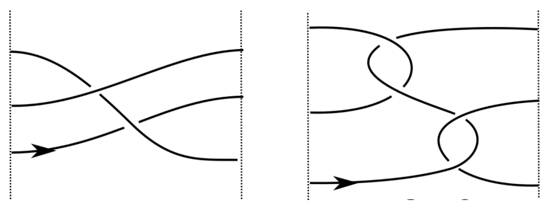


FIGURE 12. L6a2 (left) and L8a9 (right) induce pseudo-homomorphisms

Example 5.11 (L6a2 induces a pseudo-homomorphism but not a homomorphism)**.** L6a2 induces a pseudo-homomorphism since L6a2(U) is slice and the pattern L6a2 is isotopic to -L6a2. Since a single crossing change (+) to ($-$) takes L6a2 to $C_{3,-1}$, we have that we have that

(3)
$$\tau(\text{L6a2}(T_{2,3})) \leq \tau(C_{3,-1}(T_{2,3})) + 1 = (3\tau(T_{2,3}) - 2) + 1 = 3\tau(T_{2,3}) - 1 < 3\tau(T_{2,3}).$$

(This uses Theorem 5.1 and the facts that $\tau(-K) = -\tau(K)$ for all K, $C_{3,-1}(T_{2,3}) = -C_{3,1}(-T_{2,3})$, and $\epsilon(-T_{2,3}) = -1$.) So Theorem D implies that L6a2 does not induce a homomorphism.

It is not hard to generalize L6a2 to produce patterns of each odd winding number which induce pseudo-homomorphisms yet not homomorphisms of $\mathcal{C}_s$. This leads us to the following question about the existence of 'non-standard pseudo-homomorphisms'. We remark that this question also relates to whether all torsion elements of the concordance group are represented by negative amphichiral knots (Question 1.94, [Kir78].)

Question 5.12. Let P be a pattern inducing a pseudo-homomorphism on the concordance group. Must P be concordant in $(S^1 \times D^2) \times I$ to a pattern Q with the property that $Q(-K)$ is isotopic to $-Q(K)$ for all K?

By work of Hartley [Har79], the winding number of a pattern with P isotopic to $-P$ must either be 0 or odd. Since winding number is preserved by concordance, an affirmative answer to Question 5.12 would imply Conjecture 1.2 for patterns of non-zero even winding number.

Example 5.13 (The pattern induced by L8a9)**.** We are left to consider P=L8a9. Since $P = -P$ we see that this pattern induces a pseudo-homomorphism. However, $P(K)$ and K are very difficult to distinguish: in particular, since either a $(+)$ to $(-)$ or a $(-)$ to $(+)$ crossing change takes P to a core of the solid torus, we have that $\tau(P(K)) = \tau(K)$ for all knots K. One can also check that Rasmussen's s-invariant and many other smooth concordance invariants are similarly incapable of showing that K and $P(K)$ are not concordant. However, it is straightforward to verify that $P_{+1}(U)$, the knot in S^3 obtained by doing a $+1$ twist along the meridian of the solid torus that P lies within, is not even topologically slice and so P is not concordant to a core.

We are therefore left with the following questions: does L8a9 act by the identity and, if not, does it induce a non-standard homomorphism of $\mathcal{C}_s$?

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